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ON THE INTEGRAL DEGREE OF INTEGRAL RING EXTENSIONS

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Abstract Let $A \subset B$ be an integral ring extension of integral domains with fields of fractions K and L, respectively. The integral degree of $A \subset B$, denoted by $\mathrm{d}_A(B)$, is defined as the supremum of the degrees of minimal integral equations of elements of B over A. It is an invariant that lies in between $\mathrm{d}_K(L)$ and $\mu_A(B)$, the minimal number of generators of the A-module B. Our purpose is to study this invariant. We prove that it is sub-multiplicative and upper-semicontinuous in the following three cases: if $A \subset B$ is simple; if $A \subset B$ is projective and finite and $K \subset L$ is a simple algebraic field extension; or if A is integrally closed. Furthermore, d is upper-semicontinuous if A is noetherian of dimension 1 and with finite integral closure. In general, however, d is neither sub-multiplicative nor upper-semicontinuous.

Keywords: integral extension; integral degree; integral closure; field extension; Dedekind domain; Nagata ring

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1. Introduction

Let $A \subset B$ be an integral ring extension, where A and B are two commutative integral domains with fields of fractions $K = Q(A)$ and $L = Q(B)$, respectively. Then, for any element $b \in B$, there exist $n \geq 1$ and $a_i \in A$, such that

$$b^n + a_1 b^{n-1} + a_2 b^{n-2} + \cdots + a_{n-1} b + a_n = 0.$$

The minimum integer $n \geq 1$ satisfying such an equation is called the *integral degree of b over A* and is denoted by $\mathrm{id}_A(b)$. The supremum, possibly infinite, of all the integral

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degrees of elements of B over A, $\sup\{\mathrm{id}_A(b) \mid b \in B\}$, is called the *integral degree of B over A* and is denoted by $\mathrm{d}_A(B)$.

These notions are indeed very natural. They were explicitly considered in [**3**] and, previously, in a different framework, by Kurosch [**11**], Jacobson [**5**], Kaplansky [**9**] and Levitzki [**12**], and more recently by Voight [**20**].

The goal in [**3**] was to study the uniform Artin–Rees property with respect to the set of regular ideals having a principal reduction. It was proved that the integral degree, in fact, provides a uniform Artin–Rees number for such a set of ideals.

The purpose of the present paper is to investigate more deeply the invariant $\mathrm{d}_A(B)$. We first note that $\mathrm{d}_A(B)$ is between $\mathrm{d}_K(L)$, the integral degree of the corresponding algebraic field extension $K \subset L$, and $\mu_A(B)$, the minimal number of generators of the A-module B. That is,

$$\mathrm{d}_K(L) \leq \mathrm{d}_A(B) \leq \mu_A(B).$$

In a sense, $\mathrm{d}_A(B)$ can play the role of, or just substitute for, one of them. For instance, it is a central question in commutative ring theory whether the integral closure $\overline{A}$ of a domain A is a finitely generated A-module. It is well known that, even for one-dimensional noetherian local domains, $\mu_A(\overline{A})$ might be infinite (see, e.g., [**4**, § 4.9], [**15**, § 33]). However, for one-dimensional noetherian local domains $\mathrm{d}_A(\overline{A})$ is finite [**3**, Proposition 6.5]. Hence, in this situation, $\mathrm{d}_A(B)$ would be an appropriate substitute for $\mu_A(B)$. Another positive aspect of $\mathrm{d}_A(B)$, compared with $\mu_A(B)$, is good behaviour with respect to inclusion, that is, if $B_1 \subset B_2$, then $\mathrm{d}_A(B_1) \leq \mathrm{d}_A(B_2)$, while in general we cannot deduce that $\mu_A(B_1)$ is smaller than or equal to $\mu_A(B_2)$.

Similarly, $\mathrm{d}_K(L)$ is a simplification of $\mathrm{d}_A(B)$. Note that $\mathrm{d}_K(L) \leq [L : K]$, the degree of the algebraic field extension $K \subset L$. We will see that $\mathrm{d}_K(L) = [L : K]$ if and only if $K \subset L$ is a simple algebraic field extension.

Of special interest would be to completely characterise when $\mathrm{d}_A(B)$ reaches its maximal or its minimal value. We will say that $A \subset B$ has *maximal integral degree* when $\mathrm{d}_A(B) = \mu_A(B)$. Similarly, we will say that $A \subset B$ has *minimal integral degree* when $\mathrm{d}_K(L) = \mathrm{d}_A(B)$. Examples of maximal integral degree are simple integral extensions $A \subset B = A[b]$, $b \in B$ (Proposition 2.3(b)). Examples of minimal integral degree occur when $A \subset B$ is a projective finite integral ring extension with corresponding simple algebraic field extension $K \subset L$ or when A is integrally closed (cf. Theorem 5.2 and Proposition 6.1). By a projective, respectively free, finite ring extension $A \subset B$ we mean that B is a finitely generated projective, respectively free, A-module. Moreover, integral ring extensions $A \subset B$ of both at the same time minimal and maximal integral degree are precisely free finite integral ring extensions $A \subset B$ with corresponding simple algebraic field extension $K \subset L$ (see Corollary 5.3).

Considering the multiplicativity property of the degree of algebraic field extensions $K \subset L \subset M$, that is, $[M : K] = [L : K][M : L]$, and the sub-multiplicativity property of the minimal number of generators of integral ring extensions $A \subset B \subset C$, namely, $\mu_A(C) \leq \mu_A(B)\mu_B(C)$, it is natural to ask for the same property of $\mathrm{d}_A(B)$. We will say that the integral degree d is *sub-multiplicative with respect to $A \subset B$* if $\mathrm{d}_A(C) \leq \mathrm{d}_A(B)\mathrm{d}_B(C)$, for every integral ring extension $B \subset C$. We prove that d is sub-multiplicative with respect to $A \subset B$ in the following three situations: if $A \subset B$ has maximal integral degree (e.g., if

$A \subset B$ is simple); if $A \subset B$ is projective and finite and $K \subset L$ is simple; or if A is integrally closed (see Corollaries 3.4, 5.5 and 6.7). Note that in the three cases above, $A \subset B$ has either maximal integral degree, or else minimal integral degree. We do not know an instance in which d is sub-multiplicative with respect to $A \subset B$ and $\mathrm{d}_K(L) < \mathrm{d}_A(B) < \mu_A(B)$. We will prove that d is not sub-multiplicative in general. Taking advantage of an example of Dedekind, we find a non-integrally closed noetherian domain A of dimension 1, with finite integral closure B, where B is the ring of integers of a number field, and a degree-two integral extension C of B, such that $\mathrm{d}_A(C) = 6$, whereas $\mathrm{d}_A(B) = 2$ and $\mathrm{d}_B(C) = 2$. In this particular example, $\mathrm{d}_K(L) = 1$, $\mathrm{d}_A(B) = 2$ and $\mu_A(B) = 3$, so $A \subset B$ is neither of maximal nor of minimal integral degree (see Example 6.8).

Another aspect well worth considering is semicontinuity, taking into account that the minimal number of generators is an upper-semicontinuous function (see, e.g., [**10**, Chapter IV, § 2, Corollary 2.6]). Note that if $\mathfrak{p}$ is a prime ideal of A, clearly $A_\mathfrak{p} \subset B_\mathfrak{p}$ is integral. Thus one can regard the integral degree as a function $\mathrm{d} : \mathrm{Spec}(A) \to \mathbb{N}$, defined by $\mathrm{d}(\mathfrak{p}) = \mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p})$. We will say that the integral degree d is *upper-semicontinuous with respect to* $A \subset B$ if $\mathrm{d} : \mathrm{Spec}(A) \to \mathbb{N}$ is an upper-semicontinuous function, that is, if $\{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathrm{d}(\mathfrak{p}) < n\}$ is open, for all $n \geq 1$. We prove (in Proposition 7.1) that d is upper-semicontinuous with respect to $A \subset B$ in the following two situations: if $A \subset B$ is simple; or if $A \subset B$ has minimal integral degree (e.g., if $A \subset B$ is projective and finite and $K \subset L$ is simple; or if A is integrally closed). Note that in the two cases above, $A \subset B$ has either maximal integral degree, or else minimal integral degree. There is a setting in which we can prove that d is upper-semicontinuous with respect to $A \subset B$, yet $\mathrm{d}_A(B)$ might be different from $\mathrm{d}_K(L)$ and $\mu_A(B)$. This happens when A is a non-integrally closed noetherian domain of dimension 1 with finite integral closure (see Theorem 7.2). However, d is not upper-semicontinuous in general, even if A is a noetherian domain of dimension 1 (see Example 7.4).

The paper is organized as follows. In § 2 we recall some definitions and known results given in [**3**]. We also prove that $\mathrm{d}_A(B)$ is a local invariant in the following sense:

$$\mathrm{d}_A(B) = \sup\{\mathrm{d}_{A_\mathfrak{m}}(B_\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(A)\} = \sup\{\mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(A)\}.$$

Observe that the analogue for $\mu_A(B)$ is not true in general. Section 3 is mainly devoted to the sub-multiplicativity of the integral degree. Sections 4, 5 and 6 are devoted to the integral degree of, respectively, algebraic field extensions, projective finite ring extensions and integral ring extensions with base ring integrally closed. Finally, § 7 is devoted to the upper-semicontinuity of the integral degree.

Notation and conventions. All rings are assumed to be commutative and with unity. Throughout, $A \subset B$ and $B \subset C$ are integral ring extensions. Moreover, we always assume that A, B and C are integral domains, though many definitions and results can easily be extended to the non-integral domain case. The fields of fractions of A, B and C are denoted by $K = Q(A)$, $L = Q(B)$ and $M = Q(C)$, respectively. The integral closure of A in $K = Q(A)$ is denoted by $\overline{A}$ and is simply called the 'integral closure of A'. In the particular case in which A, B and C are fields, we write $A = K$, $B = L$ and $C = M$. Whenever $\{x_1, \ldots, x_r\} \subset N$ is a generating set of an A-module N, we will write $N = \langle x_1, \ldots, x_r\rangle_A$. The minimal number of generators of N as an A-module, understood as the minimum of the cardinalities of generating sets of N, is denoted by $\mu_A(N)$.

2. Preliminaries and first properties

We start by recalling and extending some definitions and results from [**3**, § 6]. Recall that $A \subset B$ is an integral ring extension of integral domains, and $K = Q(A)$ and $L = Q(B)$ are their fields of fractions.

Definition 2.1. Let $b \in B$. A *minimal degree polynomial* of b over A (which is not necessarily unique) is a monic polynomial $\mathrm{m}(T) = T^n + a_1T^{n-1} + \cdots + a_{n-1}T + a_n \in A[T]$, $n \geq 1$, with $\mathrm{m}(b) = 0$, and such that there is no other monic polynomial of lower degree in $A[T]$ and vanishing at b. The *integral degree of* b *over* A, denoted by $\mathrm{id}_A(b)$, is the degree of a minimal degree polynomial $\mathrm{m}(T)$ of b over A. In other words:

$$\mathrm{id}_A(b) = \deg \mathrm{m}(T) = \min\{n \geq 1 \mid b \text{ satisfies an integral equation over } A \text{ of degree } n\}.$$

The *integral degree of* B *over* A is defined as the value (possibly infinite)

$$\mathrm{d}_A(B) = \sup \{\mathrm{id}_A(b) \mid b \in B\}.$$

Note that $\mathrm{d}_A(B) = 1$ if and only if $A = B$.

We give a first example, which will be completed subsequently (see Corollary 5.6).

Example 2.2. Let B be an integral domain and let G be a finite group acting as automorphisms on B. Let $A = B^G = \{b \in B \mid \sigma(b) = b, \text{ for all } \sigma \in G\}$. Then $A \subset B$ is an integral ring extension and $\mathrm{d}_A(B) \leq \mathrm{o}(G)$, the order of G.

Proof. Let $G = \{\sigma_1, \dots, \sigma_n\}$. For every $b \in B$, take $p(T) = (T - \sigma_1(b)) \cdots (T - \sigma_n(b))$. Clearly $p(T) \in A[T]$ and $p(b) = 0$. Thus b is integral over A and $\mathrm{id}_A(b) \leq n = \mathrm{o}(G)$. □

The following is a first list of properties of the integral degree mainly proved in [**3**].

Proposition 2.3. *Let $A \subset B$ be an integral ring extension. The following properties hold.*

(a) $\mathrm{d}_A(B) \leq \mu_A(B)$.

(b) *If $A \subset B = A[b]$ is simple, then* $\mathrm{id}_A(b) = \mathrm{d}_A(B) = \mu_A(B)$.

(c) *If S is a multiplicatively closed subset of A, then $S^{-1}A \subset S^{-1}B$ is an integral ring extension and* $\mathrm{d}_{S^{-1}A}(S^{-1}B) \leq \mathrm{d}_A(B)$.

(d) *If $S = A \setminus \{0\}$, then $S^{-1}B = L$.*

(e) $\mathrm{d}_K(L) \leq \mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p}) \leq \mathrm{d}_A(B)$, *for every* $\mathfrak{p} \in \mathrm{Spec}(A)$.

(f) $\mathrm{d}_K(L) \leq [L : K]$.

Proof. (a), (b) and (c) can be found in [**3**, Corollary 6.3, Corollary 6.2 and Proposition 6.8]. For (d), since $K = S^{-1}A \subset S^{-1}B$ is an integral ring extension, then $S^{-1}B$ is a zero-dimensional domain, hence a field, lying inside $L = Q(B)$. Thus $S^{-1}B = L$. Let us prove (e). Take $\mathfrak{p} \in \operatorname{Spec}(A)$, so $A \setminus \mathfrak{p} \subseteq S$. Since $A \subset B$ is an integral ring extension, then $A_{\mathfrak{p}} \subset B_{\mathfrak{p}}$ and

$$K = S^{-1}A = (A_{\mathfrak{p}} \setminus \{0\})^{-1}A_{\mathfrak{p}} \subset (A_{\mathfrak{p}} \setminus \{0\})^{-1}B_{\mathfrak{p}} = S^{-1}B = L$$

are integral ring extensions. Applying (c) twice, we get (e). Finally, applying (d) and (a), one has $\mathrm{d}_K(L) = \mathrm{d}_{S^{-1}A}(S^{-1}B)) \leq \mu_{S^{-1}A}(S^{-1}B) = \mu_K(L) = [L : K]$, which proves (f). □

Notation 2.4. The following picture can help in reading the paper

$$\begin{array}{ccc} \mathrm{d}_K(L) & \leq & \mathrm{d}_A(B) \\ \mathrel{\rotatebox{90}{$\geq$}} & & \mathrel{\rotatebox{90}{$\geq$}} \\ [L : K] & \leq & \mu_A(B). \end{array}$$

We say that $A \subset B$ has *minimal integral degree* when $\mathrm{d}_K(L) = \mathrm{d}_A(B)$. Similarly, we say that $A \subset B$ has *maximal integral degree* when $\mathrm{d}_A(B) = \mu_A(B)$.

Remark 2.5. By Proposition 2.3(b), $A \subset B$ simple implies $A \subset B$ has maximal integral degree. We will see that the converse is true for finite field extensions (see Proposition 4.2). However, in general, $A \subset B$ of maximal integral degree does not imply $A \subset B$ simple. Take for instance $A = \mathbb{Z}$ and B the ring of integers of an algebraic number field L, that is, B is the integral closure of $A = \mathbb{Z}$ in L, a finite field extension of the field of rational numbers $K = \mathbb{Q}$. Then $\mathrm{d}_A(B) = \mu_A(B)$ (see Corollary 5.7). Nevertheless, not every ring of integers B is a simple extension of $A = \mathbb{Z}$. We will take advantage of this fact in Example 6.8.

Clearly, there are integral ring extensions of non-maximal integral degree. This can already happen with affine domains, as shown in the next example.

Example 2.6. Let k be a field and t a variable over k. Let $A = k[t^3, t^8, t^{10}]$ and $B = k[t^3, t^4, t^5]$. Then $A \subset B$ is a finite ring extension with $\mathrm{d}_A(B) = 2$ and $\mu_A(B) = 3$.

Proof. Since $k[t^3] \subset A$, then $B = \langle 1, t^4, t^5 \rangle_A$ and $A \subset B$ is a finite ring extension with $\mu_A(B) \leq 3$. If $x = a + bt^4 + ct^5 \in B$, with $a, b, c \in A$, then $x^2 - 2ax \in A$. Therefore $\mathrm{d}_A(B) = 2$. Let us see that $\mu_A(B) = 3$. Suppose that there exist $f, g \in B$ such that $B = \langle f, g \rangle_A$, that is, $1, t^4, t^5 \in \langle f, g \rangle_A$. Write $f = a_0 + t^3 f_1$ and $g = b_0 + t^3 g_1$, with $a_0, b_0 \in k$ and $f_1, g_1 \in k[t]$. Since $1 \in \langle f, g \rangle_A$, one can suppose that $a_0 = 1$ and $b_0 = 0$. Thus $f = 1 + a_3t^3 + a_4t^4 + a_5t^5 + \cdots$ and $g = b_3t^3 + b_4t^4 + b_5t^5 + \cdots$. In particular, every element of $\langle f, g \rangle_A$ is of the form:

$$\begin{aligned} &(\lambda_0 + \lambda_3t^3 + \lambda_6t^6 + \lambda_8t^8 + \cdots)(1 + a_3t^3 + a_4t^4 + a_5t^5 + \cdots) \\ &+ (\mu_0 + \mu_3t^3 + \mu_6t^6 + \mu_8t^8 + \cdots)(b_3t^3 + b_4t^4 + b_5t^5 + \cdots) \\ &\quad = (\lambda_0) + (\lambda_3 + \lambda_0a_3 + \mu_0b_3)t^3 + (\lambda_0a_4 + \mu_0b_4)t^4 + (\lambda_0a_5 + \mu_0b_5)t^5 + \cdots . \end{aligned}$$

From $t^4 \in \langle f, g \rangle_A$, one deduces that ($\lambda_0 = 0$ and) $b_4 \neq 0$. Hence, one can suppose that $a_4 = 0$. From $1 \in \langle f, g \rangle_A$, it follows that ($\lambda_0 = 1$, $\mu_0 = 0$ and) $a_5 = 0$. From $t^4 \in \langle f, g \rangle_A$

again, now it follows $b_5 = 0$. But from $t^5 \in \langle f, g\rangle_A$, one must have $b_5 \neq 0$, a contradiction. Hence $\mu_A(B) = 3$. □

Remark 2.7. In the example above $K = L$ and so $A \subset B$ does not have minimal integral degree. We will prove that if $A \subset B$ is projective and finite with $K \subset L$ simple, or if A is integrally closed, then $A \subset B$ has minimal integral degree (see Theorem 5.2 and Proposition 6.1).

As for the finiteness of the integral degree, we recall the following.

Remark 2.8. There exist one-dimensional noetherian local domains A with integral closure $\overline{A}$ such that $\mathrm{d}_A(\overline{A})$ is finite while $\mu_A(\overline{A})$ is infinite (see [**3**, Proposition 6.5]). There exist one-dimensional noetherian domains A such that $\mathrm{d}_A(\overline{A})$ is infinite (see [**3**, Example 6.6]).

Next we prove that the integral degree coincides with the supremum of the integral degrees of the localizations. (The analogue for $\mu_A(B)$ is not true in general.)

Proposition 2.9. *Let $A \subset B$ be an integral ring extension. For any $b \in B$, there exists a maximal ideal $\mathfrak{m} \in \mathrm{Max}(A)$ such that $\mathrm{id}_A(b) = \mathrm{id}_{A_\mathfrak{m}}(b/1)$. In particular,*

$$\mathrm{d}_A(B) = \sup\{\mathrm{d}_{A_\mathfrak{m}}(B_\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(A)\} = \sup\{\mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(A)\}.$$

Furthermore, if $\mathrm{d}_A(B)$ is finite, then there exists $\mathfrak{m} \in \mathrm{Max}(A)$ such that $\mathrm{d}_A(B) = \mathrm{d}_{A_\mathfrak{m}}(B_\mathfrak{m})$.

Proof. If $\mathrm{id}_A(b) = 1$, then $b \in A$. Thus, for any $\mathfrak{m} \in \mathrm{Max}(A)$, $b/1 \in A_\mathfrak{m}$, so $\mathrm{id}_{A_\mathfrak{m}}(b/1) = 1$ and $\mathrm{id}_A(b) = \mathrm{id}_{A_\mathfrak{m}}(b/1)$. Suppose that $\mathrm{id}_A(b) = n \geq 2$. Then

$$A[b]/\langle 1, b, \ldots, b^{n-2}\rangle \neq 0 \quad \text{and} \quad A[b]/\langle 1, b, \ldots, b^{n-1}\rangle = 0.$$

Clearly, for every $\mathfrak{p} \in \mathrm{Spec}(A)$, and for every $m \geq 1$,

$$(A[b]/\langle 1, b, \ldots, b^m\rangle)_\mathfrak{p} = A_\mathfrak{p}[b/1]/\langle 1, b/1, \ldots, b^m/1\rangle.$$

In particular, $A_\mathfrak{p}[b/1]/\langle 1, b/1, \ldots, b^{n-1}/1\rangle = 0$, for every $\mathfrak{p} \in \mathrm{Spec}(A)$. Since $A[b]/\langle 1, b, \ldots, b^{n-2}\rangle \neq 0$, then there exists a maximal ideal $\mathfrak{m} \in \mathrm{Max}(A)$ with

$$A_\mathfrak{m}[b/1]/\langle 1, b/1, \ldots, b^{n-2}/1\rangle \neq 0 \quad \text{and} \quad A_\mathfrak{m}[b/1]/\langle 1, b/1, \ldots, b^{n-1}/1\rangle = 0.$$

Therefore, $\mathrm{id}_{A_\mathfrak{m}}(b/1) = n$ and $\mathrm{id}_A(b) = \mathrm{id}_{A_\mathfrak{m}}(b/1)$. In particular,

$$\mathrm{d}_A(B) \leq \sup\{\mathrm{d}_{A_\mathfrak{m}}(B_\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(A)\} \leq \sup\{\mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(A)\}.$$

On the other hand, by Proposition 2.3, $\sup\{\mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(A)\} \leq \mathrm{d}_A(B)$. Finally, if $\mathrm{d}_A(B)$ is finite, then there exists $b \in B$ such that $\mathrm{id}_A(b) = \mathrm{d}_A(B)$. We have just shown above that there exists a maximal ideal $\mathfrak{m} \in \mathrm{Max}(A)$ with $\mathrm{id}_A(b) = \mathrm{id}_{A_\mathfrak{m}}(b/1)$. Therefore

$$\mathrm{d}_A(B) = \mathrm{id}_A(b) = \mathrm{id}_{A_\mathfrak{m}}(b/1) \leq \mathrm{d}_{A_\mathfrak{m}}(B_\mathfrak{m}) \leq \mathrm{d}_A(B)$$

and the equality holds. □

Remark 2.10. Suppose that $A \subset B$ is finite. Since A is a domain, by generic flatness, there exists $f \in A \setminus \{0\}$ such that $A_f \subset B_f$ is a finite free extension (see, e.g., [**14**, Theorem 22.A]). In particular, for every $\mathfrak{p} \in D(f) = \mathrm{Spec}(A) \setminus V(f)$, $A_\mathfrak{p} \subset B_\mathfrak{p}$ is a finite free ring extension. So $\mathrm{d}_A(B) = \max\{\mathrm{d}_1, \mathrm{d}_2\}$, where $\mathrm{d}_1 = \sup\{\mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p}) \mid \mathfrak{p} \in V(f)\}$ and $\mathrm{d}_2 = \sup\{\mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p}) \mid A_\mathfrak{p} \subset B_\mathfrak{p} \text{ is free}\}$. Therefore, if one is able to control the integral degree for finite free ring extensions, the calculation of $\mathrm{d}_A(B)$ is reduced to find $\mathrm{d}_1 = \sup\{\mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p}) \mid \mathfrak{p} \in V(f)\}$, where $V(f)$ is a proper closed set of $\mathrm{Spec}(A)$. We will come back to this question in Theorem 5.3.

3. Sub-multiplicativity

In this section we study the sub-multiplicativity of the integral degree with respect to $A \subset B$, that is, whether $\mathrm{d}_A(C) \le \mathrm{d}_A(B)\mathrm{d}_B(C)$ holds for every integral ring extension $B \subset C$. Observe that, in this situation, $A \subset C$ is an integral ring extension too and, by definition, $\mathrm{d}_B(C) \le \mathrm{d}_A(C)$. We start with a useful criterion to determine possible bounds $\nu \in \mathbb{N}$ in the inequality $\mathrm{d}_A(C) \le \nu\, \mathrm{d}_B(C)$.

Lemma 3.1. *Let $A \subset B$ and $B \subset C$ be two integral ring extensions. Set $\nu \in \mathbb{N}$. The following conditions are equivalent:*

(i) $\mathrm{d}_A(D) \le \nu\, \mathrm{d}_B(D)$, *for every ring D such that $B \subseteq D \subseteq C$;*

(ii) $\mathrm{d}_A(D) \le \nu\, \mathrm{d}_B(D)$, *for every ring D such that $D = B[\alpha]$ for some $\alpha \in C$;*

(iii) $\mathrm{id}_A(\alpha) \le \nu\, \mathrm{id}_B(\alpha)$, *for every element $\alpha \in C$.*

In particular, if (iii) holds, then $\mathrm{d}_A(C) \le \nu\, \mathrm{d}_B(C)$.

Proof. Clearly, (i) ⇒ (ii). Let $\alpha \in C$; in particular, α is integral over A. Since $A[\alpha] \subset B[\alpha]$, then $\mathrm{id}_A(\alpha) = \mathrm{d}_A(A[\alpha]) \le \mathrm{d}_A(B[\alpha])$. By hypothesis (ii), $\mathrm{d}_A(B[\alpha]) \le \nu\, \mathrm{d}_B(B[\alpha]) = \nu\, \mathrm{id}_B(\alpha)$. Therefore, $\mathrm{id}_A(\alpha) \le \nu\, \mathrm{id}_B(\alpha)$, which proves (ii) ⇒ (iii). To see (iii) ⇒ (i), take D with $B \subseteq D \subseteq C$ and $\alpha \in D$, which will be integral over B and, hence, integral over A. By hypothesis (iii), $\mathrm{id}_A(\alpha) \le \nu\, \mathrm{id}_B(\alpha) \le \nu\, \mathrm{d}_B(D)$. Taking supremum over all $\alpha \in D$, $\mathrm{d}_A(D) \le \nu\, \mathrm{d}_B(D)$.

Finally, if (iii) holds, then (i) holds for $D = C$, so $\mathrm{d}_A(C) \le \nu\, \mathrm{d}_B(C)$. □

The next result shows that we can take $\nu = \mu_A(B)$ as a particular $\nu \in \mathbb{N}$, understanding that if $A \subset B$ is not finite, then $\mu_A(B) = \infty$ and the inequality is trivial.

Theorem 3.2. *Let $A \subset B$ and $B \subset C$ be two integral ring extensions. Then, for every $\alpha \in C$,*

$$\mathrm{id}_A(\alpha) \le \mu_A(B)\mathrm{id}_B(\alpha).$$

In particular,

$$\mathrm{d}_A(C) \le \mu_A(B)\mathrm{d}_B(C).$$

Proof. Let $\alpha \in C$, which is integral over B and A. Then

$$\mathrm{id}_A(\alpha) = \mathrm{d}_A(A[\alpha]) \leq \mathrm{d}_A(B[\alpha]) \leq \mu_A(B[\alpha]) \leq \mu_A(B)\mu_B(B[\alpha]) = \mu_A(B)\mathrm{id}_B(\alpha).$$

To finish, apply Lemma 3.1. □

Remark 3.3. A proof of Theorem 3.2 using the standard 'determinantal trick' would be as follows. Suppose $B = \langle b_1, \ldots, b_n \rangle_A$, with $\mu_A(B) = n$, and consider $\alpha \in C$ with $\mathrm{id}_B(\alpha) = m$. Let X be the following $nm \times 1$ vector, whose entries form an A-module generating set of $B[\alpha]$,

$$X^\top = (1, \alpha, \ldots, \alpha^{m-1}, b_1, b_1\alpha, \ldots, b_1\alpha^{m-1}, \ldots, b_n, b_n\alpha, \ldots, b_n\alpha^{m-1}).$$

Then there exists an nm square matrix P with coefficients in A, such that $\alpha X = PX$. Therefore, $(\alpha\mathrm{I} - P)X = 0$. Multiplying by the adjugate matrix (that is, the transpose of the cofactor matrix) leads to $Q_P(\alpha) = \det(\alpha\mathrm{I} - P) = 0$, where $Q_P(T)$ is the characteristic polynomial of P (recall that C is a domain). Hence $\mathrm{id}_A(\alpha) \leq \deg Q_P(T) = nm = \mu_A(B)\mathrm{id}_B(\alpha)$.

As an immediate consequence of Theorem 3.2, we obtain the sub-multiplicativity of the integral degree with respect to integral ring extensions of maximal integral degree.

Corollary 3.4. *Let $A \subset B$ and $B \subset C$ be two integral ring extensions. If $\mathrm{d}_A(B) = \mu_A(B)$, then*

$$\mathrm{d}_A(C) \leq \mathrm{d}_A(B)\mathrm{d}_B(C).$$

To finish this section we recover part of a result shown in [**3**], but now with a slightly different proof.

Corollary 3.5 (see [3, Proposition 6.7]). *Let $A \subset B$ and $B \subset C$ be two integral ring extensions. Then, for every $\alpha \in C$,*

$$\mathrm{id}_A(\alpha) \leq \mathrm{d}_A(B)^{\mathrm{d}_B(C)}\mathrm{id}_B(\alpha).$$

In particular,

$$\mathrm{d}_A(C) \leq \mathrm{d}_A(B)^{\mathrm{d}_B(C)}\mathrm{d}_B(C).$$

Furthermore, if $A \subset B$ and $B \subset C$ have finite integral degrees, then $A \subset C$ has finite integral degree.

Proof. Let $\alpha \in C$; in particular, α is integral over B and over A. Let $\mathrm{m}(T)$ be a minimal degree polynomial of α over B, $\mathrm{m}(T) = T^n + b_1T^{n-1} + \cdots + b_n \in B[T]$, so that $n = \mathrm{id}_B(\alpha) \leq \mathrm{d}_B(C)$. Set $E = A[b_1, \ldots, b_n]$, where $A \subseteq E \subseteq B$. Therefore,

$$\mathrm{id}_A(\alpha) = \mathrm{d}_A(A[\alpha]) \leq \mathrm{d}_A(E[\alpha]) \leq \mu_A(E[\alpha]) \leq \mu_A(E)\mu_E(E[\alpha]),$$

where clearly $\mu_A(E) \leq \prod_{i=1}^n \mathrm{id}_A(b_i) \leq \mathrm{d}_A(B)^{\mathrm{d}_B(C)}$, and $\mu_E(E[\alpha]) = \mathrm{id}_E(\alpha) = \mathrm{id}_B(\alpha)$. To finish, apply Lemma 3.1. □

4. Integral degree of algebraic field extensions

In this section, we suppose that $A = K$, $B = L$ and $C = M$ are fields. For ease of reading, we begin by recalling some definitions and basic facts (see, e.g., [**2**, Chapter V] and [**6**]).

Reminder 4.1. Let $K \subset L$ be a finite field extension.

- A polynomial is separable if it has no multiple roots (in any field extension). The extension $K \subset L$ is separable if every element of L is the root of a separable polynomial of $K[T]$. A field K is perfect if either has characteristic zero or else, when it has characteristic $p > 0$, every element is a pth power in K. If K is perfect, then $K \subset L$ is separable.

- The primitive element theorem states that a finite separable field extension is simple. Even more, there exists an 'extended' version which affirms that a simple algebraic field extension of a finite separable field extension is again simple (cf. [**6**, III, Chapter I, § 11, Theorem 14]).

- Let K_s be the separable closure of K in L, that is, the set of all elements of L which are separable over K. Then K_s is a field and $K \subset K_s$ is a separable extension. Its degree $[K_s : K]$ is called the separable degree and is denoted by $[L : K]_s := [K_s : K]$.

For the rest of the reminder, suppose that K has characteristic $p > 0$ and let K_s be the separable closure of K in L.

- Then $K_s \subset L$ is a purely inseparable field extension, that is, for every element $\alpha \in L$, there exists an integer $m \geq 1$ such that $\alpha^{p^m} \in K_s$. The least such integer m is called the height of α over K_s. Let $\mathrm{ht}_{K_s}(\alpha)$ stand for the height of α over K_s. Set $h = \sup\{\mathrm{ht}_{K_s}(\alpha) \mid \alpha \in L\}$ and call h the *height of the purely inseparable extension* $K_s \subset L$.

- Given $\alpha \in L$ with $\mathrm{ht}_{K_s}(\alpha) = m$, setting $a = \alpha^{p^m}$, one proves that $T^{p^m} - a$ is irreducible in $K_s[T]$ (see, e.g., [**2**, Chapter V, § 5]). Thus $[K_s(\alpha) : K_s] = p^m$. Since $K_s \subset L$ is a finite extension, then $L = K_s(\alpha_1, \ldots, \alpha_r)$, where each α_i is purely inseparable over $K_s(\alpha_1, \ldots, \alpha_{i-1})$, $i = 2, \ldots, r$. Hence $[L : K_s] = p^e$, for some $e \geq 1$. Call e the *exponent of the purely inseparable extension* $K_s \subset L$. Note that, since $[K_s(\alpha) : K_s]$ (which is p^m) divides $[L : K_s] = [L : K_s(\alpha)][K_s(\alpha) : K_s]$ (which is p^e), then $m \leq e$ and $h \leq e$.

Our first result characterizes simple finite field extensions as finite field extensions of maximal integral degree.

Proposition 4.2. *Let $K \subset L$ be a finite field extension. Then $K \subset L$ is simple if and only if $\mathrm{d}_K(L) = [L : K]$.*

Proof. Since $K \subset L$ is an algebraic extension, $K(\alpha) = K[\alpha]$, for any $\alpha \in L$. By Proposition 2.3(b), $\mathrm{id}_K(\alpha) = \mathrm{d}_K(K(\alpha)) = [K(\alpha) : K]$. Therefore,

$$[L : K] = [L : K(\alpha)][K(\alpha) : K] = [L : K(\alpha)]\mathrm{d}_K(K(\alpha)) = [L : K(\alpha)]\mathrm{id}_K(\alpha). \quad (1)$$

If $K \subset L$ is simple, then $L = K(\alpha)$, for some $\alpha \in L$. Using (1), it follows that

$$[L : K] = [L : K(\alpha)]\mathrm{d}_K(K(\alpha)) = \mathrm{d}_K(L).$$

Conversely, if $\mathrm{d}_K(L) = [L : K] < \infty$, by definition, there exists $\alpha \in L$ with $\mathrm{id}_K(\alpha) = [L : K]$. By (1) again, it follows that $[L : K(\alpha)] = 1$ and $K \subset L$ is simple. □

Using the primitive element theorem we obtain the following result.

Corollary 4.3. *Let $K \subset L$ be a finite separable field extension. Then $\mathrm{d}_K(L) = [L : K]$.*

The 'extended' version of the primitive element theorem will be very useful in proving the next result.

Proposition 4.4. *Let $K \subset L$ be a finite field extension. Suppose that K has characteristic $p > 0$ and let K_s be the separable closure of K in L. Set h and e to be the height and exponent, respectively, of the finite purely inseparable field extension $K_s \subset L$. Then the following hold.*

(a) $\mathrm{d}_K(L) = \mathrm{d}_K(K_s)\mathrm{d}_{K_s}(L)$.

(b) *For every $\alpha \in L$, $\mathrm{d}_{K_s}(\alpha) = p^m$, where m is the height of α over K_s.*

(c) $\mathrm{d}_{K_s}(L) = p^h$ *and* $[L : K_s] = p^e$.

(d) $[L : K]_s$ *divides* $\mathrm{d}_K(L)$ *and* $\mathrm{d}_K(L)$ *divides* $[L : K]$. *Concretely,*

$$\mathrm{d}_K(L) = [L : K]_s p^h \text{ and } [L : K] = p^{e-h}\mathrm{d}_K(L).$$

Proof. By Corollary 4.3, $\mathrm{d}_K(K_s) = [K_s : K]$. By Corollary 3.4, $\mathrm{d}_K(L) \leq \mathrm{d}_K(K_s)\,\mathrm{d}_{K_s}(L)$. To see the other inequality, take $\alpha \in L$ with $\mathrm{id}_{K_s}(\alpha) = \mathrm{d}_{K_s}(L)$. By Proposition 2.3(b),

$$\mathrm{id}_{K_s}(\alpha) = \mathrm{d}_{K_s}(K_s[\alpha]) = \mu_{K_s}(K_s[\alpha]).$$

By the extended primitive element theorem, $K \subset K_s[\alpha]$ is a simple algebraic field extension (cf. [**6**, III, Chapter I, § 11, Theorem 14]). Hence, by Proposition 2.3(b), $\mathrm{d}_K(K_s[\alpha]) = [K_s[\alpha] : K]$. Since $K \subset K_s$ is a finite separable extension, by Corollary 4.3, $\mathrm{d}_K(K_s) = [K_s : K]$. Writing all together:

$$\begin{aligned}\mathrm{d}_K(K_s)\mathrm{d}_{K_s}(L) &= [K_s : K]\mathrm{id}_{K_s}(\alpha) = [K_s : K]\mu_{K_s}(K_s[\alpha]) = [K_s[\alpha] : K]\\ &= \mathrm{d}_K(K_s[\alpha]) \leq \mathrm{d}_K(L).\end{aligned}$$

This proves (a). Let $\alpha \in L$ with $\mathrm{ht}_{K_s}(\alpha) = m$. Set $a = \alpha^{p^m}$. Then $T^{p^m} - a \in K_s[T]$ is irreducible in $K_s[T]$ and hence it is the minimal polynomial of α over K_s. It follows that

$\mathrm{id}_{K_s}(\alpha) = p^m$. This proves (b). Therefore,

$$\mathrm{d}_{K_s}(L) = \sup\{\mathrm{id}_{K_s}(\alpha) \mid \alpha \in L\} = \sup\{p^{\mathrm{ht}_{K_s}(\alpha)} \mid \alpha \in L\} = p^{\sup\{\mathrm{ht}_{K_s}(\alpha) \mid \alpha \in L\}} = p^h,$$

which proves (c). Finally, (d) follows from (a), (c) and Corollary 4.3 applied repeatedly. Indeed,

$$\mathrm{d}_K(L) = \mathrm{d}_K(K_s)\mathrm{d}_{K_s}(L) = [K_s : K]p^h = [L : K]_s p^h$$

and

$$[L : K] = [L : K_s][K_s : K] = p^e \mathrm{d}_K(K_s) = p^{e-h}\mathrm{d}_{K_s}(L)\mathrm{d}_K(K_s) = p^{e-h}\mathrm{d}_K(L). \qquad \square$$

Here there is an example of a finite field extension of non-maximal integral degree.

Example 4.5. Let $p > 1$ be a prime and $K = \mathbb{F}_p(u_1^p, u_2^p)$, where u_1, u_2 are algebraically independent over $\mathbb{F}_p$. Set $L = K[u_1, u_2]$. Then $K \subset L$ is a finite purely inseparable field extension with $\mathrm{d}_K(L) = p$. However $[L : K] = p^2$.

Proof. Any $\beta \in L$ is of the form $\beta = \sum_{0 \le i,j \le p-1} a_{i,j} u_1^i u_2^j$, with $a_{i,j} \in K$. So

$$\beta^p = \sum_{0 \le i,j \le p-1} a_{i,j}^p u_1^{ip} u_2^{jp} = \sum_{0 \le i,j \le p-1} a_{i,j}^p (u_1^p)^i (u_2^p)^j,$$

which is an element of K. Therefore $\beta^p \in K$ and $\mathrm{id}_K(\beta) \le p$. Since $\mathrm{id}_K(u_1) = p$, it follows that $\mathrm{d}_K(L) = p$. Since $K \subsetneq K(u_1) \subsetneq L$ are finite field extensions, each one of degree p, by the multiplicative formula for algebraic field extensions, $[L : K] = [L : K(u_1)][K(u_1) : K] = p^2$. $\square$

Similarly, we obtain an example of an infinite field extension with finite integral degree (see also Remark 2.8).

Example 4.6. Let $p > 1$ be a prime and $K = \mathbb{F}_p(u_1^p, u_2^p, \ldots)$, where $u_1, u_2, \ldots$ are algebraically independent over $\mathbb{F}_p$. Set $L = K[u_1, u_2, \ldots]$. Then $\mathrm{d}_K(L) = p$ but $[L : K] = \infty$.

Now we prove the sub-multiplicativity of the integral degree with respect to an algebraic field extension $K \subset L$.

Theorem 4.7. *Let $K \subset L$ and $L \subset M$ be two algebraic field extensions. Then, for every $\alpha \in M$,*

$$\mathrm{id}_K(\alpha) \le \mathrm{d}_K(L)\mathrm{id}_L(\alpha).$$

In particular,

$$\mathrm{d}_K(M) \le \mathrm{d}_K(L)\mathrm{d}_L(M).$$

Proof. We can assume that $\mathrm{d}_K(L)$ and $\mathrm{d}_L(M)$ are finite.

Let $\alpha \in M$ and let $\mathrm{m}(T) = T^n + b_1 T^{n-1} + \cdots + b_n \in L[T]$ be a minimal degree polynomial of α over L. Let h be the height of the purely inseparable field extension $K_s \subset L$, where K_s is the separable closure of K in L. Let $p = \mathrm{char}(K)$. If K has characteristic 0, we understand that $p^h = 1$. Then $0 = \mathrm{m}(\alpha)^{p^h} = \alpha^{np^h} + b_1^{p^h}\alpha^{(n-1)p^h} + \cdots + b_n^{p^h}$. It follows that α is a root of a monic polynomial in $K_s[T]$ of degree $p^h\mathrm{id}_L(\alpha)$. So we have

$$\begin{aligned}\mathrm{id}_K(\alpha) = \mathrm{d}_K(K[\alpha]) &\leq \mathrm{d}_K(K_s[\alpha]) \leq \mu_K(K_s[\alpha]) \leq \mu_K(K_s)\cdot\mu_{K_s}(K_s[\alpha])\\ &= \mathrm{d}_K(K_s)\cdot\mathrm{id}_{K_s}(\alpha) \leq \mathrm{d}_K(K_s)p^h\mathrm{id}_L(\alpha) = \mathrm{d}_K(L)\mathrm{id}_L(\alpha).\end{aligned}$$

To finish, apply Lemma 3.1. □

Though sub-multiplicative, the integral degree might not be multiplicative, even for two simple algebraic field extensions.

Example 4.8. Let $p > 1$ be a prime and let $K = \mathbb{F}_p(u_1^p, u_2^p)$, where u_1, u_2 are algebraically independent over $\mathbb{F}_p$. Set $L = K[u_1]$ and $M = L[u_2]$. Then $K \subset L$ and $L \subset M$ are two finite field extensions with $\mathrm{d}_K(M) = p$ and $\mathrm{d}_K(L)\mathrm{d}_L(M) = \mathrm{id}_K(u_1)\mathrm{id}_L(u_2) = p^2$ (see Example 4.5 and Proposition 2.3).

However, for finite separable field extensions, multiplicativity holds.

Remark 4.9. Let $K \subset L$ be a finite separable field extension and $L \subset M$ be a simple algebraic field extension. Then

$$\mathrm{d}_K(M) = \mathrm{d}_K(L)\mathrm{d}_L(M).$$

Proof. By the extended primitive element theorem, $K \subset M$ is simple. Hence, by Proposition 4.2, $\mathrm{d}_K(M) = [M : K]$, $\mathrm{d}_K(L) = [L : K]$ and $\mathrm{d}_L(M) = [M : L]$. □

5. Integral degree of projective finite ring extensions

We return to the general hypotheses: $A \subset B$ and $B \subset C$ are integral ring extensions of integral domains, and K, L and M are their fields of fractions, respectively. In this section we are interested in the integral degree of projective finite ring extensions (by a projective, respectively free, ring extension $A \subset B$ we understand that B is a projective, respectively free, A-module). We begin by recalling some well-known definitions and facts (see, e.g., [**10**, Chapter IV, § 2, 3]).

Reminder 5.1. Let A be a domain and let N be a finitely generated A-module.

- N is a free A-module if it has a basis, that is, a linearly independent system of generators. The *rank of a free module* N, $\mathrm{rank}_A(N)$, is defined as the cardinality of (indeed, any) a basis. Clearly, N is free of rank n if and only if $N \cong A^n$. If N is a free A-module, the minimal generating sets are just the bases of N. In particular, $\mu_A(N) = \mathrm{rank}_A(N)$.

- N is a projective A-module if there exists an A-module N' such that $N \oplus N'$ is free. One has that N is projective if and only if N is finitely presentable and locally free. The *rank of a projective module* N *at a prime* $\mathfrak{p}$, $\mathrm{rank}_{\mathfrak{p}}(N)$, is defined as the rank of the free $A_\mathfrak{p}$-module $N_\mathfrak{p}$, that is, $\mathrm{rank}_\mathfrak{p}(N) = \mathrm{rank}_{A_\mathfrak{p}}(N_\mathfrak{p}) = \mu_{A_\mathfrak{p}}(N_\mathfrak{p}) = \dim_{k(\mathfrak{p})}(N \otimes k(\mathfrak{p}))$, where $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ stands for the residue field of A at $\mathfrak{p}$.

- If N is projective, then $\mathfrak{p} \mapsto \mathrm{rank}_\mathfrak{p}(N)$ is constant (since A is a domain, $\mathrm{Spec}(A)$ is connected) and is simply denoted by $\mathrm{rank}_A(N)$. In particular, on taking the prime ideal (0), then $\mathrm{rank}_A(N) = \mu_K(N \otimes K) = \mathrm{rank}_\mathfrak{p}(N)$, for every prime ideal $\mathfrak{p}$. Clearly, when N is free both definitions of rank coincide.

Theorem 5.2. *Let $A \subset B$ be a projective finite ring extension. Then*

$$\mathrm{d}_K(L) \leq \mathrm{d}_A(B) \leq \mathrm{rank}_A(B) = [L : K].$$

If moreover $K \subset L$ is simple, then

$$\mathrm{d}_K(L) = \mathrm{d}_A(B) = \mathrm{rank}_A(B) = [L : K].$$

Proof. By Proposition 2.9, there exists a maximal ideal $\mathfrak{m}$ of A such that $\mathrm{d}_A(B) = \mathrm{d}_{A_\mathfrak{m}}(B_\mathfrak{m})$. By Proposition 2.3 and using that $B_\mathfrak{m}$ is $A_\mathfrak{m}$-free and B is A-projective, then

$$\begin{aligned}\mathrm{d}_K(L) \leq \mathrm{d}_A(B) = \mathrm{d}_{A_\mathfrak{m}}(B_\mathfrak{m}) \leq \mu_{A_\mathfrak{m}}(B_\mathfrak{m}) = \mathrm{rank}_{A_\mathfrak{m}}(B_\mathfrak{m})\\ = \mathrm{rank}_A(B) = \mu_K(B \otimes K) = [L : K].\end{aligned}$$

To finish, recall that if $K \subset L$ is simple, then $\mathrm{d}_K(L) = [L : K]$ (see Propositions 2.3 or 4.2). □

The next result characterizes finite ring extensions of maximal and minimal integral degree at the same time.

Corollary 5.3. *Let $A \subset B$ be a finite ring extension.*

(a) *$A \subset B$ is free if and only if $[L : K] = \mu_A(B)$.*

(b) *$A \subset B$ is free and $K \subset L$ is simple if and only if $\mathrm{d}_K(L) = \mathrm{d}_A(B) = \mu_A(B)$.*

Proof. If $A \subset B$ is free, then $\mu_A(B) = \mathrm{rank}_A(B) = [L : K]$ (see Reminder 5.1 and Theorem 5.2). Reciprocally, suppose that $[L : K] = \mu_A(B)$. Set $\mu_A(B) = n$ and let $u_1, \ldots, u_n$ be a system of generators of the A-module B. Thus, $u_1, \ldots, u_n$ is a system of generators of the K-module L, where $n = [L : K]$ (recall that, if $S = A \setminus \{0\}$, then $K = S^{-1}A$ and $L = S^{-1}B$, cf. Proposition 2.3). Hence, they are a K-basis of L, so K-linearly independent. In particular, $u_1, \ldots, u_n$ are A-linearly independent. Since they also generate B, one concludes that $u_1, \ldots, u_n$ is an A-basis of B and that B is a free A-module. This shows (a). Since $\mathrm{d}_K(L) \leq \mathrm{d}_A(B) \leq \mu_A(B)$ and $\mathrm{d}_K(L) \leq [L : K] \leq \mu_A(B)$ (see Notation 2.4), part (b) follows from part (a) and Proposition 4.2. □

Corollary 5.4. *Let $A \subset B = A[b]$ be a projective simple integral ring extension. Then $A \subset B$ is free and $1, b, \ldots, b^{n-1}$ is a basis, where*

$$n = \mathrm{id}_A(b) = \mathrm{d}_A(B) = \mu_A(B) \quad \textit{and} \quad n = \mathrm{id}_K(b/1) = \mathrm{d}_K(L) = [L : K].$$

Proof. If $S = A \setminus \{0\}$, then $K = S^{-1}A$ and $L = S^{-1}B = S^{-1}A[b] = K[b/1]$. Thus $K \subset L = K[b/1]$ is a simple algebraic field extension. By Proposition 2.3, $\mathrm{id}_A(b) = \mathrm{d}_A(B) = \mu_A(B) = n$, say, and $\mathrm{id}_K(b/1) = \mathrm{d}_K(L) = [L : K] = m$, say. By Theorem 5.2 and Corollary 5.3, $n = m$ and $A \subset B$ is free. Since $\{1, b, \ldots, b^{n-1}\}$ is a minimal system of generators of $B = A[b]$, then it is a basis of the free A-module B (see Reminder 5.1). □

Sub-multiplicativity holds in the case of projective finite ring extensions $A \subset B$ with $K \subset L$ being simple.

Corollary 5.5. *Let $A \subset B$ and $B \subset C$ be two finite ring extensions. If $A \subset B$ is projective and $K \subset L$ is simple, then*

$$\mathrm{d}_A(C) \leq \mathrm{d}_A(B)\mathrm{d}_B(C).$$

If moreover, $K \subset L$ is separable, $B \subset C$ is projective and $L \subset M$ is simple, then

$$\mathrm{d}_A(C) = \mathrm{d}_A(B)\mathrm{d}_B(C).$$

Proof. By Proposition 2.9, there exists a maximal ideal $\mathfrak{m}$ of A such that $\mathrm{d}_A(C) = \mathrm{d}_{A_\mathfrak{m}}(C_\mathfrak{m})$. Since $A \subset B$ is projective, then $A_\mathfrak{m} \subset B_\mathfrak{m}$ is free with fields of fractions $Q(A_\mathfrak{m}) = Q(A) = K$ and $Q(B_\mathfrak{m}) = Q(B) = L$, respectively, where $K \subset L$ is simple by hypothesis. By Corollary 5.3, $\mathrm{d}_{A_\mathfrak{m}}(B_\mathfrak{m}) = \mu_{A_\mathfrak{m}}(B_\mathfrak{m})$. Therefore, by Corollary 3.4 and Proposition 2.3,

$$\mathrm{d}_A(C) = \mathrm{d}_{A_\mathfrak{m}}(C_\mathfrak{m}) \leq \mathrm{d}_{A_\mathfrak{m}}(B_\mathfrak{m})\mathrm{d}_{B_\mathfrak{m}}(C_\mathfrak{m}) \leq \mathrm{d}_A(B)\mathrm{d}_B(C).$$

As for the second part of the statement, by hypothesis, $A \subset C$ is projective and $K \subset M$ is simple (again, we use the extended primitive element theorem). By Theorem 5.2, $\mathrm{d}_K(L) = \mathrm{d}_A(B)$, $\mathrm{d}_L(M) = \mathrm{d}_B(C)$ and $\mathrm{d}_K(M) = \mathrm{d}_A(C)$. By Remark 4.9, $\mathrm{d}_K(M) = \mathrm{d}_K(L)\mathrm{d}_L(M)$, so $\mathrm{d}_A(C) = \mathrm{d}_A(B)\mathrm{d}_B(C)$. □

Now we can complement Example 2.2. Let $A \subset B$ a ring extension. Let G be a finite group acting as A-algebra automorphisms on B. Define B^G as the subring $B^G = \{b \in B \mid \sigma(b) = b, \text{ for all } \sigma \in G\}$. It is said that $A \subset B$ is a *Galois extension with group G* if $B^G = A$, and for any maximal ideal $\mathfrak{n}$ in B and any $\sigma \in G \setminus \{1\}$, there is a $b \in B$ such that $\sigma(b) - b \notin \mathfrak{n}$ (see, e.g., [**8**, Definition 4.2.1]).

Corollary 5.6. *Let G be a finite group and let $A \subset B$ be a Galois ring extension with group G. Then $A \subset B$ is a projective finite ring extension and $\mathrm{d}_K(L) = \mathrm{d}_A(B) = [L : K] = \mathrm{o}(G)$.*

Proof. Since $A \subset B$ is a Galois ring extension of domains with group G, then $A \subset B$ is a projective finite ring extension, $K \subset L$ is a Galois field extension with group G and $[L : K] = \mathrm{o}(G)$ (see, e.g., [**8**, subsequent Remark to Definition 4.2.1 and Lemma 4.2.5]). In particular, $K \subset L$ is separable and hence simple. By Theorem 5.2, $\mathrm{d}_K(L) = \mathrm{d}_A(B) = [L : K] = \mathrm{o}(G)$. □

Next we calculate the integral degree when A is a Dedekind domain and $K \subset L$ is simple, for example, when B is the ring of integers of an algebraic number field (see Remark 2.5).

Corollary 5.7. *Let $A \subset B$ be a finite ring extension. Suppose that A is Dedekind and that $K \subset L$ is simple. Then $A \subset B$ is projective and $\mathrm{d}_K(L) = \mathrm{d}_A(B) = \mathrm{rank}_A(B) = [L : K]$. If moreover A is a principal ideal domain, then $A \subset B$ is free and $\mathrm{d}_K(L) = \mathrm{d}_A(B) = \mu_A(B)$.*

Proof. From the structure theorem of finitely generated modules over a Dedekind domain, and since B is a torsion-free A-module, it follows that $A \subset B$ is a projective finite ring extension (see, e.g., [**16**, Corollary to Theorem 1.32, p. 30]). Since $A \subset B$ is projective finite and $K \subset L$ is simple, then $\mathrm{d}_K(L) = \mathrm{d}_A(B) = \mathrm{rank}_A(B) = [L : K]$ (see Theorem 5.2). Finally, if A is a principal ideal domain, then $A \subset B$ must be free and we apply Corollary 5.3. □

6. Integrally closed base ring

As always, $A \subset B$ and $B \subset C$ are integral ring extensions of domains, and K, L and M are their fields of fractions, respectively. Recall that $\overline{A}$ denotes the integral closure of A in K. In this section we focus our attention on the case where A is integrally closed. We begin by noting that, in such a situation, $A \subset B$ has minimal integral degree.

Proposition 6.1. *Let $A \subset B$ be an integral ring extension. Then, for every $b \in B$, $\mathrm{id}_K(b) = \mathrm{id}_{\overline{A}}(b)$. In particular, if A is integrally closed, then $\mathrm{d}_K(L) = \mathrm{d}_A(B)$.*

Proof. Since $K \supset \overline{A}$, $\mathrm{id}_K(b) \le \mathrm{id}_{\overline{A}}(b)$. On the other hand, it is well known that the minimal polynomial of b over K has coefficients in $\overline{A}$ (see, e.g., [**1**, Chapter V, § 1.3, Corollary to Proposition 11]), which forces $\mathrm{id}_{\overline{A}}(b) \le \mathrm{id}_K(b)$. So $\mathrm{id}_K(b) = \mathrm{id}_{\overline{A}}(b)$.

Suppose now that A is integrally closed. Then, for every $b \in B$, $\mathrm{id}_A(b) = \mathrm{id}_{\overline{A}}(b) = \mathrm{id}_K(b) \le \mathrm{d}_K(L)$. Thus $\mathrm{d}_A(B) \le \mathrm{d}_K(L)$. The equality follows from Proposition 2.3. □

Certainly, $\mathrm{id}_{\overline{A}}(b)$ may not be equal to $\mathrm{id}_A(b)$, as the next example shows.

Example 6.2. Let $A = \mathbb{Z}[\sqrt{-3}]$. Then $K = Q(A) = \mathbb{Q}(\sqrt{-3})$. Let $b = (1 + \sqrt{-3})/2 \in K$. Clearly, b is integral over A, and the minimal polynomial of b over A is $T^2 - T + 1$. Thus $\mathrm{id}_A(b) = 2$, whereas $\mathrm{id}_K(b) = 1$.

Recall that a simple integral ring extension $B = A[b]$ over an integrally closed domain A is free. Indeed, as said above, the minimal polynomial $p(T)$ of b over K has coefficients in A. Therefore $1, b, \ldots, b^{n-1}$ is a set of generators of the A-module $A[b]$ (where $n = \deg p(T)$). Moreover, since they are linearly independent over K, they are also linearly independent over A. The next result, which is a rephrasing of this fact, is obtained as a direct consequence of Proposition 6.1.

Corollary 6.3. *Let $A \subset B$ be a finite ring extension. Suppose that A is an integrally closed domain. Then, $\mathrm{d}_A(B) = \mu_A(B)$ is equivalent to $A \subset B$ free and $K \subset L$ simple. In particular, if $A \subset B$ is simple and A is integrally closed, then $A \subset B$ is free.*

Proof. By Proposition 6.1, one has $\mathrm{d}_K(L) = \mathrm{d}_A(B)$. Thus, $\mathrm{d}_A(B) = \mu_A(B)$ is equivalent to $\mathrm{d}_K(L) = [L : K] = \mu_A(B)$ (see Notation 2.4). The latter is equivalent to $A \subset B$ free and $K \subset L$ simple (see Corollary 5.3). To finish apply Proposition 2.3. □

This corollary suggests how to find a finite integral extension $A \subset B$ with $\mathrm{d}_K(L) = \mathrm{d}_A(B)$ and $[L : K] < \mu_A(B)$. It suffices to take, as in the next example, an extension of number fields $K \subset L$ which does not admit a relative integral basis (see also Final comments § 8).

Example 6.4. Let $K = \mathbb{Q}(\sqrt{-14})$ and $L = K(\sqrt{-7})$. Let A be the integral closure of $\mathbb{Z}$ in K and let B be the integral closure of $\mathbb{Z}$ in L. Then $A \subset B$ is a finite integral extension, A is integrally closed, $K \subset L$ is simple, but $A \subset B$ is not free (see [**13**]). Hence, by Corollary 6.3, $\mathrm{d}_A(B) < \mu_A(B)$. Note that $\mathrm{d}_K(L) = \mathrm{d}_A(B) = [L : K] = 2$. Moreover, it is well known that $A = \mathbb{Z}[\sqrt{-14}]$ and $B = \mathbb{Z}[(1 + \sqrt{-7})/2, \sqrt{2}]$ (see, e.g., [**7**, Theorem 9.5]). An easy calculation shows that $B = \langle 1, (1 + \sqrt{-7})/2, \sqrt{2}\rangle_A$. Thus $\mu_A(B) = 3$.

Now, we return to the sub-multiplicativity question.

Theorem 6.5. *Let $A \subset B$ and $B \subset C$ be two integral ring extensions. Then, for every $\alpha \in C$,*

$$\mathrm{id}_A(\alpha) \leq \mu_A(\overline{A})\mathrm{d}_A(B)\mathrm{id}_B(\alpha).$$

In particular,

$$\mathrm{d}_A(C) \leq \mu_A(\overline{A})\mathrm{d}_A(B)\mathrm{d}_B(C).$$

Proof. Let $\alpha \in C$. Consider the integral extensions $A \subset \overline{A}$ and $\overline{A} \subset \overline{A}[C]$, where $\overline{A}[C]$ stands for the $\overline{A}$-algebra generated by the elements of C. By Theorem 3.2, $\mathrm{id}_A(\alpha) \leq \mu_A(\overline{A})\mathrm{id}_{\overline{A}}(\alpha)$. But, by Proposition 6.1, $\mathrm{id}_{\overline{A}}(\alpha) = \mathrm{id}_K(\alpha)$. On the other hand, applying Theorem 4.7 and Proposition 2.3, we have

$$\mathrm{id}_K(\alpha) \leq \mathrm{d}_K(L)\mathrm{id}_L(\alpha) \leq \mathrm{d}_A(B)\mathrm{id}_B(\alpha).$$

Hence, $\mathrm{id}_A(\alpha) \leq \mu_A(\overline{A})\mathrm{d}_A(B)\mathrm{id}_B(\alpha)$. By Lemma 3.1, we are done. □

Remark 6.6. The ring $\overline{A}[C]$ appears in the proof of Theorem 6.5. A natural question is whether this ring is the tensor product $\overline{A} \otimes_A C$. Observe that indeed there is a natural surjective morphism of rings $\overline{A} \otimes_A C \to \overline{A}[C]$. However this morphism is not necessarily an isomorphism. For instance, take $A = k[t^2, t^3]$ and $C = \overline{A}$, where $\overline{A} = k[t]$. So $\overline{A}[C] = \overline{A} = k[t]$. One can check that $\overline{A} \otimes_A C$ is not a domain. Indeed, write $\overline{A} = A[X]/I$, with $I = (X^2 - t^2, t^2X - t^3, t^3X - t^4)$. Therefore $\overline{A} \otimes_A C = A[X, Y]/H$, where $H = (X^2 - t^2, t^2X - t^3, t^3X - t^4, Y^2 - t^2, t^2Y - t^3, t^3Y - t^4)$. Note that $X^2 - Y^2$ is in H, but neither $X - Y$ nor $X + Y$ are in H. Hence $\overline{A} \otimes_A C$ is not a domain and cannot be isomorphic to $\overline{A}[C] = k[T]$, which is a domain.

As an immediate consequence of Theorem 6.5, we get the sub-multiplicativity of the integral degree with respect to $A \subset B$ when A is integrally closed.

Corollary 6.7. *Let $A \subset B$ and $B \subset C$ be two integral ring extensions. Suppose that A is an integrally closed domain. Then*

$$\mathrm{d}_A(C) \leq \mathrm{d}_A(B)\mathrm{d}_B(C).$$

However, in the non-integrally closed case, this formula may fail already for noetherian domains of dimension 1, as shown below. To see this, we take advantage of an example due to Dedekind of a non-monogenic number field L. Concretely, we consider B as the ring of integers of L and find A and C such that $\mathrm{d}_A(C) > \mathrm{d}_A(B)\mathrm{d}_B(C)$.

Example 6.8. Let γ_1 be a root of the irreducible polynomial $T^3 - T^2 - 2T - 8 \in \mathbb{Q}[T]$. Let $L = \mathbb{Q}(\gamma_1)$. Let B be the integral closure of $\mathbb{Z}$ in L (i.e., the ring of integers of L). Then:

(a) B is a free $\mathbb{Z}$-module with basis $\{1, \gamma_1, \gamma_2\}$, where $\gamma_2 = (\gamma_1^2 + \gamma_1)/2$;

(b) $\mathrm{d}_{\mathbb{Q}}(L) = \mathrm{d}_{\mathbb{Z}}(B) = \mu_{\mathbb{Z}}(B) = 3$ and the extension $\mathbb{Z} \subset B$ is not simple (L is non-monogenic).

Let $A = \langle 1, 2\gamma_1, 2\gamma_2\rangle_{\mathbb{Z}} = \{a + b\gamma_1 + c\gamma_2 \in B \mid a, b, c \in \mathbb{Z}, b \equiv c \equiv 0 \pmod 2\}$. Then:

(c) A is a free $\mathbb{Z}$-module and an integral domain with field of fractions $K = Q(A) = L$;

(d) B is the integral closure of A in L, $\mathrm{d}_A(B) = 2$ and $\mu_A(B) = 3$.

Let $C = B[\alpha]$, where α is a root of $p(T) = T^2 + \gamma_1 T + (1 + \gamma_2) \in B[T]$. Then:

(e) $B \subset C$ is an integral extension with $\mathrm{d}_B(C) = 2$ and $\mathrm{d}_A(C) = 6$.

In particular, $\mathrm{d}_A(B)\mathrm{d}_B(C) < \mathrm{d}_A(C) < \mathrm{d}_A(\overline{A})\mathrm{d}_A(B)\mathrm{d}_B(C)$.

Proof. By Corollary 5.7, $\mathbb{Z} \subset B$ is free and $\mathrm{d}_{\mathbb{Q}}(L) = \mathrm{d}_{\mathbb{Z}}(B) = \mu_{\mathbb{Z}}(B)$. Moreover, since $\gamma_1 \in B$ with $\mathrm{id}_{\mathbb{Z}}(\gamma_1) = 3$, then $\mathrm{d}_{\mathbb{Z}}(B) \geq 3$. The proof that $\{1, \gamma_1, \gamma_2\}$ is a free $\mathbb{Z}$-basis of B and that $\mathbb{Z} \subset B$ is not simple is due to Dedekind (see, e.g., [**16**, p. 64]). This proves (a) and (b).

Note that, from the equalities

$$\gamma_1^2 = -\gamma_1 + 2\gamma_2, \qquad \gamma_2^2 = 6 + 2\gamma_1 + 3\gamma_2 \quad \text{and} \quad \gamma_1\gamma_2 = 4 + 2\gamma_2,$$

the product in B can be immediately computed in terms of its $\mathbb{Z}$-basis $\{1, \gamma_1, \gamma_2\}$.

Clearly $\{1, 2\gamma_1, 2\gamma_2\}$ are $\mathbb{Z}$-linearly independent. One can easily check that A is a ring and that $x^2 + x \in A$, for every $x \in B$. Hence, $A \subset B$ is an integral extension with $\mathrm{d}_A(B) = 2$. Moreover, the field of fractions of A is $K = Q(A) = L$, and the integral closure of A in K is B. Observe that $\mu_A(B) \leq \mu_{\mathbb{Z}}(B) = 3$. Below we will see that $\mu_A(B) = 3$.

Now let us prove that $\mathrm{d}_B(C) = 2$. One readily checks that the discriminant $\Delta = -\gamma_1^2 - 2\gamma_1 - 4$ of $p(T)$ has norm $N_{L/\mathbb{Q}}(\Delta) = -16$. Since -16 is not a square in $\mathbb{Q}$, then Δ cannot be a square in L. Therefore $p(T)$ is irreducible over L and $\mathrm{d}_B(C) = 2$.

Let $h(T) \in A[T]$ be a minimal degree polynomial of α over A. Since $p(T)$ is the irreducible polynomial of α over L, it follows that $h(T) = p(T)q(T)$, for some $q(T) \in L[T]$. Moreover, $q(T)$ must necessarily belong to $B[T]$, because B is integrally closed in L

(see, e.g., [1, Chapter V, § 1.3, Proposition 11]). Therefore, $q(T)$ is a monic polynomial in $B[T]$ such that $p(T)q(T) \in A[T]$. An easy computation shows that this implies that $\deg(q(T)) \geq 4$ (note that the existence of such a polynomial $q(T) = T^n + b_1T^{n-1} + \cdots + b_{n-1}T + b_n$ is equivalent to the solvability in $\mathbb{Z}$ modulo 2 of a certain system of linear equations with coefficients in $\mathbb{Z}$, in the unknowns $a_{ij} \in \mathbb{Z}$, where $b_i = a_{i,1} + a_{i,2}\gamma_1 + a_{i,3}\gamma_2$). Thus, $\mathrm{id}_A(\alpha) = \deg(h(T)) \geq 6$. By Theorem 3.2,

$$6 \leq \mathrm{id}_A(\alpha) \leq \mathrm{d}_A(C) \leq \mu_A(B)\mathrm{d}_B(C) \leq 6.$$

Hence $\mathrm{d}_A(C) = 6$ and $\mu_A(B) = 3$. □

Remark 6.9. It is not possible to construct a similar example with B having rank 2 over $\mathbb{Z}$, because $\mathrm{d}_A(B) \leq \mu_A(B) \leq \mathrm{rank}_{\mathbb{Z}}(B) = 2$ implies $\mathrm{d}_A(B) = \mu_A(B)$ and then, by Corollary 3.4, $\mathrm{d}_A(B) \leq \mathrm{d}_A(B)\mathrm{d}_B(C)$.

7. Upper-semicontinuity

Recall that $A \subset B$ is an integral ring extension of integral domains, and $K = Q(A)$ and $L = Q(B)$ are their fields of fractions. Let $\mathrm{d} : \mathrm{Spec}(A) \to \mathbb{N}$ be defined by $\mathrm{d}(\mathfrak{p}) = \mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p})$. In this section we study the upper-semicontinuity of d, that is, whether or not,

$$\mathrm{d}^{-1}([n, +\infty)) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathrm{d}(\mathfrak{p}) \geq n\}$$

is a closed set for every $n \geq 1$. There are two cases in which upper-semicontinuity follows easily from our previous results.

Proposition 7.1. *Let $A \subset B$ be an integral ring extension. Then $\mathrm{d} : \mathrm{Spec}(A) \to \mathbb{N}$, defined by $\mathrm{d}(\mathfrak{p}) = \mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p})$, is upper-semicontinuous in any of the following cases:*

(a) *$A \subset B$ is simple;*

(b) *$A \subset B$ has minimal integral degree (e.g., $A \subset B$ is projective finite and $K \subset L$ is simple; or A is integrally closed).*

Proof. If $A \subset B = A[b]$ is simple, then $A_\mathfrak{p} \subset B_\mathfrak{p} = A_\mathfrak{p}[b/1]$ is simple too, for every $\mathfrak{p} \in \mathrm{Spec}(A)$. By Proposition 2.3, it follows that $\mathrm{d}(\mathfrak{p}) = \mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p}) = \mu_{A_\mathfrak{p}}(B_\mathfrak{p})$. But the minimal number of generators is known to be an upper-semicontinuous function (see, e.g., [10, Chapter IV, § 2, Corollary 2.6]). This shows case (a). By Proposition 2.3(e), $\mathrm{d}_K(L) \leq \mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p}) \leq \mathrm{d}_A(B)$, for every $\mathfrak{p} \in \mathrm{Spec}(A)$. In case (b), that is, if $\mathrm{d}_K(L) = \mathrm{d}_A(B)$, then $\mathrm{d}(\mathfrak{p}) = \mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p}) = \mathrm{d}_K(L)$ is constant, and thus upper-semicontinuous. □

A possible way to weaken the integrally closed hypothesis is to shrink the conductor $\mathcal{C} = (A : \overline{A})$ of A in its integral closure $\overline{A}$. A first thought would be to suppose that $\mathcal{C}$ is of maximal height. However, with some extra assumptions on A, e.g., A local Cohen–Macaulay, analytically unramified and A not integrally closed, one can prove that the conductor must have height 1 (see, e.g., [4, Exercise 12.6]). In this sense, it seems appropriate to start by considering the case when $\dim A = 1$.

Theorem 7.2. *Let $A \subset B$ be an integral ring extension. Suppose that A is a noetherian domain of dimension 1 and with finite integral closure (e.g., A is a Nagata ring). Then* $\mathrm{d} : \mathrm{Spec}(A) \to \mathbb{N}$, *defined by* $\mathrm{d}(\mathfrak{p}) = \mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p})$, *is upper-semicontinuous.*

Proof. If A is integrally closed, the result follows from Proposition 7.1. Thus we can suppose that A is not integrally closed. Since $\overline{A}$ is finitely generated as an A-module, then $\mathcal{C} = (A : \overline{A}) \neq 0$. Since A is a one-dimensional domain, $\mathcal{C}$ has height 1 and any prime ideal $\mathfrak{p}$ containing $\mathcal{C}$ must be minimal over it. Therefore, the closed set $V(\mathcal{C})$ coincides with the set of minimal primes over $\mathcal{C}$, so it is finite. Note that, for any $\mathfrak{p} \in \mathrm{Spec}(A)$, $\mathrm{d}(\mathfrak{p}) = \mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p}) \geq \mathrm{d}_K(L)$ (see Proposition 2.3). Moreover, if $\mathfrak{p} \notin V(\mathcal{C})$, then $A_\mathfrak{p} = \overline{A_\mathfrak{p}}$ and $\mathrm{d}(\mathfrak{p}) = \mathrm{d}_K(L)$ (see Proposition 6.1). Now, take $n \geq 1$. If $n > \mathrm{d}_K(L)$, then $\{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathrm{d}(\mathfrak{p}) \geq n\} \subseteq V(\mathcal{C})$ is a finite set, hence a closed set. If $n \leq \mathrm{d}_K(L)$, then $\{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathrm{d}(\mathfrak{p}) \geq n\} = \mathrm{Spec}(A)$. Thus, for every $n \geq 1$, $\mathrm{d}^{-1}([n, +\infty))$ is a closed set and $\mathrm{d} : \mathrm{Spec}(A) \to \mathbb{N}$ is upper-semicontinuous. □

Remark 7.3. Note that the proof of Theorem 7.2 only uses that $V(\mathcal{C})$ is a finite set of $\mathrm{Spec}(A)$. For instance, it also holds if A is a noetherian local domain of dimension 2 and with finite integral closure $\overline{A}$. Another example where it would work would be the following: let A be the coordinate ring of a reduced and irreducible variety V over a field of characteristic zero. Then the conductor $\mathcal{C}$ contains the Jacobian ideal J. Now J defines the singular locus of V, so if we suppose that V has only isolated singularities, then J is of dimension zero, so $\mathcal{C}$ is of dimension zero also. Hence $V(\mathcal{C})$ is finite (see [**4**, Theorem 4.4.9] and [**19**, Corollary 6.4.1]).

If we skip the condition that $\overline{A}$ be finitely generated, the result may fail. The following example is inspired by [**17**, Example 1.4] (see also [**3**, Example 6.6]).

Example 7.4. There exists a noetherian domain A of dimension 1 with $\mathrm{d}_A(\overline{A}) = 2$, but $\mu_A(\overline{A}) = \infty$, and such that $\mathrm{d} : \mathrm{Spec}(A) \to \mathbb{N}$, defined by $\mathrm{d}(\mathfrak{p}) = \mathrm{d}_{A_\mathfrak{p}}(\overline{A}_\mathfrak{p})$, is not upper-semicontinuous.

Proof. Let $t_1, t_2, \ldots, t_n, \ldots$ be infinitely many indeterminates over a field k. Let

$$R = k[t_1^2, t_1^3, t_2^2, t_2^3, \ldots] \subset D = k[t_1, t_2, \ldots].$$

Clearly $\overline{R} = D$. Note that for $f \in D = k[t_1, t_2, \ldots]$, $f \in R$ if and only if every monomial $\lambda t_1^{i_1} \cdots t_r^{i_r}$ of f has each $i_j = 0$ or $i_j \geq 2$.

For every $n \geq 1$, let $\mathfrak{q}_n = (t_n^2, t_n^3)R$, which is a prime ideal of R of height 1. Note that for $f \in R$, $f \in \mathfrak{q}_n$ if and only if every monomial $\lambda t_1^{i_1} \cdots t_r^{i_r}$ of f has $i_n \geq 2$. It follows that $t_n \notin R_{\mathfrak{q}_n}$, because if $t_n = a/b$, $a, b \in R$ and $b \notin \mathfrak{q}_n$, then every monomial of $a = bt_n$ has each $i_j = 0$ or $i_j \geq 2$, so has $i_n \geq 2$. Therefore, t_n appears in each monomial of b, but since $b \in R$, the exponent of t_n in each monomial of b must be at least 2, so $b \in \mathfrak{q}_n$, a contradiction.

Now, set $R \subset D_n = k[t_1, \ldots, t_{n-1}, t_n^2, t_n^3, t_{n+1}, \ldots] \subset D$ and $S_n = R \setminus \mathfrak{q}_n$, a multiplicatively closed subset of R. Clearly $R_{\mathfrak{q}_n} = S_n^{-1} D_n$.

Claim. Let I be an ideal of R such that $I \subseteq \cup_{n\geq 1} \mathfrak{q}_n$. Then I is contained in some $\mathfrak{q}_j$.

If I is contained in a finite union of $\mathfrak{q}_i$, using the ordinary prime avoidance lemma, we are done. Suppose that I is not contained in any finite union of $\mathfrak{q}_i$ and let us reach a contradiction. Take $f \in I$, $f \neq 0$. Then $f \in k[t_1, \dots, t_n]$ for some $n \geq 1$ and f is in a finite number of $\mathfrak{q}_i$, corresponding to the variables t_i that appear in every single monomial of f. We can suppose that $f \in \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$, for some $1 \leq r \leq n$, and $f \notin \mathfrak{q}_i$, for $i > r$. Since $I \not\subset \mathfrak{q}_1 \cup \cdots \cup \mathfrak{q}_r$, there exists $g \in I$ such that $g \notin \mathfrak{q}_1 \cup \cdots \cup \mathfrak{q}_r$. Let $h = t_s^2 g \in I$, where $s > n$, so that f and h have no common monomials. Since $\mathfrak{q}_i$ are prime, then $h = t_s^2 g \notin \mathfrak{q}_1 \cup \cdots \cup \mathfrak{q}_r$. Since $f, h \in I \subseteq \cup_{n \geq 1} \mathfrak{q}_n$, then $f + h \in \mathfrak{q}_m$, for some $m \geq 1$. But since $f \in \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ and $h \notin \mathfrak{q}_1 \cup \cdots \cup \mathfrak{q}_r$, then necessarily $m > r$. Thus $f + h \in \mathfrak{q}_m$, where $m > r$. But since f and h have no common monomials, this implies that every monomial of f must contain t_m^2, so $f \in \mathfrak{q}_m$, a contradiction. Hence $I \subseteq \mathfrak{q}_j$ for some j and the Claim is proved. (An alternative proof would follow from [**18**, Proposition 2.5], provided that k is uncountable.)

Let $S = R \setminus \cup_{n \geq 1} \mathfrak{q}_n$, a multiplicatively closed subset of R. Let $A = S^{-1}R$ and $\mathfrak{p}_n = S^{-1}\mathfrak{q}_n$. If Q is a prime ideal of R such that $Q \subseteq \cup_{n \geq 1} \mathfrak{q}_n$, then, by the Claim above, $Q \subseteq \mathfrak{q}_j$, for some $n \geq 1$. In particular, $\operatorname{Spec}(A) = \{(0)\} \cup \{\mathfrak{p}_n \mid n \geq 1\}$, where each $\mathfrak{p}_n$ is finitely generated. Therefore A is a one-dimensional noetherian domain.

For every $n \geq 1$, $A_{\mathfrak{p}_n} = (S^{-1}R)_{S^{-1}\mathfrak{q}_n} = R_{\mathfrak{q}_n} = S_n^{-1} D_n$. Moreover, $t_n = t_n^3 / t_n^2$ is in the field of fractions of $A_{\mathfrak{p}_n}$ and $t_n^2 \in A_{\mathfrak{p}_n}$, that is, t_n is integral over $A_{\mathfrak{p}_n}$. Thus

$$A_{\mathfrak{p}_n}[t_n] = (S_n^{-1} D_n)[t_n] = S_n^{-1} D \quad \text{and} \quad \overline{A_{\mathfrak{p}_n}} = \overline{A_{\mathfrak{p}_n}[t_n]} = \overline{S_n^{-1} D} = S_n^{-1}(\overline{D}) = S_n^{-1} D.$$

Hence $\overline{A_{\mathfrak{p}_n}} = A_{\mathfrak{p}_n}[t_n]$. Recall that $t_n \notin R_{\mathfrak{q}_n} = A_{\mathfrak{p}_n}$ and $\mathrm{d}_{A_{\mathfrak{p}_n}}(t_n) \leq 2$. By Proposition 2.3, $\mathrm{d}_{A_{\mathfrak{p}_n}}(\overline{A_{\mathfrak{p}_n}}) = \mathrm{d}_{A_{\mathfrak{p}_n}}(A_{\mathfrak{p}_n}[t_n]) = 2$.

Consider the integral extension $A \subset \overline{A}$ and $\mathrm{d} : \operatorname{Spec}(A) \to \mathbb{N}$, defined by $\mathrm{d}(\mathfrak{p}) = \mathrm{d}_{A_\mathfrak{p}}(\overline{A}_\mathfrak{p}) = \mathrm{d}_{A_\mathfrak{p}}(\overline{A_\mathfrak{p}})$. We have just shown that, for every $n \geq 1$, $\mathrm{d}(\mathfrak{p}_n) = \mathrm{d}_{A_{\mathfrak{p}_n}}(\overline{A_{\mathfrak{p}_n}}) = 2$. On the other hand, $\mathrm{d}((0)) = \mathrm{d}_{Q(A)}(Q(\overline{A})) = 1$ because $Q(A) = Q(\overline{A})$. Therefore, $\mathrm{d}^{-1}([2, +\infty)) = \operatorname{Spec}(A) \setminus \{(0)\}$, which is not a closed set. Indeed, suppose that $\operatorname{Spec}(A) \setminus \{(0)\} = V(I)$, for some non-zero ideal I. Since A is a one-dimensional noetherian domain, I has height 1 and $V(I)$ is the finite set of associated primes to I. However, $\operatorname{Spec}(A) \setminus \{(0)\} = \operatorname{Max}(A)$, which is infinite, a contradiction. So $\mathrm{d} : \operatorname{Spec}(A) \to \mathbb{N}$ is not upper-semicontinuous. □

Remark 7.5. Contrary to the upper-semicontinuity, sub-multiplicativity does not work for one-dimensional noetherian domains with finite integral closure. See Example 6.8, where A was a noetherian domain of dimension 1 and with finite integral closure.

8. Final comments

We finish the paper by mentioning some points that we think would be worth clarifying. To simplify, suppose that $A \subset B$ and $B \subset C$ are two finite ring extensions, where, as always, A and B are two integral domains, and K and L are their fields of fractions, respectively.

(1) We have shown that $A \subset B$ of maximal integral degree implies sub-multiplicativity (cf. Corollary 3.4). Does the same work for minimal integral degree?

(2) We have shown that $A \subset B$ of minimal integral degree implies upper-semicontinuity (cf. Proposition 7.1). Does the same work for maximal integral degree?

(3) We have shown that $A \subset B$ free and $K \subset L$ simple implies $\mathrm{d}_K(L) = \mathrm{d}_A(B)$ (Corollary 5.3). Can we omit the hypothesis $K \subset L$ simple? In other words, does $[L : K] = \mu_A(B)$ imply $\mathrm{d}_K(L) = \mathrm{d}_A(B)$? If so, we would have a 'down-to-up rigidity' in the diagram of Notation 2.4. Note that the 'up-to-down rigidity' is not true (see, e.g., Example 6.4).

(4) Does the condition $\mathrm{d}_A(B) = \mu_A(B)$ localize? In particular, does $\mathrm{d}_A(B) = \mu_A(B)$ imply $\mathrm{d}_K(L) = [L : K]$? That would imply a 'right-to-left rigidity' in the diagram of Notation 2.4. If A is integrally closed, the answer is affirmative. Note that Examples 2.6 and 6.8 affirm that the 'left-to-right rigidity' is not true.

(5) It would be interesting to study the sub-multiplicativity and upper-semicontinuity properties for the specific case of affine domains A and B.

(6) Can one replace $\mu_A(\overline{A})$ by $\mathrm{d}_A(\overline{A})$ in the inequality $\mathrm{d}_A(C) \leq \mu_A(\overline{A})\mathrm{d}_A(B)\mathrm{d}_B(C)$ of Theorem 6.5?

(7) Is the integral degree upper-semicontinuous for Nagata rings of dimension greater than 1?

(8) Is there any clear relationship between $\mathrm{d}_A(B)$ and the pair of numbers $\mathrm{d}_{A/\mathfrak{p}}(B/\mathfrak{p}B)$ and $\mathrm{d}_{A_\mathfrak{p}}(B_\mathfrak{p})$? An affirmative answer could be useful in recursive arguments.

(9) Upper-semicontinuity does not imply sub-multiplicativity. We wonder to what extent sub-multiplicativity could imply upper-semicontinuity.

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