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On outindependent subgraphs of strongly regular graphs

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An outindependent subgraph of a graph Γ, with respect to an independent vertex subset $C \subset V$, is the subgraph Γ_C induced by the vertices in $V \setminus C$. We study the case when Γ is strongly regular, where the results of de Caen [1998, The spectra of complementary subgraphs in a strongly regular graph. *European Journal of Combinatorics*, **19**(5), 559–565.], allow us to derive the whole spectrum of Γ_C. Moreover, when C attains the Hoffman–Lovász bound for the independence number, Γ_C is a regular graph (in fact, distance-regular if Γ is a Moore graph). This article is mainly devoted to study the non-regular case. As a main result, we characterize the structure of Γ_C when C is the neighborhood of either one vertex or one edge.

Keywords: Strongly regular graph; Independent set; Spectrum

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1. Preliminaries

In this article we address the following question: Let $\Gamma = (V, E)$ be a strongly regular graph with a given independent set $C \subset V$. What can be said about the structure of the graph Γ_C induced by the vertices in $V \setminus C$? The study of Γ_C, which we call the *outindependent graph*, can be motivated, for instance, by the study of some possible distance-regular graphs, such as the Moore graph with degree 57 and diameter 2. Moreover, this work could also have some relevance to the study of completely regular codes. Some known results (see [1,2]), allow us to compute the whole spectrum of Γ_C, and prove that it is a regular graph precisely when the cardinality of C attains the Hoffman–Lovász (spectral) bound for the independence number. In fact when such a bound is attained in a Moore graph, the corresponding outindependent graph turns out to be a distance-regular graph. A known example of this fact occurs when

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$\Gamma = HS$ (the Hoffman–Singleton graph) and C is a maximum independent set with 15 vertices, in which case $\Gamma_C = O_4$ (the 'odd graph' [3] with degree 4); see Jeurissen [4]. The main results of this article, contained in section 3, deal with the case when Γ_C in a non-regular graph. In particular, some cases where the non-trivial component of Γ_C is a distance-regular graph are characterized. In the rest of this introductory section, we summarize some of the background used in our study.

1.1. *Graphs and their spectra*

Consider the adjacency matrix $\boldsymbol{A}$ of a *regular* graph $\Gamma = (V, E)$ on $n = |V|$ vertices, with spectrum

$$\operatorname{sp}\Gamma := \operatorname{sp}\boldsymbol{A} = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}, \tag{1}$$

where the eigenvalues λ_i, $0 \le i \le d$, are in decreasing order, $\lambda_0 > \lambda_1 > \cdots > \lambda_d$, and the superscripts denote multiplicities. The set of such eigenvalues is denoted by $\operatorname{ev}\Gamma$, and their corresponding eigenspaces are $\mathcal{E}_i := \operatorname{Ker}(\boldsymbol{A} - \lambda_i \boldsymbol{I})$, $0 \le i \le d$. The orthogonal projections onto the eigenspaces $\mathcal{E}_i$ are represented by the so-called (*principal*) *idempotents* of $\boldsymbol{A}$:

$$\boldsymbol{E}_i := \frac{1}{\phi_i} \prod_{j=0(j\ne i)}^{d} (\boldsymbol{A} - \lambda_j \boldsymbol{I}) \qquad (0 \le i \le d),$$

where $\phi_i := \prod_{j=0(j\ne i)}^{d} (\lambda_i - \lambda_j)$. Given any subset C with $r := |C| \ge 1$ vertices, we consider its normalized characteristic vector $\boldsymbol{e}_C := (1/\sqrt{r}) \sum_{u\in C} \boldsymbol{e}_u$, where $\boldsymbol{e}_u$ denotes the u-th standard unit basis vector in $\mathbb{R}^n$. Then, the *C-multiplicity* of the eigenvalue λ_i is defined by

$$m_C(\lambda_i) := \|\boldsymbol{E}_i \boldsymbol{e}_C\|^2 = \langle \boldsymbol{E}_i \boldsymbol{e}_C, \boldsymbol{e}_C \rangle = \frac{1}{r} \sum_{u,v\in C} (\boldsymbol{E}_i)_{uv} \qquad (0 \le i \le d). \tag{2}$$

Note that, since $\boldsymbol{e}_C$ is a unit vector, we have $\sum_{i=0}^{d} m_C(\lambda_i) = 1$. Moreover, we see from the above that, if Γ is connected, the C-multiplicity of λ_0 is $m_C(\lambda_0) = r/n > 0$. In particular, when C is a single vertex u, the $\{u\}$-multiplicities of λ_i correspond to the so-called "(local) u-multiplicities" (see [5]). If $\mu_0(= \lambda_0) > \mu_1 > \cdots > \mu_{d_C}$ represent the eigenvalues in $\operatorname{ev}\Gamma$ with non-zero C-multiplicities, the (*local*) *C-spectrum* of C is

$$\operatorname{sp}C := \left\{ \mu_0^{\tilde{m}_0} > \mu_1^{\tilde{m}_1} > \cdots > \mu_{d_C}^{\tilde{m}_{d_C}} \right\} \tag{3}$$

where $\tilde{m}_i := m_C(\mu_i)$, $0 \le i \le d_C$, and $d_C(\le d)$ is called the *dual degree* of C. It is known that, if Γ is connected, then the *eccentricity* of C, defined by $\operatorname{ecc}(C) := \max_{u\in V} \operatorname{dist}(u, C) = \max_{u\in V} \min_{v\in C} \operatorname{dist}(u, v)$, is bounded above by d_C. (For more details, see [6,7].)

The following theorem is usually known as "the interlacing theorem" (see, for instance [2,8]).