

An algebraic approach to lifts of digraphs *

C. Dalfo^a, M. A. Fiol^b, M. Miller^c, J. Ryan^d, J. Širáň^e^aDepartament de Matemàtiques

Universitat Politècnica de Catalunya, Barcelona, Catalonia

`crisrina.dalfo@upc.edu`^bDepartament de Matemàtiques

Universitat Politècnica de Catalunya

Barcelona Graduate School of Mathematics, Barcelona, Catalonia

`miguel.angel.fiol@upc.edu`^cSchool of Mathematical and Physical Sciences

The University of Newcastle, Newcastle, Australia

^cDepartment of Mathematics

University of West Bohemia, Plzeň, Czech Republic

^dSchool of Electrical Engineering and Computing

The University of Newcastle, Newcastle, Australia

`joe.ryan@newcastle.edu.au`^eDepartment of Mathematics and Statistics

The Open University, Milton Keynes, UK

^eDepartment of Mathematics and Descriptive Geometry

Slovak University of Technology, Bratislava, Slovak Republic

`j.siran@open.ac.uk`

September 25, 2018

This paper is dedicated to the memory of Mirka Miller,
 who enjoyed polynomial matrices very much.

*Research of the first two authors is supported by MINECO under project MTM2014-60127-P, and by AGAUR under project 2014SGR1147. The fifth author acknowledges support from the research grants APVV 0136/12, APVV-15-0220, VEGA 1/0026/16, and VEGA 1/0142/17.



The first author has also received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 734922.

Abstract

We study the relationship between two key concepts in the theory of (di)graphs: the quotient digraph, and the lift Γ^α of a base (voltage) digraph. These techniques contract or expand a given digraph in order to study its characteristics, or obtain more involved structures. This study is carried out by introducing a quotient-like matrix, with complex polynomial entries, which fully represents Γ^α . In particular, such a matrix gives the quotient matrix of a regular partition of Γ^α , and when the involved group is Abelian, it completely determines the spectrum of Γ^α . As some examples of our techniques, we study some basic properties of the Alegre digraph. In addition we completely characterize the spectrum of a new family of digraphs, which contains the generalized Petersen graphs, and that of the Hoffman-Singleton graph.

Keywords: Digraph, adjacency matrix, regular partition, quotient digraph, Abelian group, spectrum, voltage digraphs, lifted digraph, generalized Petersen graph.

Mathematics Subject Classifications: 05C20; 05C50; 15A18.

1 Introduction

In the study of interconnection and communication networks, the theory of digraphs plays a key role as, in many cases, the links between nodes are unidirectional. Within this theory, there are two concepts that have shown to be very fruitful to construct good and efficient networks. Namely, those of quotient digraphs and lifts of voltage digraphs. Roughly speaking, quotient digraphs allow us to give a simplified or ‘condensed’ version of a larger digraph, while the voltage digraph technique does the converse, by ‘expanding’ a smaller digraph into its ‘lift’. From this point of view, it is natural that both techniques have close relationships. In this paper we explore some of such interrelations.

The paper is organized as follows. In the rest of this section, we give some basic background information. In Section 2 we present the basic definition and results on regular partitions and their corresponding quotient digraphs. In Section 3, we recall the definitions of voltage and lifted digraphs. Section 4 is devoted to studying a representation of a lifted digraph with a quotient-like matrix whose size equals the order of the (much smaller) base digraph. In particular, it is shown that such a matrix can be used to deduce combinatorial properties of the lifted digraph. Following this approach, and as a main result, Section 5 present a new method to completely determine the spectrum of the lift by using only the spectrum of the quotient digraph. The results are illustrated by following some examples. The first one is the so-called Alegre digraph (first shown by Fiol, Alegre, and Yebra in [7]), which is the largest digraph (with order 25) with degree 2 and diameter 4. This digraph can be constructed as the lifted digraph of a voltage digraph, and a major part of its structure is characteristic of a line digraph. The second example is a new family of digraphs which, as a particular case, contains the well-know generalized Petersen graphs. Finally, we recalculate the spectrum of the Hoffman-Singleton graph without using its

strong regularity character.

1.1 Background

Here, we recall some basic terminology and simple results concerning digraphs and their spectra. For the concepts and/or results not presented here, we refer the reader to some of the basic textbooks on the subject; for instance, Bang-Jensen and Gutin [2], Chartrand and Lesniak [4], or Diestel [6].

Through this paper, $\Gamma = (V, E)$ denotes a digraph, with vertex set V and arc set E , that is *strongly connected*, namely, each vertex is connected to all other vertices by traversing the arcs in their corresponding direction. An arc from vertex u to vertex v is denoted by either (u, v) , uv , or $u \rightarrow v$. We allow *loops* (that is, arcs from a vertex to itself), and *multiple arcs*. The set of vertices adjacent to and from $v \in V$ is denoted by $\Gamma^-(v)$ and $\Gamma^+(v)$, respectively. Such vertices are referred to as *in-neighbors* and *out-neighbors* of v , respectively. Moreover, $\delta^-(v) = |\Gamma^-(v)|$ and $\delta^+(v) = |\Gamma^+(v)|$ are the *in-degree* and *out-degree* of vertex v , and Γ is *d-regular* when $\delta^+(v) = \delta^-(v) = d$ for all $v \in V$. Similarly, given $U \subset V$, $\Gamma^-(U)$ and $\Gamma^+(U)$ represent the sets of vertices adjacent to and from (the vertices) of U , respectively. Given two vertex subsets $X, Y \subset V$, the subset of arcs from X to Y is denoted by $e(X, Y)$.

The spectrum of Γ , denoted by $\text{sp } \Gamma = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}$, is constituted by the distinct eigenvalues λ_i with the corresponding algebraic multiplicities m_i , for $i = 0, 1, \dots, d$, of its adjacency matrix \mathbf{A} .

2 Regular partitions and quotient digraphs

Let $\Gamma = (V, E)$ be a digraph with n vertices and adjacency matrix \mathbf{A} . A partition π of its vertex set $V = U_1 \cup U_2 \cup \dots \cup U_m$, for $m \leq n$, is called *regular* if the number of arcs from a vertex $u \in U_i$ to vertices in U_j only depends on i and j . Let c_{ij} be the number of arcs that join a fixed vertex in U_i to vertices in U_j . A matrix characterization of this property is the following: Let \mathbf{S} be the 0-1 $(n \times m)$ -matrix whose i -th column is the normalized characteristic vector of U_i , so that $\mathbf{S}^\top \mathbf{S} = \mathbf{I}$ (the identity matrix), and consider the so-called *quotient matrix*

$$\mathbf{B} = \mathbf{S}^\top \mathbf{A} \mathbf{S}. \quad (1)$$

Then, it can be easily checked that π is regular if and only if

$$\mathbf{S} \mathbf{B} = \mathbf{A} \mathbf{S}. \quad (2)$$

The digraph $\pi(\Gamma)$ whose (weighted) adjacency matrix is the quotient matrix is called *quotient digraph*, and its arcs can have weight different from 1. More precisely, the vertices

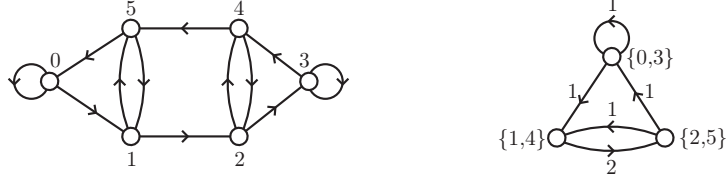


Figure 1: The digraph with adjacency matrix given by (3), and its quotient digraph with weighted arcs.

of the quotient digraph are the subsets U_i , for $i = 1, 2, \dots, m$, and the arc from vertex U_i to vertex U_j has weight c_{ij} . For the case of quotient digraphs obtained from non-directed graphs, see Godsil [9, Lemma 2.1]. We have the following basic result, where the regular partition of V is also called a *regular* partition of \mathbf{A} .

Lemma 2.1. *Every eigenvalue of the quotient matrix \mathbf{B} of a regular partition of \mathbf{A} is also an eigenvalue of \mathbf{A} , that is, $\text{sp } \mathbf{B} \subset \text{sp } \mathbf{A}$.*

By way of example, consider the digraph of Figure 1. Then, its adjacency matrix is

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3)$$

with $\text{sp } \mathbf{A} = \{-1, 0^{(3)}, 1, 2\}$. A possible regular partition is constituted by the vertex subsets $U_i = \{i, i + 3\}$, for $i = 0, 1, 2$, with characteristic matrix

$$\mathbf{S} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the corresponding quotient matrix is

$$\mathbf{B} = \mathbf{S}^\top \mathbf{A} \mathbf{S} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix},$$

with $\text{sp } \mathbf{B} = \{-1, 0, 2\} \subset \text{sp } \mathbf{A}$, as expected.

3 Voltage and lifted digraphs

Voltage (di)graphs are, in fact, a type of compounding that consists of connecting together several copies of a (di)graph by setting some (directed) edges between any two copies. Usually, the symmetry of the obtained constructions yields digraphs with large automorphism groups. As far as we know, one of the first papers where voltage graphs were used for construction of dense graphs is Alegre, Fiol and Yebra [1], but without using the name of ‘voltage graphs’. This name was coined previously by Gross [11]. For more information, see Gross and Tucker [12], Baskoro, Branković, Miller, Plesník, Ryan and Siráň [3], and Miller and Siráň [14].

Let Γ be a digraph with vertex set $V = V(\Gamma)$ and arc set $E = E(\Gamma)$. Then, given a group G with generating set Δ , a *voltage assignment* of Γ is a mapping $\alpha : E \rightarrow \Delta$. The pair (Γ, α) is often called a *voltage digraph*. The *lifted digraph* (or, simply, *lift*) Γ^α is the digraph with vertex set $V(\Gamma^\alpha) = V \times G$ and arc set $E(\Gamma^\alpha) = E \times G$, where there is an arc from vertex (u, g) to vertex (v, h) if and only if $uv \in E$ and $h = g\alpha(uv)$. Such an arc is denoted by (uv, g) .

3.1 The adjacency matrix of the lifted digraph

It is clear that the base digraph with the voltage assignment univocally determines the adjacency matrix of its lift. To define it we need to consider the following concepts. Given a (multiplicative) group G together with a given ordering of its elements $g_0 (= 1), g_1, \dots, g_{n-1}$, a *G-circulant matrix* is defined as a square matrix \mathbf{A} of order n indexed by elements of G , with first row $a_{0,0} = a_{g_0}, a_{0,1} = a_{g_1}, \dots, a_{0,n-1} = a_{g_{n-1}}$, and entries

$$(\mathbf{A})_{g,h} = a_{hg}, \quad g, h \in G.$$

Thus, the elements of row g are identical to those of the first row, but they are permuted by the action of g on G . In particular, a *circulant matrix* (see Davis [5]) corresponds to a G -circulant matrix with the cyclic group $G = \mathbb{Z}_n$ and natural ordering $0, 1, \dots, n-1$. Another example is the adjacency matrix \mathbf{A} of the Cayley digraph $\text{Cay}(G, \Delta)$ of the group G with generating set Δ , which is a G -circulant matrix whose first row has elements $a_{1,j} = 1$ if $g_j \in \Delta$, and $a_{1,j} = 0$, otherwise. The concept of block G -circulant matrix is similar, but now the elements $a_{g_0}, a_{g_1}, \dots, a_{g_{n-1}}$ of the first row (and, consequently, the other rows) are replaced by the $m \times m$ matrices (or blocks) $\mathbf{A}_0 = \mathbf{A}_{g_0}, \dots, \mathbf{A}_{n-1} = \mathbf{A}_{g_{n-1}}$.

The following result is an easy consequence of the above definitions.

Lemma 3.1. *Let Γ be a base digraph with voltage assignment α on the group $G = \{g_0 (= 1), \dots, g_{m-1}\}$. Let Γ_i be the spanning subgraph of Γ with arc set $\alpha^{-1}(g_i)$, and adjacency matrix \mathbf{A}_i , for $i = 0, \dots, m-1$. Then, the adjacency matrix \mathbf{A} of the lifted digraph Γ^α is the block G -circulant matrix with first block-row $\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_{m-1}$. Moreover, the lift Γ^α has a regular partition with quotient matrix $\mathbf{B} = \sum_{i=0}^{m-1} \mathbf{A}_i$. \square*

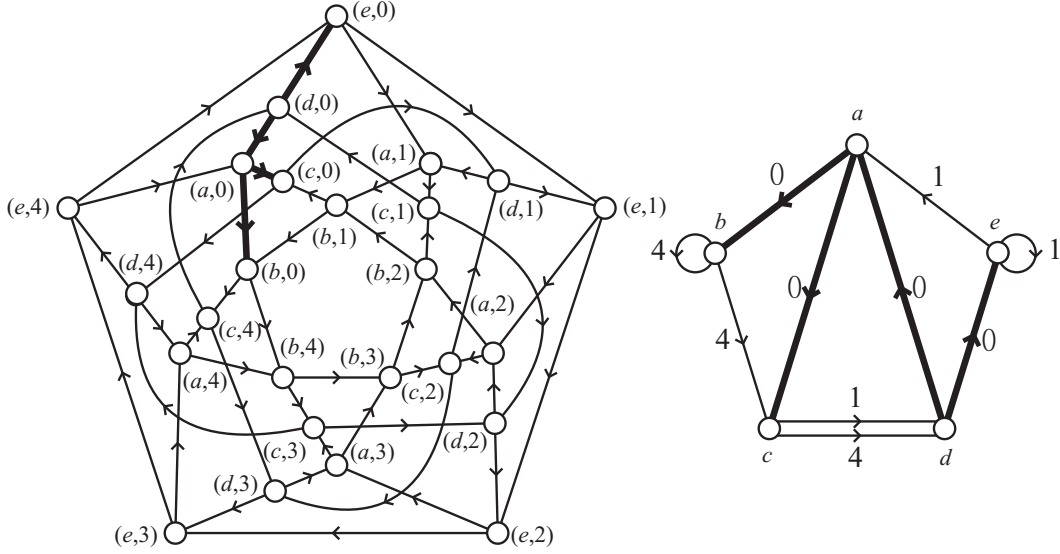


Figure 2: The Alegre digraph (left), and its base digraph (right). The adjacencies in the copy 0 are represented with a thick line.

By way of example, let us consider the Alegre digraph, which is a 2-regular digraph with $n = 25$ vertices and diameter $k = 4$ represented in Figure 2 (left). This digraph was found by Fiol, Yebra, and Alegre in [7]. The Alegre digraph can be seen as the lifted digraph Γ^α of the base digraph Γ with the voltage assignments shown in Figure 2 (right). Then, the nonzero blocks of the first row constituting the adjacency matrix of Γ^α are

$$\mathbf{A}_0 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{A}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, Γ^α has a regular partition with five sets of five vertices each, which is better illustrated in the drawing of Figure 3, with quotient matrix

$$\mathbf{B} = \sum_{i=0}^4 \mathbf{A}_i = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (4)$$

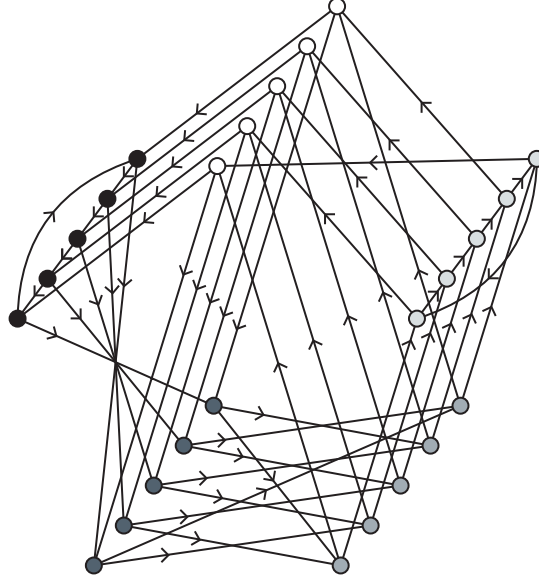


Figure 3: The Alegre digraph and its regular partition with five vertex sets.

4 Matrix representations

In this section, we study how to fully represent a lifted digraph with a matrix whose size equals the order of the base digraph.

First, we deal with the case when the group G of the voltage assignments is cyclic. Thus, let $\Gamma = (V, E)$ be a digraph with voltage assignment α on the group $G = \mathbb{Z}_k = \{0, 1, \dots, k-1\}$. Its *polynomial matrix* $\mathbf{B}(z)$ is a square matrix indexed by the vertices of Γ , and whose elements are complex valued polynomials in the quotient ring $\mathbb{R}_{k-1}[z] = \mathbb{R}[x]/(z^k)$, where (z^k) is the ideal generated by the polynomial z^k . More precisely, each entry of $\mathbf{B}(z)$ is fully represented by a polynomial of degree at most $k-1$, say $(\mathbf{B}(z))_{uv} = p_{uv}(z) = \alpha_0 + \alpha_1 z + \dots + \alpha_{k-1} z^{k-1}$, where

$$\alpha_i = \begin{cases} 1 & \text{if } uv \in E \text{ and } \alpha(uv) = i, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 0, \dots, k-1$. For example, in the case of the Alegre digraph in Figure 1, the polynomial matrix is

$$\mathbf{B}(z) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & z^4 & z^4 & 0 & 0 \\ 0 & 0 & 0 & z + z^4 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ z & 0 & 0 & 0 & z \end{pmatrix}, \quad (5)$$

where $(\mathbf{B}(z))_{ij} = \alpha_r z^r + \alpha_s z^s$, with $\alpha_r, \alpha_s \in \{0, 1\}$ and $r, s \in \mathbb{Z}_5$, means that there are

arcs from vertex (i, p) to vertices $(j, p + r)$ or/and $(j, p + s)$ if and only if $\alpha_r = 1$ or/and $\alpha_s = 1$.

Since the polynomial matrix $\mathbf{B}(z)$ fully represents the digraph Γ^α , we should be able to retrieve from the former any property of the latter. As a first example, the following result shows that the powers of $\mathbf{B}(z)$ yield the same information as the powers of the adjacency matrix of the lift Γ^α .

Lemma 4.1. *Let $(\mathbf{B}(z)^\ell)_{uv} = \beta_0 + \beta_1 z + \dots + \beta_{k-1} z^{k-1}$. Then, for every $i = 0, \dots, k-1$, the coefficient β_i equals the number of walks of length ℓ in the lifted digraph Γ^α , from vertex (u, h) to vertex $(v, h + i)$ for every $h \in G$. In particular, $\mathbf{B}(1)$ is the quotient matrix of the corresponding regular partition of Γ^α .*

Proof. The result is clear for $\ell = 0, 1$. Then, the result follows easily by using induction. \square

For instance, in the case of Alegre digraph, we get

$$\mathbf{B}(z)^4 = \begin{pmatrix} 2 + z^2 + z^3 & z + z^2 + z^4 & z + z^2 + z^4 & z^2 + z^4 & 2 + z^2 + z^3 \\ z + z^2 + 2z^4 & 1 + z + z^3 & 1 + z + z^3 & z + z^3 & z + z^2 + 2z^4 \\ z + z^3 & 2 + z^2 + z^3 & 2 + z^2 + z^3 & 2 + z^2 + z^3 & z + z^3 \\ z + z^3 + z^4 & 1 + z^2 + z^3 & 1 + z^2 + z^3 & 2 + z^2 + z^3 & z + z^3 + z^4 \\ 1 + z^2 + z^4 & z + z^3 + z^4 & z + z^3 + z^4 & 2z + z^3 + z^4 & 1 + z^2 + z^4 \end{pmatrix}.$$

Moreover, the first row of $\mathbf{I} + \mathbf{B}(z) + \mathbf{B}(z)^2 + \mathbf{B}(z)^3 + \mathbf{B}(z)^4$ has entries: $3 + z + z^2 + z^3 + z^4$, $1 + z + z^2 + z^3 + 2z^4$, $1 + z + z^2 + z^3 + 2z^4$, $1 + z + z^2 + z^3 + 2z^4$, $2 + z + z^2 + z^3 + z^4$. Note that all coefficients α_i , for $i = 0, \dots, 4$, of these polynomials are non-zero. This shows that the eccentricities of the vertices $(u, 0)$, for $u \in V$, are 4, as expected since this is the diameter of Γ^α . By reading as columns, this means that if \mathbf{A} is the adjacency matrix of the Alegre digraph, then, the first row of $\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \mathbf{A}^4$ is $3, 1, 1, 1, 2, 1, 1, \dots, 1, 2, 2, 2, 1$.

Note that, in the above result, the products of the entries (polynomials) of $\mathbf{B}(z)$ must be understood in the ring $\mathbb{R}_{k-1}[z]$. In terms of the vectors of the polynomial coefficients, this means that, in fact, we are carrying out a circular convolution. Thus, an efficient way of doing so is by using the discrete Fourier transform, since convolution in one variable is equivalent to entrywise product in the other. In the next subsection, we give details and one example of this procedure.

4.1 Using the discrete Fourier transform

Recall that the *discrete Fourier transform* (DFT) of a vector $\mathbf{z} = (z_0, z_1, \dots, z_{n-1}) \in \mathbb{C}^n$ is the vector $\mathbf{Z} = (Z_0, Z_1, \dots, Z_{n-1}) \in \mathbb{C}^n$ with components

$$Z_k = \frac{1}{\sqrt{n}} \sum_{\ell=0}^{n-1} z_\ell \omega^{-ik\ell}, \quad k = 0, \dots, n-1,$$

where $\omega = e^{i\frac{2\pi}{n}}$ is the n -th root of the unity. Then the inverse transform is

$$z_\ell = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} Z_k \omega^{ik\ell}, \quad \ell = 0, \dots, n-1,$$

This is usually written as $\mathbb{Z} = \mathcal{F}\{\mathbf{z}\} = \mathbf{F}\mathbf{z}$ and $\mathbf{z} = \mathcal{F}^{-1}\{\mathbf{z}\} = \overline{\mathbf{F}}\mathbf{z}$, where \mathbf{F} is the matrix with entries $f_{k\ell} = \frac{1}{\sqrt{n}}\omega^{-k\ell}$. Here we use the property of the discrete Fourier transform

$$\mathcal{F}(\mathbf{z} * \mathbf{y}) = \sqrt{n}(\mathcal{F}\{\mathbf{z}\} \circ \mathcal{F}\{\mathbf{y}\})$$

where $*$ is the cyclic convolution, defined as

$$(\mathbf{z} * \mathbf{y})_i = \sum_{j=0}^{n-1} z_j \mathbf{y}_{i-j \pmod{n}}, \quad i = 0, \dots, n-1,$$

and \circ is the entrywise product of vectors

$$(\mathbf{z} \circ \mathbf{y})_i = z_i \mathbf{y}_i, \quad i = 0, \dots, n-1.$$

The Alegre digraph revisited

Let us see the application of this method by using again the Alegre digraph. First, we consider the vectors of coefficients of the involved polynomials:

$$\begin{aligned} 1 &\rightarrow \mathbf{z}_0 = (1, 0, 0, 0, 0); \\ z &\rightarrow \mathbf{z}_1 = (0, 1, 0, 0, 0); \\ z^4 &\rightarrow \mathbf{z}_4 = (0, 0, 0, 0, 1); \\ z + z^4 &\rightarrow \mathbf{z}_{1,4} = \mathbf{z}_1 + \mathbf{z}_4. \end{aligned}$$

with corresponding Fourier transforms (with entries rounded to three decimals):

$$\begin{aligned} \mathbf{y}_0 &= \mathcal{F}\{\mathbf{z}_0\} = (0.447, 0.447, 0.447, 0.447, 0.447); \\ \mathbf{y}_1 &= \mathcal{F}\{\mathbf{z}_1\} = (0.447, 0.138 - 0.425i, -0.362 - 0.263i, -0.362 + 0.263i, 0.138 + 0.425i); \\ \mathbf{y}_4 &= \mathcal{F}\{\mathbf{z}_4\} = (0.447, 0.138 + 0.425i, -0.362 + 0.263i, -0.362 - 0.263i, 0.138 - 0.425i); \\ \mathbf{y}_{1,4} &= \mathbf{y}_1 + \mathbf{y}_4. \end{aligned}$$

This leads to the matrices:

$$\mathbf{Y}_0 = \begin{pmatrix} 0 & 0.447 & 0.447 & 0 & 0 \\ 0 & 0.447 & 0.447 & 0 & 0 \\ 0 & 0 & 0 & 0.894 & 0 \\ 0.447 & 0 & 0 & 0 & 0.447 \\ 0.447 & 0 & 0 & 0 & 0.447 \end{pmatrix};$$

$$\begin{aligned}
\mathbf{Y}_1 &= \begin{pmatrix} 0 & 0.447 & 0.447 & 0 & 0 \\ 0 & 0.138 + 0.425i & 0.138 + 0.425i & 0 & 0 \\ 0 & 0 & 0 & 0.276 & 0 \\ 0.447 & 0 & 0 & 0 & 0.447 \\ 0.138 - 0.425i & 0 & 0 & 0 & 0.138 - 0.425i \end{pmatrix}; \\
\mathbf{Y}_2 &= \begin{pmatrix} 0 & 0.447 & 0.447 & 0 & 0 \\ 0 & -0.362 + 0.263i & -0.362 + 0.263i & 0 & 0 \\ 0 & 0 & 0 & -0.7236 & 0 \\ 0.447 & 0 & 0 & 0 & 0.447 \\ -0.362 - 0.263i & 0 & 0 & 0 & -0.362 - 0.263i \end{pmatrix}; \\
\mathbf{Y}_3 &= \begin{pmatrix} 0 & 0.447 & 0.447 & 0 & 0 \\ 0 & -0.362 - 0.263i & -0.362 - 0.263i & 0 & 0 \\ 0 & 0 & 0 & -0.7236 & 0 \\ 0.447 & 0 & 0 & 0 & 0.447 \\ -0.362 + 0.263i & 0 & 0 & 0 & -0.362 + 0.263i \end{pmatrix}; \\
\mathbf{Y}_4 &= \begin{pmatrix} 0 & 0.447 & 0.447 & 0 & 0 \\ 0 & 0.138 - 0.425i & 0.138 - 0.425i & 0 & 0 \\ 0 & 0 & 0 & 0.276 & 0 \\ 0.447 & 0 & 0 & 0 & 0.447 \\ 0.138 + 0.425i & 0 & 0 & 0 & 0.138 + 0.425i \end{pmatrix};
\end{aligned}$$

Then, for instance, the $(0,0)$ -entries of the matrices $\mathbf{Y}_0^4, \mathbf{Y}_1^4, \mathbf{Y}_2^4, \mathbf{Y}_3^4, \mathbf{Y}_4^4$ are the entries of the vector $\mathbf{y}_{0,0} = (0.160, 0.015, 0.105, 0.105, 0.015)$, with inverse Fourier transform $(0.179, 0, 0.089, 0.089, 0)$. Hence, since $(\sqrt{5})^3(0.179, 0, 0.089, 0.089, 0) \approx (2, 0, 1, 1, 0)$, we conclude that

$$(\mathbf{B}(z)^4)_{0,0} = 2 + z^2 + z^3,$$

in concordance with (4).

4.2 The case of non-cyclic groups

In the case when G is not a cyclic group, we can generalize the above matrix representation in the following manner: Let G be a group with generating set $\Delta = \{g_0, \dots, g_{m-1}\}$. Given two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{N}^m$ (where $\mathbb{N} = \{0, 1, 2, \dots\}$), their G -convolution $\mathbf{a} *_G \mathbf{b}$ is defined to be the vector of \mathbb{N}^m with components

$$(\mathbf{a} *_G \mathbf{b})_i = \sum_{j,k: g_j g_k = g_i} a_j b_k.$$

Let $\Gamma = (V, E)$ be a digraph with voltage assignment α on the group G . Its G -representation matrix \mathbf{B}_G is a square matrix indexed by the vertices of Γ , and whose elements are vectors of \mathbb{N}^m , $(\mathbf{B}_G)_{uv} = \mathbf{b}_{uv}$, where

$$(\mathbf{b}_{uv})_i = \begin{cases} 1 & \text{if } \exists uv \in E : \alpha(uv) = g_i, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, \dots, m$. The product of \mathbf{B}_G by itself, denoted by \mathbf{B}_G^2 , has entries

$$\mathbf{b}_{uv}^{(2)} = \sum_{w \in V} \mathbf{b}_{uw} *_G \mathbf{b}_{wv},$$

and the power matrix \mathbf{B}_G^ℓ is computed as expected. The following result shows how such a power matrix contains the information about the walks in the lifted digraph Γ^α .

Lemma 4.2. *If the uv -entry of \mathbf{B}_G^ℓ is $\mathbf{b}_{uv}^{(\ell)} = (\beta_1, \beta_2, \dots, \beta_n)$, then, for every $i = 1, \dots, n$, there are β_i walks of length ℓ from vertex (u, h) , $h \in G$, to vertex (v, hg_i) of the lifted graph Γ^α . \square*

In particular, if G is an Abelian group, say $G = \mathbb{Z}_{k_1} \times \dots \times \mathbb{Z}_{k_n}$, with $m = |G| = \prod_{i=1}^n k_i$, the vectors representing the entries of \mathbf{B}_G can be replaced by polynomials with n variables z_1, \dots, z_n . Namely, $(\mathbf{B}_G)_{uv} = \sum_{i_1, \dots, i_n} \alpha_{i_1, \dots, i_n} z_1^{i_1} \dots z_n^{i_n}$, where

$$\alpha_{i_1, \dots, i_n} = \begin{cases} 1 & \text{if } \exists uv \in E : \alpha(uv) = (g_1, \dots, g_m) \in G, \\ 0 & \text{otherwise.} \end{cases}$$

Then, as in the case of cyclic groups, we compute the powers \mathbf{B}_G^ℓ using the standard polynomial multiplication, and the coefficients of the resulting polynomial entries gives the same information described in Lemmas 4.1 and 4.2. As an example of this case, in the next section we use the known fact that the Hoffman-Singleton graph can be constructed as a lift of a base graph on two vertices, with voltages in the group $\mathbb{Z}_5 \times \mathbb{Z}_5$.

5 The spectrum of the lifted digraph

Apart from the obvious approach of computing the characteristic polynomial of the adjacency matrix, we aim to get a more simple method of computing the whole spectrum of the lifted digraph Γ^α . First, we have the following simple result, which is a consequence of Lemma 2.1.

Corollary 5.1. *Let Γ be a base digraph with vertices u_1, \dots, u_n , and a given voltage assignment α on the group G with generating set $\{g_0, \dots, g_{m-1}\}$. Let $\mathbf{B} = \sum_{i=0}^{m-1} \mathbf{A}_i$, where \mathbf{A}_i is the adjacency matrix of the subgraph of Γ with arc set $\alpha^{-1}(g_i)$. Then,*

$$\text{sp } \mathbf{B} \subset \text{sp } \Gamma^\alpha. \quad \square$$

For example, the Alegre digraph has quotient matrix $\mathbf{B} = \mathbf{B}(1)$ given in (4), with spectrum $\text{sp } \mathbf{B} = \{2, 0^{(2)}, i, -i\}$. In fact, the spectrum of the Alegre digraph is

$$\text{sp } \Gamma^\alpha = \left\{ 2, 0^{(10)}, i^{(5)}, -i^{(5)}, \frac{1}{2}(-1 + \sqrt{5})^{(2)}, \frac{1}{2}(-1 - \sqrt{5})^{(2)} \right\}, \quad (6)$$

where, in agreement with Corollary 5.1, we observe that $\text{sp } \mathbf{B} \subset \text{sp } \Gamma^\alpha$. Notice also that, in this case, the other eigenvalues of Γ are those ($\neq 2$) of the undirected cycle C_5 , whose spectrum is

$$\text{sp } C_5 = \left\{ 2, \frac{1}{2}(-1 + \sqrt{5})^{(2)}, \frac{1}{2}(-1 - \sqrt{5})^{(2)} \right\}.$$

5.1 The case of cyclic groups

The following result shows how the spectrum of the lift Γ^α can be completely determined from the spectrum of the polynomial matrix $\mathbf{B}(z)$ in the case when voltages are taken in a cyclic group. Here we assume that entries of $\mathbf{B}(z)$ are elements of the polynomial ring $\mathbb{C}(z)$. If Γ has r vertices and the cyclic group has order k , the characteristic polynomial $P(\lambda, z) = \det(\lambda I - \mathbf{B}(z))$ is, technically, a polynomial in two complex variables λ and z , of degree r in λ and at most $k - 1$ in z . As we shall see, however, later we will be interested only in the corresponding polynomials in λ arising by substituting suitable complex roots of unity for z .

Proposition 5.2. *Let $\Gamma = (V, E)$ be a base digraph on r vertices, with a voltage assignment α in \mathbb{Z}_k . Let $P(\lambda, z) = \det(\lambda I - \mathbf{B}(z))$ be the characteristic polynomial of the polynomial matrix $\mathbf{B}(z)$ of the voltage digraph (Γ, α) . For $j = 0, \dots, k - 1$, let ω_j be the distinct k -th complex roots of unity. Then, the spectrum of the lift Γ^α is the multiset of kr roots λ of the k polynomials $P(\lambda, \omega_j)$ of degree r each, where $0 \leq j \leq k - 1$; formally,*

$$\text{sp } \Gamma^\alpha = \{ \lambda_{i,j} : P(\lambda_{i,j}, \omega_j) = 0, 1 \leq i \leq r, 0 \leq j \leq k - 1 \}.$$

Proof. Although entries of $\mathbf{B}(z)$ are polynomials in a complex variable z , in what follows let z be any fixed complex number for which we will make appropriate choices later. For our $z \in \mathbb{C}$, let $\mathbf{x}(z) = (x_u(z))_{u \in V}$ be an eigenvector corresponding to an eigenvalue $\lambda(z)$ of the matrix $\mathbf{B}(z)$ of Γ ; that is,

$$\sum_{uv \in E} z^{\alpha(uv)} x_v(z) = \lambda(z) x_u(z). \quad (7)$$

Let $R(k)$ denote the set of k -th roots of unity. Making now the choice $z = \omega \in R(k)$ in (7) we obtain

$$\sum_{uv \in E} \omega^{\alpha(uv)} x_v(\omega) = \lambda(\omega) x_u(\omega).$$

Multiplying by ω^i for any (fixed) $i \in \mathbb{Z}_k$, we have

$$\sum_{uv \in E} \omega^{i+\alpha(uv)} x_v(\omega) = \lambda(\omega) \omega^i x_u(\omega). \quad (8)$$

Now, for every pair $(u, j) \in V \times \mathbb{Z}_k$, let the map $\phi_{(u,j)} : R(k) \rightarrow \mathbb{C}$ be defined as

$$\phi_{(u,j)}(\omega) = \omega^j x_u(\omega).$$

Then, as $\phi_{(v,i+\alpha(uv))}(\omega) = \omega^{i+\alpha(uv)}x_v(\omega)$, we can rewrite (8) in the form

$$\sum_{uv \in E} \phi_{(v,i+\alpha(uv))}(\omega) = \lambda(\omega)\phi_{(u,i)}(\omega).$$

But this means that $\lambda(\omega)$ is an eigenvalue of the lift, corresponding to the eigenvector $\phi(\omega) := (\phi_{(u,i)}(\omega))_{(u,i) \in V \times \mathbb{Z}_k}$.

Since $\lambda = \lambda(\omega)$ is a root of the characteristic polynomial $P(\lambda, \omega)$, we obtain, in this way, a total of rk eigenvalues (including repetitions), which is the number of eigenvalues of the adjacency matrix \mathbf{A} of the lift Γ^α .

According to the properties of the polynomial matrix, if

$$(\mathbf{B}(z)^\ell)_{uu} = \alpha_{u,0}^{(\ell)} + \alpha_{u,1}^{(\ell)}z + \cdots + \alpha_{u,k-1}^{(\ell)}z^{k-1}$$

then, the total number of rooted closed ℓ -walks in Γ^α is

$$\text{tr}(\mathbf{A}^\ell) = \sum_{\lambda \in \text{sp } \Gamma^\alpha} \lambda^\ell = k \sum_{u \in V} \alpha_{u,0}^{(\ell)}.$$

But, since $\sum_{j=0}^{k-1} \omega_j^\ell = 0$ for every $j, \ell \neq 0$, we have that

$$\alpha_{u,0}^{(\ell)} = \frac{1}{k} \sum_{j=0}^{k-1} (\mathbf{B}(\omega_j))_{uu}.$$

Then,

$$\sum_{\lambda \in \text{sp } \Gamma^\alpha} \lambda^\ell = \text{tr}(\mathbf{A}^\ell) = \sum_{u \in V} \sum_{j=0}^{k-1} (\mathbf{B}(\omega_j))_{uu} = \sum_{j=0}^{k-1} \text{tr}(\mathbf{B}(\omega_j)) = \sum_{j=0}^{k-1} \sum_{\mu \in \text{sp } \mathbf{B}(\omega_j)} \mu^\ell.$$

Since this is true for any value of $\ell \geq 0$, both (multi)sets of eigenvalues must coincide (see for example Gould [10]). \square

Returning to the example of the Alege digraph, its polynomial matrix $\mathbf{B}(z)$ in (5) has eigenvalues 0, with multiplicity 2, and i , $-i$, and $z + \frac{1}{z}$ with multiplicity 1. Then, from Proposition 5.2, evaluating them at the 5-th roots of unity ω_i , for $i = 0, 1, 2, 3, 4$, we get the complete spectrum (6) of the digraph, see Table 1.

5.2 The digraphs $\mathcal{P}(n, p_1, p_2)$

As another example of application, consider the following family of digraphs, which contains, as a particular case, the well-known generalized Petersen graphs; see for example Gera and Stănică [8]. (Notice that all of our results apply also to graphs, since they can

$z \setminus \lambda(z)$	$0^{(2)}$	$i^{(1)}$	$-i^{(1)}$	$(z + \frac{1}{z})^{(1)}$
1	$0^{(2)}$	$i^{(1)}$	$-i^{(1)}$	$2^{(1)}$
ω	$0^{(2)}$	$i^{(1)}$	$-i^{(1)}$	$\frac{1}{2}(-1 + \sqrt{5})^{(1)}$
ω^2	$0^{(2)}$	$i^{(1)}$	$-i^{(1)}$	$\frac{1}{2}(-1 - \sqrt{5})^{(1)}$
ω^3	$0^{(2)}$	$i^{(1)}$	$-i^{(1)}$	$\frac{1}{2}(-1 + \sqrt{5})^{(1)}$
ω^4	$0^{(2)}$	$i^{(1)}$	$-i^{(1)}$	$\frac{1}{2}(-1 - \sqrt{5})^{(1)}$

Table 1: The eigenvalues of the Alegre digraph.

be just considered as symmetric digraphs where each digon represents an edge.) Given an integer n and two polynomials $p_1, p_2 \in \mathbb{R}_{n-1}[z]$, the digraph $\mathcal{P}(n, p_1, p_2)$ is obtained by ‘cyclically joining’ n (undirected) edges. More precisely, $\mathcal{P}(n, p_1, p_2)$ is the lift of the base digraph with polynomial matrix

$$\mathbf{B}(z) = \begin{pmatrix} p_1(z) & 1 \\ 1 & p_2(z) \end{pmatrix}.$$

Then, from Proposition 5.2, the eigenvalues of $\mathcal{P}(n, p_1, p_2)$ are

$$\lambda_{i,j} = \frac{1}{2} \left[p_1(\omega_j) + p_2(\omega_j) \pm \sqrt{(p_1(\omega_j) + p_2(\omega_j))^2 + 4} \right], \quad i = 0, 1, \quad j = 0, \dots, n-1, \quad (9)$$

where ω_j is the j -th n -root of unity. This includes the results of Gera and Stănică [8] about the spectra of the generalized Petersen graphs, denoted by $P(n, k)$, where $n \in \mathbb{N}$ and $k \in \mathbb{Z}_n$. Recall that $P(n, k)$ has $2n$ vertices, u_0, \dots, u_{n-1} and v_0, \dots, v_{n-1} , with edges (u_i, v_i) , $(u_i, u_{i\pm 1})$, and $(v_i, v_{i\pm k})$ (arithmetic understood modulo n). Then, in our context, $P(n, k)$ corresponds to the case when we take the polynomials $p_1(z) = z + z^{-1}$ and $p_2(z) = z^k + z^{-k}$. Thus, when we evaluate these polynomials at ω_j , we get

$$p_1(\omega_j) = 2 \cos\left(\frac{j2\pi}{n}\right) \quad \text{and} \quad p_2(\omega_j) = 2 \cos\left(\frac{jk2\pi}{n}\right),$$

and (9) yields the result in [8] (Theorem 2.4 and Corollary 2.5).

5.3 The Hoffman-Singleton graph

As commented in the previous section, when the group G is Abelian, the lift can be constructed similarly by using multivariate polynomials as the entries of \mathbf{B}_G . Moreover the above spectral theory also works with the natural changes. Instead of ‘repeating ourselves’ by giving the analogous of Proposition 5.2, we follow an example with the Hoffman-Singleton (HS) graph, first discovered in [13]. As it is common knowledge, this is a Moore 7-regular graph with 50 vertices, and diameter 2 (for more details, see e.g. Hoffman and Singleton [13] or Godsil [9]). As it was shown by Šiagiová [15] (see also Mirka

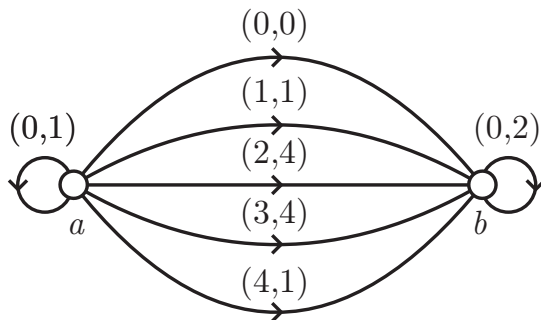


Figure 4: A base graph for the HS graph.

and Širáň [14]), the HS graph can be obtained as the lift of a base graph G consisting of two vertices with voltages in the group $\mathbb{Z}_5 \times \mathbb{Z}_5$, see Figure 4, and polynomial matrix

$$\mathbf{B}(w, z) = \begin{pmatrix} w + \frac{1}{w} & 1 + zw + z^2w^4 + z^3w^4 + z^4w^4 \\ 1 + \frac{1}{zw} + \frac{1}{z^2w^4} + \frac{1}{z^3w^4} + \frac{1}{z^4w^4} & w^2 + \frac{1}{w^2} \end{pmatrix}.$$

Then, by giving to (w, z) the 25 possible values in $R(5) \times R(5)$, the eigenvalues of $\mathbf{B}(w, z)$ are shown in Table 2.

$z \backslash w$	1	ω	ω^2	ω^3	ω^4
1	7, -3	2, -3	2, -3	2, -3	2, -3
ω	2, 2	2, -3	2, -3	2, -3	2, -3
ω^2	2, 2	2, -3	2, -3	2, -3	2, -3
ω^3	2, 2	2, -3	2, -3	2, -3	2, -3
ω^4	2, 2	2, -3	2, -3	2, -3	2, -3

Table 2: The eigenvalues of the HS graph.

As a consequence, we have that the spectrum of the HS graph is

$$\text{sp HS} = \{7^{(1)}, 2^{(28)}, -3^{(21)}\},$$

as it is well known, as the HS graphs is strongly regular.

References

- [1] I. Alegre, M. A. Fiol, and J. L. A. Yebra, Some large graphs with given degree and diameter, *J. Graph Theory* **10** (1986) 219–224.
- [2] J. Bang-Jensen and G. Gutin, *Digraphs. Theory, Algorithms and Applications*. Second edition. Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2009.
- [3] E. T. Baskoro, L. Branković, M. Miller, J. Plesník, J. Ryan, and J. Širáň, Large digraphs with small diameter: A voltage assignment approach, *JCMCC* **24** (1997) 161–176.
- [4] G. Chartrand and L. Lesniak, *Graphs & Digraphs*, third ed., Chapman and Hall, London, 1996.
- [5] P. J. Davis, *Circulant Matrices*, John Wiley & Sons, New York, 1979.
- [6] R. Diestel, *Graph Theory* (4th ed.), Graduate Texts in Mathematics **173**, Springer-Verlag, Heilderberg, 2010.
- [7] M. A. Fiol, J. L. A. Yebra, and I. Alegre, Line digraph iterations and the (d, k) digraph problem, *IEEE Trans. Comput.* **C-33** (1984) 400–403.
- [8] R. Gera and P. Stănică, The spectrum of generalized Petersen graphs, *Australasian J. Combin.* **49** (2011) 39–45.
- [9] C. D. Godsil, *Algebraic Combinatorics*, Chapman and Hall, New York, 1993.
- [10] H. W. Gould, The Girard-Waring power sum formulas for symmetric functions and Fibonacci sequences, *Fibonacci Quart.* **37** (1999), no. 2, 135–140.
- [11] J. J. Gross, Voltage graphs, *Discrete Math.* **9** (1974) 239–246.
- [12] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, Wiley, New York, 1987.
- [13] A.J. Hoffman and R.R. Singleton, On Moore graphs with diameters 2 and 3, *IBM J. Res. Develop.* **4** (1960) 497–504.
- [14] M. Miller and J. Širáň, Moore graphs and beyond: A survey of the degree/diameter problem, *Electron. J. Combin.* **20(2)** (2013) #DS14v2.
- [15] J. Šiagiová, A note on the McKay-Miller-Širáň graphs, *J. Combi. Theory Ser B* **81** (2001), 205–208.