Stochastic Theory of Synchronization Transitions in Extended Systems

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We propose a general Langevin equation describing the universal properties of synchronization transitions in extended systems. By means of theoretical arguments and numerical simulations we show that the proposed equation exhibits, depending on parameter values: (i) a continuous transition in the bounded Kardar-Parisi-Zhang universality class, with a zero largest Lyapunov exponent at the critical point; (ii) a continuous transition in the directed percolation class, with a negative Lyapunov exponent, or (iii) a discontinuous transition (that is argued to be possibly just a transient effect). Cases (ii) and (iii) exhibit coexistence of synchronized and unsynchronized phases in a broad (fuzzy) region. This reproduces almost all of the reported features of synchronization transitions, providing a unified theoretical framework for the analysis of synchronization transitions in extended systems.

Arrays of coupled oscillators appear ubiquitously in nature, and their mutual synchronization is undoubtedly one of the most intriguing phenomena appearing in complex systems [1]. Some well-known instances of synchronizing systems are chemical reactions [2], neuronal networks [3], flashing fireflies, Josephson junctions [4], electronic circuits [5], and semiconductor lasers [6]. In particular, coupled map lattices (CMLs) [7], initially introduced as simple models of spatiotemporal chaos [8], have received a great deal of attention, as models of synchronization in spatially extended systems (another possibility is to analyze discrete cellular automata [9,10]). In this context, it was observed that distant patches of a given CML can oscillate in phase [8], a phenomenon related to the Kardar-Parisi-Zhang (KPZ) equation [11] describing the roughening of growing interfaces [12]. Afterwards, it was also realized that when two different replicas of a same CML are *locally* coupled, they become synchronized if the coupling strength is large enough [5,9,13]. Also *globally* coupled CML can achieve mutual synchronization [14] with interesting implications in neuroscience [15]. Last but not least, different replicas of a CML can be synchronized if they are coupled to a sufficiently large common external random noise, even if they are not directly coupled to each other [16–18].

In all these examples, there is a transition from a chaotic or unsynchronized phase in which perturbations grow, and two replicas evolve independently, to a synchronized phase in which memory of the initial difference is asymptotically lost and replicas synchronize. This synchronization transition (ST) very much resembles other nonequilibrium critical phenomena, particularly transitions into *absorbing states* [19], and the determination of its universal properties has become the subject of many recent studies [9,18,20,21] as well as the main motivation of this Letter. Within this context, a major contribution was made by Pikovsky and Kurths (PK) who proposed a stochastic model for the dynamics of perturbations in locally coupled CML, in which, under very general conditions, the difference-between-two-replicas field can be described by the so-called multiplicative noise (MN) Langevin equation [20]:

$$\partial_t \phi(x, t) = -a\phi - b\phi^2 + D\nabla^2\phi(x, t) + \sigma\phi(x, t)\eta, \tag{1}$$

where $\phi(x, t)$ is the difference field (or "synchronization error"), a, b, σ^2, and $D > 0$ are parameters, and η a delta correlated Gaussian white noise with zero average. It is worth remarking that the nonlinear saturation term was added in [20] by hand, in order to ensure that the synchronization error remains bounded from above (i.e., of the order of the original field variables) [20,21].

Equation (1) has been extensively studied, unveiling a rich phenomenology, including a novel nonequilibrium phase transition [22]. This equation is related to (i) the problem of directed polymers in random media [12,20], and (ii) the KPZ equation in the presence of a limiting upper wall [22]. To verify this last, particularly illuminating relation, it is useful to perform a Cole-Hopf transformation $\phi(x, t) = \exp[h(x, t)]$ [21,22], obtaining

$$\partial_t h(x, t) = -a - b\exp h + D\nabla^2 h + D(\nabla h)^2 + \sigma\eta, \tag{2}$$

which is indeed a KPZ equation plus a bounding term— the upper wall—pushing positive values of the height h towards negative values [20,22]. The synchronized or absorbing phase corresponds in this language to a de-pinned interface, falling continuously towards minus infinity, governed by KPZ dynamics. The unsynchronized or active phase, on the other hand, translates into a pinned-to-the-wall interface.

Using the PK theory, it could be naively concluded that all ST belong to the MN universality class [20]. On the other hand, it has been known for some time that many ST in discrete cellular automata (CA) belong generically to the directed percolation (DP) universality class [9].

This difference led Grassberger to conjecture that the discreteness of the local field variables is the factor that determines a ST universality class: MN for continuous variables and DP for discrete ones [9]. Following Grassberger, the key difference lies in the fact that while for continuous variables one has "incomplete death" (fluctuations can always generate unsynchronized regions from synchronized ones [10]), discrete systems do not allow such a process ("complete death"). This difference between MN and DP has been characterized in Ref. [23].

Some counterexamples, however, have recently shown that this conjecture does not apply all the way: in some cases, CML (continuous variables) can exhibit DP behavior [18,21] or present a discontinuity at the transition point [17]. Based on these observations a new global picture has emerged: there are two types of ST depending on whether the largest Lyapunov exponent, Λ, is zero or negative at the transition. In the first case, the transition is MN-like and controlled by linear stability effects (Λ changes sign at the transition). In the second case the transition is either DP or discontinuous, depending on microscopic details, and the transition is located where the propagation velocity of the nonlinear perturbations changes sign, while Λ is negative.

The aim of this Letter is to construct a stochastic model, including Eq. (1) as a particular case, able to reproduce in a unified framework all the aforementioned phenomenology. With this purpose, we consider the following Langevin equation:

$$\partial_t \phi(x, t) = -\frac{\delta V(\phi)}{\delta \phi} + D\nabla^2 \phi(x, t) + \phi(x, t)\eta(x, t),$$
$$V(\phi) = a\phi^2 + b\phi^3 + c\phi^4, \tag{3}$$

which has appeared in the literature in the context of nonequilibrium wetting and depinning transitions [24,25]. This is a generalization of Eq. (1) including saturation effects in a more general way. In particular, for $b > 0$, the parameter c is irrelevant in the renormalization group sense, and Eq. (3) behaves as Eq. (1) (i.e., it belongs to the MN universality class). For $b < 0$, however, new physics emerges [24] (note that in the KPZ language $b < 0$ translates into the presence of an *attractive* upper wall). This second possibility is required to describe transitions in which a nonvanishing value of the synchronization-error field is a local attractor of the dynamics. For example, for discrete boolean CA, the value $\phi(x) = 1$ should be a minimum of the local deterministic dynamics, and therefore $b < 0$ is required to describe them within this formalism. Also, for discontinuous CML [18,21] this term describes finite jumps from one map sector to another.

In Fig. 1 we show a plot of the exact single-site effective potential in the zero-dimensional limit in the interface representation (i.e., the stationary solution of the associated Fokker-Planck equation after performing the Cole-

Hopf transformation; see Ref. [23]). For $b < 0$ a new minimum (absent for $b > 0$), present even in the synchronized phase, appears—the KPZ wall becomes attractive. As long as $b > 0$, c is irrelevant [26], the transition is in the MN universality class, and it occurs where the depinned interface loses its stability, implying that the average velocity $\langle \dot{h} \rangle$ changes sign at the transition point. It is also well known [20] that $\langle \dot{h} \rangle$ coincides exactly with the largest Lyapunov exponent, Λ. Therefore, for $b > 0$, the situation is completely analogous to ST in the MN class where Λ changes sign at the transition. On the other hand, for $b < 0$ [26] the transition can be easily shown not to occur at $\langle \dot{h} \rangle = 0$ [24]. Instead, there is a finite interval of values of a, $[a^*, a_c(b)]$, in which phase coexistence is found, ergodicity is broken, and the final state depends upon initial conditions [24]. Within this interval, pinned interfaces remain usually pinned, and depinned interfaces escape towards minus infinity at a finite velocity. Below (above) $a = a^*$ [$a_c(b)$] depinned (pinned) interfaces lose their stability. Consequently, at a_c the depinned phase is stable (contrary to what happens for $b > 0$) and $\langle \dot{h}(a_c) \rangle = \Lambda$ is negative, in analogy with ST in either the DP class or discontinuous transitions where $\Lambda < 0$ (as explained afterwards the first-order transition could be a transient effect).

In order for Eq. (3) to constitute a valid theory for ST, a DP critical point should be observed in some parameter range for $b < 0$. To the best of our knowledge DP behavior has not been reported for this equation. As we will show, DP behavior may emerge for $b < 0$ as the forthcoming numerical results show.

To simulate Eq. (3), it has been discretized, using the Ito prescription, in a one-dimensional lattice, using 1 and 0.001 as discretization space and time meshes, respectively. Systems sizes run from $L = 50$ to $L = 2000$, reaching up to 2×10^8 iteration time steps in the longest runs. Averages are performed over 100 to 10 000 runs (up to 2×10^6 in some cases). Depending on the values explored, different universality classes have been found:

(i) MN.—Setting $b = c = D = \sigma^2 = 1$, we observe a transition around $a_{mn} \approx -0.01$ in the MN class [22]. We
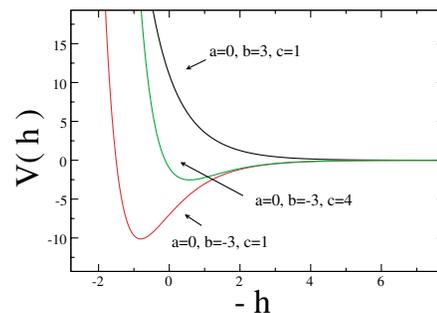

FIG. 1 (color online). Effective potential in the 0-dimensional case in terms of $-h$ for values of a in the depinned phase. Note the attractive effect for $b < 0$.

have also verified that the asymptotic properties do not depend on parameter values as long as $b > 0$.

(ii) First-order transition.—Changing the sign of b (setting it to -3), while keeping the rest of the parameters as in the previous case, we find a discontinuous transition at $a_{\mathrm{ft}} = 1.85(1)$. In this regime we recover all the phenomenology reported in Ref. [24]: an apparent first-order phase transition, a broad region of phase coexistence, triangular patterns in the interface representation, etc. However, by enlarging the value of c the attractive well becomes less deep (see Fig. 1), the discontinuity of the transition is reduced, and it becomes more and more difficult to clearly establish the order of the transition. Indeed, following a claim by Hinrichsen [27] it could well be the case that this discontinuous transition is only a transient effect, asymptotically crossing over to DP for very large system sizes.

(iii) DP.—In order to weaken the mechanism leading to a first-order transition—the mechanism responsible for destabilizing depinned patches [25]—we consider a shallow well, $c = 4$, reduce the diffusion constant to $D = 0.1$, and keep $b = -3$ and $\sigma = 1$. In Fig. 2(a) we show how the order parameter decays at criticality [$a_c = 0.1865(3)$] with a DP exponent $\theta \approx 0.16$ for $L = 1000$, and also how it saturates for smaller system sizes. From the scaling of the saturation values at criticality, we estimate $\beta/\nu = 0.245(15)$, in good agreement with the DP value $\beta/\nu = 0.252 \cdots$ [19]. As an independent check of universality, we have also studied the dependence of $\langle \phi \rangle$ on a at a fixed system size $L = 1000$. A fit to the form $\langle \phi \rangle \sim (a_c - a)^\beta$ provides the independent estimates $a_c = 0.1871(5)$ and $\beta = 0.28(1)$, again compatible with DP; see Fig. 2(b). To further confirm DP scaling, we have

computed different moment and cumulant ratios at criticality. It has been recently shown that moment and cumulant ratios are universal at the DP fixed point, and they are not strongly affected by finite size effects. In particular, we have measured $r_1 = (m_2 - m_1^2)/m_1^2$ (analogous to Binder's cumulant for these transitions) and $r_2 = m_4/m_2^2$, where m_i is the ith moment. In Fig. 3 we show their time evolution at criticality for different system sizes. Observe the convergence to the extrapolated DP values for infinite lattices $r_1 \approx 0.173$ and $r_2 \approx 1.554$ [28]. Slight deviations from the values reported in Ref. [28] are due to the lack of precision in the determination of the critical point.

We have also measured a different order parameter: the density of lattice sites with h smaller than a certain reference threshold—i.e., the number of sites pinned at the attractive wall. We have checked that it scales as $\langle \phi \rangle$ (DP) at criticality, independently of the arbitrary threshold. Also, as a complementary test to verify the generation of DP scaling for $b < 0$, we have studied the nature of fluctuations. In particular, we determine the distribution of the output order parameter, $P(\langle \phi(t + 1) \rangle)$, after one iteration as a function of the input order parameter value $\langle \phi(t) \rangle$. It is not difficult to verify that its variance is proportional to $\sqrt{\phi(t)}$, as expected for the DP class [19]. In other words, the presence of an attractive wall generates a DP-like noise $\sqrt{\phi(x,t)}\eta$, to be included in Eq. (1). While in the absence of such a wall the leading fluctuations are those intrinsic to the interface (linear in ϕ), and the attractive wall generates a new noise term, related to the fluctuation of the number of sites pinned by the potential well. This noise dominates the asymptotic behavior, driving it from a MN type to DP. Let us remark that even though the transition is DP like in this regime, it has some first-order flavor. Indeed, as stressed before, there is a finite phase coexistence interval $[a^*, a_c]$, within which pinned and depinned interfaces coexist with a phenomenology very similar to spatiotemporal intermittency [24]. For $a < a^*$ ($a > a_c$) only the pinned
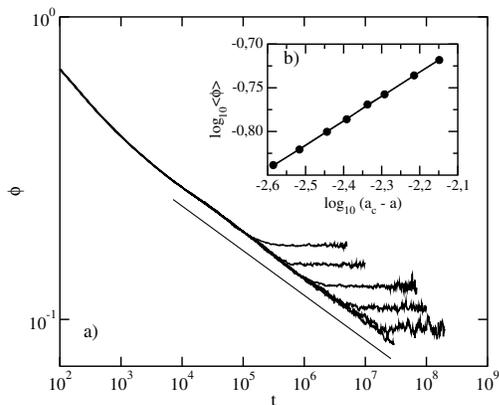

FIG. 2. (a) Log-log plot of the order parameter time evolution (averaged over surviving, pinned, runs) at the critical point, for system sizes (top to bottom) $L = 25, 50, 100, 200, 400$, and 1000. Both the main slope, $\theta = 0.159(5)$, and the scaling of saturation values, $\beta/\nu = 0.245(15)$, reveal DP-scaling behavior. (b) Log-log plot of the average order parameter as a function of $a_c - a$. The solid line is a fit to the form $\langle \phi \rangle \sim (a_c - a)^\beta$, with an exponent $\beta = 0.28(1)$, also compatible with DP.
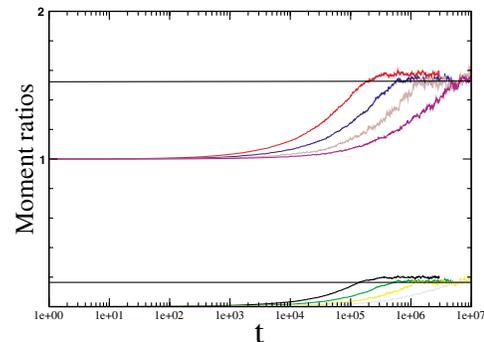

FIG. 3 (color online). Time evolution of the moment ratios reported in the text for $L = 25, 50, 100$, and 200 at the critical point. Note the convergence for large sizes to the expected DP values (straight lines).

(depinned) phase is stable. A qualitatively similar ST has been found for CMLs and the term *fuzzy transition* has been coined to describe it [16].

From the theoretical side, we have performed a renormalization group perturbative calculation of Eq. (3) along the same lines followed to study Eq. (1) [22]. We find a weak coupling trivial fixed point (above $d = 2$), and runaway trajectories above some separatrix analogously to what happens for Eq. (1) [22] (in particular, in $d = 1$ these are the only solutions). The possible existence of a DP strong-coupling fixed point cannot therefore be settled within this perturbative approach. This calls for new theoretical studies (see [29]).

To sum up, all the observed universal features of synchronizing CML are well reproduced by a simple Langevin equation, Eq. (3), including a potential with up to quartic terms, diffusion, and multiplicative noise, that is equivalent to a KPZ equation with a bounding upper wall. A MN transition is found for $b > 0$ (nonattractive wall), while for $b < 0$ (attractive wall) firstorder or DP transitions are observed depending upon parameter values. In the MN case the scaling is controlled by interface fluctuations, while in the second case it is dominated by fluctuations of the number of sites pinned by the attracting wall (generating a DP type of noise). This second mechanism has been shown to exhibit DP scaling (probably, DP with many absorbing states [19,30]) or, alternatively, to induce (apparently) discontinuous transitions; in both cases, the transition is "fuzzy": there is a finite coexistence region between synchronized and unsynchronized phases. In this way we have constructed a general framework to study ST in spatially extended systems and, in particular, in coupled map lattices, successfully reproducing all their universal critical features. It would be very interesting to derive analytically the Reggeon field theory Langevin equation describing DP [19] from Eq. (1) in the adequate parameter range, and to develop clean-cut criteria establishing the nature of the transition as a function of bare parameter values (i.e., the full phase diagram). These issues will be addressed in a future publication.

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