

ON DOUBLING AND VOLUME: CHAINS

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ABSTRACT. The well-known Freiman–Ruzsa Theorem provides a structural description of a set A of integers with $|2A| \leq c|A|$ as a subset of a d -dimensional arithmetic progression P with $|P| \leq c'|A|$, where d and c' depend only on c . The estimation of the constants d and c' involved in the statement has been the object of intense research. Freiman conjectured in 2008 a formula for the largest volume of such a set. In this paper we prove the conjecture for a general class of sets called chains.

1. INTRODUCTION

The Freiman–Ruzsa theorem giving the structure of sets of integers with small doubling is one of the deep results in Additive Number Theory:

Theorem 1.1 (Freiman–Ruzsa). *Let A be a finite set of integers. If $|2A| \leq c|A|$ then there are constants d, c' depending only on c such that A is contained in a multidimensional arithmetic progression P with dimension d and cardinality $|P| \leq c'|A|$.*

The estimation of the constants d and c' involved in the statement has been the object of a long series of papers. From the first proof of Freiman [4] and Bilu [1] one can obtain a fourth exponential dependence of c' on c . The proof by Ruzsa [9], which provided an estimation of the form $c' \leq \exp(c^c)$, was subsequently refined by Chang [2], giving $d \leq c^{2+o(1)}$ and $c' \leq \exp(c^2 + o(1))$, further improved by Sanders to $d \leq c^{4/7+o(1)}$ and $c' \leq \exp(c^{4/7+o(1)})$ and eventually brought to its essentially best values $d \leq c^{1+k(\log c)^{-1/2}}$ and $c' \leq \exp(c^{1+k(\log c)^{-1/2}})$, k an absolute constant, by Schoen [11].

In a conference in Toronto in 2008, Freiman proposed a precise formula for the largest possible volume of a set A with given doubling $T = |2A|$ in terms of a specific parametrization of the value of T , see e.g. Freiman [6]. We next recall some definitions in order to give this conjectured formula for the maximum volume.

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Let G, G' be two abelian groups. Two finite sets $A \subset G$ and $B \subset G'$ are *Freiman isomorphic* (F -isomorphic for short) if there is a bijection $\phi : A \rightarrow B$ such that, for every $x, y, z, t \in G$,

$$x + y = z + t \Leftrightarrow \phi(x) + \phi(y) = \phi(z) + \phi(t),$$

in which case we write $A \cong_F B$.

A set A is in *normal form* if $\min(A) = 0$ and $\gcd(A) = 1$. Every set A is F -isomorphic to $\tilde{A} = (A - \min(A))/\gcd(A)$ which is in normal form, and \tilde{A} is the *normalization* of A . If $\min(A) = 0$, then the *reflexion* of A is defined as $A^- = -A + \max(A)$.

The *additive dimension* $\dim(A)$ of a set $A \subset G$ is the largest d such that there is a set $B \subset \mathbb{Z}^d$ not contained in a hyperplane of \mathbb{Z}^d which is F -isomorphic to A .

The *volume* $\text{vol}(A)$ of a d -dimensional set A is the minimum cardinality, among all sets $B \subset \mathbb{Z}^d$ which are F -isomorphic to A , of the convex hull of B . In particular, if A is a 1-dimensional set in normal form, then

$$\text{vol}(A) = \max(A) + 1.$$

We are interested in obtaining upper bounds for the volume of a set A of integers in terms of its cardinality $|A|$ and the cardinality of its doubling $|2A|$. We denote by

$$\text{vol}(k, T) = \max\{\text{vol}(A) : A \subset \mathbb{N}, |A| = k, |2A| = T\},$$

the maximum value of the volume of a set A of integers among all sets with cardinality k and doubling T .

A set A is *extremal* if $\text{vol}(A) = \text{vol}(|A|, |2A|)$. The following conjecture is stated in Freiman [6].

Conjecture 1 (Freiman). *Let A be a set of integers with cardinality $k = |A| \geq 4$. If*

$$(1) \quad |2A| = ck - \binom{c+1}{2} + b + 2,$$

where $2 \leq c \leq k - 2$ and $1 \leq b \leq k - c - 1$, then

$$(2) \quad \text{vol}(A) \leq 2^{c-2}(k - c + b + 1) + 1.$$

Moreover the inequality is tight and it is reached by 1-dimensional sets.

Conjecture 1 is proved for $2k - 1 \leq |2A| \leq 3k - 4$. For these values it is known that A is 1-dimensional and that its largest element is at most $k + b - 1$, the bound being tight.

Moreover the structure of extremal sets with maximum element $k+b-1$ can be described in detail (see Freiman [5] and Section 3).

According to the notation in Conjecture 1, given k and $T \in [2k-1, \binom{k}{2} + 2]$ there are uniquely defined

$$c = c(k, T) \text{ and } b = b(k, T)$$

subject to the boundary conditions $2 \leq c \leq k-2$ and $1 \leq b \leq k-c-1$, or $b=0$ and $c=2$, such that T can be expressed as the right-hand side of (1). If A has cardinality k and doubling $|2A| = T$ then we call $c = c(k, T)$ the *doubling constant* of A . Thus, for example, the doubling constant is 2 if $|2A| = 3|A| - 4$ and it is 3 if $|2A| = 3|A| - 3$, according to the structural change on A on these values of its doubling. We also denote by

$$(3) \quad \mu(k, T) = 2^{c-2}(k - c + b + 1),$$

the conjectured maximum volume minus one of a set with cardinality k and doubling T .

The main result of this paper proves Conjecture 1 for a general class of sets called chains. Let $[A] = [\min(A), \max(A)]$ denote the convex hull of a set A of integers. A set A is a *chain* if there is a sequence

$$A_3 \subset A_4 \subset \cdots \subset A_k = A$$

such that

- (i) $A_3 \cong_F \{0, 1, 2\}$,
- (ii) each A_i , $4 \leq i \leq k$, is a 1-dimensional set with i elements, doubling $|2A_i| \leq \binom{i}{2} + 1$ and $A_i \cap [A_{i-1}] = A_{i-1}$,
- (iii) A_i has largest volume among all sets B with the same cardinality and doubling as A_i and such that $B \cap [A_{i-1}] = A_{i-1}$.

Thus chains belong to a sequence of stretched out sets and are most natural candidates to be extremal sets. The validity for chains gives strong evidence to Conjecture 1, and this is the main motivation of this paper.

Moreover, the structure of chains can be described. We define the following operations on a set A :

- (i) $D(A) = \tilde{A} \cup \{2 \max(\tilde{A})\}$.
- (ii) $D_x(A) = 2 \cdot \tilde{A} \cup \{x\} = \{2a : a \in \tilde{A}\} \cup \{x\}$, x odd and $x \in 2\tilde{A} \setminus \tilde{A}$.

We also denote by $D^-(A) = D((\tilde{A})^-)$ and $D_x^-(A) = D_x((\tilde{A})^-)$. The main result in this paper is the following one:

Theorem 1.2. *Let A be a chain with $k = |A|$ and $T = |2A| \leq \binom{k}{2} + 2$. Then*

$$\text{vol}(A) = \mu(k, T) + 1.$$

Moreover, there are subchains $B \subseteq B' \subseteq A$ with $|2B| \leq 3|B| - 4$ and an integer $s \geq 0$ such that

$$A \cong_F \phi_s \phi_{s-1} \cdots \phi_1(B'),$$

where each $\phi_i \in \{D, D^-, D_x, D_x^-\}$ and $|B'| \leq |B| + 1$.

As we have already mentioned, the structure of B in Theorem 1.2 is already well-known by the so-called $(3k - 4)$ -Theorem, which we recall in Section 3. Therefore Theorem 1.2 gives a precise structural description of chains (the case in which $|B'| = |B| + 1$ is clarified as well in Section 5.) One remarkable feature of Theorem 1.2 is that it holds for chains A with $|2A| = c|A|$ for the doubling constant c up to $(|A| - 1)/2$.

One consequence of Theorem 1.2 is to prove Conjecture 1 for the class \mathcal{C} of chains. In particular, for every $k \geq 4$ and $T \in [2k - 1, \binom{k}{2} + 2]$, we have

$$(4) \quad \text{vol}(k, T) \geq \mu(k, T) + 1.$$

The paper is organized as follows. We give general terminology and basic results in Section 2. The $(3k - 4)$ -Theorem of Freiman is recalled in Section 3. In Section 4 we give a construction of sets with volume $\mu(k, T) + 1$ for every suitable value of T , giving the lower bound (4) for $\text{vol}(k, T)$. Some discussion on chains and the lemmas leading to the proof of the main result are given in Section 5. The proof of Theorem 1.2 is given in Section 6. The proof is based on structural properties of chains, and the nature of the result calls for elementary methods. We conclude the paper with some final remarks in Section 7.

2. NOTATION AND PRELIMINARY RESULTS

A set $A = \{a_0 < a_1 < \cdots < a_{k-1}\}$ of integers is in *normal* form if $a_0 = 0$ and $\gcd(A) = 1$. The convex hull of a set of integers is the smallest interval containing it. We call a *hole* of A an element in its convex hull not contained in A .

A d -progression is a set of the form $\{a, a + d, a + 2d, \dots, a + (k - 1)d\}$ for some $d \geq 1$ and $k \geq 1$. We note that, with this definition, a singleton set $\{a\}$ is a d -progression for each $d \geq 1$.

We parametrize the values of the doubling $T = T(A) = |2A|$ of a set A of integers with cardinality $k = |A|$ as follows. For each $k \geq 4$ and each $c \in [2, k - 2]$ we denote by I_c the integer interval

$$I_{c,k} = ck - \binom{c+1}{2} + 2 + [1, k - c - 1],$$

including the value $2k - 1$ in $I_{2,k}$, so that their disjoint union

$$\bigcup_{c=2}^{k-1} I_{c,k} = \left[2k - 1, \binom{k}{2} + 2 \right],$$

is a range of values of $T(A)$. Given k and T we denote by $c(k, T)$ the value of c for which $T \in I_{c,k}$ and we write

$$(5) \quad T = ck - \binom{c+1}{2} + b + 2,$$

where $b = b(k, T) \in [1, k - c - 1]$. We call $c = c(k, T)$ the doubling constant of the set A .

We next recall a key result on the additive dimension of a set. Let $A = \{a_1, \dots, a_k\} \subset G$, be a finite subset in an abelian group G . Let e_1, \dots, e_k denote a basis of the k -dimensional real vector space \mathbb{R}^k . To each relation of the form $a_i + a_j = a_r + a_s$ satisfied by the elements of A we associate the vector $e_i + e_j - e_r - e_s \in \mathbb{R}^k$. Let us denote by $\lambda(A)$ the dimension of the subspace of \mathbb{R}^k generated by all these vectors. This dimension is related to the additive dimension of A by the following Theorem of Konyagin and Lev [8].

Theorem 2.1 (Konyagin, Lev). *The additive dimension of a set A with cardinality k is*

$$\dim(A) = k - 1 - \lambda(A).$$

For example, the set $\{a_1, a_2, a_3, a_4\} = \{0, 1, 2, 4\}$ has dimension one since it contains the relations $a_1 + a_3 = 2a_2$ and $a_1 + a_4 = 2a_3$ which correspond to two independent vectors in \mathbb{R}^4 , while $\{a_1, a_2, a_3, a'_4\} = \{0, 1, 2, 5\}$ is 2-dimensional.

We will often use the following simple consequence of Theorem 2.1.

Corollary 2.2. *Let A be a 1-dimensional set in normal form and $x > \max(A)$. Then $A \cup \{x\}$ is 1-dimensional if and only if $x \in 2A - A$.*

Proof. According to Theorem 2.1, the set $A \cup \{x\}$ is 1-dimensional if and only if x is involved in an additive relation with the elements of A , that is, $a + x = a' + a''$ for some $a, a', a'' \in A$. \square

3. STABLE SETS AND THE $(3k - 4)$ -THEOREM

We recall in this Section the $(3k - 4)$ -Theorem of Freiman giving the structure of extremal sets whose doubling is at most $3k - 4$. We first need some definitions.

By a segment we mean a set of consecutive integers, denoted by $[a, a + k - 1] = \{a, a + 1, \dots, a + k - 1\}$. The length of a set A is

$$\ell(A) = \max(A) - \min(A) + 1.$$

Given two sets A, B in normal form, we denote their concatenation by

$$A \circ B = A \cup (\max(A) + B).$$

We thus have

$$\begin{aligned} \ell(A \circ B) &= \ell(A) + \ell(B) - 1, \\ |A \circ B| &= |A| + |B| - 1. \end{aligned}$$

A set $A = \{a_0 = 0 < a_1 < \dots < a_{k-1}\}$, $k \geq 2$, of integers is *stable* if

$$2A \cap [0, a_{k-1}] = A \quad \text{and} \quad \{1, a_{k-1} - 1\} \cap A = \emptyset.$$

By convention we say that $A = \{0\}$ is also a stable set. We say that a set A is *right stable* if its reflexion $A^- = -A + a_{k-1}$ is stable. The typical examples of stable sets are d -progressions with $d \geq 2$, which are also right-stable. Actually, a stable set is always a union of d -progressions with difference $d = a_1$. For example, $\{0, 4, 5, 8, 9, 12\}$ is stable (but not right-stable) and $\{0, 2, 3, 6\}$ is right-stable (but not stable).

In fact, stability concerns the property that the doubling of A leaves A invariant in the interval $[0, \max(A)]$. We remark that, in our current definition of stable sets with more than one element, we impose the additional property that A does not contain 1 (so that A is not a segment) and A does not contain $a_{k-1} - 1$. One reason to include this additional condition to the definition of stable sets is the following simple Lemma, which gives the maximum density of initial segments of an stable set.

Lemma 3.1. *Let $A = \{0 = a_0 < a_1 \cdots < a_{k-1}\}$ be a stable set. For each $x \in [0, a_{k-1}]$ we have*

$$|A \cap [0, x]| \leq \left\lceil \frac{x+1}{2} \right\rceil.$$

Proof. Let $A(x) = |A \cap [0, x]|$. If x is a hole in A then, since A is stable, $x \notin 2A$. Hence, at most one among i and $x - i$ belongs to A for each $0 \leq i \leq x$, and $A(x) \leq \lceil x/2 \rceil$.

In particular, since $a_{k-1} - 1$ is a hole in A , we have $A(a_{k-1}) \leq A(a_{k-1} - 1) + 1 \leq \left\lceil \frac{a_{k-1} + 1}{2} \right\rceil$.

Suppose that $x \in A$, $x < a_{k-1}$, and let $y > x$ be the smallest hole in A . We have,

$$A(x) = A(y) - (y - x - 1) \leq \left\lceil \frac{y}{2} \right\rceil - (y - x - 1) \leq \left\lceil \frac{x + 1}{2} \right\rceil.$$

□

We say that a stable set A is *dense* if $|A| = \lceil (\max(A) + 1)/2 \rceil$. Arithmetic progressions of difference two are examples of dense stable sets. The stable set $\{0\}$ is also dense.

We shall use the following structural characterization from Freiman [5] of extremal sets with doubling at most $3k - 4$.

Theorem 3.2 (Freiman). *Let $A \subset \mathbb{Z}$ be an extremal set in normal form with $|A| = k$ and*

$$|2A| = 2k - 1 + b, \quad 0 \leq b \leq k - 3.$$

Then

$$(6) \quad A = A_1 \circ P \circ A_2,$$

where

- (i) A_1 is stable and A_2 is right stable,
- (ii) P is a segment with $|P| \geq k - b$, and
- (iii) $\ell(A) = k + b$. In particular A is 1-dimensional.

Moreover

$$(7) \quad 2A = A_1 \circ P' \circ A_2,$$

where P' is a segment with $|P'| \geq |2A| - b$.

The canonical decomposition of a set A under the conditions of Theorem 3.2 described in (6) will be called the *stable decomposition* of A . It is uniquely defined by our definition of

stable sets of cardinality at least two having a hole (non element) in the one before the last position.

We observe that, for each $T \in [2k - 1, 3k - 4]$ there is an extremal set A with $k = |A|$ and $|2A| = T$. For example, for each $b \in [1, k - 3]$ the set

$$\{0\} \cup [b + 1, k + b - 1],$$

is an extremal set with doubling $T = |2A| = 2k - 1 + b$.

4. A LOWER BOUND FOR THE VOLUME

We will show that

$$\text{vol}(k, T) \geq \mu(k, T) + 1,$$

by describing, for each $k \geq 4$ and $2k - 1 \leq T \leq \binom{k}{2} + 2$, a family of normalized 1-dimensional sets with cardinality k , doubling T and maximum element $\mu(k, T)$.

We recall the following operations on set A of integers:

- (i) $D(A) = \tilde{A} \cup \{2 \max(\tilde{A})\}$.
- (ii) $D_x(A) = 2 \cdot \tilde{A} \cup \{x\} = \{2a : a \in \tilde{A}\} \cup \{x\}$, x odd and $x \in 2\tilde{A} \setminus \tilde{A}$.

Lemma 4.1. *Let A be a 1-dimensional set in normal form with $T = |2A|$. If $\max(A) = \mu(k, T)$ and x is an odd number in $2A \setminus A$, then both $D(A)$ and $D_x(A)$ are normal 1-dimensional sets with*

$$\max(D(A)) = \max(D_x(A)) = \mu(k + 1, T + k).$$

Proof. Let $c = c(k, T)$ and $b = b(k, T)$ so that

$$T = ck - \binom{c+1}{2} + b + 2.$$

Let $B = D(A)$ and $a = \max(A)$. We have

$$|2B| = |2A \cup (2a + A) \cup \{4a\}| = |2A| + k,$$

and

$$\max(B) = 2a = 2^{c-1}((k+1) - (c+1) + b + 1).$$

Since

$$c(k+1, T+k) = c(k, T) + 1 \text{ and } b(k+1, T+k) = b(k, T),$$

we have $\max(B) = \mu(k+1, T+k)$. In order to show that B is one-dimensional we just observe that the new additive relation corresponding to the 3-term arithmetic progression

$\{0, a, 2a\}$ is linearly independent from the existing ones in A . It follows that $\lambda(B) = \lambda(A) + 1$ and therefore, according to Theorem 2.1, B remains one-dimensional.

Let now $C = D_x(A)$. Since x is the only odd number in C we have

$$|2C| = |2A| + k,$$

and, since $x \in 2A$, we also have $\max(C) = 2\max(A) = \mu(k+1, T+k)$. We clearly have $\lambda(2 \cdot A) = \lambda(A)$ (the two sets are F -isomorphic.) Since $x \in 2A$ we have a relation of the form $a_i + a_j = x$ for some i, j . Hence $2a_i + 2a_j = 2x$ gives a relation in C involving $x \notin A$, which is therefore linearly independent of the existing relations in $2 \cdot A$. Hence $\lambda(C) = \lambda(A) + 1$ and C is one dimensional. We note that the only relation involving x in C must be of the form $2a_i + 2a_j = 2x$, since x is odd. Thus, if $x \notin 2A \setminus A$ then $\dim(C) = \dim(A) + 1$. \square

By using the above operators D and D_x one can construct normal 1-dimensional sets A with $k = |A|, T = |2A|$ and $\max(A) = \mu(k, T)$ for all suitable values of T .

Corollary 4.2. *For each $k \geq 4$ and each $T \in [2k - 1, \binom{k}{2} + 2]$ there is a normal 1-dimensional set A with $k = |A|, T = |2A|$ and $\max(A) = \mu(k, T)$*

Proof. By Theorem 3.2 the statement holds for every $k \geq 3$ and every $T \in [2k - 1, 3k - 4]$. By induction on $k \geq 4$ and T , there is a normal 1-dimensional set A' with cardinality $k - 1$ with $\max(A') = \mu(k - 1, T - k)$. By Lemma 4.1, $A = D(A')$ is a normal 1-dimensional set with $|A| = k$ and $|2A| = T$ and $\max(A) = \mu(k, T)$. \square

In particular we have shown that the value for the maximum volume in Conjecture 1 is tight.

Corollary 4.3. *Let $k \geq 4$ and $T \in [2k - 1, \binom{k}{2} + 2]$. Then*

$$\text{vol}(k, T) \geq \mu(k, T) + 1.$$

Equivalently, every extremal set A with cardinality k and maximum element $\max(A) \leq \mu(k, T)$ satisfies

$$|2A| \leq T.$$

The main result in the paper essentially states that chains are obtained by iterate application of operators of the form D and D_x to some extremal set with doubling constant $c = 2$.

5. CHAINS

Theorem 1.2 gives a structural characterization of a general class \mathcal{C} of sets that we call chains. We denote by $[A] = [\min(A), \max(A)]$ the convex hull of a set of integers.

We recall that a set A is a *chain* if there is a sequence

$$A_3 \subset A_4 \subset \cdots \subset A_k = A$$

such that

- (i) $A_3 \cong_F \{0, 1, 2\}$,
- (ii) each A_i , $4 \leq i \leq k$, is a 1-dimensional set with i elements with doubling $|2A_i| \leq \binom{i}{2} + 1$, and $A_i \cap [A_{i-1}] = A_{i-1}$,
- (iii) A_i has largest volume among all sets B with the same cardinality and doubling as A_i and such that $B \cap [A_{i-1}] = A_{i-1}$.

For example, up to isomorphism, the class of chains in normal form with 5 elements consist of the following sets:

A									$ 2A $
•	•	•	•	•	○	○	○	○	9
•	○	•	•	•	•	○	○	○	10
•	○	•	○	•	•	•	○	○	11
•	○	○	•	•	•	•	○	○	11
•	○	•	•	•	○	•	○	○	11
•	○	○	○	•	•	•	○	•	12
•	○	○	○	•	○	•	•	•	12
0	1	2	3	4	5	6	7	8	

For every chain A , the deletion of either its larger element or its smaller element is also a chain. The set $\{0, 3, 4, 6, 7, 8\}$, according to Theorem 3.2, is an extremal set (it has doubling 14 and largest element 8) but it is not a chain: by removing the last element we obtain a two-dimensional set, while by removing the first element we obtain a set which does not satisfy condition (iii) above.

We denote by $\text{vol}_{\mathcal{C}}(k, T)$ the maximum volume of a chain with cardinality k and doubling T . We say that a chain A is *extremal* if $\text{vol}(A) = \text{vol}_{\mathcal{C}}(|A|, |2A|)$.

Arithmetic progressions are chains. The examples of extremal sets given after Theorem 3.2 are chains (by removing its smaller element we obtain an arithmetic progression.) We

note that, if A is a chain, then $D(A)$ is also a chain. It follows from the above remarks and Corollary 4.3 that, for each k and each $T \in [2k, \binom{k}{2} + 1]$, we have

$$\text{vol}_c(k, T) \geq \mu(k, T) + 1.$$

Theorem 1.2 states that equality holds. The remainder of this Section contains a sequence of Lemmas which provide the proof of Theorem 1.2. Some of the statements apply to the larger class of extremal one-dimensional sets. We say that a set A is *1-extremal* if it has largest volume among all one-dimensional sets with its same cardinality and doubling.

We collect some observations in the following Lemma for future reference.

Lemma 5.1. *Let A and $A_x = A \cup \{x\}$, $x > \max(A)$, be two one-dimensional sets. Set $k = |A|, T = |2A|, T_x = |2A_x|$ and $\Delta T = T_x - T$. The following holds:*

$$(8) \quad \Delta T = k + 1 - |2A \cap (A + x)|,$$

and

$$(9) \quad 2 \leq \Delta T \leq k,$$

and

$$(10) \quad c(k, T) - 1 \leq c(k + 1, T_x) \leq c(k, T) + 1.$$

Proof. We have

$$T_x = |2A \cup (x + A) \cup \{2x\}| = |2A| + |A| - |2A \cap (x + A)| + 1.$$

Since A_x is one-dimensional, x must be involved in an additive relation with the elements in A , which implies $|2A \cap (x + A)| \geq 1$. On the other hand, $x > \max(A)$ implies $|2A \cap (x + A)| \leq |A| - 1$. This gives (8) and (9).

For the last part, write $c = c(k, T)$ and just note that, for $c = 2$, $T \in [2k - 1, 3k - 4]$ and $T + [2, k] \subset [2(k + 1) - 1, 4(k + 1) - 8] = I_{2, k+1} \cup I_{3, k+1}$, while, for $c \geq 3$,

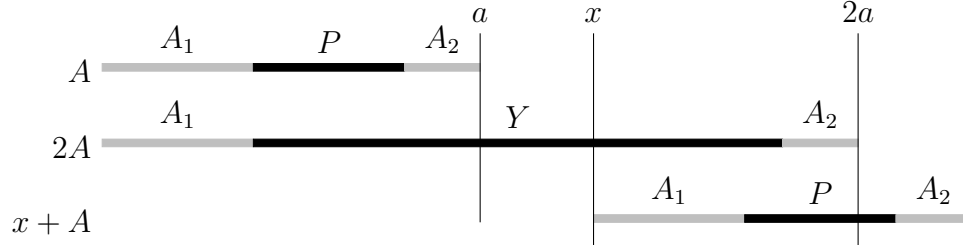
$$T \in I_{c, k} = \left[ck - \binom{c+1}{2} + 3, ck - \binom{c+1}{2} + (k - c) + 1 \right],$$

and the interval

$$I_{c-1, k+1} \cup I_{c, k+1} \cup I_{c+1, k+1} = \left[(c-1)(k+1) - \binom{c}{2} + 3, (c+1)(k+1) - \binom{c+2}{2} + (k-c) + 1 \right],$$

contains the values of $T + \Delta T$ for each $\Delta T \in [2, k]$. \square

Our next Lemma gives a lower bound on the values of integers x which can be added to a set A which admits a stable decomposition, if one expects to obtain an extremal set.

FIGURE 1. Illustration of the evaluation of $|2A \cap (x + A)|$.

Lemma 5.2. *Let $A = A_1 \circ P \circ A_2$ be a set with $k = |A|$, $T = |2A|$ and $a = \max(A)$, where A_1 is an stable set and A_2 is a right stable set and P is a segment.*

Let $A_x = A \cup \{x\}$ with $a < x \leq 2a$ and $T_x = |2A_x|$.

Assume that $T_x > 3(k + 1) - 4$ and that A_x is 1-dimensional.

If $a = \mu(k, T)$ and $x \geq \mu(k + 1, T_x)$ then

$$x \geq 2a - (a_1 + a_2 - 2),$$

where $a_1 = \max(A_1)$ and $a_2 = \max(A_2)$.

Proof. Since A_1 is stable and A_2 is right stable, we have

$$2A = A_1 \circ Y \circ A_2,$$

for some set Y . From (8),

$$(11) \quad \Delta T = T_x - T = |A| + 1 - |2A \cap (x + A)|.$$

The intersection $2A \cap (x + A)$ is contained in the interval $[x, 2a]$. If $x < 2a - (a_1 + a_2 - 2)$ then all holes of A are also holes in $2A \cap (x + A)$ (see Figure 1 for an illustration). It follows that

$$(12) \quad |2A \cap (x + A)| \leq 2a - x + 1 - (a - |A| + 1) = a - x + |A|.$$

Let $\Delta\mu = \mu(k + 1, T_x) - \mu(k, T)$. By combining (11) and (12) and using the hypothesis on x and a we obtain

$$(13) \quad \Delta T \geq x - a + 1 \geq \Delta\mu + 1.$$

Let $c = c(k, T)$, $b = b(k, T)$ and $c_x = c(k + 1, T_x)$, $b_x = b(k + 1, T_x)$. It follows from (10) that $c - 1 \leq c_x \leq c + 1$.

If $c_x = c$ then we have, by using (3) and (5),

$$\Delta T = c + b_x - b \quad \text{and} \quad \Delta \mu = 2^{c-2}(b_x - b + 1).$$

Since $T_x > 3(k+1) - 4$ we have $c_x \geq 3$. Moreover, $c_x = c$ implies $b_x \geq b$. Since $c \leq 2^{c-2} + 1$ for all $c \geq 3$ it follows that (13) does not hold, a contradiction.

Suppose now $c_x = c + 1$. In this case,

$$\Delta T = k + b_x - b \quad \text{and} \quad \Delta \mu = 2^{c-2}(k - c + 2b_x - b + 1).$$

If $c = 2$ then $\Delta \mu + 1 = \Delta T + b_x > \Delta T$ and (13) does not hold. Suppose $c \geq 3$. Since $b \leq k - c - 1$, we have $k - c + 2b_x - b + 1 \geq 2b_x + 2 \geq 4$. Hence,

$$\Delta \mu + 1 \geq (2^{c-2} - 1)4 + \Delta T - c + 2 \geq \Delta T + 2^c - c - 2 > \Delta T,$$

which again contradicts (13).

Finally, if $c_x = c - 1$ then $c \geq 3$ and

$$\Delta T = 2c + b_x - b - (k + 1) \quad \text{and} \quad \Delta \mu = 2^{c-3}(c + b_x - 2b - (k + 1)).$$

Hence we can write $\Delta \mu = 2^{c-3}(\Delta T - c - b)$. It follows from the proof of Lemma 5.1 that $c_x = c - 1$ implies $\Delta T < c$ so that we get $\Delta \mu < 0$, a contradiction. This completes the proof. \square

5.1. Crossing $3k - 4$. The next Lemma gives tight conditions on an extremal set with doubling smaller than $3k - 4$ to be extendable to a 1-extremal set with doubling larger than $3(k + 1) - 4$.

Lemma 5.3. *Let A be a extremal set with $k = |A|$, $a = \max(A)$ and*

$$T = |2A| = 2k - 1 + b, \quad 1 \leq b \leq k - 3.$$

Let $A_1 \circ P \circ A_2$ be the stable decomposition of A with $a_1 = \max(A_1)$ and $a_2 = \max(A_2)$.

If $A_x = A \cup \{x\}$, $a < x \leq 2a$, is 1-extremal with $T_x = |2A_x| \geq 3(k + 1) - 3$ then

$$(14) \quad x = \mu(k + 1, T_x).$$

Moreover,

$$(15) \quad |A \cap (x - a + A)| = \left\lceil \frac{2a - x + 1}{2} \right\rceil.$$

Proof. Assume that A_x is 1-extremal. Let $T = |2A|$ and $T_x = |2A_x|$. By Theorem 3.2 we have

$$(16) \quad a = k - 1 + b,$$

and

$$(17) \quad 2A = A_1 \circ P' \circ A_2,$$

where P' is a progression with length

$$(18) \quad |P'| = 2a + 1 - (a_1 + a_2).$$

Moreover, since $T_x > 3(k + 1) - 4$, we have $x > 2(k + 1) - 4$. We write the integers in the interval $[2(k + 1) - 3, 4(k + 1) - 3]$ as

$$(19) \quad x = 2(k + 1) - 4 + 2b_x + \delta, \quad 1 \leq b_x \leq (k + 1) - 4, \quad 0 \leq \delta \leq 1.$$

By Corollary 4.3, since A_x is 1-extremal, we have

$$(20) \quad T_x \leq 3(k + 1) - 4 + b_x = T + k + b_x - b.$$

Indeed, the above inequality holds when $\delta = 0$ in (19) by Corollary 4.3. When $\delta = 1$ then the inequality also holds because $\mu(k + 1, T_x + 1) = \mu(k + 1, T_x) + 2$ for our range of T_x .

We have

$$(21) \quad \Delta T = T_x - T = (k + 1) - |2A \cap (x + A)|.$$

It follows from inequality (20) and (21) that

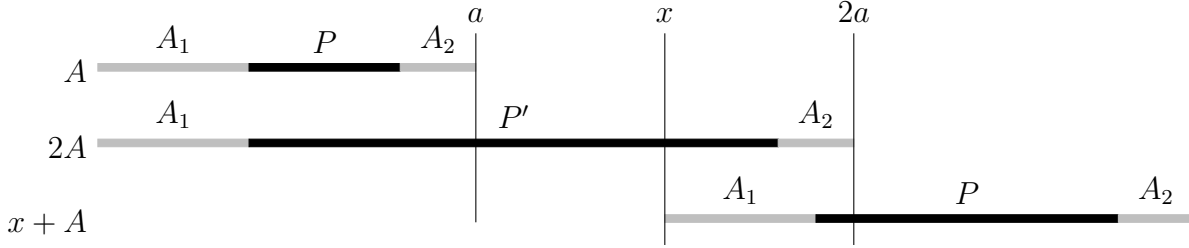
$$(22) \quad |2A \cap (x + A)| \geq b - b_x + 1.$$

We note that $2A \cap (x + A)$ is contained in the interval $[x, 2a]$ whose length, by (16) and (19), is

$$(23) \quad 2a - x + 1 = 2(b - b_x) + 1 - \delta.$$

By Lemma 5.2 we have $x \geq 2a - (a_1 + a_2 - 2)$. Therefore, $2A \cap (x + A)$ contains at most elements coming from the initial segment of the stable set A_1 or from the final segment of the right stable set A_2 (see Figure 2 for an illustration). Therefore, Lemma 3.1 gives

$$(24) \quad |2A \cap (x + A)| \leq \left\lceil \frac{2a - x + 1}{2} \right\rceil = b - b_x + 1 - \delta.$$


 FIGURE 2. Illustration of the evaluation of $|2A \cap (x + A)|$.

Hence, according to (22), the only possibility for A_x to be extremal is that

$$\delta = 0,$$

and equality holds in (24), which can be rewritten as (15) in the Lemma; indeed, by (18), we have $2A \cap [x, 2a] = [x, 2a - a_2] \circ A_2$ and hence $2A \cap (x + A) = (a + A) \cap (x + A)$. Moreover, equality in (24) implies that there is also equality in (20) which implies that x is given precisely by $\mu(k + 1, T_x)$, proving (14). \square

In the case that the extremal set A in Lemma 5.3 is a chain then the next lemmas further show that $D(A)$ is the only possible extension of A to a larger chain unless both stable sets in the stable decomposition of A are 2-progressions.

Lemma 5.4. *Let A be a normal set of integers with $k = |A|$ and $|2A| \leq 3k - 4$ with stable decomposition $A = A_1 \circ P \circ A_2$. If A is a chain then A_1 and A_2 contain no pair of consecutive elements.*

Proof. Suppose that A_1 or A_2 contain a pair of consecutive elements. By replacing A by its reflexion A^- if necessary, we may assume that A_1 contains two consecutive elements. Let $a_1 = \max(A_1)$. Let y be the largest element in A_1 such that $y + 1 \in A_1$ and $y + 2 \notin A_1$ (such element exists since $a_1 - 1 \notin A_1$.) Since $|P| \geq 3$, we have $a_1 - 1 \notin A$, and $\{a_1, a_1 + 1\} \subset A$.

Let A' be the minimal subchain of A containing $\{y, a_1 + 1\}$. Then $|2A'| \leq 3|A'| - 4$ and $\min(A') = y$ or $\max(A') = a_1 + 1$. Assume $\min(A') = y$, the other case being similar. By Theorem 3.2, A' can not be extremal since, once normalized, it has the form $\{0, 1, 3, \dots\}$ and hence, it has no stable decomposition. This contradicts that A is a chain. \square

Lemma 5.5. *Let A and A_x be as in Lemma 5.3.*

Assume that A is a chain.

Then A_x is 1-extremal for some $x < 2a$ if and only if x is even, $2a - x \geq a_1 + a_2 - 2$, both A_1 and A_2 are 2-progressions, and $(a - x + A) \cap A$ is a 2-progression. In this case,

$$A_x = A_1 \circ P \circ A'_2,$$

where $A'_2 = A_2 \cup \{x - (a - a_2)\}$ is right-stable.

Proof. Suppose that A_x is extremal for some $x < 2a$.

By Lemma 5.3 we have $2a - x \leq a_1 + a_2 - 2$ and $x = \mu(k + 1, T_x)$, which is an even number, and $X = A \cap (a - x + A) \subset [0, 2a - x]$ contains $(2a - x)/2 + 1$ elements.

Since $2a - x$ is even either X is a 2-progression or it contains two consecutive elements. Suppose the latter holds. As the length of X is at most $2a - x \leq a_1 + a_2 - 2$, the set X consists of an initial segment of A_1 and a final segment of A_2 (see Figure 2 for an illustration.) Hence A_1 or A_2 contain two consecutive elements. By Lemma 5.4 this contradicts that A is a chain.

Hence X is a 2-progression, which implies that each of A_1 and A_2 must be a 2-progression. This completes the proof of the ‘if’ part of the statement.

Suppose now that each of A_1 , A_2 and X is a 2-progression, x is even and $2a - x \geq a_1 + a_2 - 1$. Then there is equality in (24) and (22), which implies that A_x is 1-extremal.

We note that $A_x = (A_1 \circ P \circ A_2) \cup \{x\} = A_1 \circ P \circ (A_2 \cup \{x - (a - a_2)\})$. Since each of A_2 and $X = (a - x + A) \cap A$ is a 2-progression, then $(A'_2)^- = \{0\} \cup ((x - a) + A_2^-)$ is also stable. This completes the proof. \square

5.2. Beyond $3k - 4$. The next step towards the proof of Theorem 1.2 is to show that, under mild conditions, a 1-extremal set of the form $D(A)$ can be only extended to a larger chain by iterating the operator D .

Lemma 5.6. *Let A be a set with $k = |A|$, $T = |2A|$ and $a = \max(A)$. Let $B = D(A)$.*

If $B_x = B \cup \{x\}$ is 1-extremal for some $2a < x \leq 4a$ then $x \in \{3a, 4a\}$.

Moreover, if $a \leq \mu(k, T)$ and $\mu(k, T) > 2^c$, where $c = c(k, T)$, then

$$B_x = D^2(A).$$

Proof. Suppose that $B_x = B \cup \{x\}$ is extremal for some $2a < x \leq 4a$. As usual we consider

$$(25) \quad |2B_x| = |2B| + |B| + 1 - |(x + B) \cap 2B|.$$

Since $B = D(A) = A \cup \{2a\}$, we have

$$(26) \quad |(x + B) \cap 2B| = |(x + A) \cap (2a + A)|$$

If $x > 3a$ then $|(x + B) \cap 2B| \leq 1$ (the intersection contains at most $4a$) and, according to (25), B_x has not smaller doubling than $D(B)$ but has smaller volume unless $x = 4a$. On the other hand, if $2a < x < 3a$ then $B' = A \cup \{x - a\}$ has doubling (see Figure 3 for an illustration)

$$\begin{aligned} |2B'| &= |2A| + |A| + 1 - |(x - a + A) \cap 2A| \\ &\leq |2A| + |A| + 1 - |(x - a + A) \cap (a + A)| \\ &\stackrel{(26)}{=} |2B| + 1 - |(x + B) \cap 2B| \\ &\stackrel{(25)}{=} |2B_x| - |B|. \end{aligned}$$

It follows that $D(B')$ has the same cardinality as B_x and $|2D(B')| = |2B'| + |B| \leq |2B_x|$, while its larger element is $2(x - a) = x + (x - 2a) > x$, contradicting that B_x is extremal.

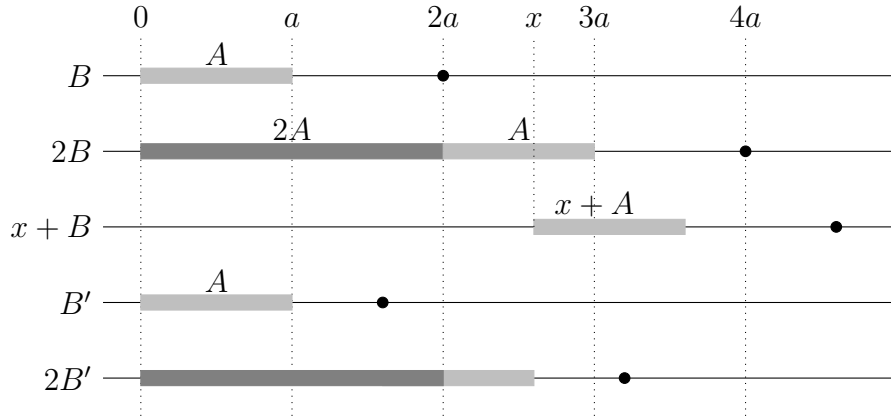


FIGURE 3. An illustration of the computation of $2B_x$ and $2B'$.

If $x = 3a$, then $|(x + B) \cap 2B| = 2$ which yields

$$|T_x| = |2B_x| = |2B| + |B| - 1 = |T| + 2k.$$

Assume now that $a \leq \mu(k, T)$ and $\mu(k, T) > 2^c$. We have $2a \leq \mu(k + 1, T + k)$ and

$$4a \leq \mu(k + 2, T + 2k + 1) = \mu(k + 2, T_x + 1) = \mu(k + 2, T_x) + 2^{c'-2},$$

where $c' = c(k + 2, T_x) \leq c + 2$. If B_x with $x = 3a$ is extremal, we have

$$3a \geq \mu(k + 2, T_x) \geq 4a - 2^c,$$

and hence $a \leq 2^c$, a contradiction. It follows that B_x is not extremal for $x = 3a$ and the only choice left is $x = 4a$. Thus $B_x = D^2(A)$. \square

We remark that the condition $\mu(k, T) = 2^{c-2}(k - c + b + 1) > 2^c$, $c = c(k, T)$, in Lemma 5.6 is only violated when $k \leq c - b + 3$, namely when $b = 1$ and $c = k - 2$. In such cases, indeed, the choice $x = 3a$ in Lemma 5.6 can give rise to an extremal set which is not of the form $D(B)$ for some B . A simple example is $B = \{0, 1, 2, 4\} = D(\{0, 1, 2\})$ which is contained in the 1-extremal set $\{0, 1, 2, 4, 8\} \neq D(B)$. By Lemma 5.1, if the doubling constant of a chain A is smaller than $k - 2$ then any chain B containing A has also doubling constant smaller than $|B| - 2$. Therefore all examples of chains for which the condition $\mu(k, T) = 2^{c-2}(k - c + b + 1) > 2^c$, $c = c(k, T)$, in Lemma 5.6 does not apply are the ones obtained from $\{0, 1, 2, 4\}$. We shall discuss the structure of chains arising from this kind of examples later on.

Lemma 5.6 only handles the case when a 1-extremal set of the form $B = D(A)$ is extended by adding one element to the right. Next Lemma considers extending B to the left, namely, extending its reflexion B^- by adding one element to the right. As it happens, showing that again $D(B^-)$ is the only extension of B^- to a larger chain, requires the use of the full strength of the assumptions, namely, that $A \cup \{y\}$ is not a chain for $\max(A) < y < 2 \max(A)$.

Lemma 5.7. *Let A be a set with $k = |A|, T = |2A|$ and $a = \max(A)$. Assume that $a \leq \mu(k, T)$ and that $\mu(k, T) > 2^c$, $c = c(k, T)$, and*

$$y < \min\{\mu(k + 1, |2(A \cup \{y\})|), \mu(k + 1, |2(A^- \cup \{y\})|)\}, \text{ for all } a < y < 2a.$$

Let $B = (D(A))^-$.

Then $B_x = B \cup \{x\}$ is extremal for some $2a < x \leq 4a$ if and only if

$$B_x = D(B).$$

Proof. Suppose that $B_x = B \cup \{x\}$ is extremal for some $2a < x \leq 4a$. We have

$$|2B_x| = |2B| + |B| + 1 - |(x + B) \cap 2B|.$$

We again consider three cases (see an illustration in Figure 4).

Suppose first that $3a < x \leq 4a$. Then $(x + B) \cap 2B$ contains at most the point x . Hence, B_x has no smaller doubling than $D(B)$ but has a smaller volume unless $x = 4a$.

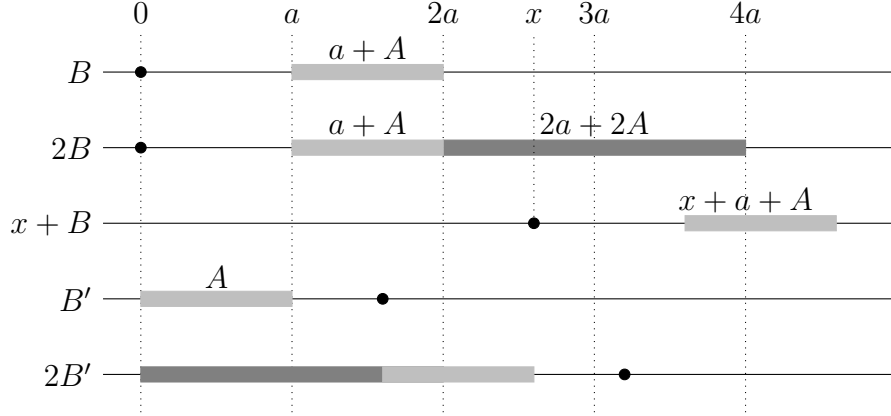


FIGURE 4. An illustration of the computation of $2B_x$ and $2B'$.

Consider now $2a < x < 3a$. Set $k = |A|$, $T = |2A|$, $c = c(k, T)$ and $b = b(k, T)$. Since B_x is 1-dimensional we have $x \in 2(\{0\} \cup (a + A))$, which implies $x \in 2a + 2A$. Hence,

$$\begin{aligned} T_x = |2B_x| &= |2(\{0\} \cup (a + A) \cup \{x\})| \\ &= 1 + |(a + A^-) \cup (x + a + A) \cup (2a + 2A)| + 1 \\ &= 1 + |a + A| + |(x + a + A) \cup (2a + 2A)|. \end{aligned}$$

On the other hand, for $B' = A \cup \{x - a\}$ we have

$$T' = |2B'| = |(x - a + A) \cup 2A| + 1,$$

and

$$|2D(B')| = |B'| + |2B'| = 1 + |A| + |(x + a + A) \cup (2a + 2A)| + 1 = T_x + 1.$$

By assumption, $x - a < \mu(k + 1, T')$ and hence,

$$\begin{aligned} 2(x - a) &< \mu(k + 2, T' + k + 1) \\ &= \mu(k + 2, T_x + 1) \\ &= \mu(k + 2, T_x) + 2^{c_x - 2} \\ (27) \quad &\leq x + 2^{c_x - 2}, \end{aligned}$$

where $c_x = c(k + 2, T_x)$ satisfies

$$c + 1 \leq c_x \leq c + 2.$$

It follows that

$$\mu(k + 2, T_x) \leq x < 2a + 2^{c_x - 2},$$

the lower bound since B_x is extremal, the upper bound from (27). By plugging in the values of $\mu(k+2, T_x) = 2^{c_x-2}(k+2 - c_x + b_x + 1)$ and $a = \mu(k, T) = 2^{c-2}(k - c + b + 1)$ we obtain

$$2^{c_x-2}(k+2 - c_x + b_x + 1) < 2^{c-1}(k - c + b + 1).$$

If $c_x = c + 2$ the above inequality leads to $b_x < b - (k + 1) < 0$, which is not possible. If $c_x = c + 1$ then we get $b_x < b - 1$, and hence $T_x < T + k$, again a contradiction.

Finally, if $x = 3a$, then $|(x + B) \cap 2B| = 2$ which yields

$$|T_x| = |2B_x| = |2B| + |B| - 1 = |T| + 2k.$$

Assume now that $\mu(k, T) > 2^c$. We have $2a = \mu(k + 1, T + k)$ and

$$4a = \mu(k + 2, T + 2k + 1) = \mu(k + 2, T_x + 1) = \mu(k + 2, T_x) + 2^{c_x-2}.$$

If B_x with $x = 3a$ is 1-extremal, we have

$$3a \geq \mu(k + 2, T_x) \geq 4a - 2^c,$$

and hence $a \leq 2^c$, a contradiction. It follows that B_x is not 1-extremal for $x = 3a$ and the only choice left is $x = 4a$. Thus $B_x = D^2(A)$. \square

5.3. The case of 2-progressions. It follows from Lemma 5.5 that a chain A with doubling $T \leq 3|A| - 4$ can only be extended to $D(A)$ unless both stable sets in its stable decomposition are 2-progressions. If this is not the case then lemmas 5.6 and 5.7 show that every further extension is obtained by iterate applications of the operator D (or D^-), which proves Theorem 1.2 in this situation. In order to complete the proof of Theorem 1.2 it remains to analyze the case of stable sets which are 2-progressions. This is the purpose of the final part of this Section.

Lemma 5.8. *Let $A = A_1 \circ P \circ A_2$ and $k = |A|$. Assume that both of A_1 and A_2 are 2-progressions and $|P| \geq 4$.*

Let $y > x > a = \max(A)$ such that $T_x = |2A_x| > 3(k + 1) - 4$, $A_x = A \cup \{x\}$.

- (i) *If $A_{xy} = A_x \cup \{y\}$ is a chain then $A_{xy} = D(A_x)$.*
- (ii) *If $A'_{xy} = (A_x)^- \cup \{y\}$ is a chain and $y > x + 2$ then $A_{xy} = D((A_x)^-)$.*

Proof. By the structure of A we have $|2A| \leq 3k - 4$ and A is extremal. By Lemma 5.3 we have $x = \mu(k + 1, T_x) \geq 2a - (a_1 + a_2 - 2) = a + |P|$, an even number. Hence, since $|P| \geq 4$ and a is odd when $|P| = 4$, we have

$$(28) \quad x > a + 4.$$

Let

$$T = |2D(A_x)| = |2D((A_x)^-)| = T_x + k + 1.$$

(i) Let $g(y) = |(y + A_x) \cap 2A_x|$, so that

$$T_{xy} = |2A_{xy}| = T_x + (k + 2) - g(y) = T + 1 - g(y).$$

We have $A_x = A_1 \circ P \circ A_2'$ where A_2' is right stable, and therefore $2A_x = A_1 \circ Y \circ A_2'$ for some Y . By Lemma 5.2 applied to A_x and A_{xy} we have $y \geq x + a - (a_1 + a_2 - 2)$. Moreover, since each of A_1, A_2 is a 2–progression, by the structure of $2A_x$ we have (see Figure 5 for an illustration)

$$(29) \quad g(y) \leq \begin{cases} 1, & x + a < y \leq 2x \\ \lfloor h/2 \rfloor + 2, & y = x + a - h, 0 \leq h \leq a_1 + a_2 - 2 \end{cases}.$$

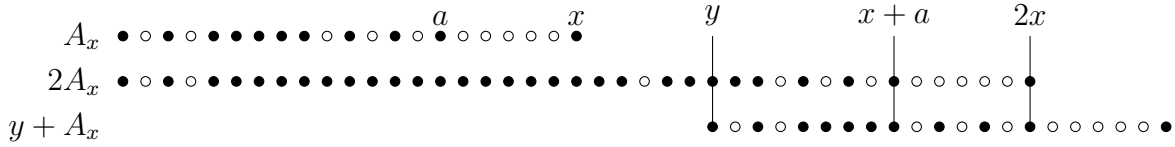


FIGURE 5. An illustration of the computation of $g(y) = |(y + A_x) \cap 2A_x|$.

If $y = 2x$ then A_{xy} is 1–extremal with $T_{xy} = T$ and $g(y) = 1$.

According to (30), the largest value of y for which $T_{xy} = T - 1$, namely $g(y) = 2$, is $y = x + a$. Since $c(k + 2, T) = 4$, we have $\mu(k + 2, T - 1) = \mu(k + 2, T) - 4 = 2x - 4$. By (28) we have $x + a < 2x - 4$. Hence A_{xy} is not 1–extremal for $y = x + a$. Moreover, we have $c(k + 2, T - h') = 3$ and $\mu(k + 2, T - h' - 1) = \mu(k + 2, T - h') - 2$ for each h' such that $T_x < T - h' - 1 < T - 1$. According to (30), the largest y for which $T_{xy} = T - h' - 1$, namely $g(y) = h' + 2$, is

$$y = x + a - 2h' < \mu(k + 2, T - 1) - 2h' = \mu(k + 2, T - h' - 1).$$

Hence A_{xy} is not 1–extremal for all the remaining values of y . Hence, if A_{xy} is 1–extremal we must have $y = 2x$ and $A_{xy} = D(A_x)$. This completes the proof of (i).

(ii) The proof follows the same lines as the above one. Let $g(y) = |(y + (A_x)^-) \cap 2(A_x)^-|$, so that

$$T'_{xy} = |2A'_{xy}| = T_x + (k + 2) - g(y) = T + 1 - g(y).$$

We have $(A_x)^- = (A'_2)^- \circ P \circ A_1$ where A_1 is right stable, and therefore $2(A_x)^- = (A'_2)^- \circ Y \circ A_1$ for some Y . By Lemma 5.2 applied to $(A_x)^-$ and A'_{xy} we have $y \geq x + a - (a_1 + a_2 - 2)$. Moreover, since each of A_1, A_2 is a 2–progression, by the structure of $2(A_x)^-$ we have (see Figure 6 for an illustration)

$$(30) \quad g(y) \leq \begin{cases} 1, & x + a < y \leq 2x \\ \lfloor h/2 \rfloor + 2, & y = x + a - h, 0 \leq h \leq a_1 + a_2 - 2 \end{cases} .$$

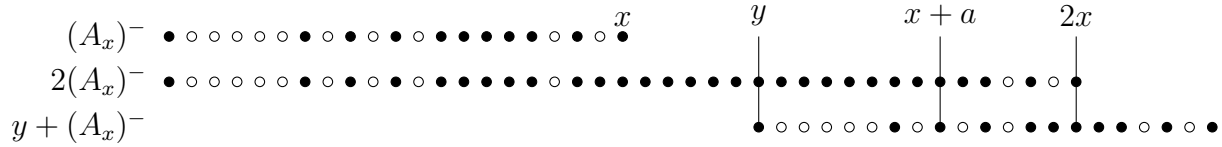


FIGURE 6. An illustration of the computation of $g(y) = |(y + (A_x)^-) \cap 2(A_x)^-|$.

The proof of part (ii) is now completed in the same way as part (i). □

Our final Lemma concerns the sets of the form $A = A_1 \circ P \circ A_2$ where A_1 and A_2 are 2–progressions and $|P| = 3$. This is a rather special case where the second operator D_x introduced in Section 4 will be used. We observe that in this case A has an only odd number.

Lemma 5.9. *Let A be a chain with an only odd number x . The following hold:*

- (i) $A = D_x(A')$ where A' is a chain.
- (ii) Every chain B containing A with $|2B| > 3|B| - 4$ has also an only odd number.

Proof. (i) The fact that $A = D_x(A')$ follows from the definition of the operator D_x . Let us show that A' is a chain.

Since A is a chain, it contains three consecutive elements, one of them the only odd number in A . It follows that A' contains two consecutive elements and hence $\gcd(A') = 1$. Therefore A' is in normal form. Moreover, if B' is a chain with the same cardinality and doubling as A' but with larger volume, then $B = D(B')$ has the same cardinality and doubling as A but with larger volume, contradicting that A is extremal. It remains to show that A' is a chain. Let $A_3 \subset A_4 \cdots \subset A_k = A$ be the sequence of chains contained in

A (up to translation). We observe that $x \in A_3$. If $A'_i = D_x(A_i \setminus \{x\})$ then $A'_4 \subset \cdots \subset A'_k$ is a sequence of chains contained in A' .

(ii) In order to prove (ii), suppose that $A \cup \{y\}$ is a 1-dimensional set with $y > \max(A)$ odd. We have

$$|2(A \cup \{y\})| = |2A| + k + 1 - |(y + A) \cap 2A|.$$

Since all elements in $y + A$ except $y + x$ are odd numbers, we have

$$(31) \quad |(y + A) \cap 2A| = |(y + A_0) \cap (x + A_0)| + |(y + x) \cap 2A_0|.$$

We show that there is an even number $y' > y$ such that $A \cup \{y'\}$ is 1-dimensional and has no larger doubling than $A \cup \{y\}$, so that the latter set is not 1-extremal.

We consider two cases:

Case 1. $|(y + A_0) \cap (x + A_0)| \neq 0$.

Since A is 1-dimensional, x must be involved in one elementary relation. Being the only odd number in A the relation must be $2x = z + z'$ for some $z < z' \in A_0$. Let $\alpha = z' - x = x - z$ and $y' = y + \alpha$, an even number.

Since $|(y + A_0) \cap (x + A_0)| \neq 0$, we have $y + u = x + u'$ for some $u, u' \in A_0$ and hence

$$y' + u = z' + u' \in 2A_0,$$

which shows that $A \cup \{y'\}$ is also 1-dimensional. Moreover,

$$(32) \quad |(y' + A) \cap 2A| \geq |(y' + A_0) \cap (z' + A_0)| = |(y + A_0) \cap (x + A_0)|.$$

If $|(x + y) \cap 2A_0| = 0$ then (31) and (32) show that $A \cup \{y\}$ is not 1-extremal.

If on the contrary $x + y = w + w'$ for some $w, w' \in A_0$ then $y' + z = (y + \alpha) + (x - \alpha) = w + w' \in |(y' + A) \cap 2A|$. We observe that, since $\alpha = z' - z$ is odd, $y' + z \notin z' + A_0$ and therefore the inequality in (32) is strict. Again (31) shows that $A \cup \{y\}$ is not 1-extremal.

Case 2. $|(y + A_0) \cap (x + A_0)| = 0$.

In this case we take $y' = 2 \max(A) > y$ which has doubling $|2A| + k$ as $A \cup \{y\}$, again contradicting that $A \cup \{y\}$ is extremal. \square

6. PROOF OF MAIN RESULT

Proof of Theorem 1.2 Suppose first that A contains an only odd number. By Lemma 5.9(i) we can write $A = D_x(A')$ where x is the odd number in A and A' is a chain. Hence we can write $A = \phi_1 \cdots \phi_k(A_0)$ where each ϕ_i is of the form D_{x_i} for some odd number x_i

and either A_0 has more than one odd number or $A_0 \cong_F \{1, 2, 3\}$. In the latter case the Theorem holds with $B = \{1, 2, 3\}$. In what follows we assume that A contains more than one odd number.

Let $A_3 \subset A_4 \subset \dots \subset A_k = A$ be a chain sequence of A . Let t be the largest integer s for which $|2A_s| \leq 3|A_s| - 4$ and set $B = A_t$. By Lemma 5.9(ii) B contains more than one odd number.

By Lemma 5.5, we have $A_{t+1} \cong_F D(A_t)$ or $A_{t+1} \cong_F D^-(A_t)$ unless $B = B_0 \circ P \circ B_2$ where B_1 and B_2 are 2-progressions. If this is not the case then lemmas 5.6 and 5.7 show that $A_i \cong_F D(A_{i-1})$ for each $t < i \leq k$ proving the statement of the Theorem.

The last case to consider is $B = B_0 \circ P \circ B_2$ where B_1 and B_2 are 2-progressions. Since B contains more than one odd number we have $|P| \geq 4$. By Lemma 5.8 we have $A_{t+2} \cong_F D(A_{t+1})$. Lemmas 5.6 and 5.7 show that $A_i \cong_F D(A_{i-1})$ for each $t < i \leq k$ proving the statement of the Theorem in this case. This completes the proof.

7. FINAL REMARKS

The notion of chains for which Conjecture 1 is proved in this paper is a quite natural one, and gives evidence to the conjecture. As observed in the Introduction, chains are most natural candidates to be extremal sets. Chains have the strong structure described in Theorem 1.2. Proving the conjecture for general sets has the difficulty of loosing these strong structural properties along subsets, and additional techniques have to be used. So far we have only been able to address the case of doubling constant $c = 3$, the contents of a forthcoming paper, which nevertheless opens a path which has been the object of many attempts for several decades. The best current result in this direction is obtained by R. Jin [7] which proves the conjecture for doubling $3k - 4 + b$ and small values of b , by using nonstandard analysis. It seems unlikely that the conjecture can be proved for all values of the doubling constant c up to $|A|/2$ as it has been for chains. It would be a significant breakthrough to know something about the structure of sets with doubling constant some growing function of $|A|$.

In the definition of chains we introduced the notion of 1-extremal sets, sets with largest volume for given cardinality and doubling among 1-dimensional sets. A partial result towards the proof that extremal sets are in fact 1-dimensional, as asserted in Conjecture 1, is given in Freiman [6]. A natural question arises as if the conjectured maximum volume could be smaller if we restrict our sets to be d -dimensional with $d > 1$. The following

simple example shows that the lower bound $\mu(k, T)$ given in (4) for the largest volume of a set with given cardinality k and doubling T is not far from the truth when restricted to d -dimensional sets. Let $A = A_1 \cup \{e_2, \dots, e_d\}$ where $A_1 \subset \mathbb{R}e_1$ is a 1-extremal set and e_1, \dots, e_d is the standard basis of the d -dimensional space \mathbb{R}^d . If $k = |A|$ and $T = |2A|$ then

$$T = |2A_1| + k + 1,$$

while

$$\text{vol}(A) = \text{vol}(A_1) + (d - 1) \geq \mu(k - (d - 1), T - (k + 1)) + d,$$

giving a lower bound on the volume of the d -dimensional set A in terms of the function $\mu(k, T)$.

The Freiman–Ruzsa theorem is instrumental in many applications. It would be interesting to feed the quantitative and structural information provided by Theorem 1.2 to these applications to assess the relevance of the result.

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