Antimagic Labelings of Caterpillars

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Abstract

A k-*antimagic labeling* of a graph G is an injection from $E(G)$ to $\{1, 2, \dots, |E(G)|+k\}$ such that all vertex sums are pairwise distinct, where the *vertex sum* at vertex u is the sum of the labels assigned to edges incident to u. We call a graph k-*antimagic* when it has a k-antimagic labeling, and *antimagic* when it is 0-antimagic. Hartsfield and Ringel conjectured that every simple connected graph other than K_2 is antimagic, but the conjecture is still open even for trees. Here we study k-antimagic labelings of *caterpillars*. We use algorithmic and constructive techniques, instead of the standard Combinatorial NullStellenSatz method, to prove our results: (i) any caterpillar of order n is $(\lfloor (n-1)/2 \rfloor - 2)$-antimagic; (ii) any caterpillar with a spine of order s with either at least $\lfloor (3s+1)/2 \rfloor$ leaves or $\lfloor (s-1)/2 \rfloor$ consecutive vertices of degree at most 2 at one end of a longest path, is antimagic; and (iii) if p is a prime number, any caterpillar with a spine of order p, $p-1$ or $p-2$ is 1-antimagic.

1 Introduction

All graphs considered in this paper are finite, undirected, connected and simple. Given a graph $G = (V(G), E(G))$ and a vertex $v \in V(G)$, $E_G(v)$ denotes the set of edges incident to v and $d_G(v) = |E_G(v)|$ stands for the degree of v in G (we will just write $E(v)$ and $d(v)$ when G is clear from the context).

A k-*antimagic labeling* of G is an injection $f : E(G) \to \{1, 2, \dots, |E(G)| + k\}$ such that all vertex sums are pairwise distinct, where the *vertex sum* at vertex v is defined as $\sum_{e \in E(v)} f(e)$. When G has a k-antimagic labeling, it is said to be k-*antimagic*. For $k = 0$, the labeling as well as the graph G are called *antimagic*. The following conjecture, posed by Hartsfield and Ringel [12] in 1990, has attracted much attention over the years.

Conjecture 1. *([12]) Every connected graph other than K_2 is antimagic.*

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Classes of graphs which are known to be antimagic include: paths, stars, complete graphs, cycles, wheels, and bipartite graphs $K_{2,m}$, $m \geq 3$ [12]; graphs of order n with maximum degree at least $n-3$ [23]; dense graphs (i.e., graphs having minimum degree $\Omega(\log n)$) and complete partite graphs but K_2 [2, 10]; toroidal grid graphs [20]; lattice grids and prisms [6]; k-regular bipartite graphs of degree $k \geq 2$ [8]; k-regular graphs, $k \geq 3$, with odd degree [9]; even degree regular graphs [5]; cubic graphs [16]; generalized pyramid graphs [3]; graph products [21]; or Cartesian product of graphs [7, 24].

Concerning trees, Kaplan, Lev, and Roditty [14] proved that any tree having more than two vertices and at most one vertex of degree 2 is antimagic (see also [17]). Complete m-ary trees are also known to be antimagic [4]. Nevertheless, the conjecture is still open for the general class of trees.

Conjecture 2. *([12]) Every tree other than K_2 is antimagic.*

Hefetz [13] introduced the related concept of (w, k)-antimagic labeling, which will be used in Subsection 3.3 as a fundamental tool. Given a graph G and a vertex weight function $w : V(G) \to \mathbb{N}$, an injection $f : E(G) \to \{1, 2, \ldots, |E(G)| + k\}$ is called a (w, k)-*antimagic labeling* of G if all vertex sums are pairwise distinct, where the *vertex sum* at vertex v is defined, in this context, as $w(v) + \sum_{e \in E(v)} f(e)$. We say that G is *weighted-k-antimagic* if for any vertex weight function w, G has a (w, k)-antimagic labeling. Wong and Zhu [22] proved that every connected graph of order $n \geq 3$ is weighted-$(\lceil 3n/2 \rceil - 2)$-antimagic. For more details on antimagic labelings for particular classes of graphs see the dynamic survey by Gallian [11].

In this paper we focus on *caterpillars*, which constitute a well-known subclass of trees for which the above Conjecture 2 is still open. A *caterpillar* C is a tree of order at least 3 the removal of whose leaves produces a path, called the *spine* of C. A relevant fact is that in our proofs we always use algorithmic techniques, i.e., *constructive approaches*, instead of using the Combinatorial NullStellenSatz method which is the regular technique used in most of the references above (see also [19] for another use of the algorithmic approach). Thus, we obtain a different way for solving antimagicness problems which can give a clear insight on the difficulties for getting solutions.

Our contribution. We show in Section 2, using constructive techniques, that caterpillars of order $n > 2$ are $(\lfloor \frac{n-1}{2} \rfloor - 2)$-antimagic. Section 3 is devoted to the study of some classes of caterpillars. Concretely, we prove that if C is a caterpillar with a spine of order s with at least $\lfloor (3s+1)/2 \rfloor$ leaves, then C is antimagic (Subsection 3.1); if C is a caterpillar with a spine of order s and $\lfloor (s-1)/2 \rfloor$ consecutive vertices of degree at most 2 at one end of a longest path, then C is antimagic (Subsection 3.2); and if p is a prime number, any caterpillar with a spine of order p, $p-1$ or $p-2$ is 1-antimagic (Subsection 3.3).

2 Caterpillars

We introduce the following concept for caterpillars. The *leaves excess* of a caterpillar C is $\mathcal{E}(C) = \sum_{d(v) \geq 4} (d(v) - 3)$. Figure 1 shows an example of leaves excess.

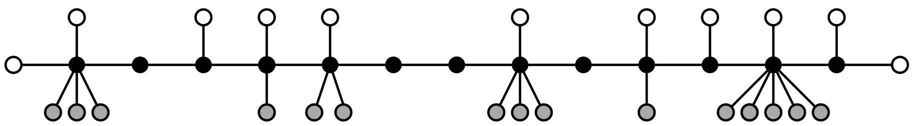

Figure 1: The number of grey leaves equals the leaves excess $\mathcal{E}(C)$.

Theorem 1. *A caterpillar C with a spine of order s is* $\max(0, \lfloor \frac{s-1}{2} \rfloor - 1 - \mathcal{E}(C))$*-antimagic.*

Proof. Given a caterpillar C with m edges and ℓ leaves, let P be a longest path in C. It is clear that P has $m-\ell+3$ vertices which will be called *path vertices* and $m-\ell+2$ edges which will be called *path edges*. Also, the number of edges of the spine is $m-\ell = s-1$. In order to define an injection from $E(C)$ to a label set, say L, we first choose the values:

$$i = \left\lceil \frac{m-\ell}{2} \right\rceil + 1, \quad j = \left\lceil \frac{m+\ell}{2} \right\rceil - 1, \quad k = \max\left(\mathcal{E}(C), \left\lfloor \frac{m-\ell}{2} \right\rfloor - 1\right),$$

and take $L = \bigcup_{r=1}^{4} L_r$, where:

- $L_1 = \{1, \ldots, k\}$,
- $L_2 = \{k+1, \ldots, k+i\}$,
- $L_3 = \{k+i+1, \ldots, k+j-\mathcal{E}(C)\}$,
- $L_4 = \{k+j-\mathcal{E}(C)+1, \ldots, k+m-\mathcal{E}(C)\}$.

Since labels are consecutive across label sets L_1, L_2, L_3, and L_4, the total number of labels $|L|$ coincides with the value of the largest label, which equals:

$$m + \max\left(\mathcal{E}(C), \left\lfloor \frac{m-\ell}{2} \right\rfloor - 1\right) - \mathcal{E}(C) = m + \max\left(0, \left\lfloor \frac{m-\ell}{2} \right\rfloor - 1 - \mathcal{E}(C)\right).$$

Therefore, C will be $\max(0, \lfloor \frac{m-\ell}{2} \rfloor - 1 - \mathcal{E}(C))$-antimagic if there is an injection from $E(C)$ to L. We next show how to construct such an injection.

Labels in L_2 and L_4 will be assigned to the path edges, while labels in L_1 and L_3 will be assigned to the non-path edges[1]. We consider two phases to complete the assignment:

- *Phase 1: Labeling the path edges.* Assign alternately labels in L_2 and L_4 to the path edges, starting with the largest label in each set, and keep the alternation of labels until the first ones are reached. We consider two cases depending on whether s, the order of the spine, is odd or even. Note that $s = m-\ell+1$, $|L_2| = i$, and $|L_4| = m-j$.
  - If s is odd, then $m-\ell$ is even and from the definition of i and j we have that $i+j = m$. Thus, $|L_2| = i = m-j = |L_4|$ and we can alternate labels along the path P in the following way:

$$k+i, k+m-\mathcal{E}(C), \ldots, k+1, k+j-\mathcal{E}(C)+1.$$

[1]For a fast understanding of the assignment method we recommend the reader to simultaneously follow the example given after the proof.

- If s is even, then $m-\ell$ is odd and $i+j=m+1$. Then, $|L_2|=i=m-j+1=|L_4|+1$. The alternation of labels in P now ends with the first label in L_2:

$$k+i, k+m-\mathcal{E}(C), \ldots, k+2, k+j-\mathcal{E}(C)+1, k+1.$$

The previous partial assignment uses all the labels in L_2 and L_4 to produce partial sums at the inner path vertices ranging between $2k+j+2-\mathcal{E}(C)$ and $2k+i+m-\mathcal{E}(C)$. In addition, the endpoints of path P have the sums $k+i$ and $k+j-\mathcal{E}(C)+1$ (if s is odd), or $k+i$ and $k+1$ (if s is even).

Clearly, all partial sums at the path vertices are different. The vertex sums at the end of the path are smaller than the sums at the inner path vertices, which are obtained summing up two consecutive labels (the smallest sum is $2k+j+2-\mathcal{E}(C)$, which is greater than the possible sums at the extremes of the path: $k+i$, $k+j-\mathcal{E}(C)+1$, and $k+1$). On the other hand, the vertex sums at the inner vertices of P of degree 2 are all different since, by the way the assignment is defined, sums are strictly decreasing.

- *Phase 2: Labeling the non-path edges.* Now, we add the labels in L_1 and L_3 to the non-path edges in the following way. First, for every path vertex u with degree $d(u)>3$, we randomly assign labels from L_1 to $d(u)-3$ non-path edges incident to u. After this step, all the path vertices are incident with at most one non-path edge that has not yet been assigned a label.

  Let E' be the list of the still unlabeled non-path edges in non-increasing order of the partial sum of the only path-vertex incident to each of them. Let L_3' be the list of still unused labels of L_3 in decreasing order. Assign each label of L_3' to the edge of E' at the same position in the lists. This assignment guarantees all path vertices of degree at least 3 to have different sums.

  Note that vertex sums at non-path vertices, which correspond to unique labels in L_1 or L_3, are smaller than those at the path vertices of degree at least 2 as they include at least one label from L_4.

Finally, we check that sums (of Phase 1) at vertices of degree 2 cannot coincide with those (of Phase 2) at vertices of degree at least 3. The largest vertex sum that can be achieved at a vertex of degree 2 is obtained summing up the largest labels in L_2 and L_4:

$$(k+i)+(k+m-\mathcal{E}(C)) = 2k+i+m-\mathcal{E}(C)$$

while the smallest vertex sum that can be achieved at a vertex of degree at least 3 is obtained summing up the smallest labels in L_2, L_3, and L_4:

$$(k+1)+(k+i+1)+(k+j-\mathcal{E}(C)+1) = 3k+i+j+3-\mathcal{E}(C).$$

But we have that $2k+i+m-\mathcal{E}(C) < 3k+i+j+3-\mathcal{E}(C)$ if and only if $m-3<k+j$. By the definition of j and k, this last inequality is true if

$$m-3 < \left\lfloor \frac{m-\ell}{2} \right\rfloor - 1 + \left\lceil \frac{m+\ell}{2} \right\rceil - 1 = m-2,$$

which is true. Therefore, the sums at both kinds of vertices cannot coincide and the labeling we have constructed is an injection.

Recall that the order of the spine of C is $s = m-\ell+1$. Since according to our labeling, C is $\max(0, \lfloor\frac{m-\ell}{2}\rfloor - 1 - \mathcal{E}(C))$-antimagic, we conclude that it is $\max(0, \lfloor\frac{s-1}{2}\rfloor - 1 - \mathcal{E}(C))$-antimagic and the proof is complete. $\square$

Example. As a detailed example of Theorem 1, consider the caterpillar illustrated in Figure 1. This caterpillar has 39 vertices, 38 edges, 26 leaves, and leaves excess equal to 15. Thus, $i = \left\lceil\frac{38-26}{2}\right\rceil + 1 = 7$, $j = \left\lceil\frac{38+26}{2}\right\rceil - 1 = 31$ and $k = \max\left(15, \lfloor\frac{38-26}{2}\rfloor - 1\right) = 15$. The label sets are $L_1 = \{1, \ldots, 15\}$; $L_2 = \{16, \ldots, 22\}$; $L_3 = \{23, \ldots, 31\}$; $L_4 = \{32, \ldots, 38\}$.

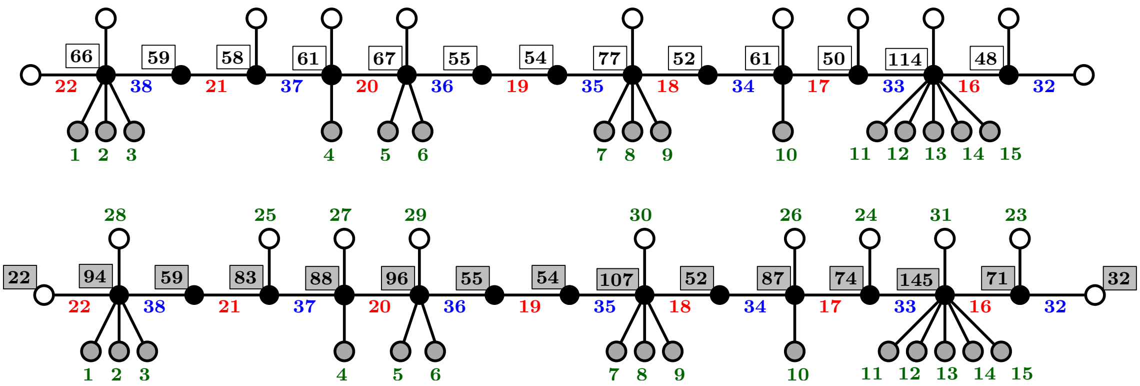

Figure 2: Above, red labels belong to L_2; blue labels belong to L_4 and green labels belong to L_1. For each spine vertex, the partial sum is calculated. Below, labels from L_3 (also in green) are assigned to the remaining edges (only the sums are shown) and the final sum of each non-leaf vertex is shown (squared).

First, we assign the labels of the sets L_2 and L_4 to the edges of a longest path in the way pointed out in the proof of the theorem. Secondly, we consider a set of "leaves excess" by choosing $d(u)-3$ leaves for each vertex u having degree $d(u) \geq 4$, and we assign randomly the labels of the set L_1 to the edges incident to them. Finally, we assign the labels of L_3 to the remaining edges in decreasing order of the partial sum of their incident vertices from the longest path (see Figure 2).

Since any caterpillar of order n with a spine of order s satisfies $s - 1 \leq n - 3$, the biggest value of $\max(0, \lfloor\frac{s-1}{2}\rfloor - 1 - \mathcal{E}(C))$ in Theorem 1 is $(\lfloor\frac{n-1}{2}\rfloor - 2)$. So we get the following result.

Corollary 1. *Caterpillars of order n are $(\lfloor\frac{n-1}{2}\rfloor - 2)$-antimagic.*

Using the Combinatorial Nullstellensatz polynomial method of Alon [1], Lladó and Miller [18] proved the following result about trees.

Theorem 2. *([18, Thm. 7]) Trees of order $n \geq 3$ having exactly k vertices of degree at least 2 with no leaf adjacent to them are k-antimagic.*

The preceding theorem is stated in terms of the *base inner vertices*, which are those with no adjacent leaf. Theorem 1 and Theorem 2 for caterpillars do not imply each other, as can be seen in the examples shown in Figure 2.

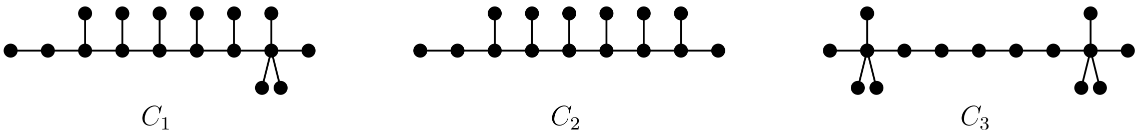

Figure 3: We can derive that C_1 is antimagic (from Theorems 1 and 2); C_2 is antimagic (from Theorem 2), and 2-antimagic (from Theorem 1); C_3 is antimagic (from Theorem 1) and 5-antimagic (from Theorem 2).

3 Special subclasses

3.1 Caterpillars with many leaves

In the case that the parameter $\mathcal{E}(C)$ is large enough, it is easy to observe that caterpillar C is antimagic by direct application of Theorem 1, i.e., caterpillars with a spine of order s and $\mathcal{E}(C) \geq \lfloor \frac{s-1}{2} \rfloor - 1$ are antimagic. We obtain the following corollary (see also [15]).

Corollary 2. *Caterpillars with a spine of order s and at least $\lfloor \frac{3s+1}{2} \rfloor$ leaves are antimagic.*

Proof. Let C be a caterpillar with ℓ leaves and a spine of order s. Let s_3 be the number of vertices of the spine with degree at least 3. From the definition of $\mathcal{E}(C)$, we have that $\mathcal{E}(C) = \ell - s_3 - 2$ (see Figure 1). Then, since $s_3 \leq s$, we obtain

$$\mathcal{E}(C) = \ell - s_3 - 2 \geq \left\lfloor \frac{3s+1}{2} \right\rfloor - s - 2 = \left\lfloor \frac{3s+1}{2} - s - 1 \right\rfloor - 1 = \left\lfloor \frac{s-1}{2} \right\rfloor - 1.$$

The result thus follows by Theorem 1. □

3.2 Caterpillars with a long tail

The technique used in Theorem 1 can be adapted to the case when a caterpillar has a *tail* getting an improvement of that theorem for this type of caterpillar. A *tail* of order t in a caterpillar C is a path containing one endpoint of a longest path in C and $t-1$ more vertices with degree 2.

Theorem 3. *Caterpillars with a spine of order s and a tail of order $\lfloor \frac{s-1}{2} \rfloor$ are antimagic.*

Proof. Let C be a caterpillar with m edges and ℓ leaves. Let P be a longest path in C. Using the same terminology as in Theorem 1, we observe that C has $m-\ell+3$ path vertices and $m-\ell+2$ path edges. Let T be a tail of order $\lfloor \frac{s-1}{2} \rfloor$. Without loss of generality, T can be chosen in such a way that it is a subgraph of P and its endpoint coincides with one of the endpoints of P. Recall that $m-\ell=s-1$. In order to define the injection from $E(C)$ to a label set, we choose the values:

$$i=\left\lceil\frac{m-\ell}{2}\right\rceil+1, \quad j=\left\lceil\frac{m+\ell}{2}\right\rceil-1,$$

and consider a label set $L=\bigcup_{r=1}^{3}L_r$, where:

- $L_1=\{1,\dots,i\}$
- $L_2=\{i+1,\dots,j\}$
- $L_3=\{j+1,\dots,m\}$

Since labels are consecutive across label sets L_1, L_2, and L_3, the total number of labels is $|L|=m$, the value of the largest one. Therefore, C will be antimagic if there is an injection from $E(C)$ to L. Now we show how to construct such an injection.

Labels in L_1 and L_3 will be assigned to the path edges, while labels in L_2 will be assigned to the non-path edges. We consider two phases to complete the assignment:

- *Phase 1: Labeling the path edges.* By assumption, the vertex having degree 1 in the tail T is one of the endpoints of P. We choose the other endpoint in P, say u, to start our labeling. We assign labels to the path P starting with the largest label in L_1, which will be assigned to the edge incident to u, then the largest label in L_3 will be assigned to the next edge in P, then the previous ones and so on, keeping the alternation of labels until the first ones are reached. Later on, we will use the fact that the smallest labels in L_1 and L_3 have been assigned to the tail T. Note that $|L_1|=i$ and $|L_3|=m-j$. Moreover,

  - If s is odd, then $m-\ell$ is even and from the definition of i and j we have that $i+j=m$. Thus, $|L_1|=i=m-j=|L_3|$ and we can alternate labels along the path P in the following way:

  $$i, m, \dots, 1, j+1.$$

  - If s is even, then $m-\ell$ is odd and $i+j=m+1$. Then, $|L_1|=i=m-j+1=|L_3|+1$. The alternation of labels in P now ends with the first label in L_1:

  $$i, m, \dots, 2, j+1, 1.$$

  The previous partial assignment uses all the labels in L_1 and L_3 to produce partial sums at the inner path vertices ranging between $j+2$ and $i+m$. In addition, the endpoints of path P have the sums i and $j+1$ (in case s is odd), or i and 1 (in case s is even).

  Clearly, all partial sums at the path vertices are different. The vertex sums at the endpoints of the path are smaller than the sums at the inner path vertices, which are obtained summing up two consecutive labels (the smallest sum is $j+2$, which is greater than the possible sums at the extremes of the path: i, $j+1$, and 1). On the other hand, the vertex sums at the inner vertices of P of degree 2 in C are all different since, by the way the assignment is defined, sums are strictly decreasing.

- *Phase 2: Labeling the non-path edges.* Now, we add the labels in L_2 to the non-path edges in the following way. In the first place, for every path vertex u having degree $d(u) > 3$, we randomly assign labels from L_2 to $d(u) - 3$ non-path edges incident to u. After this step, all the path vertices are incident with at most one non-path edge which has not yet been assigned a label.

  Let E' be the list of the still unlabeled non-path edges in non-increasing order of the partial sum of the only path-vertex incident to each of them. Let L_2' be the list of still unused labels of L_2 in decreasing order. Assign each label of L_2' to the edge of E' at the same position in the lists. This assignment guarantees all path vertices of degree at least 3 to have different sums.

  Also note that the vertex sums obtained at the non-path vertices (each of them from a unique label in L_2) are smaller than the vertex sums at the path vertices of degree at least 2, which contain at least one label from L_3 and, hence, they are greater than any label in L_2.

Finally, we check that sums (of Phase 1) at vertices of degree 2 cannot coincide with those (of Phase 2) at vertices of degree at least 3. On the one hand, the largest vertex sum that can be achieved at a vertex of degree 2 is $i+m$, since only labels in L_1 and L_3 are used. On the other hand, the smallest sum that can be obtained at a vertex of degree at least 3 will be achieved in a vertex belonging to a path vertex but not to the tail. This sum will be composed of two parts:

1. The smallest label in L_2, $i+1$, and
2. The smallest sum obtained in Phase 1 at a vertex not belonging to the tail. The sums obtained at the tail vertices are (from the vertex in T having degree 1 in C) $1, j+2, j+3, j+4, \ldots$ or $j+1, j+2, j+3, j+4, \ldots$ depending on the parity of the tail order. In any case, the sum $j + \lfloor \frac{m-\ell}{2} \rfloor + 1$ is the smallest one obtained in Phase 1 for a vertex not belonging to the tail (which has order $\lfloor \frac{m-\ell}{2} \rfloor$).

Therefore, the smallest sum of a path vertex with degree at least 3 and not belonging to the tail is $(i+1) + (j + \lfloor \frac{m-\ell}{2} \rfloor + 1)$. Then, we need that $i + m < i + j + \lfloor \frac{m-\ell}{2} \rfloor + 2$, which is true if and only if

$$m < \left\lceil \frac{m+\ell}{2} \right\rceil - 1 + \left\lfloor \frac{m-\ell}{2} \right\rfloor + 2 = m+1,$$

which is true. Therefore, the sums at both kinds of vertices cannot coincide and the labeling we have constructed is an injection.

The order of the spine of C is $s = m - \ell + 1$. Since we have assumed that the tail T has order $\lfloor \frac{m-\ell}{2} \rfloor$, this is equivalent to T having order $\lfloor \frac{s-1}{2} \rfloor$, as it is assumed in the statement of the theorem. □

Theorems 1, 2, and 3 for caterpillars do not imply each other, as can be seen in the example shown in Figure 4. An antimagic labeling for the caterpillar of this example obtained with the procedure described in the proof of Theorem 3 is also shown.

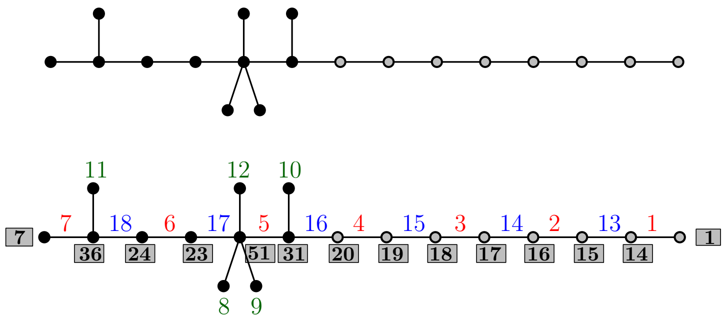

Figure 4: A caterpillar with a tail of order 8 (gray vertices). Theorem 3 implies that it is antimagic. Nevertheless, we can only derive that it is 2-antimagic from Theorem 1 and 9-antimagic from Theorem 2. An antimagic labeling is given: red labels correspond to L_1, blue labels to L_3, and green labels to L_2. The vertex sums at vertices belonging to a longest path are shown (squared numbers).

3.3 Caterpillars with a spine of prime order

The following theorem is proved by Wong and Zhu [22] using the Combinatorial Nullstellensatz method, and it will be a fundamental tool for our result below.

Theorem 4. *([22, Thm. 15]) If n is a prime number, then P_n (the path on n vertices) is weighted-1-antimagic.*

By converting a caterpillar into a weighted path, we are able to obtain a 1-antimagic labeling for caterpillars whose spine order is close to a prime number.

Corollary 3. *Let p be a prime number. Any caterpillar with a spine of order p, $p-1$ or $p-2$ is 1-antimagic.*

Proof. Suppose that p is a prime number. Let C be a caterpillar with m edges, ℓ leaves, and a spine S of order $s = p - k$, where $0 \leq k \leq 2$. Let $L = \{1, \ldots, m+1\}$ be the label set.

Note that the spine S is part of a longest path with two more vertices than S; let P be defined by S plus k more vertices of such longest path of C to which S belongs. Thus, P has order p, which is a prime number. Now, we assign weights to the vertices of this path P.

Let $r = \ell - k$, i.e., r is the number of leaves of C which are not endpoints of P. We assign a distinct label in $\{1, \ldots, r\}$ to each of the edges incident to these leaves. This labeling produces distinct labels at the r leaves and also a partial sum, called $f(u)$, at each vertex u of C (i.e., the sum of the labels of the non-path edges incident to u). Define the weight $w(u)$ for each path vertex u of P as follows:

$$w(u) = \begin{cases} f(u) + r, & \text{if } u \text{ is an endpoint of } P \\ f(u) + 2r, & \text{otherwise.} \end{cases}$$

Now, by Theorem 4, we get that the path P is weighted-1-antimagic. Moreover, let $h : E(P) \to \{1, \ldots, p\}$ be a 1-antimagic labeling for P according to Theorem 4. Then, we can argue that

$$h'(e) = \begin{cases} h(e) + r, & \text{if } e \in E(P) \\ f(u), & \text{otherwise, where } u \notin V(P) \text{ is incident to } e \end{cases}$$

is a 1-antimagic labeling for C. First, notice that for any $e \in E(P)$,

$$h(e) + r \leq p + r = (s + k) + (\ell - k) = s + \ell = (m - \ell + 1) + \ell = |L|$$

and, therefore, $h' : E(C) \to L$. Furthermore, all vertex sums are pairwise distinct due to the following facts:

1. Vertex sums for vertices in $V(C) \setminus V(P)$ have a value at most r and are pairwise distinct since they correspond to the initial labeling of the leaves of C which are not endpoints of P.

2. Vertex sums for vertices in $V(P)$ have a value at least $r+1$ since every path vertex in $V(P)$ is adjacent to some path edge e, for which we have defined $h'(e) = h(e)+r > r$. Additionally, the vertex sum at a vertex $u \in V(P)$ using labeling h' coincide with the vertex sum at u using labeling h and weight function w. We consider two cases:

   (a) u is an endpoint of P. Let e^u be the edge in P incident to u. The vertex sum at u in C with labeling h' is $h(e^u) + r + f(u)$, while the vertex sum at u in P with weight function w and labeling h is $h(e^u) + w(u) = h(e^u) + f(u) + r$.

   (b) u is not an endpoint of P. Let e_1^u and e_2^u be the edges on the path P incident to u. The vertex sum at u in C with labeling h' is

   $$(h(e_1^u) + r) + (h(e_2^u) + r) + f(u) = h(e_1^u) + h(e_2^u) + f(u) + 2r.$$

   On the other hand, the vertex sum at u in P with weight function w and labeling h is $h(e_1^u) + h(e_2^u) + w(u) = h(e_1^u) + h(e_2^u) + f(u) + 2r$.

   Since P is weighted-1-antimagic with the weight function w via labeling h, all vertex sums at the vertices in P must be pairwise distinct. Therefore, all vertex sums for vertices in $V(P)$ are also pairwise distinct with labeling h' (and no weights on the vertices).

We conclude that h' is a 1-antimagic labeling for C. □

4 Open problem

Although the main problem we leave open in this paper is whether caterpillars are antimagic, an intriguing particular case is the following.

Open problem 1. *Are caterpillars with maximum degree* 3 *antimagic?*

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