

COMMUTATORS IN GROUPS OF PIECEWISE PROJECTIVE HOMEOMORPHISMS.

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ABSTRACT. In [7] Monod introduced examples of groups of piecewise projective homeomorphisms which are not amenable and which do not contain free subgroups, and in [6] Lodha and Moore introduced examples of finitely presented groups with the same property. In this article we examine the normal subgroup structure of these groups. Two important cases of our results are the groups H and G_0 . We show that the group H of piecewise projective homeomorphisms of \mathbb{R} has the property that H'' is simple and that every proper quotient of H is metabelian. We establish simplicity of the commutator subgroup of the group G_0 , which admits a presentation with 3 generators and 9 relations. Further, we show that every proper quotient of G_0 is abelian. It follows that the normal subgroups of these groups are in bijective correspondence with those of the abelian (or metabelian) quotient.

INTRODUCTION

In [7] Monod proved that the group H of piecewise projective homeomorphisms of the real line is non-amenable and does not contain non-abelian free subgroups. This provides a new counterexample to the so called von Neumann–Day problem [8, 2]. In fact, Monod introduced a family of groups $H(A)$ for a subring A of \mathbb{R} . In the case where A is strictly larger than \mathbb{Z} , they were all demonstrated to be counterexamples. The group H is the case in which $A = \mathbb{R}$.

The subgroups G_0 and G of H were introduced by Lodha and Moore in [6] as finitely presented counterexamples. The groups G_0 and G share many features with Thompson’s group F . They can be viewed as groups of homeomorphisms of the Cantor set of infinite binary sequences, and as groups of homeomorphisms of the real line. They admit small finite presentations, and symmetric infinite presentations with a natural normal form [6, 5]. Further, they are of type F_∞ [5]. Viewed as homeomorphisms of the Cantor set, the elements can be represented by tree diagrams.

Thompson’s group F satisfies the property that F' is simple, and every proper quotient of F is abelian [1]. In this article we examine the normal subgroup structure, and in particular the commutator subgroup structure of G , G_0 , and $H(A)$ for a subring A of \mathbb{R} , and obtain properties similar to F . We prove the following.

Theorem 1. *Let A be a subring of \mathbb{R} . If A has units other than ± 1 , then:*

- (1) $H(A)' \neq H(A)''$.

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- (2) $H(A)''$ is simple.
- (3) Every proper quotient of $H(A)$ is metabelian.

If the only units in A are ± 1 , then:

- (1) $H(A)'$ is simple.
- (2) Every proper quotient of $H(A)$ is abelian.
- (3) All finite index subgroups of $H(A)$ are normal in $H(A)$.

We show the following for the finitely presented groups G_0 and G (defined in Section 1).

Theorem 2. *The group G_0 satisfies the following:*

- (1) G'_0 is simple.
- (2) Every proper quotient of G_0 is abelian.
- (3) All finite index subgroups of G_0 are normal in G_0 .

The group G satisfies the following:

- (1) $G'' \neq G'$.
- (2) G'' is simple and $G'' = G'_0$.
- (3) Every proper quotient of G is metabelian.

One interesting feature of this article is that although the proofs of Theorems 1 and 2 both use a theorem of Higman, the proofs are different in the following sense. The arguments of the proof of Theorem 2 are intrinsic to the combinatorial model developed for G_0, G in [6] using continued fractions. The arguments of the proof of Theorem 1 arise in the setting of the action of the groups $H(A)$ on the real line.

1. BACKGROUND

Actions considered will be from the right except when we explicitly use function notation or represent elements by matrices. The conjugate of g by h is $h^{-1}gh$. All groups under study here are subgroups of $PPSL_2(\mathbb{R})$, the group of piecewise projective homeomorphisms of $\mathbb{R} \cup \{\infty\}$ preserving orientation. Namely, for each element in $PPSL_2(\mathbb{R})$ there are finitely many points t_1, \dots, t_n such that in each of the intervals $(-\infty, t_1]$, $[t_i, t_{i+1}]$ and $[t_n, \infty)$ the map is of the form $t \mapsto (at + b)/(ct + d)$ for some matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with determinant one. The group H is the subgroup of $PPSL_2(\mathbb{R})$ formed of those maps that stabilize infinity, and that hence give homeomorphisms of \mathbb{R} . Observe that elements of H have affine germs at $\pm\infty$, that is, in the interval $[t_n, \infty)$ the map is $(at + b)/d$ with $ad = 1$, and similarly on the interval $(-\infty, t_1]$. Given a subring A of \mathbb{R} , we denote by P_A the set of fixed points of hyperbolic elements of $PSL_2(A)$. Then $H(A)$ is defined to be the subgroup of H consisting of elements that are piecewise $PSL_2(A)$ with breakpoints in P_A . See [7] for details on these groups.

The two Lodha–Moore groups [6] are finitely presented subgroups of H and will be denoted by G and G_0 . The group G_0 is the group of homeomorphisms of \mathbb{R} generated by the following three maps:

$$a(t) = t + 1 \quad b(t) = \begin{cases} t & \text{if } t \leq 0 \\ \frac{t}{1-t} & \text{if } 0 \leq t \leq \frac{1}{2} \\ \frac{3t-1}{t} & \text{if } \frac{1}{2} \leq t \leq 1 \\ t+1 & \text{if } 1 \leq t \end{cases} \quad c(t) = \begin{cases} \frac{2t}{t+1} & \text{if } 0 \leq t \leq 1 \\ t & \text{otherwise} \end{cases}$$

As is done with Thompson’s group F , elements will be given an interpretation in terms of maps of binary sequences and tree diagrams. A *binary sequence* is a (finite or infinite) sequence of 0 and 1. The set of infinite binary sequences will be denoted by $2^{\mathbb{N}}$, and the set of finite ones by $2^{<\mathbb{N}}$. We will use \mathbf{s} and \mathbf{t} to denote finite binary sequences (with a distinctive font), and Greek letters ξ , ζ or η for infinite ones. Two binary sequences can be concatenated as long as the first one is finite, such as 010ξ , $\mathbf{s}01$ or $\mathbf{s}\zeta$.

We define the binary sequence map:

$$\begin{aligned} x : 2^{\mathbb{N}} &\longrightarrow 2^{\mathbb{N}} \\ (00\xi) \cdot x &= 0\xi \\ (01\xi) \cdot x &= 10\xi \\ (1\xi) \cdot x &= 11\xi \end{aligned}$$

and also, recursively, the pair of mutually inverse maps

$$\begin{aligned} y : 2^{\mathbb{N}} &\longrightarrow 2^{\mathbb{N}} & y^{-1} : 2^{\mathbb{N}} &\longrightarrow 2^{\mathbb{N}} \\ (00\xi) \cdot y &= 0(\xi \cdot y) & (0\xi) \cdot y^{-1} &= 00(\xi \cdot y^{-1}) \\ (01\xi) \cdot y &= 10(\xi \cdot y^{-1}) & (10\xi) \cdot y^{-1} &= 01(\xi \cdot y) \\ (1\xi) \cdot y &= 11(\xi \cdot y) & (11\xi) \cdot y^{-1} &= 1(\xi \cdot y^{-1}). \end{aligned}$$

Each of these maps will give rise to a family of maps of binary sequences defined in the following way. Given a finite binary sequence \mathbf{s} , the map $x_{\mathbf{s}}$ is the identity except on the infinite sequences starting with \mathbf{s} , where it acts as x on the tail. That is,

$$(\mathbf{s}\xi) \cdot x_{\mathbf{s}} = \mathbf{s}(\xi \cdot x)$$

and as the identity if the sequence does not start with \mathbf{s} . The maps $y_{\mathbf{s}}$ or $y_{\mathbf{s}}^{-1}$ are defined in an analogous way. The map $x_{\mathbf{s}}$ (and also similarly $y_{\mathbf{s}}$) also admits a partial action on the set $2^{<\mathbb{N}}$ of finite binary sequences in the natural way: if a sequence extends $\mathbf{s}00$, $\mathbf{s}01$ or $\mathbf{s}1$, we have $(\mathbf{s}u) \cdot x_{\mathbf{s}} = \mathbf{s}(u \cdot x)$, and $x_{\mathbf{s}}$ is the identity on those sequences which are incompatible with \mathbf{s} . The map $x_{\mathbf{s}}$ is not defined on the other sequences, namely, $x_{\mathbf{s}}$ is not defined on $\mathbf{s}0$ or on the prefixes of \mathbf{s} , in particular, observe that x is not defined on the sequence 0.

The reason for these definitions is that they represent elements of G_0 under an identification given by the following maps:

$$\begin{aligned} \varphi : 2^{\mathbb{N}} &\longrightarrow [0, \infty] & \Phi : 2^{\mathbb{N}} &\longrightarrow \mathbb{R} \\ \varphi(0\xi) &= \frac{1}{1 + \frac{1}{\varphi(\xi)}} & \Phi(0\xi) &= -\varphi(\tilde{\xi}) \\ \varphi(1\xi) &= 1 + \varphi(\xi) & \Phi(1\xi) &= \varphi(\xi) \end{aligned}$$

where $\tilde{\xi}$ is the sequence obtained from ξ by replacing all symbols 0 by 1 and viceversa. Under these definitions, the maps a , b and c are represented by the binary sequence maps x , x_1 and y_{10} , respectively. We have the following result (Proposition 3.1 in [6]):

Proposition 1.1. *For all ξ in $2^{\mathbb{N}}$ we have*

$$a(\Phi(\xi)) = \Phi(x(\xi)) \quad b(\Phi(\xi)) = \Phi(x_1(\xi)) \quad c(\Phi(\xi)) = \Phi(y_{10}(\xi))$$

For all details and proofs, see [6]. Hence, we can consider that the group G_0 is the group of maps of $2^{\mathbb{N}}$ generated by x, x_1, y_{10} .

The group G is defined to be the group generated by all $x_{\mathbf{s}}$ and $y_{\mathbf{s}}$. The group G_0 is generated by all $x_{\mathbf{s}}$ and by all those $y_{\mathbf{s}}$ where \mathbf{s} is not constant, that is, \mathbf{s} is not 0^n nor 1^n . We have a series of relations which are satisfied by these generators:

- (1) $x_{\mathbf{s}}^2 = x_{\mathbf{s}0}x_{\mathbf{s}}x_{\mathbf{s}1}$,
- (2) if $\mathbf{t} \cdot x_{\mathbf{s}}$ is defined, then $x_{\mathbf{t}}x_{\mathbf{s}} = x_{\mathbf{s}}x_{\mathbf{t} \cdot \mathbf{s}}$,
- (3) if $\mathbf{t} \cdot x_{\mathbf{s}}$ is defined, then $y_{\mathbf{t}}x_{\mathbf{s}} = x_{\mathbf{s}}y_{\mathbf{t} \cdot \mathbf{s}}$,
- (4) if \mathbf{s} and \mathbf{t} are incompatible, then $y_{\mathbf{s}}y_{\mathbf{t}} = y_{\mathbf{t}}y_{\mathbf{s}}$,
- (5) $y_{\mathbf{s}} = x_{\mathbf{s}}y_{\mathbf{s}0}y_{\mathbf{s}10}^{-1}y_{\mathbf{s}11}$.

The relations (1) and (2) are part of a known presentation for Thompson's group F given by Dehornoy in [3]. The key relation for the Lodha–Moore groups is the relation (5), which represents algebraically the recursive definition of y given above.

Finally, the relations satisfied by these generators, and in particular the Thompson's group relations, allow us to obtain finite presentations. The group G_0 is generated only by x, x_1 and y_{10} with a set of 9 relations, whereas G is generated by x, x_1, y_0, y_1 and y_{10} .

In [6] it is shown that every element of G can be written in a standard form. Recall that G is generated by the set $X \cup Y$ where $X = \{x_{\mathbf{s}} : \mathbf{s} \in 2^{<\mathbb{N}}\}$ and $Y = \{y_{\mathbf{s}} : \mathbf{s} \in 2^{<\mathbb{N}}\}$. A word over $X \cup Y$ is said to be in *standard form* if it is of the form

$$hy_{\mathbf{s}_1}^{a_1} \dots y_{\mathbf{s}_n}^{a_n}$$

where h is a word over X , \mathbf{s}_j is a prefix of \mathbf{s}_i only if $j \geq i$, and the a_i are arbitrary, non-zero integers. It is shown in [6, Lemma 5.4] that every element of G can be written in standard form. The standard form is, however, not unique.

2. COMMUTATORS FOR G_0

We first study the abelianization of the group G_0 . The partial action of F on the set of all non-constant binary sequences is transitive. Therefore, from relation (3) it follows that for all non-constant \mathbf{s} , $y_{\mathbf{s}}$ is a conjugate of y_{10} .

Lemma 2.1. *The map $\{x, x_1, y_{10}\} \rightarrow \mathbb{Z}^3$ given by*

$$x \mapsto (1, 0, 0) \quad x_1 \mapsto (0, 1, 0) \quad y_{10} \mapsto (0, 0, 1)$$

extends to a surjective homomorphism $\pi : G_0 \rightarrow \mathbb{Z}^3$, with kernel being the commutator subgroup $G'_0 = [G_0, G_0]$.

Proof. We use the infinite presentation of G_0 having generating set $S_0 = X \cup Y_0$ where $X = \{x_{\mathbf{s}} : \mathbf{s} \in 2^{<\mathbb{N}}\}$ and $Y_0 = \{y_{\mathbf{s}} : \mathbf{s} \in 2^{<\mathbb{N}}, \mathbf{s} \text{ is not a constant word}\}$. The set of relations is given by those relations of the form (1) to (5) above that involve only elements of S_0 . That this gives a presentation of G_0 is Theorem 3.3 in [6]. One could, of course, also use the finite presentation for G_0 on the generating set $\{x, x_1, y_{10}\}$ (see [6]), but we shall not do so.

Noting that each element $y_{\mathbf{s}} \in Y_0$ is conjugate in G_0 to y_{10} , that $x_{0^m} \in X$ is conjugate to x_0 , that $x_{1^m} \in X$ is conjugate to x_1 and that $x_{\mathbf{s}} \in X$ is conjugate to x_{10} when \mathbf{s} is non-constant, we consider the map from $S_0 \rightarrow \mathbb{Z}^3$ given by

$$y_{\mathbf{s}} \mapsto (0, 0, 1) \quad x \mapsto (1, 0, 0) \quad x_{\mathbf{s}} \mapsto \begin{cases} (1, -1, 0) & \text{if } \mathbf{s} = 0^m \text{ for some } m \geq 1 \\ (0, 1, 0) & \text{if } \mathbf{s} = 1^m \text{ for some } m \geq 1 \\ (0, 0, 0) & \text{if } \mathbf{s} \text{ is non-constant} \end{cases}$$

That this map extends to a homomorphism $\pi : G_0 \rightarrow \mathbb{Z}^3$ can be readily seen by considering the effect on each of the relations (1) to (5).

Since $\{x, x_1, y_{10}\}$ projects to a generating set for the abelianization G_0/G'_0 and the image $\{\pi(x), \pi(x_1), \pi(y_{10})\}$ is a basis for \mathbb{Z}^3 we conclude that π induces an isomorphism from G_0/G'_0 to \mathbb{Z}^3 . \square

The restriction of π to the subgroup F generated by $\{x, x_1\}$ gives (after a change of basis) the abelianization map for Thompson's group F , where elements are evaluated by the two germs at $\pm\infty$. The third component of the map π represents the total exponent for the y -generators in a word representing an element. Hence, we have the following result.

Proposition 2.2. *The commutator G'_0 contains exactly those elements in G_0 which have compact support, and which have a total exponent in the y -generators equal to zero. \square*

The goal of this section is to prove the first part of Theorem 2, which concerns G_0 . To proceed with the proof, we will use the following theorem, due to Higman. Let Γ be a group of bijections of some set E . For $g \in \Gamma$ define its *moved set* $D(g)$ as the set of points $x \in E$ such that $g(x) \neq x$. This is analogous to the support, but since *a priori* there is no topology on E , we do not take the closure.

Theorem 2.3. *Suppose that for all $\alpha, \beta, \gamma \in \Gamma \setminus \{1_{\Gamma}\}$, there is a $\rho \in \Gamma$ such that the following holds: $\gamma(\rho(S)) \cap \rho(S) = \emptyset$ where $S = D(\alpha) \cup D(\beta)$. Then Γ is simple.*

The proof can be seen in [4].

Corollary 2.4. *Let Γ be a group of compactly supported homeomorphisms of \mathbb{R} that contains F' . Then Γ' is simple.*

This is the consequence of the high transitivity of F' which ensures that the conditions of the theorem are satisfied. We shall apply this corollary to several groups in the paper.

This corollary cannot be applied directly to G_0 , since this group contains elements (e.g., x) whose support is all of \mathbb{R} . So we will apply this corollary to the group G'_0 , whose elements have compact support. We conclude that G''_0 is simple. The proof of Theorem 2 for G_0 will now be complete with the following result.

Proposition 2.5. $G'_0 = G''_0$.

Proof. Consider an element $g \in G'_0$ and write the element in standard form as $g = hz$, where $h \in F'$ and

$$z = y_{\mathbf{s}_1}^{a_1} \dots y_{\mathbf{s}_n}^{a_n}$$

for some binary sequences \mathbf{s}_i , and with $a_1 + \dots + a_n = 0$. Since h has compact support, we have $h \in F' = F'' \subset G''_0$. So we only need to show that $z \in G''_0$. Note that for any generator $x_{\mathbf{s}}$ such that \mathbf{s} is a non constant sequence, $x_{\mathbf{s}} \in G''_0$. We will make use of this fact repeatedly in what follows.

The proof proceeds by induction on $k = |a_1| + \dots + |a_n|$ which is clearly an even number. For the element z there will be some a_i positive and some negative. Since G''_0 is normal, we can cyclically conjugate and assume that the word starts with a subword of the type $y_{\mathbf{s}}y_{\mathbf{t}}^{-1}$. As the starting point of the induction, just take $1 \in G''_0$. We just need to prove that $y_{\mathbf{s}}y_{\mathbf{t}}^{-1} \in G''_0$ and using the induction hypothesis for the rest of the word, the proof will be complete. We have three cases.

Case (1): \mathbf{s} and \mathbf{t} are consecutive. This just means that the corresponding intervals in \mathbb{R} are adjacent, or that if \mathbf{s} and \mathbf{t} are leaves in a tree, they are consecutive in the natural order of the leaves.

Take the word $w = y_{100}y_{101}^{-1} \in G'_0$. Construct an element $f \in F'$ such that

$$fwf^{-1} = y_{10011}y_{101}^{-1},$$

which is possible because these two sequences are also consecutive and any of these can be F' -conjugated to any other. Now we have that $[w, f] = wf w^{-1}f^{-1} \in G''_0$, since it is the commutator of two elements in G'_0 . But clearly

$$[w, f] = y_{100}y_{10011}^{-1},$$

Now apply relation (5) to y_{100} to get

$$[w, f] = x_{100}y_{1000}y_{10010}^{-1}y_{10011}y_{10011}^{-1} = x_{100}y_{1000}y_{10010}^{-1}.$$

As mentioned before, we know that $x_{100} \in F' \subset F'' \subset G''_0$, so this implies that $y_{1000}y_{10010}^{-1} \in G''_0$, and these are two consecutive binary sequences. Hence, by conjugation, any $y_{\mathbf{s}}y_{\mathbf{t}}^{-1}$ with consecutive binary sequences is in G''_0 .

Case (2): \mathfrak{s} and \mathfrak{t} are not consecutive and also not comparable. Assume $\mathfrak{s} < \mathfrak{t}$, as the other case reduces to this by inverting the element. In that case, just write

$$y_{\mathfrak{s}}y_{\mathfrak{t}}^{-1} = y_{\mathfrak{s}}y_{\mathfrak{s}_1}^{-1}y_{\mathfrak{s}_1}y_{\mathfrak{s}_2}^{-1}y_{\mathfrak{s}_2} \cdots y_{\mathfrak{s}_m}^{-1}y_{\mathfrak{s}_m}y_{\mathfrak{t}}^{-1}$$

such that the pairs

$$y_{\mathfrak{s}}y_{\mathfrak{s}_1}^{-1} \quad y_{\mathfrak{s}_i}y_{\mathfrak{s}_{i+1}}^{-1} \quad y_{\mathfrak{s}_m}y_{\mathfrak{t}}^{-1}$$

lie in the previous case.

Case (3): \mathfrak{s} and \mathfrak{t} are comparable, so assume $\mathfrak{t} = \mathfrak{su}$. The case $\mathfrak{s} = \mathfrak{tu}$ reduces to this by taking an inverse. We apply relation (5) to $y_{\mathfrak{s}}$ again and find pairs which now correspond to cases (1) or (2). Cases $y_{\mathfrak{s}_1}^{-1}y_{\mathfrak{s}_2}$ are cyclically permuted to $y_{\mathfrak{s}_2}y_{\mathfrak{s}_1}^{-1}$. We distinguish all four easy cases for clarity:

- $u = 0$ or $u = 11$. Since $y_{\mathfrak{s}}y_{\mathfrak{su}}^{-1} = x_{\mathfrak{s}}y_{\mathfrak{s}0}y_{\mathfrak{s}_{10}}^{-1}y_{\mathfrak{s}_{11}}y_{\mathfrak{su}}^{-1}$, $y_{\mathfrak{su}}^{-1}$ cancels with one of the results of relation (5) applied to $y_{\mathfrak{s}}$. We are left with the product of an x -generator (in G_0'') with a word that falls in case (1).
- $u = 1$. Applying relation (5) we obtain

$$x_{\mathfrak{s}}y_{\mathfrak{s}0}y_{\mathfrak{s}_{10}}^{-1}y_{\mathfrak{s}_{11}}y_{\mathfrak{s}_1}^{-1}$$

and we apply case (1) to the pairs

$$y_{\mathfrak{s}0}y_{\mathfrak{s}_1}^{-1} \quad y_{\mathfrak{s}_{10}}^{-1}y_{\mathfrak{s}_{11}}.$$

- $u = 10$. Using relations (4), (5) we obtain:

$$x_{\mathfrak{s}}y_{\mathfrak{s}0}y_{\mathfrak{s}_{10}}^{-2}y_{\mathfrak{s}_{11}}$$

and we just need to apply case (1) to $y_{\mathfrak{s}0}y_{\mathfrak{s}_{10}}^{-1}$ and $y_{\mathfrak{s}_{10}}^{-1}y_{\mathfrak{s}_{11}}$.

- $u \neq 0, 1, 10, 11$. Using relation (5) we obtain

$$x_{\mathfrak{s}}y_{\mathfrak{s}0}y_{\mathfrak{s}_{10}}^{-1}y_{\mathfrak{s}_{11}}y_{\mathfrak{su}}^{-1}.$$

It suffices to show that $y_{\mathfrak{s}0}y_{\mathfrak{s}_{10}}^{-1}y_{\mathfrak{s}_{11}}y_{\mathfrak{su}}^{-1} \in G_0''$. If u begins with a 1, by cyclic conjugation we obtain $(y_{\mathfrak{su}}^{-1}y_{\mathfrak{s}0})(y_{\mathfrak{s}_{10}}^{-1}y_{\mathfrak{s}_{11}})$. The word $y_{\mathfrak{su}}^{-1}y_{\mathfrak{s}0}$ falls in cases (1) or (2) and the word $y_{\mathfrak{s}_{10}}^{-1}y_{\mathfrak{s}_{11}}$ falls in (1), so we are done. For the case where u begins with a 0 we express the word as a product of $y_{\mathfrak{s}0}y_{\mathfrak{s}_{10}}^{-1}$ and $y_{\mathfrak{s}_{11}}y_{\mathfrak{su}}^{-1}$, both words fall in previous case.

□

Our main theorem for G_0 has some important corollaries.

Corollary 2.6. *The finite-index subgroups of G_0 are in bijection with the finite-index subgroups of \mathbb{Z}^3 . Every subgroup H of finite index is normal, and $H' = G_0'$.*

Proof. If H is not normal, then take the intersection K of all its conjugates, which is now normal. Consider $K \cap G_0'$. This is a finite-index normal subgroup of G_0' , and since this group is simple and infinite, it has to be that $K \cap G_0' = G_0'$ and hence $G_0' \subset K \subset H$. Since every finite-index subgroup contains G_0' , now all of them correspond to those of the abelianization map, and hence they are all normal. The last assertion is true because $H' \subseteq G_0'$, and since H' is characteristic in H and hence normal in G_0 , we have that $G_0' \subseteq H'$. □

For our next corollary we will need the following fact.

Proposition 2.7. *The center of G_0 is trivial.*

Proof. Let $g \in G_0$ be in the center of G_0 . In particular, g commutes with integer translations.

Let I be an interval on which the action of g is not affine. Now the action of any piecewise projective homeomorphism near infinity is affine. We conjugate g by $x + n$ for $n \in \mathbb{N}$ to obtain a map g' which is not affine on the interval $n + I$. By our hypothesis $g = g'$, so it follows that g is in fact piecewise affine.

Moreover, by our hypothesis it follows that the set of breakpoints of our piecewise affine g is invariant under translation, hence empty. So g is in fact an affine map. The only affine maps that commute with integer translations are themselves translations, so g is of the form $x + t$ for $t \in \mathbb{R}$. Now our lemma follows from the fact that b, c do not commute with $x + t$ for $t \neq 0$. \square

We remark that the above argument is quite general. Given any group of piecewise projective homeomorphisms that contains both a translation and a non translation, then the center of the group is trivial.

So we have now the following.

Corollary 2.8. *Every proper quotient of G_0 is abelian.*

Proof. Let $p : G_0 \rightarrow Q$ be a proper quotient map, and let $K = \ker p$. Since the quotient is not G_0 , there exists $x \in K$ with $x \neq 1$. Since the center is trivial, then there exists $y \in G_0$ such that $[x, y] \neq 1$. But then $[x, y] \in K \cap G'_0$, which is a normal subgroup of G'_0 , so it follows that $G'_0 \subset K$ and Q is abelian. \square

3. COMMUTATORS FOR G

In this section we consider the commutator subgroups G' and G'' . Recall that G is generated by the set $\{x, x_1, y_{10}, y_0, y_1\}$, that for all $n \in \mathbb{N}$, y_{0^n} is a conjugate of y_0 and y_{1^n} is a conjugate of y_1 and that for all non-constant \mathfrak{s} , $y_{\mathfrak{s}}$ is a conjugate of y_{10} .

From the relations of the form $y_{\mathfrak{s}} = x_{\mathfrak{s}} y_{\mathfrak{s}0} y_{\mathfrak{s}10}^{-1} y_{\mathfrak{s}11}$ we see that, unlike the situation for G_0 , the elements x and x_1 lie in the kernel of the abelianization map. In more detail, we have the following relation in G :

$$y_1 = x_1 y_{10} y_{110}^{-1} y_{111}$$

which, combined with

$$y_{111} = x^{-2} y_1 x^2 \quad y_{110} = x^{-1} y_{10} x$$

shows that x_1 is in the kernel of the abelianization. Similarly x_0 is in the kernel of the abelianization because of the relation:

$$y_0 = x_0 y_{00} y_{010}^{-1} y_{011},$$

and observing that, in an analogous way as above, y_{00} is a conjugate of y_0 by an element of F , and similarly, y_{010} and y_{011} are also conjugate by an element of F . Hence when we

abelianize this relation and simplify, we obtain just $x_0 = 1$. That x is also in the kernel then follows from the relation

$$x_0 = x^2 x_1^{-1} x^{-1}$$

which is a consequence of relation (1). Since x and x_1 together generate F , the whole of F lies in G' .

We obtain the following.

Lemma 3.1. *The map given by:*

$$x \mapsto (0, 0, 0) \quad x_1 \mapsto (0, 0, 0) \quad y_{10} \mapsto (1, 0, 0) \quad y_0 \mapsto (0, 1, 0) \quad y_1 \mapsto (0, 0, 1).$$

extends to a surjective homomorphism $G \rightarrow \mathbb{Z}^3$, and its kernel is exactly the commutator subgroup G' .

Proof. As in the proof of Lemma 2.1 we consider the infinite presentation and show that the given map extends to a homomorphism by first noting the extension to the infinite generating set. The infinite generating set for G we consider is $S = X \cup Y$ where $X = \{x_{\mathbf{s}} : \mathbf{s} \in 2^{<\mathbb{N}}\}$ and $Y = \{y_{\mathbf{s}} : \mathbf{s} \in 2^{<\mathbb{N}}\}$. The set of relations is given by (1) to (5) above. That this gives a presentation of G is Theorem 3.3 in [6].

Consider the map from $S \rightarrow \mathbb{Z}^3$ given by

$$x_{\mathbf{s}} \mapsto (0, 0, 0) \quad y \mapsto (-1, 1, 1) \quad y_{\mathbf{s}} \mapsto \begin{cases} (0, 1, 0) & \text{if } \mathbf{s} = 0^m \text{ for some } m \geq 1 \\ (0, 0, 1) & \text{if } \mathbf{s} = 1^m \text{ for some } m \geq 1 \\ (1, 0, 0) & \text{if } \mathbf{s} \text{ is non-constant} \end{cases}$$

That this map extends to a homomorphism $\pi : G \rightarrow \mathbb{Z}^3$ can be readily seen by considering the effect on each of the relations (1) to (5).

Since $\{y_0, y_1, y_{10}\}$ projects to a generating set for the abelianization G/G' and the image $\{\pi(y_0), \pi(y_1), \pi(y_{10})\}$ is a basis for \mathbb{Z}^3 we conclude that π induces an isomorphism from G/G' to \mathbb{Z}^3 . \square

Given a word in standard form $h y_{\mathbf{s}_1}^{a_1} \dots y_{\mathbf{s}_n}^{a_n}$, define the *left y -exponent sum* to be the integer given by summing the elements of $\{a_i : \mathbf{s}_i = 0^n, n \geq 1\}$. Similarly, define the *right y -exponent sum* and *central y -exponent sum* as the sums of the sets $\{a_i : \mathbf{s}_i = 1^n, n \geq 1\}$ and $\{a_i : \mathbf{s}_i \text{ is non-constant}\}$ respectively. The above discussion established the following.

Proposition 3.2. *The commutator subgroup G' consists precisely of those elements of G that have a standard form expression with left y -exponent sum, right y -exponent sum and central y -exponent sum all equal to zero.* \square

Note that elements of G' need not be compactly supported (e.g., x). An element of G' is compactly supported precisely when h is compactly supported and all \mathbf{s}_i are non-constant. Elements of G'' have compact support. In fact, we have the following.

Proposition 3.3. $G'' = G'_0$

Proof. One inclusion is clear since $G'_0 = G''_0 \subseteq G''$. For the reverse inclusion recall that the action of any piecewise projective homeomorphism is affine near infinity. The elements

of G'' have compact support and therefore we claim they have a standard form that does not contain any elements of the form $y_{\mathbf{s}}$ with \mathbf{s} constant.

It is shown in [5] (Section 5) that any element $g \in G$ can be represented as a normal form $g = fy_{s_1}^{t_1} \dots y_{s_n}^{t_n}$ with the property that it does not admit potential cancellations or potential contractions. (These notions are defined in [5], we do not recall them here since we only need a corollary of these. Also, this is proved for the group G_0 but the same proof, line by line, applies to G .)

We use from [5] the notion of *calculation* of a standard form on binary sequences. This is defined for infinite sequences in Definition 3.13 in [5] and the definition for finite sequences occurs in the paragraph after Lemma 3.14 in [5].

Let u_0 be a finite binary sequence that contains as a prefix s_i for some $1 \leq i \leq n$. Moreover, assume that for any sequence s_j in the set $\{s_1, \dots, s_n\}$ either $s_j \subset u_0$ or $u_0 \not\subset s_j$. This holds for instance whenever u_0 is longer than all sequences in $\{s_1, \dots, s_n\}$.

Let Λ be the associated calculation of $fy_{s_1}^{t_1} \dots y_{s_n}^{t_n}$ on u_0 . Since $fy_{s_1}^{t_1} \dots y_{s_n}^{t_n}$ does not contain potential cancellations, Λ does not contain potential cancellations. So by Lemma 3.15 in [5], there is a finite binary sequence u_1 such that one can perform a sequence of moves on the calculation Λu_1 to obtain a string of the form $u_2 y^n$ for some $u_2 \in 2^{<\mathbb{N}}$ and $n \in \mathbb{N} \setminus \{0\}$. Here n is the number of occurrences of y^{\pm} in the calculation Λu_1 , which is positive since u_0 contains s_i as a prefix.

The calculation of $fy_{s_1}^{t_1} \dots y_{s_n}^{t_n}$ on $u_0 u_1 0^{2^n} 10^{2^n} 1 \dots$ equals

$$\Lambda u_1 0^{2^n} 10^{2^n} 1 \dots$$

By our hypothesis, upon performing moves this simplifies to

$$u_2 y^n 0^{2^n} 10^{2^n} 1 \dots$$

and finally to

$$u_2 0 1^{2^n} 0 1^{2^n} 0 \dots$$

Hence the action of $fy_{s_1}^{t_1} \dots y_{s_n}^{t_n}$ on the sequence $u_0 u_1 0^{2^n} 10^{2^n} 1 \dots$ does not preserve tail equivalence.

In particular, the element g does not preserve tail equivalence on a dense subset of $\text{Supp}(y_{s_i}^{t_i})$ for any $1 \leq i \leq n$. If $g \in G''$, it is compactly supported, and hence must preserve tail equivalence outside a compact interval. This means that the support of each $y_{s_i}^{t_i}$ is compact. In particular, $fy_{s_1}^{t_1} \dots y_{s_n}^{t_n}$ does not contain elements of the form $y_{\mathbf{s}}^{\pm}$ with \mathbf{s} constant.

The total y -exponent sum of such a standard form is zero since this is true for G' . It then follows from Proposition 2.2 that $G'' \subseteq G'_0$. \square

In particular G'' is simple. Note that G'' is strictly smaller than G' since elements of G'' have compact support.

Recall that x and x_1 (hence any element of F) are in G' .

Proposition 3.4. *The group G'/G'' is generated by the cosets of x and x_1 . There is an isomorphism $G'/G'' \rightarrow \mathbb{Z}^2$ given by the images of the generators:*

$$xG'' \mapsto (1, 0) \quad x_1G'' \mapsto (0, 1).$$

Proof. Any element of G' has y_0 exponent sum equal to zero and y_1 exponent sum equal to zero. The relations $y_{0^{n+1}} = x^n y_0 x^{-n}$ and $y_{1^{n+1}} = x^{-n} y_1 x^n$ then show that any element of G'/G'' can be written as

$$h y_{\mathbf{s}_1}^{a_1} \dots y_{\mathbf{s}_n}^{a_n} G''$$

with $h \in F$, each \mathbf{s}_i non-constant and the sum of the a_i equal to zero. Hence, $y_{\mathbf{s}_1}^{a_1} \dots y_{\mathbf{s}_n}^{a_n} \in G'_0$ by Proposition 2.2. Therefore, as $G'_0 = G''$, any element of G'/G'' can be written in the form $x^m x_1^n G''$ with $m, n \in \mathbb{Z}$.

That there is a homomorphism sending xG'' to $(1, 0)$ and x_1G'' to $(0, 1)$ is clear from the above discussion. To see that this is surjective, note that there is a homomorphism $G' \rightarrow \mathbb{Z}^2$, given by the germs at infinity. This map must therefore be precisely G'/G'' . \square

We have seen that the derived series for G is $G'' \triangleleft G' \triangleleft G$ with $G/G' \cong \mathbb{Z}^3$, $G'/G'' \cong \mathbb{Z}^2$ and G'' perfect.

4. COMMUTATORS FOR $H(A)$

In this section we consider the group $H(A)$, where A is a subring of \mathbb{R} . We observe a basic fact about P_A .

Lemma 4.1. *Let A be a subring of \mathbb{R} .*

- (1) *If A has a unit $c \neq \pm 1$, then $\infty \in P_A$.*
- (2) *If the only units in A are ± 1 , then $\infty \notin P_A$.*

Proof. First we consider the case where A has a unit $c \neq \pm 1$. Consider an affine map of the form $t \rightarrow c^2 t$. Then, this map fixes 0 and ∞ . The corresponding matrix in $PSL_2(A)$

$$\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}$$

is a hyperbolic matrix which fixes ∞ . So it follows that $\infty \in P_A$.

Now we consider the case when the only units in A are ± 1 . Any hyperbolic matrix that fixes ∞ must be the form

$$\begin{pmatrix} u & v \\ 0 & u^{-1} \end{pmatrix}$$

so that u is a unit that does not equal ± 1 . Since there are no such units in A , there are no such matrices in $PSL_2(A)$, and so $\infty \notin P_A$. \square

The fact that ∞ is such an important point and that it belongs to P_A only in the case where there are nontrivial units is the reason why the proof of the theorem is split in these two cases.

So we consider first the case in which A has units other than ± 1 . We define $H_c(A)$ as the group of compactly supported elements of $H(A)$. Since the elements of $H(A)$ are affine near infinity, it follows that $H(A)'' \subseteq H_c(A)$, because two elements of the type $y = ax + b$ have a commutator of the type $y = x + k$, and two of these commute. Due to the high transitivity of the action of $PSL_2(\mathbb{Z})$ on the real line, a variation of Corollary 2.4 applies to the groups $H_c(A)$ and $H(A)''$ and therefore the groups $H_c(A)'$ and $H(A)'''$ are simple.

From our hypothesis on A it is clear that $H(A)' \neq H(A)''$, since all elements of $H(A)''$ are compactly supported, whereas there are maps in $H'(A)$ that are not compactly supported. For instance, consider a commutator of an integer translation with a map of the form $t \mapsto p^2 t$, where $p \neq \pm 1$ is a unit in A . Moreover, since $H(A)'' \subseteq H_c(A)$ it follows that $H(A)'''$ is a normal subgroup of $H_c(A)'$ and by the simplicity of these groups we deduce that $H_c(A)' = H(A)'''$. To establish simplicity of $H(A)''$ it suffices to show that $H(A)'' = H_c(A)'$. Indeed it suffices to show that if $g, h \in H(A)'$, then $[g, h] \in H_c(A)'$.

Before we show this, we first build some generic elements of $H(A)$ which will be used in the proof.

Definition 4.2. *Given any positive real number $r \in A$, there is an $x \in (0, 1)$ such that $\frac{x}{1-x} = x + r$, because the graphs of $t \rightarrow \frac{t}{1-t}$ and $t \rightarrow t + r$ must intersect in $(0, 1)$. We define a map:*

$$t \cdot \gamma_r = \begin{cases} t & \text{if } t \leq 0 \\ \frac{t}{1-t} & \text{if } 0 \leq t \leq x \\ t + r & \text{if } x \leq t \end{cases}$$

Now the matrices associated to the maps $t \rightarrow t + r, \frac{t}{1-t}$ are

$$\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

respectively. It follows that x is fixed by

$$\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1}$$

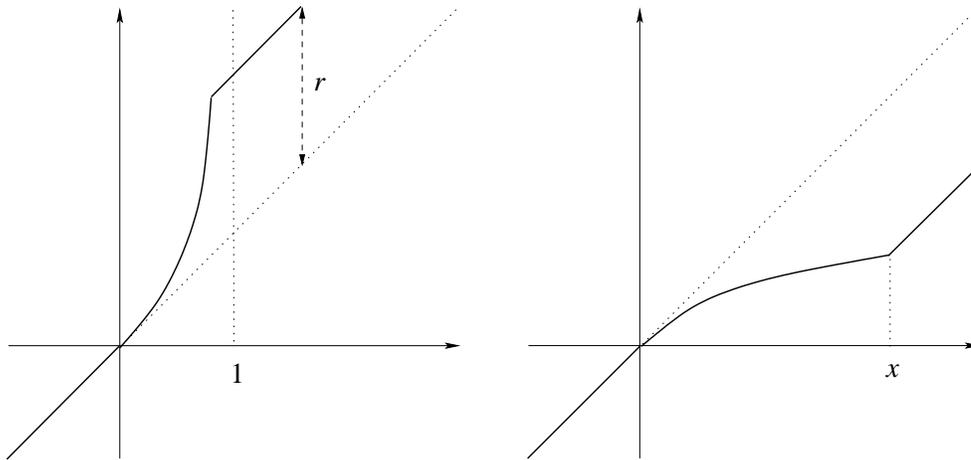
which is a hyperbolic matrix. Therefore $x \in P_A$, and hence $\gamma_r \in H(A)$.

Now given any $n \in \mathbb{Z}, r \in A, r > 0$, we define the map:

$$t \cdot \gamma_{n,r} = \begin{cases} t & \text{if } t \leq n \\ \frac{t-n}{1-(t-n)} + n & \text{if } n \leq t \leq n + x \\ t + r & \text{if } n + x \leq t \end{cases}$$

This map is obtained by conjugating γ_r by the map $t \rightarrow t + n$. Clearly, $\gamma_{n,r} \in H(A)$.

Definition 4.3. *Let $r \in A$ be a negative real number. The graph of the map $t \rightarrow t + r$ meets the map $\frac{t}{1+t}$ at some number x contained in the interval $[-r, -r + 1]$. We define the map*


 FIGURE 1. The maps γ_r and λ_r .

$$t \cdot \lambda_r = \begin{cases} t & \text{if } t \leq 0 \\ \frac{t}{1+t} & \text{if } 0 \leq t \leq x \\ t + r & \text{if } x \leq t \end{cases}$$

Just as in the previous definition, one checks that $x \in P_A$ and so $\lambda_r \in H(A)$. For $n \in \mathbb{Z}$, the map $\lambda_{n,r}$ is obtained by conjugating λ_r by $t \rightarrow t + n$.

$$t \cdot \lambda_{n,r} = \begin{cases} t & \text{if } t \leq n \\ \frac{t-n}{1+(t-n)} + n & \text{if } n \leq t \leq n + x \\ t + r & \text{if } n + x \leq t \end{cases}$$

It follows that $\lambda_{n,r} \in H(A)$.

The idea of the construction of these elements is to provide “bump” or “step” functions between the identity and $t + r$, always within $H(A)$. Our maps in $H(A)$ are translations $t + r$ near infinity, so these functions will be used to provide steps to the identity which will transform them into compactly supported maps. See Figure 1.

Stated in general, the generic elements constructed above allow us to observe the following.

Lemma 4.4. *For each $r \in A$ and $p \in \mathbb{R}$ there is a $f \in H(A)$ such that:*

- (1) f is supported on $[y, \infty)$ for some $y < p$.
- (2) The restriction of f to (p, ∞) equals addition by r .

In an analogous fashion, one establishes the following.

Lemma 4.5. *For each $r \in A$ and $p \in \mathbb{R}$ there is a $f \in H(A)$ such that:*

- (1) f is supported on $(-\infty, z]$ for some $z > p$.
- (2) The restriction of f to $(-\infty, p)$ equals addition by r .

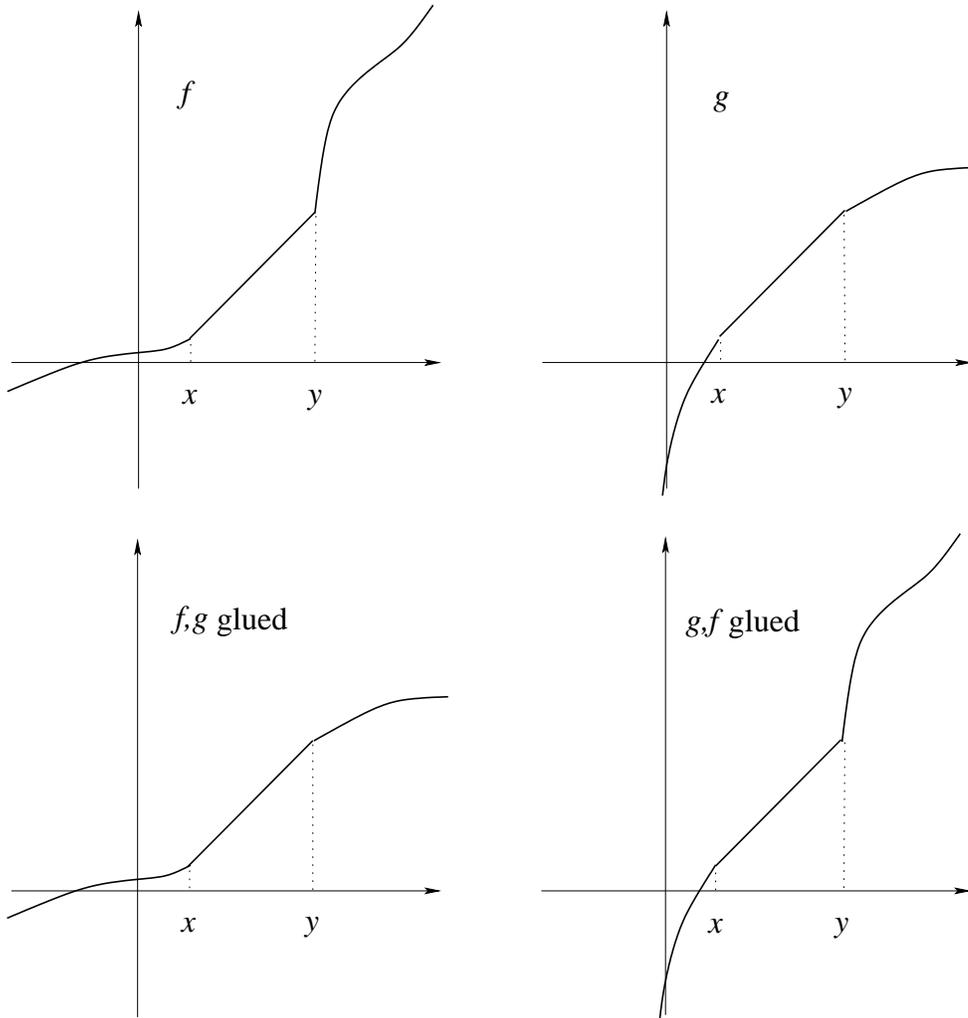


FIGURE 2. An example of gluing.

We now provide an elementary *gluing construction* that allows one to build piecewise projective maps, by gluing pieces of piecewise projective maps provided they agree on a suitable interval. If two maps agree in an interval, we can take a hybrid of the two which consists of one on one side, and another in the other side. See Figure 2.

The proof of this lemma is straightforward.

Lemma 4.6. (*Gluing*) *Let f, g be an ordered pair of elements in $H(A)$ and let $I = [x, y]$ be such that the restrictions of f, g on $[x, y]$ agree. Then there is an element $h \in H(A)$ such that:*

- (1) *The restriction of f, h on $(-\infty, x]$ agree.*
- (2) *The restrictions of g, h on $[y, \infty)$ agree.*
- (3) *The restriction of h on $[x, y]$ agrees with the restrictions of both f, g on $[x, y]$.*

Note that in the gluing construction the order of the pair f, g is essential in determining how the elements are glued, the resulting maps are different if we glue f, g or if we glue

g, f . The interval $[x, y]$ in Lemma 4.6 will be called the *gluing interval*. Now we are ready to prove the main lemma for the case where A has units other than ± 1 .

Lemma 4.7. *Let $g, h \in H(A)'$. Then $[g, h] \in H_c(A)'$.*

Proof. The main idea of the proof is to find elements $h_1, h_2, k_1, k_2 \in H(A)$ such that:

- (1) $[h_1, h_2], [k_1, k_2] \in H_c(A)'$.
- (2) $[h_1, h_2][g, h][k_1, k_2] \in H_c(A)'$.

This will finish the proof. The construction of these elements will be done in four steps.

Step 1: The maps $f, g \in H(A)'$, they are translations near infinity. This allows us to choose a sufficiently large interval $[r, s]$ for which the restriction of each element of $\{f, g, f^{-1}, g^{-1}\}$ to $(-\infty, r)$ and (s, ∞) are translations. Outside of this interval, we will glue the step functions constructed above so our maps become compactly supported.

Step 2: Applying Lemma 4.4, we find elements h_1, h_2 such that:

- (1) h_1, h_2 are supported on an interval $[x, \infty)$ for $x < r$.
- (2) There is a real $x < x_1 < r$ such that the restrictions of h_1, f on $[x_1, r]$ agree.
- (3) There is a real $x < x_2 < r$ such that the restrictions of h_2, g on $[x_2, r]$ agree.
- (4) Let j_1, j_2 be the elements obtained by gluing h_1, f and h_2, g along $[x_1, r], [x_2, r]$ respectively. Then $[j_1, j_2] = [h_1, h_2][f, g]$.

The final condition above is satisfied if the gluing intervals are sufficiently large. Since translations commute, as we apply the sequence of elements of the commutator

$$j_1, j_2, j_1^{-1}, j_2^{-1}$$

in that order, one by one, we notice that if the gluing interval is large enough, it contains a subinterval on which each element acts like a translation, and hence the net result is the identity map. On the right side of this piece, the commutator acts like $[f, g]$, and on the left side it acts like $[h_1, h_2]$. See Figure 3.

Step 3: In this step we will do the same procedure as in step 2, but now on the right hand side of the maps. Applying Lemma 4.5, we find elements k_1, k_2 such that:

- (1) k_1, k_2 are supported on an interval $(-\infty, y)$.
- (2) There is a real $s < x_1 < y$ such that the restrictions of k_1, f on $[s, x_1]$ agree.
- (3) There is a real $s < x_2 < y$ such that the restrictions of k_2, g on $[s, x_2]$ agree.
- (4) Let l_1, l_2 be the elements obtained by gluing f, k_1 and g, k_2 along $[s, x_1], [s, x_2]$ respectively. Then $[l_1, l_2] = [f, g][k_1, k_2]$.

Step 4: We glue j_1, l_1 along $[r, s]$ to obtain f' , and we glue j_2, l_2 along $[r, s]$ to obtain g' . It follows that $[f', g'] = [h_1, h_2][f, g][k_1, k_2]$, and since f' and g' are constructed to be in $H_c(A)$, we conclude that $[f', g'] \in H_c(A)'$.

This finishes the proof of the fact that $[h_1, h_2][f, g][k_1, k_2] \in H_c(A)'$. To prove our lemma it suffices to show that $[h_1, h_2], [k_1, k_2] \in H_c(A)'$. We will show this for $[h_1, h_2]$. The other case is completely analogous. We assume that h_1, h_2 are supported on $[0, \infty)$ for

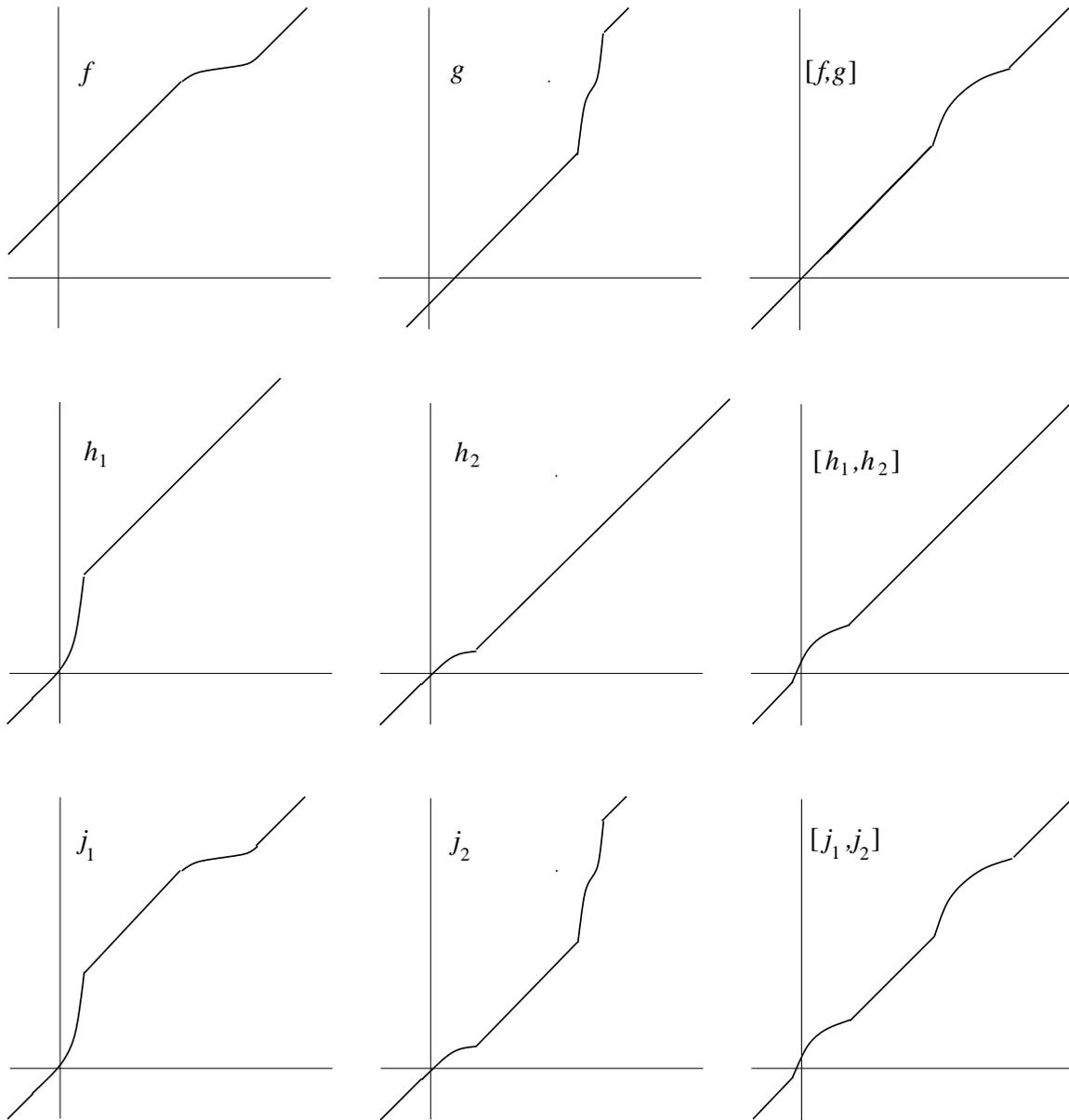


FIGURE 3. Step 2 in the proof that $[f, g] \in H_c(A)'$. The maps f and g do not have compact support, but they are glued to the maps h_1 and h_2 in such a way that the resulting maps j_1 and j_2 are the identity near $-\infty$ and their commutator $[j_1, j_2]$ agrees with $[f, g]$ except that $[h_1, h_2]$ has appeared. The next step is to perform this procedure also near $+\infty$, to produce maps f', g' which have compact support, and whose commutator agrees with $[f, g]$ except that $[h_1, h_2]$ and $[k_1, k_2]$ appear, one above, and one below $[f, g]$. The proof ends when these two latter commutators are also shown to be in $H_c(A)'$.

simplicity. (If this is not the case, we can conjugate the elements h_1, h_2 by a sufficiently large integer translation p , and then establish that $[h_1^p, h_2^p] = [h_1, h_2]^p \in H'_c$.)

The map $t \cdot M = \frac{t}{1+t}$ fixes 0 and maps $(0, \infty)$ to $(0, 1)$. So the map

$$h_3 = [M^{-1}h_1M, M^{-1}h_2M] = M^{-1}[h_1, h_2]M$$

is clearly in $H_c(A)'$. In fact, the closure of the support of h_3 is contained in an interval $[0, t] \subset [0, 1)$.

Now we will construct an element $m \in H(A)$ such that m agrees with M^{-1} on the support of h_3 . For a sufficiently large $k \in \mathbb{N}$, $\exists x \in (t, 1)$ such that $x + k = x \cdot M^{-1} = \frac{x}{1-x}$. It follows that x is fixed by

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1}$$

which is a hyperbolic matrix. So we define m as:

$$t \cdot m = \begin{cases} t & \text{if } t \leq 0 \\ \frac{t}{1-t} & \text{if } 0 \leq t \leq x \\ t + k & \text{if } x \leq t \end{cases}$$

The breakpoints of m are $0, x, \infty$. This means that $m \in H(A)$. We remark that this is the part of the argument where the existence of units in A other than ± 1 is used.

However this means that $m^{-1}h_3m = [h_1, h_2]$. Since $H_c(A)'$ is characteristic in $H_c(A)$, it is invariant under conjugation by elements of $H(A)$. Since $h_3 \in H_c(A)'$ it follows that $[h_1, h_2] \in H_c(A)'$ as desired. \square

We conclude the following.

Corollary 4.8. *$H(A)''$ is simple.*

The proof of Proposition 2.7 applies to both $H(A)$ and $H(A)'$, so the center of these groups is trivial. So we obtain the following.

Proposition 4.9. *Every proper quotient of $H(A)$ is metabelian.*

Proof. Let N be a normal subgroup of $H(A)$. Let $N_1 = H(A)' \cap N$ and $N_2 = H(A)'' \cap N$. Since $H(A)''$ is simple, either N_2 is trivial or $N_2 = H(A)''$. In the former case it follows that N_1 is in the center of $H(A)'$, which means that N_1 is trivial. This means that N is in the center of $H(A)$ which means that N is trivial. So indeed it follows that either N is trivial or N contains $H(A)''$. \square

This finishes the proof of Theorem 1 for the case where A contains units other than ± 1 . Now we turn our attention to the case where A does not contain units other than ± 1 . This condition on the units forces the elements of $H(A)$ to be translations near infinity, since the only affine maps in $PSL_2(A)$ are translations. It follows that elements of $H(A)'$ are compactly supported. By applying Higman's theorem to $H(A)'$ and $H_c(A)$ we obtain

that the groups $H(A)''$ and $H_c(A)'$ are simple, and as a consequence $H_c(A)' = H(A)''$. To prove our claim it suffices to show that $H_c(A)' = H(A)'$.

Recall that because of Lemma 4.1, we now have that ∞ is not in P_A . It follows that for any $f \in H(A)$, the translations on both germs at ∞ are the same. Moreover, $\mathbb{Q} \cap P_A = \emptyset$, since the orbit of ∞ under the action of $PSL_2(\mathbb{Z})$ is $\mathbb{Q} \cup \{\infty\}$.

We shall need an analogue of Lemmas 4.4 and 4.5 in which the resulting functions satisfy that their breakpoints that lie in \mathbb{R} are elements of P_A .

Consider the hyperbolic matrix

$$\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$$

and the corresponding fractional linear transformation $t \rightarrow \frac{-2t+1}{t-1}$. The fixed points $p < q$ of this map are elements of $P_{\mathbb{Z}} \subset P_A$ and the map tends to infinity as $t \rightarrow 1$. Moreover, $p < 0 < q < 1$.

Given any $r \in A, r > 0$, the curves of $t \rightarrow t + r$ and $t \rightarrow \frac{-2t+1}{t-1}$ meet in a real number $s \in (q, 1)$. Moreover, $s \in P_A$ since s is a fixed point of the hyperbolic matrix

$$\begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -2-r & 1+r \\ 1 & -1 \end{pmatrix}$$

We define the map:

$$t \cdot \zeta_r = \begin{cases} t & \text{if } t \leq q \\ \frac{-2t+1}{t-1} & \text{if } q \leq t \leq s \\ t+r & \text{if } s \leq t \end{cases}$$

Upon considering inverses and conjugation of this function with integer translations, as well as the analogous functions with non-trivial germs at $-\infty$, we obtain the following analogues of Lemmas 4.4 and 4.5.

Lemma 4.10. *For each $r \in A$ and $p \in \mathbb{R}$ there is a map f such that:*

- (1) f is a piecewise $PSL_2(A)$ homeomorphism of \mathbb{R} .
- (2) f is supported on $[y, \infty)$ for some $y < p$.
- (3) The restriction of f to (p, ∞) equals addition by r .
- (4) The breakpoints of f , besides ∞ , all lie in P_A .

Lemma 4.11. *For each $r \in A$ and $p \in \mathbb{R}$ there is a map f such that:*

- (1) f is a piecewise $PSL_2(A)$ homeomorphism of \mathbb{R} .
- (2) f is supported on $(-\infty, z]$ for some $z > p$.
- (3) The restriction of f to $(-\infty, p)$ equals addition by r .
- (4) The breakpoints of f , besides ∞ , all lie in P_A .

Now we are ready to prove the main Lemma.

Lemma 4.12. *Let A be a subring of \mathbb{R} whose only units are ± 1 . Let $f, g \in H(A)$. Then $[f, g] \in H_c(A)'$.*

Proof. The proof will follow analogous lines to the proof of Lemma 4.7. We assume for the rest of the proof that the translation near infinity for f is $t \rightarrow t + c_1$ and for g is $t \rightarrow t + c_2$. The main idea of the proof is to find piecewise $PSL_2(A)$ elements h_1, h_2, k_1, k_2 which are step functions which will transform f and g into compactly supported versions of themselves by stepping down the translations $t + c_i$ to the identity. Precisely, the elements h_1, h_2, k_1, k_2 will satisfy

- (1) $[h_1, h_2][k_1, k_2] \in H_c(A)'$
- (2) $[f, g][h_1, h_2][k_1, k_2] \in H_c(A)'$

and this will finish the proof. We shall follow a five step procedure to construct the required elements.

Step 1: Choose a sufficiently large interval $[r, s]$ so that the restriction of each element of $\{f, g, f^{-1}, g^{-1}\}$ to $\mathbb{R} \setminus [r, s]$ is a translation. That is, all the nontranslation action for f and g happens well inside $[r, s]$.

Step 2: The idea now is to construct the elements h_1, h_2 to provide the stepping down to the identity right outside $[r, s]$, or more precisely, to the left of r . Applying Lemma 4.10, we find elements h_1, h_2 such that:

- (1) h_1, h_2 are supported on an interval $[x, \infty)$.
- (2) There is a real number x_1 , with $x < x_1 < r$, such that the restrictions of h_1, f on $[x_1, r]$ agree.
- (3) There is a real number x_2 , with $x < x_2 < r$, such that the restrictions of h_2, g on $[x_2, r]$ agree.
- (4) Let j_1, j_2 be the elements obtained by gluing h_1, f and h_2, g along $[x_1, r], [x_2, r]$ respectively. Then $[j_1, j_2] = [h_1, h_2][f, g]$.

The last condition above is satisfied if the gluing intervals are sufficiently large, just as in the analogous case in the proof of Lemma 4.7. Observe that the elements h_1, h_2 are not in $H(A)$, because they have a breakpoint at ∞ . The germ at $-\infty$ is the identity whereas the germ at $+\infty$ is a translation by c_1 or c_2 .

Step 3: Analogously to the previous step, applying Lemma 4.11, we find elements k_1, k_2 which bring the germ at $+\infty$ down to the identity. That is:

- (1) k_1, k_2 are supported on an interval $(-\infty, y)$.
- (2) There is a real number y_1 , with $s < y_1 < y$, such that the restrictions of k_1, f on $[s, y_1]$ agree.
- (3) There is a real number y_2 , with $s < y_2 < y$, such that the restrictions of k_2, g on $[s, y_2]$ agree.
- (4) Let l_1, l_2 be the elements obtained by gluing f, k_1 and g, k_2 along $[s, y_1], [s, y_2]$ respectively, then $[l_1, l_2] = [f, g][k_1, k_2]$.

At this point we remark that by construction, the restriction of the maps h_1, k_1 on $[r, s]$ equals translation by c_1 , and the restriction of the maps h_2, k_2 on $[r, s]$ equals translation by c_2 . All four maps will work as step functions, having a step outside $[r, s]$ to have the

appropriate identity germ at ∞ . As pointed out in step 2, these maps do not belong to $H(A)$.

Step 4: We glue j_1, l_1 along $[r, s]$ to obtain s_1 , and glue j_2, l_2 along $[r, s]$ to obtain s_2 .

Step 5: We glue h_1, k_1 along $[r, s]$ to obtain t_1 , and glue h_2, k_2 along $[r, s]$ to obtain t_2 .

We would like to emphasise that all four maps s_1, s_2, t_1, t_2 do belong to $H(A)$, and in fact, they belong to $H_c(A)$. This is because after gluing, for all four maps, both germs at ∞ are equal to the identity.

Finally, by construction, from the fact that the steps have been constructed far away from the nontranslation part of f and g , it follows that $[s_1, s_2] = [h_1, h_2][f, g][k_1, k_2]$ and that this element belongs to $H_c(A)'$ because $s_1, s_2 \in H_c(A)$. Also by construction, same as before, we see that the supports of $[h_1, h_2], [f, g]$ are disjoint, so we conclude that

$$[s_1, s_2] = [h_1, h_2][f, g][k_1, k_2] = [f, g][h_1, h_2][k_1, k_2] \in H_c(A).$$

Since

$$[t_1, t_2] = [h_1, h_2][k_1, k_2] \in H_c(A)'$$

because the maps t_1, t_2 are in $H_c(A)$, this finishes the proof. \square

It follows from similar arguments, as in the case of A with units other than ± 1 , that the center of $H(A)$ is trivial, and every proper quotient of $H(A)$ is abelian. This concludes the proof of Theorem 1.

The groups $H(A)$ are not finitely presented, and a presentation for these groups would involve infinitely many maps similar to the map y and their interactions. Writing down these generators and relations would be quite complicated. Hence this makes it difficult to compute the quotients of $H(A)$ by the commutators $H(A)'$ or $H(A)''$ to get its abelianization and metabelianization. We believe that, unlike the cases for G and G_0 , it is difficult to find easy expressions for these quotients.

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