Computing denumerants in numerical 3–semigroups*

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Abstract

As far as we know, usual computer algebra packages can not compute denumerants for almost medium (about a hundred digits) or almost medium–large (about a thousand digits) input data in a reasonably time cost on an ordinary computer. Implemented algorithms can manage numerical n–semigroups for small input data.

Here we are interested in denumerants of numerical 3–semigroups which have almost medium input data. A new algorithm for computing denumerants is given for this task. It can manage almost medium input data in the worst case and medium–large or even large input data in some cases.

Keywords: Denumerant, numerical semigroup, L–shape.

1 Introduction

Let $\mathbb{N}$ be the set of non negative integers. We denote the equivalence class of k modulo m as $[k]_m$. Given $n_1,\ldots,s_k\in\mathbb{N}$, $1<n_1<\cdots<n_k$ and $\gcd(n_1,\ldots,n_k)=1$, the *numerical k–semigroup* T generated by $G=\{n_1,\ldots,n_k\}$ is defined by

$$T=\langle n_1,\ldots,n_k\rangle=\{x_1n_1+\cdots+x_kn_k:\ x_1,\ldots,x_k\in\mathbb{N}\}.$$

The *generating set* G has not necessarily be minimal. The cardinality of a minimal generating set is the *embedding dimension*, $e(T)$, of the semigroup. Given an element $m\in T\setminus\{0\}$, the *Apéry* set of T with respect to m is the set $\mathrm{Ap}(m,T)=\{s\in T:\ s-m\notin T\}$. It is well known the equivalence $s\in\mathrm{Ap}(m,T)\Leftrightarrow s=\min([s]_m\cap T)$ and so, $\mathrm{Ap}(m,T)=\{s_0,\ldots,s_{m-1}\}$ with $s_i\equiv i\,(\mathrm{mod}\ m)$.

Given $s\in T$, a vector $(x_1,\ldots,x_n)\in\mathbb{N}^k$ such that $x_1n_1+\ldots+x_kn_k=s$ is called a *factorization* of s in T. Let us denote the set of factorizations of s in T by

$$\mathcal{F}(s,T)=\{(x_1,\ldots,x_k)\in\mathbb{N}^k:\ x_1s_1+\cdots+x_ks_k=s\}.$$

The *denumerant of s in T* is defined as the cardinality of the set $\mathcal{F}(s,T)$, denoted by $\mathrm{d}(s,T)=|\mathcal{F}(s,T)|$. The *Frobenius number* of T is defined by $\mathfrak{f}(T)=\max(\mathbb{N}\setminus T)$. Detailed results on

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numerical semigroups can be found in the book of J. C. Rosales and P. A. García–Sánchez [15]. It is also interesting the book of Ramírez Alfonsín [14] where it can be found a complet source of results related to Frobenius number.

Sylvester [20] in 1882 gave the generating function $\phi(z)$ of $\mathrm{d}(m, \langle n_1, \ldots, n_k \rangle)$

$$\phi(z) = \frac{1}{(1 - z^{n_1})(1 - z^{n_2}) \cdots (1 - z^{n_k})}.$$

Schur [17] in 1926 studied the asymptotic behaviour of the denumerant,

$$\limsup_{m \to \infty} \frac{\mathrm{d}(m, \langle n_1, \ldots, n_k \rangle)}{\frac{m^{k-1}}{n_1 \cdots n_k (k-1)!}} = 1.$$

Sylvester [19] in 1857 and Cayley [6] in 1860 gave the expression $\mathrm{d}(m, \langle n_1, \ldots, n_k \rangle) = P_k(m) + Q_k(m)$ where $P_k(m)$ is a polynomial of degree $k-1$ and $Q_k(m)$ is a periodic function in the variable m. Beck, Gessel and Komatsu [5] in 2001 found an expression for $P_k(m)$ that depends upon Bernoulli numbers.

Popoviciu [13] in 1953 found an efficient semi–closed expression[1] for $n = 2$ of $\mathrm{d}(m, \langle p, q \rangle)$

$$\mathrm{d}(m, \langle p, q \rangle) = \frac{m + pf(m) + qg(m)}{\mathrm{lcm}(p, q)} - 1,$$

where $f(m) \equiv -mp^{-1} \pmod{q}$ with $1 \le f(m) \le q$ and $g(m) \equiv -mq^{-1} \pmod{p}$ with $1 \le g(m) \le p$. Ehrhart [8] in 1967 and Sertöz and Özlük in 1991 gave recursive denumerant formulae for $2 \le k \le 4$. You can find an exhaustive set of results on denumerants in the book of J. Ramírez Alfonsín [14].

No similar efficient semi–closed expressions are known for $k \ge 3$, however there are some known numerical algorithms to find the set of factorizations $\mathcal{F}(m, T)$ in the general case. Unfortunately, as far as we know, usual computer algebra systems have implemented no command for denumerant. Thus, the calculation of denumerant turns to be a time consuming task. Taking for instance, $n_1 = 7^k$, $n_2 = 11^k$, $n_2 = \mathfrak{f}(\langle 7^k, 11^k \rangle)$, $P_k = n_1 n_2 n_3$, $S_k = n_1 + n_2 + n_3$ and $m_k = P_k - S_k - k$, we obtain the figures of Table 1 for $\mathrm{d}(m_k, T_k)$ and $T_k = \langle n_1, n_2, n_3 \rangle$. The reason why we choose m_k is clear by Theorem 2.

k	m_k	$\mathrm{d}(m_k, T_k)$	Mathematica 8	Sage 7.3	GAP 1.5.1
1	4465	2232	0.011311	0.019617	0.009835
2	34139180	17069589	177.318173	535.270590	6.100101

Table 1: Time in seconds using an i5@1.3Ghz processor

Table 1 shows how popular CAS programs[2] can not manage almost medium (about half a hundred digits). Clearly the Gap package takes advantage for these input instances. From now on, we focus our attention to denumerants of numerical 3–semigroups and the notation $n_1 = a$, $n_2 = b$, $n_3 = c$ and $T = \langle a, b, c \rangle$ will be used here.

[1] This expression only requires $O(\log \max\{p, q\})$ arithmetic operations to be applied.

[2] The commands for computing the denumerant $\mathrm{d}(m, \langle a, b, c \rangle)$ are `Length[FrobeniusSolve[{a,b,c},m]]` for Mathematica 8, `WeightedIntegerVectors(m,[a,b,c]).cardinality()` for Sage 7.3 and `NrRestrictedPartitions(m,[a,b,c])` for GAP 1.5.1

Popoviciu [13, page 27] gave an $O(c\log c)$ algorithm, in the worst case, for computing $\mathrm{d}(m,T)$ when $\{a,b,c\}$ are pairwise coprime numbers (pcn). Lisoněk [11, page 230] in 1995 gave an $O(ab\log b)$ algorithm, in the worst case for pcn (this time cost can be reduced to $O(ab)$ provided that a number of $\max\{O(a^2b^2),O(abc)\}$ precomputed values, related to T, can be stored in the computer memory for later usage). Brown, Chou and Shiue in 2003 [4, page 199] gave an $O(ab\log c)$ algorithm, in the worst case. This last work also contains interesting results on denumerants that can be taken into account for numerical calculations. We refer to these algorithms as P, L and BCS, respectively. Notice that the speed of Algorithm P versus Algorithm L depends on the ratio $\frac{c\log c}{ab\log b}$.

Algorithms P, L and BCS calculate the denumerants of Table 1 significantly faster. A non-compiled Sage 7.3 implementations of them give the figures in Table 2 (using the same processor of Table 1).

k	m_k	$\mathrm{d}(m_k,T_k)$	P	L	BCS
1	4465	2232	0.003994	0.004331	0.011661
2	34139180	17069589	0.083913	0.138889	0.920211
3	207657687311	103828843654	5.063864	9.495915	63.647251

Table 2: Time in seconds obtained by P, L and BCS

Nonetheless, these algorithms do not reach the necessary efficiency for managing almost medium input. The goal of this work is to provide a reasonably efficient new algorithm which allows such kind of inputs when working on ordinary computers.

Our algorithm has a theoretical time cost of $O(b+\log c)$, in the worst case. However, numerical evidences suggest that, in some cases, it can have a smaller cost[3]. This algorithm is based on a semi-closed denumerant expression given in [2] which is included here in Theorem 4.

The summary of the paper is the following: Section 2 contains the basic known tools, mainly Theorem 4 and expression (4). Section 3 developes expression (4) to be used for numerical purposes. In this developing it is apparent that the main computation depends on the so called $S^{\pm}$ *discrete sums*. Some tools to calculate $S^{\pm}$ sums are developed in Section 4, mainly the so called *hS-type sets*. Section 5 contains the main algorithm and Section 6 analyzes the time cost, in the worst case. Finally, in Section 7, several instances of time tests are given.

2 Some definitions and known results

In this section we give the main known results that allow us to reach our goal. The usual notation for semigroups will be $T=\langle a,b,c\rangle$ with $1\le a<b<c$ and $\gcd(a,b,c)=1$. Also the product $P=abc$ and sum $S=a+b+c$ of the generators are used.

Although algorithms P and L act over pairwise coprime generators, this condition can be removed by the following result due to Brown, Shou and Shiue [4]. Here the integer $u'_v(t)$ is defined to be the unique integer value $1\le u'_v(t)\le v$ such that $uu'_v(t)\equiv -t \pmod v$ with $u,v\ge 1$ and $\gcd(u,v)=1$.

[3] As an instance, the same data of Table 2 for $k=10^3$ is calculated in 0.009836 seconds and for $k=10^5$ in 2.110464 seconds.

Lemma 1 (Brown, Chou and Shiue 2003 [4, Lemma 4.5]**)** *Consider the semigroup* $T = \langle a, b, c\rangle$ *with* $\gcd(a,b,c) = 1$*. Set* $g_a = \gcd(b,c)$*,* $g_b = \gcd(a,c)$ *and* $g_c = \gcd(a,b)$*. For any integer* $n > 0$*, the integer value* $n' = n - (g_a - a'_{g_a}(n))a - (g_b - b'_{g_b}(n))b - (g_c - c'_{g_c}(n))c$ *is multiple of* $g_a g_b g_c$ *and the denumerant's identity* $\mathrm{d}(n,T) = \mathrm{d}(\frac{n'}{g_a g_b g_c}, T')$ *holds with* $T' = \langle \frac{a}{g_b g_c}, \frac{b}{g_a g_c}, \frac{c}{g_a g_b}\rangle$*. Here it is understood that* $\mathrm{d}(0,T') = 1$ *and* $\mathrm{d}(\frac{n'}{g_a g_b g_c}, T') = 0$ *whenever* $n' < 0$*.*

By the following theorem, due to Ehrhart in 1967, we only need to compute denumerants in the range of values $m \in \{0, 1, \ldots, P-1\}$.

Theorem 1 (Ehrhart 1967 [8, Theorem 10.5]**)** *Consider* $T = \langle a, b, c\rangle$ *with* a*,* b *and* c *pcn. Set* $P = abc$*,* $S = a+b+c$ *and* $m = qP + r$ *with* $0 \le r < P$*. Then,*

$$\mathrm{d}(m,T) = \mathrm{d}(r,T) + \frac{q(m+r+S)}{2}.$$

In particular,

$$\mathrm{d}(P,T) = \frac{P+S}{2} + 1.$$

The range $\{0, \ldots, P-1\}$ can be reduced to $\{0, \ldots, P-S\}$ by the following theorem due to Sertöz and Özluk in 1991.

Theorem 2 (Sertöz and Özlük 1991 [18, page 4]**)** *Consider* $T = \langle a, b, c\rangle$ *with* a*,* b *and* c *pcn. Set* $P = abc$ *and* $S = a+b+c$*. Then, for* $1 \le x \le S-1$ *we have*

$$\mathrm{d}(P-x,T) = \frac{P+S}{2} - x.$$

In particular,

$$\mathrm{d}(P-S+1) = \frac{P-S}{2} + 1.$$

Remark 1 *The time cost, in the worst case, of the algorithms* P*,* L *and* BCS *for computing the denumerant* $\mathrm{d}(m, \langle a,b,c\rangle)$ *have been given for the largest value of* m *(by theorems 1 and 2), that is* $m \approx P = abc$*.*

We use the concept of *L–shape* as a main tool for the new algorithm. Thus, we include here some known results for this geometrical discrete structure. Denote the *interval* $[s,t) = \{x \in \mathbb{R} : \ s \le x < t\}$, the *unitary square* $[\![m,n]\!] = [m, m+1) \times [n, n+1) \in \mathbb{R}^2$ and the *discrete backwards cone* $\Delta(u,v) = \{[\![m,n]\!] : \ (m,n) \in \mathbb{N}^2, 0 \le m \le u, 0 \le n \le v\}$ for each $u, v \in \mathbb{N}$. We also denote the equivalence class of u modulo v by $[u]_v$.

Consider each unitary square $[\![m,n]\!]$, for $(m,n) \in \mathbb{N}^2$, labelled by the equivalence class $[ma + nb]_c$. Define the minimum values

$$M_n = \min\{sa + tb : \ (s,t) \in \mathbb{N}^2, [sa+tb] = [n]_c\}. \tag{1}$$

Definition 1 (Minimum distance diagram) *Consider a numerical* 3*–semigroup* $T = \langle a, b, c\rangle$*. A minimum distance diagram (MDD),* $\mathcal{H}$*, related to* T *is a set of* c *unitary squares that fulfils the following properties*

(a) for each $n \in \{0, \ldots, c-1\}$*, there is some unitary square* $[\![s,t]\!] \in \mathcal{H}$ *such that* $[sa+tb]_c = [n]_c$*,*

(b) $\Delta(s,t) \subseteq \mathcal{H}$ *for each* $[\![s,t]\!] \in \mathcal{H}$,

(c) *if* $[\![s,t]\!] \in \mathcal{H}$, *then* $sa+tb = M_n$ *with* $[sa+tb]_c = [n]_c$ *and* M_n *defined by* (1).

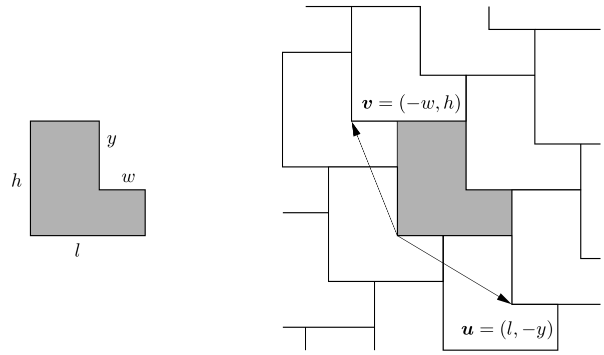

Figure 1: Generic L-shape and its related tessellation

Minimum distance diagrams related to numerical 3–semigroups are known to be L–shapes or rectangles (that will be considered as degenerated L-shapes). For this reason we also refer to MDD as L-shapes and they are denoted by the lengths of their sides $\mathrm{L}(l,h,w,y)$, see Figure 1, with $0 \le w < l$, $0 \le y < h$ and $lh - wy = c$. An L–shape tessellates the plane by translation through the vectors $\boldsymbol{u} = (l,-y)$ and $\boldsymbol{v} = (-w,h)$. The following result characterizes the L-shapes related to $T = \langle a,b,c\rangle$. From now on we assume $0 < a < b < c$ and $\gcd(a,b,c) = 1$.

Theorem 3 (A. and Marijuán 2014 [3]) *Consider the numerical* 3*–semigroup* $T = \langle a,b,c\rangle$. *An L-shape* $\mathcal{H} = \mathrm{L}(l,h,w,y)$ *is related to* T *if and only if*

(a) $lh - wy = c$ *and* $\gcd(l,h,w,y) = 1$,

(b) $la - yb \equiv 0 \pmod{c}$ *and* $hb - wa \equiv 0 \pmod{c}$,

(c) $la - yb \ge 0$, $hb - wa \ge 0$ *and both expressions can't vanish at the same time.*

Each numerical 3–semigroup has two related L-shapes at most (either one if $(la-yb)(hb-wa) > 0$ or two whenever $(la-yb)(hb-wa) = 0$, see [3, theorems 2 and 3]). L-shapes contain main information of the related semigroup. For instance, if a semigroup $T = \langle a,b,c\rangle$ has related the L-shape $\mathcal{H}$, we have $\mathrm{Ap}(c,T) = \{ia + jb : [\![i,j]\!] \in \mathcal{H}\}$.

A classification of 3–semigroups was given in terms of its related L–shapes in [3]. The tessellation of the plane associated with each L–shape was used to derive the semi-closed expression (4) for the denumerant in [2].

Given $T = \langle a,b,c\rangle$ and a related L–shape $\mathcal{H} = \mathrm{L}(l,h,w,y)$, let us denote $\delta = (la-yb)/c$ and $\theta = (hb-wa)/c$. From the definition of $\mathcal{H}$ and Theorem 3, it follows that

- $a = h\delta + y\theta$ and $b = w\delta + l\theta$,
- $\delta, \theta \in \mathbb{N}$ and $\delta + \theta > 0$,
- $\delta = 0 \Rightarrow y > 0$ and $\theta = \frac{b}{l} = \frac{a}{y} > 0$,
- $\theta = 0 \Rightarrow w > 0$ and $\delta = \frac{a}{h} = \frac{b}{w} > 0$,
- $w = 0 \Rightarrow \theta = \frac{b}{l} > 0$,

- $y = 0 \Rightarrow \delta = \frac{a}{h} > 0 > 0$.

All these properties will be used along this work.

Given $m \in T$, it is called the *basic factorization of m with respect to $\mathcal{H}$*, $(x_0, y_0, z_0) \in \mathcal{F}(m, T)$, the unique factorization such that $[\![x_0, y_0]\!] \in \mathcal{H}$. This factorization can be computed in time cost $O(\log c)$ [1].

Theorem 4 (A. and P.A. García Sánchez 2010 [2]) *Given $T = \langle a, b, c\rangle$ and a related L–shape $\mathcal{H} = \mathrm{L}(l, h, w, y)$, assume $m \in T$. Define $A_m = \left\lfloor \frac{z_0}{\delta+\theta} \right\rfloor$, where (x_0, y_0, z_0) is the basic factorization of m wrt $\mathcal{H}$. For each $0 \le k \le A_m$, set*

$$S_k = \begin{cases} \left\lfloor \frac{y_0+k(h-y)}{y} \right\rfloor & \text{if } \delta = 0, \\ \left\lfloor \frac{z_0-k(\delta+\theta)}{\delta} \right\rfloor & \text{if } y = 0, \\ \min\{\left\lfloor \frac{y_0+k(h-y)}{y} \right\rfloor, \left\lfloor \frac{z_0-k(\delta+\theta)}{\delta} \right\rfloor\} & \text{if } \delta y \neq 0, \end{cases} \tag{2}$$

and

$$T_k = \begin{cases} \left\lfloor \frac{x_0+k(l-w)}{w} \right\rfloor & \text{if } \theta = 0, \\ \left\lfloor \frac{z_0-k(\delta+\theta)}{\theta} \right\rfloor & \text{if } w = 0, \\ \min\{\left\lfloor \frac{x_0+k(l-w)}{w} \right\rfloor, \left\lfloor \frac{z_0-k(\delta+\theta)}{\theta} \right\rfloor\} & \text{if } \theta w \neq 0. \end{cases} \tag{3}$$

Then, the denumerant of m in T is

$$\mathrm{d}(m, T) = 1 + A_m + \sum_{k=0}^{A_m} (S_k + T_k). \tag{4}$$

The sum appearing in this theorem is known as *the basic sum* of the denumerant with respect to the L–shape $\mathcal{H}$. The direct computation of this sum does not give an efficient algorithm for calculating the denumerant. However, as it will be seen later, a detailed analysis of this expression does it.

Example 1 Take $T = \langle 5, 7, 11\rangle$ and $m = 87$. A related L–shape is $\mathcal{H} = \mathrm{L}(5, 3, 2, 2)$, with $\delta = \theta = 1$. The basic factorization of 87 in T is $(x_0, y_0, z_0) = (2, 0, 7)$. Thus, we have $A_m = 3$. Then, it follows that $S_0 + T_0 = 0 + 1 = 1$, $S_1 + T_1 = 0 + 2 = 2$, $S_2 + T_2 = 1 + 3 = 5$, $S_3 + T_3 = 1 + 1 = 2$ and so

$$\mathrm{d}(87, T) = 1 + 3 + (1 + 2 + 5 + 2) = 13.$$

A geometric representation of the plane projection of the set $\mathcal{F}(87, T)$, $\pi(\mathcal{F}(87, T))$, is depicted in Figure 2. It has a tree-like structure, given by the vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{u} + \boldsymbol{v}$ (and so, it follows the tessellation of the plane by $\mathcal{H}$). Each unitary square $[\![s, t]\!]$ is labelled with the value $5s + 7t$ (notice that values corresponding to unitary squares in the gray L–shape form the Apéry set $\mathrm{Ap}(87, T) = \{0, 12, 24, 14, 15, 5, 17, 7, 19, 20, 10\}$). The unitary squares corresponding to the first two coordinates of each factorization are circled. From the coordinates of a circled unitary square $[\![s, t]\!]$ follows the related factorization $(s, t, \frac{87-3s-7t}{11})$. The set of factorizations is

$$\begin{aligned} \mathcal{F}(87, T) = &\{(2, 0, 7), (0, 3, 6), (5, 1, 5), (3, 4, 4), (1, 7, 3), (8, 2, 3), (13, 0, 2), \\ &(6, 5, 2), (4, 8, 1), (2, 11, 0), (11, 3, 1), (16, 1, 0), (9, 6, 0)\}. \end{aligned}$$

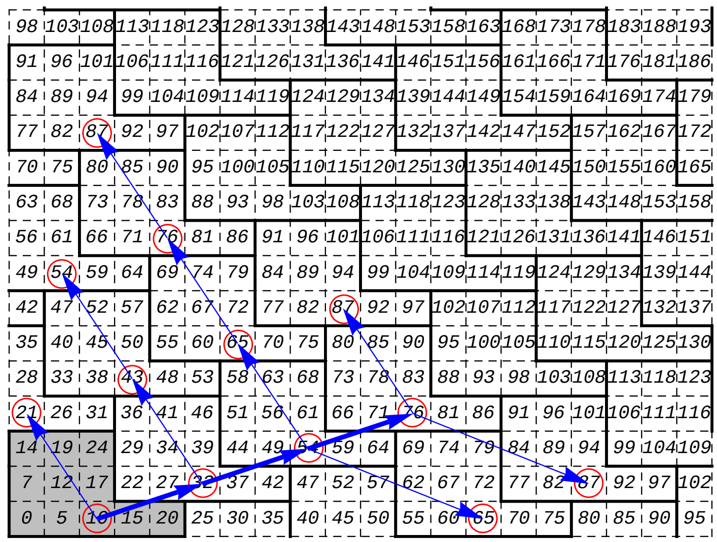

Figure 2: Tree-like structure of $\pi(\mathcal{F}(87, \langle 5, 7, 11\rangle))$

3 Developing the basic sum

As it has been commented before, the basic sum (4) does not provide a direct efficient algorithm for calculating denumerants. Thus, a detailed analysis is needed. We consider three main cases: case (i) $\delta = 0$, case (ii) $\theta = 0$ and case (iii) $\delta\theta > 0$.

The analysis of these cases reveals that the basic sum depends on several sums of the same kind. These sums will be referred to as $S^{\pm}$ sums and will be studied in the next section. These sums have the form $S^{\pm}(s,t,q,N) = \sum_{k=0}^{N} \left\lfloor \frac{s \pm kt}{q} \right\rfloor$ with $0 \le s, t < q$.

In this section we assume that $\mathcal{H} = \mathrm{L}(l,h,w,y)$ is an L–shape related to the numerical 3–semigroup $T = \langle a,b,c\rangle$. We also assume that $m \in T$ and (x_0, y_0, z_0) is the basic factorization of m with respect to T.

3.1 Case (i) $\delta = 0$

This case leads to the following expressions of the denumerant.

Theorem 5 *Let us assume $\delta = 0$. Set $A_m = \left\lfloor \frac{z_0}{\theta} \right\rfloor$. Then,*

(i.1) if $w = 0$, then

$$\mathrm{d}(m,T) = (1 + A_m)(1 + A_m + \overline{y_0}) + (\overline{h} - 2)\frac{A_m(1 + A_m)}{2} + \sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{y_0} + k\widehat{h}}{y} \right\rfloor, \qquad (5)$$

where $y_0 = \overline{y_0} y + \widehat{y_0}$ with $0 \le \widehat{y_0} < y$ and $h = \overline{h} y + \widehat{h}$ with $0 \le \widehat{h} < y$.

(i.2) if $w > 0$, set $k_0 = \left\lceil \frac{z_0 w - x_0 \theta}{b} \right\rceil$. Then,

(i.2.1) if $k_0 = 0$, then $\mathrm{d}(m,T)$ has the same expression as in (5).

(i.2.2) if $1 \le k_0 \le A_m$,

$$\mathrm{d}(m,T) = (1+A_m)(1+A_m+\overline{y_0}) + k_0(\overline{x_0}-A_m) + (\overline{h}-2)\frac{A_m(1+A_m)}{2} + \overline{l}\frac{(k_0-1)k_0}{2} + \sum_{k=0}^{A_m}\left\lfloor \frac{\widehat{y_0}+k\widehat{h}}{y}\right\rfloor + \sum_{k=0}^{k_0-1}\left\lfloor \frac{\widehat{x_0}+k\widehat{l}}{w}\right\rfloor, \tag{6}$$

where $\overline{y_0}, \widehat{y_0}, \overline{h}, \widehat{h}$ are defined as in the previous case, $x_0 = \overline{x_0}w + \widehat{x_0}$ with $0 \le \widehat{x_0} < w$ and $l = \overline{l}w + \widehat{l}$ with $0 \le \widehat{l} < w$.

(i.2.3) if $k_0 > A_m$,

$$\mathrm{d}(m,T) = (1+A_m)(1+\overline{x_0}+\overline{y_0}) + (\overline{l}+\overline{h}-2)\frac{A_m(1+A_m)}{2} + \sum_{k=0}^{A_m}\left\lfloor \frac{\widehat{x_0}+k\widehat{l}}{w}\right\rfloor + \sum_{k=0}^{A_m}\left\lfloor \frac{\widehat{y_0}+k\widehat{h}}{y}\right\rfloor, \tag{7}$$

where $\overline{y_0}, \widehat{y_0}, \overline{h}, \widehat{h}, \overline{x_0}, \widehat{x_0}, \overline{l}$ and $\widehat{l}$ are defined as in the previous case.

Remark 2 *Notice that $k_0 \ge 0$. Indeed, let us see $-1 < \frac{z_0 w - x_0\theta}{b}$ (recall that we have $w > 0$). From $b = l\theta$ (recall that $\delta = 0$), we have (recalling $b = l\theta$)*

$$-1 < \frac{z_0 w - x_0\theta}{b} \Leftrightarrow 0 < z_0 w + (l - x_0)\theta.$$

Now, as $[\![x_0, y_0]\!] \in \mathcal{H}$, it follows that $0 \le x_0 < l$ and so the inequality $0 < z_0 w + (l-x_0)\theta$ holds.

Proof of Theorem 5: If $\delta = 0$, then we have $y \neq 0$, $\theta > 0$. From Theorem 4, we have $A_m = \left\lfloor \frac{z_0}{\theta} \right\rfloor$ and $S_k = \left\lfloor \frac{y_0 + k(h-y)}{y} \right\rfloor = \left\lfloor \frac{y_0+kh}{y} \right\rfloor - k$ for all $0 \le k \le A_m$. Now two subcases appear, (i.1) $w = 0$ and (i.2) $w > 0$.

(i.1) Assume $w = 0$. From (3), $T_k = \left\lfloor \frac{z_0 - k\theta}{\theta} \right\rfloor = A_m - k$ for all $0 \le k \le A_m$. From (4),

$$\mathrm{d}(m,T) = 1 + A_m + \sum_{k=0}^{A_m}\left(\left\lfloor \frac{y_0+kh}{y} \right\rfloor + A_m - 2k \right).$$

Setting $y_0 = \overline{y_0}y + \widehat{y_0}$ with $0 \le \widehat{y_0} < y$ and $h = \overline{h}y + \widehat{h}$ with $0 \le \widehat{h} < y$, the above expression of $\mathrm{d}(m,T)$ turns to be

$$\begin{aligned}\mathrm{d}(m,T) &= 1 + A_m + \sum_{k=0}^{A_m}\left(\left\lfloor \frac{\widehat{y_0}+k\widehat{h}}{y} \right\rfloor + \overline{y_0} + k(\overline{h}-2) + A_m \right) \\ &= (1+A_m)(1+A_m+\overline{y_0}) + (\overline{h}-2)\frac{A_m(1+A_m)}{2} + \sum_{k=0}^{A_m}\left\lfloor \frac{\widehat{y_0}+k\widehat{h}}{y}\right\rfloor.\end{aligned}$$

(i.2) Assume now $w > 0$. Then,

$$T_k = \min\left\{ \left\lfloor \frac{x_0 + k(l-w)}{w} \right\rfloor, \left\lfloor \frac{z_0 - k\theta}{\theta} \right\rfloor \right\} = \min\left\{ \left\lfloor \frac{x_0+kl}{w} \right\rfloor, \left\lfloor \frac{z_0}{\theta} \right\rfloor \right\} - k.$$

The inequality $\left\lfloor \frac{x_0+kl}{w} \right\rfloor \geq \left\lfloor \frac{z_0}{\theta} \right\rfloor$ holds when either $\frac{x_0+kl}{w} \geq \frac{z_0}{\theta}$ or $n \leq \frac{x_0+kl}{w} < \frac{z_0}{\theta} < n+1$ for some $n \in \mathbb{N}$. The former holds when $k \geq k_0 = \left\lceil \frac{z_0 w - x_0\theta}{b} \right\rceil$, the latter holds whenever $0 < \frac{z_0}{\theta} - \frac{x_0+kl}{w} < 1 \Leftrightarrow \frac{z_0 w - x_0\theta}{b} - \frac{w}{l} < k < \frac{z_0 w - x_0\theta}{b}$ (and, in this case, $\left\lfloor \frac{x_0+kl}{w} \right\rfloor = \left\lfloor \frac{z_0}{\theta} \right\rfloor$ holds). Notice that, from $0 < \frac{w}{l} < 1$, if there exists some $k_1 \in \mathbb{N}$ such that $\frac{z_0 w - x_0\theta}{b} - \frac{w}{l} < k_1 < \frac{z_0 w - x_0\theta}{b}$, this value k_1 must be unique and equality $\left\lfloor \frac{x_0+k_1 l}{w} \right\rfloor = \left\lfloor \frac{z_0}{\theta} \right\rfloor$ holds. Thus, it follows that

$$T_k = \begin{cases} \left\lfloor \frac{x_0+kl}{w} \right\rfloor - k & \text{if } k < k_0, \\ A_m - k & \text{if } k \geq k_0. \end{cases} \tag{8}$$

By Remark 2 we have $k_0 \geq 0$ and we consider three possible options.

(i.2.1) Assume $k_0 = 0$. Then, $k \geq k_0$ for all k and $T_k = A_m - k$ for all k. Therefore, the expression of T_k is the same as in the previous case for all k. Thus the denumerant has the same expression as the previous case.

(i.2.2) Assume $1 \leq k_0 \leq A_m$. Now, the expression of T_k changes upon the value of $k < k_0$ and $k \geq k_0$ according to (8). Thus,

$$\begin{aligned}
\mathrm{d}(m,T) &= 1 + A_m + \sum_{k=0}^{A_m} \left(\left\lfloor \frac{y_0+kh}{y} \right\rfloor - k \right) + \sum_{k=0}^{k_0-1} \left(\left\lfloor \frac{x_0+kl}{w} \right\rfloor - k \right) + \sum_{k=k_0}^{A_m} (A_m - k) \\
&= 1 + A_m + \sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{y_0}+k\widehat{h}}{y} \right\rfloor + \sum_{k=0}^{A_m} (\overline{y_0} + k(\overline{h}-2)) + \sum_{k=0}^{k_0-1} \left\lfloor \frac{\widehat{x_0}+k\widehat{l}}{w} \right\rfloor \\
&\quad + \sum_{k=0}^{k_0-1} (\overline{x_0} + k\overline{l}) + \sum_{k=k_0}^{A_m} A_m \\
&= (1+A_m)(1+A_m+\overline{y_0}) + k_0(\overline{x_0} - A_m) + (\overline{h}-2)\frac{A_m(1+A_m)}{2} \\
&\quad + \overline{l}\frac{(k_0-1)k_0}{2} + \sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{y_0}+k\widehat{h}}{y} \right\rfloor + \sum_{k=0}^{k_0-1} \left\lfloor \frac{\widehat{x_0}+k\widehat{l}}{w} \right\rfloor,
\end{aligned}$$

where $\overline{y_0}, \widehat{y_0}, \overline{h}, \widehat{h}, \overline{x_0}, \widehat{x_0}, \overline{l}$ and $\widehat{l}$ are those parameters defined in the statement (i.2.2) of the theorem.

(i.2.3) Assume $k_0 > A_m$. Now, following (8), we have $T_k = \left\lfloor \frac{x_0+kl}{w} \right\rfloor - k$ for all k. Then,

$$\begin{aligned}
\mathrm{d}(m,T) &= 1 + A_m + \sum_{k=0}^{A_m} \left(\left\lfloor \frac{y_0+kh}{y} \right\rfloor - k + \left\lfloor \frac{x_0+kl}{w} \right\rfloor - k \right) \\
&= 1 + A_m + \sum_{k=0}^{A_m} (\overline{y_0} + k\overline{h} + \overline{x_0} + k\overline{l} - 2k) + \sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{y_0}+k\widehat{h}}{y} \right\rfloor + \sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{x_0}+k\widehat{l}}{w} \right\rfloor \\
&= (1+A_m)(1+\overline{x_0}+\overline{y_0}) + (\overline{l}+\overline{h}-2)\frac{A_m(1+A_m)}{2} \\
&\quad + \sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{x_0}+k\widehat{l}}{w} \right\rfloor + \sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{y_0}+k\widehat{h}}{y} \right\rfloor,
\end{aligned}$$

where $\overline{y_0}, \widehat{y_0}, \overline{h}, \widehat{h}, \overline{x_0}, \widehat{x_0}, \bar{l}$ and $\widehat{l}$ are the same as those defined in (i.2.2). □

Notice how all the expressions of denumerant given by Theorem 5 contain sums of type $S^{\pm}$.

3.2 Case (ii) $\theta = 0$

This case is similar to the case (i). Now we have $w \neq 0$, $\delta \neq 0$, $A_m = \left\lfloor \frac{z_0}{\delta} \right\rfloor$ and $T_k = \left\lfloor \frac{x_0+kl}{w} \right\rfloor - k$ for all k. As in the previous case, we consider the following parameters

$$\begin{aligned} x_0 &= \overline{x_0} w + \widehat{x_0}, & 0 \leq \widehat{x_0} < w, \\ l &= \bar{l} w + \widehat{l}, & 0 \leq \widehat{l} < w, \\ y_0 &= \overline{y_0} y + \widehat{y_0}, & 0 \leq \widehat{y_0} < y, \\ h &= \overline{h} y + \widehat{h}, & 0 \leq \widehat{h} < y. \end{aligned} \tag{9}$$

Defining $k_1 = \left\lceil \frac{z_0 y - y_0 \delta}{a} \right\rceil$ (recall that $a = h\delta$) and using similar arguments of (i.2) in the proof of Theorem 5, we have S_k in (2) turns to be

$$S_k = \begin{cases} \left\lfloor \frac{y_0+kh}{y} \right\rfloor - k & \text{if } k < k_1, \\ A_m - k & \text{if } k \geq k_1. \end{cases} \tag{10}$$

The following result can be obtained using similar arguments like in the proof of Theorem 5.

Theorem 6 *Let us assume $\theta = 0$. Set $A_m = \left\lfloor \frac{z_0}{\delta} \right\rfloor$. Then,*

(ii.1) if $y = 0$, then

$$\mathrm{d}(m,T) = (1 + A_m)(1 + A_m + \overline{x_0}) + (\bar{l} - 2)\frac{A_m(1 + A_m)}{2} + \sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{x_0} + k\widehat{l}}{w} \right\rfloor, \tag{11}$$

(ii.2) if $y > 0$, set $k_1 = \left\lceil \frac{z_0 y - y_0 \delta}{a} \right\rceil$. Then,

(ii.2.1) if $k_1 = 0$, then $\mathrm{d}(m,T)$ has the same expression as in (11).

(ii.2.2) if $1 \leq k_1 \leq A_m$,

$$\begin{aligned} \mathrm{d}(m,T) = {} & (1 + A_m)(1 + A_m + \overline{x_0}) + k_1(\overline{y_0} - A_m) + (\bar{l} - 2)\frac{A_m(1 + A_m)}{2} \\ & + \overline{h}\frac{(k_1 - 1)k_1}{2} + \sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{x_0} + k\widehat{l}}{w} \right\rfloor + \sum_{k=0}^{k_1-1} \left\lfloor \frac{\widehat{y_0} + k\widehat{h}}{y} \right\rfloor, \end{aligned} \tag{12}$$

(ii.2.3) if $k_1 > A_m$, then the denumerant has the same expression as (7).

Remark 3 *Although some expressions of Theorem 6 seem to be the same as some expressions of Theorem 5, the value A_m is not the same. In the former case we have $A_m = \left\lfloor \frac{z_0}{\theta} \right\rfloor$ and $A_m = \left\lfloor \frac{z_0}{\delta} \right\rfloor$ in the latter.*

Remark 4 *In Theorem 6 we have $k_1 \geq 0$. Indeed, $k_1 \geq 0 \Leftrightarrow -1 < \frac{z_0 y - y_0 \delta}{h\delta}$ (recall that $h\delta = a$) and $-1 < \frac{z_0 y - y_0 \delta}{h\delta} \Leftrightarrow 0 < z_0 y + \delta(h - y_0)$. The factorization (x_0, y_0, z_0) is the basic one of m with respect to the L–shape $\mathcal{H}$. Thus, $[\![x_0, y_0]\!] \in \mathcal{H} \Rightarrow h > y_0$.*

3.3 Case (iii) $\delta\theta > 0$

Now we have $A_m = \left\lfloor \frac{z_0}{\delta+\theta} \right\rfloor$, $a = h\delta + y\theta$ and $b = w\delta + l\theta$. There are four different options that give different expressions of $A = \sum_{k=0}^{A_m} S_k$ and $B = \sum_{k=0}^{A_m} T_k$ in (4),

(iii.1) $w = y = 0$,

(iii.2) $w \neq 0$ and $y = 0$,

(iii.3) $w = 0$ and $y \neq 0$,

(iii.4) $wy \neq 0$.

Here we use the same notation as in (9) plus the following one

$$\begin{aligned} z_0 &= \overline{z_{0,1}}\delta + \widehat{z_{0,1}}, & 0 \leq \widehat{z_{0,1}} < \delta, \\ z_0 &= \overline{z_{0,2}}\theta + \widehat{z_{0,2}}, & 0 \leq \widehat{z_{0,2}} < \theta, \\ \delta &= \overline{\delta}\theta + \widehat{\delta}, & 0 \leq \widehat{\delta} < \theta, \\ \theta &= \overline{\theta}\delta + \widehat{\theta}, & 0 \leq \widehat{\theta} < \delta, \end{aligned} \tag{13}$$

Theorem 7 *Let us assume the numerical 3–semigroup $T = \langle a, b, c\rangle$ has related the L–shape $\mathcal{H} = \mathrm{L}(l, h, w, y)$. Consider $m = x_0 a + y_0 b + z_0 c$, where (x_0, y_0, z_0) is the basic factorization of m with respect to $\mathcal{H}$ in T. Then,*

(iii.1) if $w = y = 0$,

$$\begin{aligned} \mathrm{d}(m,T) = &(1 + A_m)(1 + \overline{z_{0,1}} + \overline{z_{0,2}}) - (\overline{\delta} + \overline{\theta} + 2)\frac{A_m(1 + A_m)}{2} \\ &+ \sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{z_{0,1}} - k\widehat{\theta}}{\delta} \right\rfloor + \sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{z_{0,2}} - k\widehat{\delta}}{\theta} \right\rfloor, \end{aligned} \tag{14}$$

(iii.2) if $w \neq 0$ and $y = 0$, set $k_0 = \left\lceil \frac{z_0 w - x_0 \theta}{b} \right\rceil$; then,

(iii.2.1) if $k_0 = 0$, the denumerant has the same expression as in (14). Otherwise, when $k_0 > 0$, we have

$$\mathrm{d}(m,T) = (1 + A_m)(1 + \overline{z_{0,1}}) - (1 + \overline{\theta})\frac{A_m(1 + A_m)}{2} + \sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{z_{0,1}} - k\widehat{\theta}}{\delta} \right\rfloor + B \tag{15}$$

where B is defined through the rules of (iii.2.2) or (iii.2.3).

(iii.2.2) if $1 \le k_0 \le A_m$, we have

$$B = (1+A_m)\overline{z_{0,2}} + k_0(\overline{x_0} - \overline{z_{0,2}}) + (\bar{l}+\bar{\delta})\frac{(k_0-1)k_0}{2} - (\bar{\delta}+1)\frac{A_m(1+A_m)}{2} + \sum_{k=0}^{k_0-1}\left\lfloor \frac{\widehat{x_0}+k\widehat{l}}{w} \right\rfloor + \sum_{k=k_0}^{A_m}\left\lfloor \frac{\widehat{z_{0,2}}-k\widehat{\delta}}{\theta} \right\rfloor \tag{16}$$

(iii.2.3) if $k_0 > A_m$, then

$$B = (1+A_m)\overline{x_0} + (\bar{l}-1)\frac{A_m(1+A_m)}{2} + \sum_{k=0}^{A_m}\left\lfloor \frac{\widehat{x_0}+k\widehat{l}}{w} \right\rfloor, \tag{17}$$

(iii.3) if $w = 0$ and $y \neq 0$, define $k_1 = \left\lceil \frac{z_0 y - y_0\delta}{a} \right\rceil$; then,

(iii.3.1) if $k_1 = 0$, the denumerant has the same expression as in (14).

Otherwise, when $k_1 > 0$, we have

$$\mathrm{d}(m,T) = (1+A_m)(1+\overline{z_{0,2}}) - (\bar{\delta}+1)\frac{A_m(1+A_m)}{2} + \sum_{k=0}^{A_m}\left\lfloor \frac{\widehat{z_{0,2}}-k\widehat{\delta}}{\theta} \right\rfloor + A \tag{18}$$

where A is defined by (iii.3.2) or (iii.3.3).

(iii.3.2) if $1 \le k_1 \le A_m$, then

$$A = (1+A_m)\overline{z_{0,1}} + k_1(\overline{y_0} - \overline{z_{0,1}}) + (\bar{\theta}+\bar{h})\frac{(k_1-1)k_1}{2} - (\bar{\theta}+1)\frac{A_m(1+A_m)}{2} + \sum_{k=0}^{k_1-1}\left\lfloor \frac{\widehat{y_0}+k\widehat{h}}{y} \right\rfloor + \sum_{k=k_1}^{A_m}\left\lfloor \frac{\widehat{z_{0,1}}-k\widehat{\theta}}{\delta} \right\rfloor \tag{19}$$

(iii.3.3) if $k_1 > A_m$, then

$$A = (1+A_m)\overline{y_0} + (\bar{h}-1)\frac{A_m(1+A_m)}{2} + \sum_{k=0}^{A_m}\left\lfloor \frac{\widehat{y_0}+k\widehat{h}}{y} \right\rfloor, \tag{20}$$

(iii.4) if $wy \neq 0$, define k_0 and k_1 as in (iii.2) and (iii.3), respectively; then

$$\mathrm{d}(m,T) = 1 + A_m + A + B, \tag{21}$$

where A and B are ruled by the following expressions, depending on k_0 and k_1.

- *If $k_1 = 0$, then*

$$A = (1+A_m)\overline{z_{0,1}} - (\bar{\theta}+1)\frac{A_m(1+A_m)}{2} + \sum_{k=0}^{A_m}\left\lfloor \frac{\widehat{z_{0,1}}-k\widehat{\theta}}{\delta} \right\rfloor. \tag{22}$$

- *If* $1 \le k_1 \le A_m$*, then*

$$A = (1+A_m)\overline{z_{0,1}} - (\overline{\theta}+1)\frac{A_m(1+A_m)}{2} + k_1(\overline{y_0} - \overline{z_{0,1}}) + (\overline{h}+\overline{\theta})\frac{(k_1-1)k_1}{2} + \sum_{k=0}^{k_1-1}\left\lfloor \frac{\widehat{y_0}+k\widehat{h}}{y} \right\rfloor + \sum_{k=k_1}^{A_m}\left\lfloor \frac{\widehat{z_{0,1}}-k\widehat{\theta}}{\delta} \right\rfloor. \quad (23)$$

- *If* $k_1 > A_m$*, then* A *has the same expression as in* (20)*.*
- *If* $k_0 = 0$*, then*

$$B = (1+A_m)\overline{z_{0,2}} - (\overline{\delta}+1)\frac{A_m(1+A_m)}{2} + \sum_{k=0}^{A_m}\left\lfloor \frac{\widehat{z_{0,2}}-k\widehat{\delta}}{\theta} \right\rfloor. \quad (24)$$

- *If* $1 \le k_0 \le A_m$*, then* B *has the same expression as that in* (16)*.*
- *If* $k_0 > A_m$*, then* B *has the same expression as in* (17)*.*

Proof: For the stated values of k_0 and k_1 (recall that now we have $a = h\delta + y\theta$ and $b = w\delta + l\theta$), from (2) and (3), we still have

$$S_k = \begin{cases} \left\lfloor \frac{y_0+kh}{y} \right\rfloor - k & \text{if } k < k_1, \\ A_m - k & \text{if } k \ge k_1, \end{cases} \quad \text{and} \quad T_k = \begin{cases} \left\lfloor \frac{x_0+kl}{w} \right\rfloor - k & \text{if } k < k_0, \\ A_m - k & \text{if } k \ge k_0. \end{cases}$$

Then, all the expressions of the statement are obtained using the same arguments of the proof of Theorem 5. □

Remark 5 *Similar arguments of remarks 2 and 4 leave to* $k_0 \ge 0$ *and* $k_1 \ge 0$*.*

Remark 6 *Although in the statement of Theorem 7 appear sums like* $s = \sum_{k=k_0}^{A_m}\left\lfloor \frac{\widehat{z_{0,2}}-k\widehat{\delta}}{\theta} \right\rfloor$ *that is not of type* $S^{\pm}$ *(the sum do not begin at* $k=0$*), we can reduce it to one sum of type* $S^{\pm}$*. Indeed, taking a generic sum* $\sum_{k=n_1}^{n_2}\left\lfloor \frac{s \pm kt}{q} \right\rfloor$ *with* $0 \le s,t < q$*, and changing the summation index,* $u = k - n_1$*, we obtain an* $S^{\pm}$ *sum*

$$\sum_{u=0}^{n_2-n_1}\left\lfloor \frac{s \pm n_1 t \pm ut}{q} \right\rfloor = \overline{\alpha}(1+n_2-n_1) + \sum_{u=0}^{n_2-n_1}\left\lfloor \frac{\widehat{\alpha} \pm ut}{q} \right\rfloor$$

with $\alpha = s \pm n_1 t$*,* $\alpha = \overline{\alpha}q + \widehat{\alpha}$ *and* $0 \le \widehat{\alpha} < q$*.*

4 Discrete sums $S^{\pm}$

Let us denote the *discrete sum* $S^{\pm}$ by

$$S^{\pm}(s,t,q,N) = \sum_{k=0}^{N}\left\lfloor \frac{s \pm kt}{q} \right\rfloor, \qquad 0 \le s,t < q. \quad (25)$$

These type of sums appear to be a main tool for computing denumerants, as it has been seen in the previous section. In this section we study some properties of $S^{\pm}$ in order to obtain an efficient numerical calculation of it. This calculation will be done in a *discrete Lebesgue–like* sense.

4.1 S^+ sums

Consider the function $f(x) = \left\lfloor \frac{s+xt}{q} \right\rfloor$ that defines the general term of an $S^+(s,t,q,N)$ sum.

Definition 2 *Let us define the k–interval $I_k \subset [0,N]$ by $I_k = \{x \in [0,N] : \ f(x) = k\}$. A k–interval I_k is called an hS-type interval if $|I_k \cap \mathbb{N}| = \lceil \frac{q}{t} \rceil$.*

Lemma 2 *Given a k–interval I_k, we have*

(i) $I_k = [x_k, x_{k+1})$ with $x_k = \frac{kq-s}{t}$.

(ii) $\lfloor \frac{q}{t} \rfloor \leq |I_k \cap \mathbb{N}| \leq \lceil \frac{q}{t} \rceil$ holds except, eventually, the first and/or last intervals.

Proof: Item (i) comes directly from the expression of f. A real interval, $I = [\alpha, \beta)$ of length $\ell = \beta - \alpha$, contains at least $\lfloor \ell \rfloor$ integers and it contains at most $\lceil \ell \rceil$ integers. Item (ii) comes from the length of $|I_k| = x_{k+1} - x_k = \frac{q}{t}$. $\square$

We discuss the value of $S^+(s,t,q,N)$ depending on the following three subcases

(a) $t \mid q$,

(b) $t \nmid q$ and $\gcd(t,q) = 1$,

(c) $t \nmid q$ and $\gcd(t,q) = g > 1$.

4.1.1 Assume $t \mid q$

The maximum value attained by f in $[0,N]$ is

$$M = \left\lfloor \frac{s+Nt}{q} \right\rfloor \quad \text{at} \quad x_M = \frac{Mq-s}{t}. \tag{26}$$

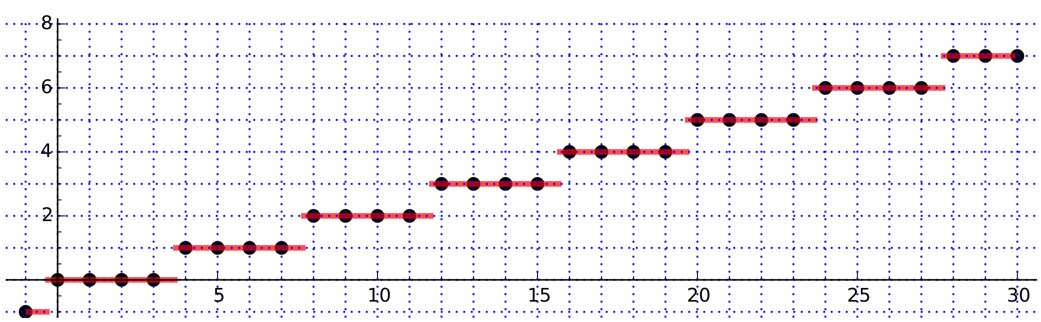

Figure 3: k-intervals for $s=1$, $t=3$, $q=12$ and $N=30$

Example 2 *Figure 3 shows the case $s=1$, $t=3$, $q=12$ and $N=30$. All k-intervals are hS-type ones. The distribution of integrals values is the same in each interval.*

Theorem 8 *Let assume $t \mid q$. Then,*

$$S^+(s,t,q,N) = \frac{q}{t}\frac{M(M-1)}{2} + M(N - \lceil x_M \rceil + 1).$$

Proof: Let us denote $q = t\overline{q}$. By Lemma 2, each I_k interval is a hS–type interval, that is $|I_k| = \overline{q}$ (except, perhaps, the first I_0 and the last one I_M). We divide the interval $I = [0, N]$ in three regions $I = I_0 \cup J \cup I_M$, where M is the maximum value attained by the function f in I.

Let us denote $n = \frac{x_M - x_1}{\overline{q}}$. Then, there are n k-intervals different from I_0 and I_M in I, i.e. $I_1 = [x_1, x_2), \ldots, I_{M-1} = [x_{M-1}, x_M)$ (here $n = M - 1$ holds). The last interval I_M can be eventually one point (that is $I_M = \{N\}$). The sum is

$$S^+(s,t,q,N) = \sum_{k=0}^{N} f(k) = 0 + \sum_{k=x_1}^{x_M-1} f(k) + \sum_{k=x_M}^{N} f(k).$$

Now we add these values like a discrete Lebesgue–like sum

$$\sum_{k=x_1}^{x_M-1} f(k) = \sum_{j=1}^{M-1} |I_j \cap \mathbb{N}| j = \sum_{j=1}^{M-1} \overline{q} j = \overline{q}\, \frac{(M-1)M}{2}.$$

Finally, we have to add $\sum_{k=x_M}^{N} f(k) = |I_M \cap \mathbb{N}| M$. The number of integral points in I_M is $|I_M \cap \mathbb{N}| = N - \lceil x_M \rceil + 1$. Thus, the value of $S^+(s,t,q,N)$ is the stated one. □

4.1.2 Assume $t \nmid q$ and $\gcd(t, q) = 1$

Assume $t \nmid q$. We use the notation

$$q = \overline{q}t + \hat{q}, \quad 1 \le \hat{q} < t, \tag{27}$$
$$s = \overline{s}t + \hat{s}, \quad 0 \le \hat{s} < t, \tag{28}$$
$$S(s,t,q,N) = \overline{q}\frac{(M-1)M}{2} + M(N - \lceil x_M \rceil + 1). \tag{29}$$

Definition 3 *Given a set $A \subset \mathbb{N}$, a subset $J \subset A$ of hS indices is a set of ordered indices in A of hS–type intervals. We define $S_J = \sum_{k \in J} k$.*

Lemma 3 *Assume $t \nmid q$. Then, $I_k \subset [0, N]$ is an hS–type interval if and only if*

$$(\hat{s} - k\hat{q}) \pmod{t} < \hat{q}.$$

Proof: The modulo in the statement is taken from the set of residues $\{0, 1, \ldots, t-1\}$. Notice that $|I_k| = x_{k+1} - x_k = \overline{q} + \frac{\hat{q}}{t}$. The interval I_k is hS–type if and only if $\lceil x_k \rceil - x_k < \frac{\hat{q}}{t}$ (so, the maximum number of integral values are located in I_k). This condition can be restated in a more numerically stable relation. From

$$x_k = \frac{kq - s}{t} = k\overline{q} - \overline{s} - \frac{\hat{s} - k\hat{q}}{t},$$

putting $\hat{s} - k\hat{q} = \alpha t + \beta$ with $0 \le \beta < t$, we have $x_k = n - \frac{\beta}{t}$. Thus, inequality $\lceil x_k \rceil - x_k < \frac{\hat{q}}{t}$ holds if and only if $\beta < \hat{q}$. Equivalently $(\hat{s} - k\hat{q}) \pmod{t} < \hat{q}$. □

Example 3 *Let us consider $s = 0$, $t = 3$, $q = 11$ and $N = 30$. Figure 4 shows all k-intervals in this case. Notice the hS-type intervals for $k \in \{0, 1, 3, 4, 6, 7\}$ (when $0 \le k \le M = 8$ is a solution of $-2k \pmod{3} < 2$). Thus, only the intervals I_2 and I_5 are not of type hS in Figure 4.*

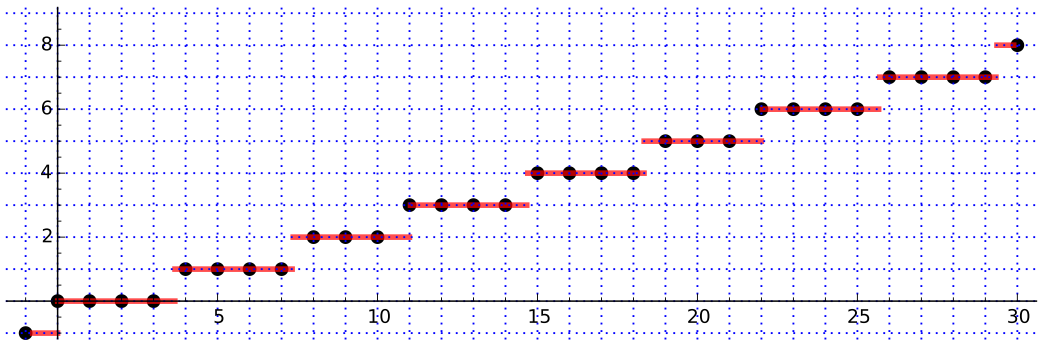

Figure 4: k-intervals for $s = 0$, $t = 3$, $q = 11$ and $N = 30$

The distribution pattern of integral values inside the k-intervals is also ruled modulo t. This fact is detailed in the following result.

Lemma 4 *Let I_k and I_{k+T}, $T > 0$, be two intervals with the same distribution of integral values. Then,*

(i) $T \equiv 0 \pmod{t}$*.*

(ii) The minimum value of T is t.

Proof: In particular, $\lceil x_{k+T} \rceil - x_{k+T} = \lceil x_k \rceil - x_k$ holds. Then, $x_{k+T} - x_k \in \mathbb{Z}$, that is (recalling that $\gcd(q, t) = 1$)

$$x_{k+T} - x_k = T\,\frac{q}{t} \in \mathbb{Z} \Leftrightarrow T \equiv 0 \pmod{t}.$$

The minimum $T > 0$ for the value $T\frac{q}{t}$ to be an integer is $T = t$. $\square$

Corollary 1 *The distribution of integral values in the k-intervals has period t.*

Although Lemma 3 and Lemma 4 give a characterization of hS–intervals, we need a more accurate description of these intervals. This description will be used to efficiently obtain a subset of hS–indices J of Definition 3. Indeed, from Lemma 3, the set J can be parameterized by

$$J = \{\widehat{q}^{-1}(\widehat{s} - i) \pmod{t} \mid 0 \leq i \leq \widehat{q} - 1\}. \tag{30}$$

Notice that $J \neq \emptyset$ because $\widehat{q} \geq 1$ ($t \nmid q$ and $\gcd(t, q) = 1$) and $\widehat{q}^{-1}\widehat{s} \in J$ always. This parameterization is useless, from the point of view of numerical efficiency, whenever we need the elements of J to be sorted. Noting that elements of J are sorted by a rule defined by two moduli, $\widehat{q}$ and t, we can obtain a sorted parameterization of J. For instance (notice that $\gcd(t, \widehat{q}) = 1$)

$$J = \{[\widehat{q}^{-1}(\widehat{s} - (\widehat{s} + iu) \pmod{\widehat{q}})] \pmod{t} \mid 0 \leq i \leq \widehat{q} - 1\}, \qquad u \equiv t \pmod{\widehat{q}} \tag{31}$$

is an example of such parameterization.

Theorem 9 *Assume $t \nmid q$ and $\gcd(t, q) = 1$. Consider the set of hS-type indices $J = \{j_0, \ldots, j_{\widehat{q}-1}\} \subset \{0, \ldots, t-1\}$ and $S(s, t, q, N)$ given by expression (29). Then,*

(a) If $j_0 \geq M$, then $S^+(s, t, q, N) = S(s, t, q, N)$ holds.

(b) If $j_0 < M$, there are three different cases.

(b.1) If $j_{\widehat{q}-1} \geq M$, consider the subset of hS indices $K \subset \{0, \dots, M-1\}$. Then, $S^+(s,t,q,N) = S(s,t,q,N) + S_K$ holds.

(b.2) If $j_{\widehat{q}-1} < M$ and $j_0 + t \geq M$, then $S^+(s,t,q,N) = S(s,t,q,N) + S_J$ holds.

(b.3) If $j_{\widehat{q}-1} < M$ and $j_0 + t < M$, set $u = \lfloor \frac{M-1}{t} \rfloor$ and consider the set of hS-type indices $K \subset \{j_0 + ut, \dots, M-1\}$. Then,

$$S^+(s,t,q,N) = S(s,t,q,N) + uS_J + \widehat{q}t\frac{(u-1)u}{2} + S_K.$$

Proof: $S^+(s,t,q,N)$ can be calculated from S plus all additional summands corresponding to hS-type intervals. That is, each hS-type interval I_j has an additional value j which must be added to S.

(a) When $j_0 \geq M$, there is no hS-type interval in $[0, M)$. Thus, all k-intervals in this region has $\overline{q}$ integral values. Then, S^+ has the same expression as in Theorem 8 replacing $\frac{q}{t}$ by $\overline{q}$, that is $S^+(s,t,q,N) = S(s,t,q,N)$.

(b) When $j_0 < M$ there are hS-type intervals in $[0, M)$. So, we also have to add all hS-type indices contained in $[0, M)$ for obtaining S^+.

(b.1) Assume $j_{\widehat{q}-1} \geq M$. Consider the set of hS-type indices $K \subset \{0, \dots, M-1\}$. There are no more hS-type indices to consider and $S^+(s,t,q,N) = S(s,t,q,N) + S_K$.

(b.2) Assume $j_{\widehat{q}-1} < M$ and $j_0 + t \geq M$. Then, by Lemma 3, all hS-type indices are J. Thus, $S^+(s,t,q,N) = S(s,t,q,N) + S_J$ holds.

(b.3) Assume $j_{\widehat{q}-1} < M$ and $j_0 + t < M$. By Lemma 3, the behaviour of the k-intervals is t-periodic. The maximum number of periods included in the set of indices $A = \{0, \dots, M-1\}$ is $u = \lfloor \frac{M-1}{t} \rfloor$. That is, all the elements in $\{j_0, \dots, j_{\widehat{q}-1}, j_0 + t, \dots, j_{\widehat{q}-1} + t, \dots, j_0 + (u-1)t, \dots, j_{\widehat{q}-1} + (u-1)t\}$ are hS-type indices. The remaining hS-type indices are located in the set of hS indices $K \subset \{ut, \dots, M-1\}$. Therefore,

$$\begin{aligned} S^+(s,t,q,N) &= S + \sum_{l=0}^{u-1}\left(\sum_{j\in J}(lt+j)\right) + S_K = S + \sum_{l=0}^{u-1}(lt|J| + S_J) + S_K \\ &= S + |J|t\sum_{l=0}^{u-1} l + uS_J + S_K = S + |J|t\frac{(u-1)u}{2} + uS_J + S_K. \end{aligned}$$

The statement follows from the identity $|J| = \widehat{q}$. □

Remark 7 *The sets of hS-type indices J and K of Theorem 9 are obtained at time cost $O(\widehat{q})$, in the worst case. The first and last elements of J, j_0 and $j_{\widehat{q}-1}$, can be obtained at constant time cost from the (sorted) parameterization* (31) *of J.*

4.1.3 Assume $t \nmid q$ and $\gcd(t,q) = g > 1$

When $t \nmid q$ and $\gcd(t,q) = g > 1$, we have $x_{h+1} - x_k = \frac{q}{t} = \frac{\tilde{q}}{\tilde{t}}$, where $\tilde{t} = \frac{t}{g}$ and $\tilde{q} = \frac{q}{g}$.

Lemma 5 *Assume $t \nmid q$ and $\gcd(t,q) = g > 1$. Let's assume I_k and I_{k+T}, $T > 0$, are two intervals with the same distribution of integral values. Then,*

(i) $T \equiv 0 \pmod{\tilde{t}}$.

(ii) The minimum value of T is $\tilde{t}$.

Proof: This lemma follows from the proof of Lemma 4 with the additional identity $\frac{q}{t} = \frac{\tilde{q}}{\tilde{t}}$, $\gcd(\tilde{t}, \tilde{q}) = 1$. □

In particular, Lemma 5 ensures that the distribution of integrals values of k-intervals in $[0, N]$ has period $\tilde{t}$. Now, by Lemma 5, detecting hS-type intervals is done as follows. Set

$$\begin{aligned} s &= \overline{s}g + s_g, & 0 \le s_g < g, \\ \overline{s} &= \overline{\overline{s}}\tilde{t} + \hat{\hat{s}}, & 0 \le \hat{\hat{s}} < \tilde{t}, \\ \tilde{q} &= \overline{\overline{q}}\tilde{t} + \hat{\hat{q}}, & 1 \le \hat{\hat{q}} < \tilde{t}. \end{aligned} \tag{32}$$

Then, I_k is an hS interval if and only if

$$(\hat{\hat{s}} - k\hat{\hat{q}}) \pmod{\tilde{t}} < \hat{\hat{q}}, \tag{33}$$

that is similar to the characterization given in Lemma 3.

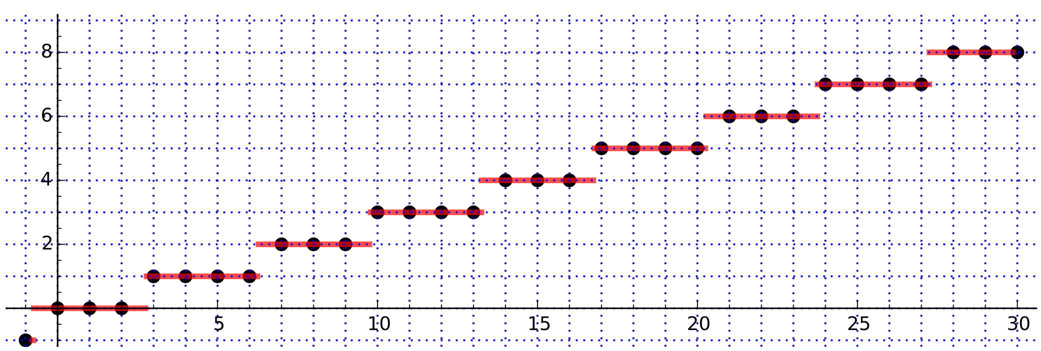

Figure 5: k-intervals for $s = 3$, $t = 4$, $q = 14$ and $N = 30$

Example 4 *Figure 5 shows k-intervals in the case $s = 3$, $t = 4$, $q = 14$ and $N = 30$. Here the distribution of integrals values in k-intervals has period $\tilde{t} = 2$, that is hS-type intervals follow the rule* (33)*, i.e. the set hS-type indices in $[0, 8)$ is $\{1, 3, 5, 7\}$.*

Remark 8 *Sorted and non sorted parameterization of J can also be obtained as in* (30) *and* (31)*. The same expressions hold replacing t by $\tilde{t}$, $\widehat{s}$ by $\hat{\hat{s}}$ and $\widehat{q}$ by $\hat{\hat{q}}$.*

Now, we denote

$$S(s, t, q, N) = \left\lfloor \frac{\tilde{q}}{\tilde{t}} \right\rfloor \frac{M(M-1)}{2} + M(N - \lceil x_M \rceil + 1) \tag{34}$$

and similar results are obtained from the $\tilde{t}$–periodicity of the hS-type intervals ruled by (33).

Theorem 10 *Assume $t \nmid q$ and $\gcd(t,q) = g > 1$. Set $\tilde{t} = t/g$ and $\tilde{q} = q/g$. Then, interchanging t by $\tilde{t}$ and $\widehat{q}$ by $\hat{\hat{q}}$, statements of Theorem 9 hold.*

Remark 9 *Notice that M and x_M are also calculated like in (26), i.e. using t and q (not $\tilde{t}$ and $\tilde{q}$).*

Remark 10 *Now, the sets of hS indices J and K are computed using (33) at time cost $O(\tilde{t})$.*

4.2 S^- sums

The *minus sums*

$$S^-(s,t,q,N) = \sum_{k=0}^{N} \left\lfloor \frac{s-kt}{q} \right\rfloor \tag{35}$$

share some behaviour with plus sums $S^+(s,t,q,N)$. We can define by analogy k-intervals $I_k = (x_k, x_{k+1}] \subset \mathbb{R}$ (those intervals such that $g(x) = \left\lfloor \frac{s-xt}{q} \right\rfloor = k$ for $x \in I_k$) with $x_k = \frac{s-(k+1)q}{t}$ and $x_{k+1} = \frac{s-kq}{t}$. hS-type intervals are also defined to be those I_k with $|I_k \cap \mathbb{Z}| = \lceil \frac{q}{t} \rceil$. We denote now

$$M = -\left\lfloor \frac{s-Nt}{q} \right\rfloor \quad \text{and} \quad x_M = \frac{s+(M-1)q}{t}, \tag{36}$$

that are the analog to (26) for S^+. Also three cases are taken into account now, i.e. $t \mid q$, $t \nmid q$ with $\gcd(t,q) = 1$ and $t \nmid q$ with $\gcd(t,q) = g > 1$. We give here, without proof, the main results for computing S^- sums.

When $t \mid q$, all intervals are hS-type ones and have the same distribution of integral values. The following result can be proved using similar arguments as in Theorem 8.

Theorem 11 *Assume $t \mid q$. Then,*

$$S^-(s,t,q,N) = -\frac{q}{t}\,\frac{(M-1)M}{2} - M(N - \lfloor x_M \rfloor).$$

When $t \nmid q$ and $\gcd(t,q) = 1$, we also use the notation $\widehat{q}$ and $\widehat{s}$ defined in (27) and (28).

Lemma 6 *Assume $t \nmid q$. Then, $I_k \subset [0,N]$ is an hS–type interval if and only if*

$$(\hat{s} + k\hat{q}) \pmod{t} < \hat{q}.$$

Lemma 6 allows a non sorted parameterization of the set of hS-type indices $J \subset \{0,\ldots,t-1\}$, that is

$$J = \{\widehat{q}^{-1}(i-\widehat{s}) \pmod{t} |\; 0 \le i \le \widehat{q}-1\}, \tag{37}$$

which is an analogous expressions to (30) for plus sums. A sorted parameterization of J is given by (now $u \equiv -t \,(\mathrm{mod}\ \widehat{q})$)

$$J = \left\{\widehat{q}^{-1}[(\widehat{s}+iu) \pmod{\widehat{q}} - \widehat{s}] \pmod{t} |\; 0 \le i \le \widehat{q}-1\right\} \quad \text{if } \widehat{s} < \widehat{q} \tag{38}$$

and

$$J = \{\widehat{q}^{-1}[(\widehat{s}+(i+1)u) \pmod{\widehat{q}} - \widehat{s}] \pmod{t} |\; 0 \le i \le \widehat{q}-1\} \quad \text{if } \widehat{s} \ge \widehat{q}. \tag{39}$$

In any case, as it has been done before, the sorted elements of J will be denoted by $J = \{j_0, \ldots, j_{\widehat{q}-1}\}$.

The distribution of integral values in k-intervals also has period t on the indices k like in the plus sums. Let us denote the sum

$$S(s,t,q,N) = -\overline{q}\,\frac{(M-1)M}{2} - M(N - \lfloor x_M \rfloor) \tag{40}$$

which corresponds to S^- when there is no hS-type k–interval, similar to (29) for S^+. The following result is the analog of Theorem 9 for S^+.

Theorem 12 *Assume $t \nmid q$ and $\gcd(t,q) = 1$. Consider the set of hS-type indices $J = \{j_0, \ldots, j_{\widehat{q}-1}\} \subset \{0, \ldots, t-1\}$ and $S(s,t,q,N)$ given by expression (40). Then,*

(a) If $j_0 \geq M$, then $S^+(s,t,q,N) = S(s,t,q,N)$ holds.

(b) If $j_0 < M$, there are three different cases:

(b.1) If $j_{\widehat{q}-1} \geq M$, consider the subset of hS indices $K \subset \{0, \ldots, M-1\}$. Then, $S^+(s,t,q,N) = S(s,t,q,N) - S_K$ holds.

(b.2) If $j_{\widehat{q}-1} < M$ and $j_0 + t \geq M$, then $S^+(s,t,q,N) = S(s,t,q,N) - S_J$ holds.

(b.3) If $j_{\widehat{q}-1} < M$ and $j_0 + t < M$, set $u = \left\lfloor \frac{M-1}{t} \right\rfloor$ and consider the set of hS-type indices $K \subset \{j_0 + ut, \ldots, M-1\}$. Then,

$$S^+(s,t,q,N) = S(s,t,q,N) - uS_J - \widehat{q}t\frac{(u-1)u}{2} - S_K.$$

When $t \nmid q$ and $\gcd(t,q) = g > 1$, we denote $\tilde{t} = t/g$ and $\tilde{q} = q/g$. The value $\overline{q}$ in (40) is the same. i.e. $\overline{q} = \lfloor \tilde{q}/\tilde{t} \rfloor = \lfloor q/t \rfloor$. Using the same notation as in (32), the analog to Lemma 6 is

$$(\hat{\hat{s}} + k\hat{\hat{q}}) \pmod{\tilde{t}} < \hat{\hat{q}} \tag{41}$$

and non sorted and sorted characterizations of J, (37), (38) and (39), have the same expressions by replacing u by $u \equiv -\tilde{t} (\mathrm{mod}\ \hat{\hat{q}})$, t by $\tilde{t}$, $\widehat{s}$ by $\hat{\hat{s}}$ and $\widehat{q}$ by $\hat{\hat{q}}$.

Theorem 13 *Assume $t \nmid q$ and $\gcd(t,q) = g > 1$. Set $\tilde{t} = t/g$ and $\tilde{q} = q/g$. Then, interchanging t by $\tilde{t}$ and $\widehat{q}$ by $\hat{\hat{q}}$, statements of Theorem 12 hold.*

Remarks 9 and 10 have their analogs here.

5 Algorithm

Let us consider any numerical 3–semigroup $N = \langle n_1, n_2, n_3 \rangle$ and $n \in N$. By Lemma 1, there is another semigroup $T = \langle a, b, c \rangle$, with $1 \leq a < b < c$ and $\gcd(a,b) = \gcd(a,c) = \gcd(b,c) = 1$, and $m \in T$ such that $\mathrm{d}(n,N) = \mathrm{d}(m,T)$. Lemma 1 only requires a time cost of $O(\log n_3)$, in the worst case. Moreover, by Theorem 1 and Theorem 2, it can be assumed that $m \in \{0, \ldots, P - S\}$ with $P = abc$ and $S = a + b + c$.

There are three possible cases for the semigroup T,

(1) $a > 1$ and $c \notin \langle a, b \rangle$,

(2) $a > 1$ and $c \in \langle a, b \rangle$,

(3) $a = 1$.

Now we analyze each case for finding the related L-shapes. Then, the time cost of the related $S^{\pm}$ sums will be studied.

5.1 Case 1: $a > 1$ and $c \notin \langle a, b\rangle$

In this case we have $e(T) = 3$. From $c \notin \langle a, b\rangle$, we also have $c \le \mathfrak{f}(a, b) < (a-1)(b-1) < ab$.

Lemma 7 (Rosales and García-Sánchez [15, Chap. 9]**)** *Let $\langle a, b, c\rangle$ be a numerical 3–semigroup with $1 < a < b < c$. Assume that $\mathcal{H} = \mathrm{L}(l, h, w, y)$ is related to T with $wy \ne 0$. Then,*

$$M_c = (l-w)a + (h-y)b = (\delta+\theta)c = \min\{kc : \ k \ge 1, kc \in \langle a, b\rangle\}.$$

Proof. Assume $k_0c < M_c$ for some $k_0 \in \mathbb{N}$ with $k_0 \ge 1$. Then, $k_0c = \alpha a + \beta b$ with $\alpha, \beta \in \mathbb{N}$ and $\alpha + \beta \ge 2$ (the identity $\alpha + \beta = 1$ leads to $k_0c = a$ or $k_0c = b$, a contradiction to $a < b < c$).

Assume $\alpha \ge 1$. Then, the squares $[\![\alpha-1, \beta]\!]$ and $[\![l-w-1, h-y]\!]$ represent the same equivalence class $[0]_c$. From $[\![l-w-1, h-y]\!] \in \mathcal{H}$ (because of $y > 0$) and $[\![\alpha-1, \beta]\!] \notin \mathcal{H}$ (only one square in $\mathcal{H}$ for each equivalence class), we have $(l-w-1)a + (h-y)b \le (\alpha-1)a + \beta b$. Thus, $M_c \le k_0c$ holds and makes a contradiction.

The case $\beta \ge 1$ also makes a contradiction by similar arguments. □

Lemma 8 *Let $T = \langle a, b, c\rangle$ be a numerical 3-semigroup with $1 < a < b < c$, $\gcd(a, b) = \gcd(a, c) = \gcd(b, c) = 1$ and $c \notin \langle a, b\rangle$. Then, only one L-shape $\mathrm{L}(l, h, w, y)$ is related to T and $wy \ne 0$.*

Proof. Assume $w = 0$. Then $hb = 0 + \theta c$ holds with $\theta \ge 1$. Thus, $c \mid h$ ($\gcd(b, c) = 1$) and $b \mid \theta$. Now, from $c = lh$, it follows that $h = c$ and $l = 1$. So, $a = yb + \delta c$ holds and makes a contradiction. Indeed, either $\delta = 0$ we have $y \ne 0$ and $a \mid b$ or $\delta > 0$ and $a \ge c$, a contradiction. The case $y = 0$ also leads to contradiction by similar arguments.

According to [3, theorems 2 and 3], T has only one related L-shape $\mathcal{H} = \mathrm{L}(l, h, w, y)$ iff $(la - yb)(hb - wa) > 0$. Assume $la = yb$ holds. Then, $a \mid y$ and $b \mid l$ ($\gcd(a, b) = 1$) and $hb = wa + \theta c$ with $\theta \ge 1$ (recall that $\delta + \theta \ge 1$). From $c = lh - wy = \frac{yb}{a}h - wy = \frac{y}{a}(hb - wa)$, we have $ac = y\theta c$. So, $y \mid a$ holds and so $y = a$. Therefore, $\theta = 1$ and $l = b$ hold.

Let us consider now M_c defined in Lemma 7. Then,

$$M_c = (l-w)a + (h-y)b = (b-w)a + (h-a)b = hb - wa = \theta c = c.$$

So, $c \in \langle a, b\rangle$ holds and makes a contradiction. The assumption $hb = wa$ also leads to contradiction by similar arguments. □

This lemma ensures that $yb < la$ and $wa < hb$ hold for $\mathrm{L}(l, h, w, y)$ related to T. A direct consequence of Lemma 8 is the non-symmetry of T.

Lemma 9 *Let $T = \langle a, b, c\rangle$ be a numerical 3-semigroup with $1 < a < b < c$ and $\gcd(b, c) = \gcd(a, c) = 1$. Assume $\mathcal{H} = \mathrm{L}(l, h, w, y)$ is an L-shape related to T. Then,*

$$la = \min\{ka : \ k \geq 1, ka \in \langle b, c\rangle\},$$
$$hb = \min\{kb : \ k \geq 1, kb \in \langle a, c\rangle\}.$$

Proof. Here we prove the first equality. The second one is proved by similar arguments.

As $\mathcal{H}$ is an L-shape related to T, we have $\mathrm{Ap}(c, T) = \{ia + jb : \ [\![i, j]\!] \in \mathcal{H}\}$. In particular, it follows that $la = \min\{ka : \ ka \notin \mathrm{Ap}(c, T)\}$. Using the same notation of [15, Lemma 10.18], we have $c_1a = r_{12}b + r_{13}c$ with $r_{12}, r_{13} > 0$, where $c_1a = \min\{ka : \ k \geq 1, ka \in \langle b, c\rangle\}$.

As $la \notin \mathrm{Ap}(c, T)$, we have $la - c \in T$ and so $la - c = x_1a + x_2b + x_3c$. Assuming $x_1 \neq 0$, $(l - x_1)a - c = x_2b + x_3c \in T$ holds and then $(l - x_1)a \notin \mathrm{Ap}(c, T)$. This is a contradiction to the minimality of la. Therefore, $x_1 = 0$ holds. Thus, $la - c \in \langle b, c\rangle$ and $l \geq c_1$ from the minimality of c_1a.

Now, from $c_1a = r_{12}b + r_{13}c$ with $r_{13} > 0$, it follows that $c_1a - c \in T$. Then, $c_1a \notin \mathrm{Ap}(c, T)$ and $l \leq c_1$ from the minimality of la. $\square$

Lemma 10 *Let $T = \langle a, b, c\rangle$ be a numerical 3-semigroup with $1 < a < b < c$, $\gcd(a, b) = \gcd(a, c) = \gcd(b, c) = 1$ and $c \notin \langle a, b\rangle$. Assume T has only one related L-shape $\mathrm{L}(l, h, w, y)$. Then, $h < a$ and $l < b$.*

Proof. By Lemma 9, it follows that $la \leq ab$ and $l \leq b$ holds. Similarly, $h \leq a$ also holds.

Assume $l = b$. So, $ab = la = yb + \delta c$ with $\delta \geq 1$ (by Lemma 8 we have $la > yb$). Then, $b(a - y) = \delta c$ holds and thus $c \mid (a - y)$ ($\gcd(b, c) = 1$). That is, $a = y + \alpha c$ with $\alpha \geq 1$ which contradicts inequality $a < c$. Similar arguments lead to contradiction assuming $h = a$. $\square$

In this case the sides of the L-shape are bounded by $w < l < b$ and $y < h < a$.

5.2 Case 2: $a > 1$ and $c \in \langle a, b\rangle$

Identities $\gcd(a, b) = \gcd(b, c) = \gcd(a, c) = 1$ also hold.

Lemma 11 (A. and Marijuán [3, Theorem 8-(d)]**)** *Assume $c = \lambda a + \mu b$, $\lambda, \mu \in \mathbb{N}$, $0 < \mu < a$, $\gcd(a, \mu) = \gcd(b, \lambda) = 1$. Then, $\lambda \neq b$ and there are two L-shapes related to $T = \langle a, b, c\rangle$, $\mathcal{H}_1 = \mathrm{L}(\lambda + b, a, b, a - \mu)$ with $(\delta, \theta) = (1, 0)$ and $\mathcal{H}_2$ with $(\delta, \theta) = (0, 1)$ given by*

$$\mathcal{H}_2 = \begin{cases} \mathrm{L}(b, a + \mu, b - \lambda, a) & \text{if } \lambda < b, \\ \mathrm{L}(b, (1 + \lfloor \lambda/b \rfloor)a + \mu, b - s, a) & \text{if } \lambda > b \text{ where } \lambda = \lfloor \lambda/b \rfloor b + s, 0 \leq s < b. \end{cases}$$

5.3 Case 3: $a = 1$

Consider a semigroup $T = \langle 1, b, c\rangle$ with $1 < b < c$ and $\gcd(b, c) = 1$.

Lemma 12 *Consider the numerical semigroup $T = \langle 1, b, c\rangle$ with $\gcd(b, c) = 1$. Then, there are two related L-shapes $\mathcal{H}_1 = \mathrm{L}(c, 1, b, 0)$ with parameters $(\delta, \theta) = (1, 0)$ and $\mathcal{H}_2$ with parameters $(\delta, \theta) = (0, 1)$ given by*

$$\mathcal{H}_2 = \begin{cases} \mathrm{L}(b, 2, 2b - c, 1) & \text{if } c < 2b, \\ \mathrm{L}(b, 1 + \lfloor c/b\rfloor, b - r, 1) & \text{if } c > 2b \text{ where } c = \lfloor c/b\rfloor b + r, 0 \le r < b. \end{cases}$$

Proof. $\mathcal{H}_1$ is related to T by Theorem 3. As $\theta = 0$, using the transformation of L-shapes defined in [3, Theorem 3], we obtain $\mathcal{H}_2$ from $\mathcal{H}_1$. $\square$

6 Time cost

Let us analyze now the time cost, in the worst case. This analysis will be done under the assumption of $m \approx P = abc$. This is the same assumption as the one made in the analysis of algorithms P, L and BCS.

Applying Theorem 5, Theorem 6 or Theorem 7 requires the calculation of the L-shape $\mathcal{H} = \mathrm{L}(l, h, w, y)$, the related basic factorization (x_0, y_0, z_0) and all the related $S^{\pm}$ sums. The first two calculations have a time cost of $O(\log c)$ [1]. Then, all $S^{\pm}$ have to be calculated.

Consider a generic sum $S^{\pm}(s, t, q, N) = \sum_{k=0}^{N} \left\lfloor \frac{s \pm kt}{q} \right\rfloor$, with $0 \le s, t < q$. Using the same notation of Section 4, we have

- If $t \mid q$, Theorem 8 for S^+ and Theorem 11 for S^- ensure a constant time cost.
- If $t \nmid q$ and $\gcd(t, q) = 1$, Theorem 9 for S^+ or Theorem 12 for S^- has to be applied. All computations are focused on finding the subsets of hS indices $K_1 \subset \{0, \ldots, M - 1\}$, $K_2 \subset \{j0 + ut, \ldots, M - 1\}$ or $J \subset \{0, \ldots, t - 1\}$. As $|K_1|, |K_2| \le |J|$, the cost has order $O(\widehat{q}) \le O(t)$, in the worst case.
- If $t \nmid q$ and $\gcd(t, q) = g > 1$, consider $\tilde{t} = t/g$. Then, Theorem 10 or Theorem 13 and similar arguments as in the previous case ensure a time cost upper bounded by $O(\tilde{t})$.

Remark 11 *Previous comments point to the fact that the higher cost of computation is reached when $t \nmid q$ and $\gcd(t, q) = 1$. In this case, the time cost is upperbounded by $O(t)$.*

Given a semigroup $S = \langle n_1, n_2, n_3\rangle$ and $n \in S$, apply Lemma 1 at constant time cost for obtaining $T = \langle a, b, c\rangle$ with $1 \le a < b < c$ and $\gcd(a, b) = \gcd(b, c) = \gcd(a, c) = 1$ and $m \in T$ such that $\mathrm{d}(n, S)$ can be calculated from $\mathrm{d}(m, T)$. Then, we have to analyze the time cost of each case given in the previous section.

The worst case for the calculation of $S^{\pm}(s, t, q, N)$, as it is highlighted in Remark 11, appears when $t \nmid q$ and $\gcd(t, q) = 1$. This case will be assumed in all cases in the following analysis. Thus, the resulting worst case order will be a pessimistic estimation.

- Case 1 ($a > 1$, $c \notin \langle a, b\rangle$). By Lemma 8, the L-shape $\mathcal{H}$ belongs to the case (iii) $\delta\theta > 0$, subcase (iii.4) $wy \ne 0$. The following sums have to be evaluated
  - If $k_1 = 0$, there is one sum $S_1^- = \sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{z_{0,1}} - k\widehat{\theta}}{\delta} \right\rfloor$. As $0 \le \widehat{\theta} < \delta = \frac{la - yb}{c} < \frac{la}{c} < \frac{ab}{c} < a$, the cost of calculating S_1^- is upperbounded by $O(a)$.

- If $1 \le k_1 \le A_m$, there are two sums $S_1^+ = \sum_{k=0}^{k_1-1} \left\lfloor \frac{\widehat{y_0}+k\widehat{h}}{y} \right\rfloor$ and $S_1^- = \sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{z_{0,1}}-k\widehat{\theta}}{\delta} \right\rfloor$. The calculation of S_1^- is $O(a)$. As $\widehat{h} < y < a$, the order for calculating S_1^+ is also upperbounded by $O(a)$. Thus, the worst case order of this case is $O(a)$.
- If $k_1 > A_m$, there is one sum $S_2^+ = \sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{y_0}+k\widehat{h}}{y} \right\rfloor$. This sum has the same order as S_1^+, that is $O(a)$.
- If $k_0 = 0$, there is one sum $S_2^- = \sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{z_{0,2}}-k\widehat{\delta}}{\theta} \right\rfloor$. From $\widehat{\delta} < \theta = \frac{hb-wa}{c} < \frac{hb}{c} < \frac{ab}{c} < a$, the order is upperbounded by $O(a)$.
- If $1 \le k_0 \le A_m$, there are two sums $S_3^+ = \sum_{k=0}^{k_0-1} \left\lfloor \frac{\widehat{x_0}+k\widehat{l}}{w} \right\rfloor$ and S_2^-. From $\widehat{l} < w < b$, the cost of both calculations is $O(a) + O(b) = O(b)$.
- If $k_0 > A_m$, we have $S_4^+ = \sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{x_0}+k\widehat{l}}{w} \right\rfloor$ with the same cost of S_3^+, that is $O(b)$.

Therefore, the overall cost of the Case 1 is $O(b)$.

- Case 2 ($a > 1$, $c \in \langle a, b \rangle$). Let us consider $c = \lambda a + \mu b$ with $1 \le \mu < a$. By Lemma 11 there are three possible cases to be examined.

  Consider the L-shape $\mathcal{H}_1 = \mathrm{L}(\lambda + b, a, b, a - \mu)$ with $\delta = 1$ and $\theta = 0$. Then, $A_m = z_0$ and $k_1 = \left\lceil \frac{z_0 a - y_0}{a} \right\rceil \le z_0 = A_m$. Thus, the case $k_1 > A_m$ never appears. So,

  - If $k_1 = 0$, there is one sum $S_1^+ = \sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{x_0}+k\widehat{l}}{w} \right\rfloor$ with $w = b$. Then, the order is upperbounded by $O(b)$.
  - If $1 \le k_1 \le A_m$, there are two sums S_1^+ and $S_2^+ = \sum_{k=0}^{k_1-1} \left\lfloor \frac{\widehat{y_0}+k\widehat{h}}{y} \right\rfloor$, From $y = a - \mu < a$, the order is upperbounded by $O(a) + O(b) = O(b)$.

  Let us examine the other related L-shape $\mathcal{H}_2$ which have an expression depending on λ. Assume $\lambda < b$. Then, we have $\mathcal{H}_2 = \mathrm{L}(b, a + \mu, b - \lambda, a)$ with $\delta = 0$ and $\theta = 1$. As $\delta = 0$, this is the Case-(i) with $w = b - \lambda > 0$. So, $A_m = z_0$ holds and the case $k_0 > A_m$ never appears. Then,

  - If $k_0 = 0$, there is one sum $S_1^+ = \sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{y_0}+k\widehat{h}}{y} \right\rfloor$. From $y = a$, we have order upperbounded by $O(a)$.
  - If $1 \le k_0 \le A_m$, there are two sums S_1^+ and $S_2^+ = \sum_{k=0}^{k_0-1} \left\lfloor \frac{\widehat{x_0}+k\widehat{l}}{w} \right\rfloor$. From $w = b - \lambda < b$, the order is upperbounded by $O(a) + O(b) = O(b)$.

  Assume now $\lambda > b$. Then, the related L-shape is $\mathcal{H}_2 = \mathrm{L}(b, (1 + \lfloor \lambda/b \rfloor)a + \mu, b - s, a)$ with $0 \le s < b$, $\delta = 0$ and $\theta = 1$. We also are in the Case (i) with $w = b - s > 0$. So, $A_m = z_0$ and $k_0 \le A_m$ always holds. Then,

  - If $k_0 = 0$, there is one sum $S_1^+ = \sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{y_0}+k\widehat{h}}{y} \right\rfloor$. From $y = a$, the order is upperbounded by $O(a)$.
  - If $1 \le k_0 \le A_m$, there are two sums S_1^+ and $S_2^+ = \sum_{k=0}^{k_0-1} \left\lfloor \frac{\widehat{x_0}+k\widehat{l}}{w} \right\rfloor$. The order is upperbounded by $O(a) + O(b) = O(b)$.

In any case, using either $\mathcal{H}_1$ or $\mathcal{H}_2$, the overall order is upperbounded by $O(b)$.

- Case 3 ($a = 1$). We have $\gcd(b, c) = 1$. By Lemma 12, there are three possibilities to analyze.

  Consider $\mathcal{H}_1 = \mathrm{L}(c, 1, b, 0)$, with $\delta = 1$ and $\theta = 0$. This L-shape can be used in the two cases $c < 2b$ and $c > 2b$ (note that $c \neq 2b$). Look at the Case (ii) in Section 3.2 and Theorem 6. As $y = 0$, from (11) we have to calculate one sum $S_1^+ = \sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{x_0}+k\widehat{l}}{w} \right\rfloor$. From $\widehat{l} < w = b$, the order is upperbounded by $O(b)$.

  Let us consider now the case $c < 2b$. Again by Lemma 12, we can use the L-shape $\mathcal{H}_2 = \mathrm{L}(b, 2, 2b - c, 1)$ with $\delta = 0$ and $\theta = 1$. Note that $A_m = z_0$. We have to look at Theorem 5-(i.2) ($y = 1 > 0$). We have $k_0 \leq A_m$, then the case (i.2.3) never appears. Then,

  – If $k_0 = 0$, there is only one sum given by (5) $S_1^+ = \sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{y_0}+k\widehat{h}}{y} \right\rfloor$. From $y = 1$, the cost is constant.

  – If $1 \leq k_0 \leq A_m$, there are two sums S_1^+ and $S_2^+ = \sum_{k=0}^{k_0-1} \left\lfloor \frac{\widehat{x_0}+k\widehat{l}}{w} \right\rfloor$. From $\widehat{l} < w = 2b - c < b$, the cost is upperbounded by $O(b)$. Then, the total cost is upperbounded by $O(b) + O(1) = O(b)$.

  When $c > 2b$, we can use the L-shape $\mathcal{H}_2 = \mathrm{L}(b, 1 + \lfloor c/b \rfloor, b - r, 1)$ with $c = \lfloor c/b \rfloor b + r$ and $0 \leq r < b$. The related parameters are $\delta = 0$ and $\theta = 1$. Using the same arguments of the previous case, it follows that the overall order is upperbounded by $O(b)$.

  Therefore, using any admissible L-shape, the total cost of this case is also $O(b)$.

So, we need a cost of $O(\log c)$ to calculate the related L-shape and the basic coordinates plus $O(b)$ to calculate the involved $S^{\pm}$ sums. Hence, our algorithm has a time cost of $O(b + \log c)$. Common instances of semigroups are such that $O(\log c) \ll O(b)$. Then, we have the following result.

Theorem 14 *The time cost, in the worst case, for computing the denumerant* $\mathrm{d}(m, T)$ *is upperbounded by* $O(b + \log c)$.

Remark 12 *When many instances of* $m \in T$ *are given and the semigrup* T *is fixed, the related L-shape is computed only once. Thus, the first calculated denumerant has a time cost of* $O(b + \log c)$*. The subsequent instances only need a time cost of* $O(b)$.

7 Some time tests

All the computations of this section have been made using SageMath 7.3 [16] and non compiled code on a i5@1.3Ghz processor. Here we test our algorithm, denoted by AL, versus the algorithms P, L and BCS. In the following, we use the notation $P = abc$ and $S = a + b + c$. The time required to calculate denumerants highly depends on the selected semigroup. This fact is reflected in the following subsections. All semigroups in this section will meet the property $\gcd(a, b) = \gcd(a, c) = \gcd(b, c) = 1$. By Lemma 1 of Brown, Chou and Shiue, this restrictions does not represent any loose of generality.

Time costs of the involved algorithms are by Table 3. It is assumed that $T = \langle a, b, c\rangle$ and $m \approx P = abc$.

Algorithm	Time cost
P	$O(c \log c)$
L	$O(ab \log b)$
BCS	$O(ab \log c)$
AL	$O(b + \log c)$

Table 3: Time costs for $T = \langle a, b, c\rangle$ and $m \approx P$

Remark 13 *According to Table 3 there are some generic behaviours to be highlighted:*

(i) Algorithm AL has the best time cost.

(ii) When $a = 1$, algorithms L and BCS are faster than Algorithm P when $b \ll c$. However, when $c \approx b$, algorithms P, L and BCS run at similar speed.

(iii) When $a \neq 1$, there are two different behaviours,

(iii.1) if $ab < c$, Algorithm L is faster than algorithms BCS and P,

(iii.2) otherwise, when $ab > c$, Algorithm P wins L and BCS.

(iv) When $b \approx c$, Algorithm P is faster than algorithms L and BCS provided that $a \gg 1$.

In the following subsections we take elements m of the semigroup that are closed to $P - S$.

7.1 $a > 1$, $c \notin \langle a, b\rangle$

k	m_k	$\mathrm{d}(m_k, T_{1,k})$	P	L	BCS	AL
1	4465	2232	0.002304	0.003423	0.010860	0.000291
2	34139180	17069589	0.078596	0.116350	1.025509	0.000615
3	207657687311	103828843654	5.291058	9.787089	68.891171	0.000533
4	1235137178269914	617568589134955	424.791713	740.592501	5275.727091	0.000376

Table 4: $T_{1,k} = \langle 7^k, 11^k, \mathfrak{f}(7^k, 11^k)\rangle$, $m_k = P_k - S_k - k$

k	m_k	$\mathrm{d}(m_k, T_{2,k})$	P	L	BCS	AL
1	893	446	0.001188	0.003566	0.006692	0.000545
2	723044	361521	0.005573	0.152317	0.419318	0.000326
3	608098947	304049472	0.045436	10.102134	31.251389	0.000402

Table 5: $T_{2,k} = \langle 7^k, 11^k, 11^k + 1\rangle$, $m_k = P_k - S_k - k$

In this case, inequality $c < ab$ always holds. Then, as it has been comment in Remark 13-(iii.2), Algorithm P is faster than Algorithm L and Algorithm BCS. Table 4 shows how this assertion is kept for the semigroups $T_{1,k} = \langle 7^k, 11^k, \mathfrak{f}(7^k, 11^k)\rangle$ and $m_k = P_k - S_k - k$ for $k \in \{1, 2, 3, 4\}$. The non increasing sequence of times in the column of Algorithm AL is because the corresponding L-shapes. Their entries do not always increase as the value of k does.

Table 5, for the semigroups $T_{2,k} = \langle 7^k, 11^k, 11^k + 1\rangle$, shows an instance of the case $c \approx b$ and, as it has been noticed in Remark 13-(iv), Algorithm P is faster than algorithms L and BCS.

7.2 $a > 1$, $c \in \langle a, b\rangle$

In this subsection we take the semigroups $T_{3,k} = \langle 7^k, 11^k, 7^k + 11^{2k}\rangle$ for the case $ab < c$ and $T_{4,k} = \langle 7^k, 11^k, 7^k + 11^k\rangle$ for $ab > c$. Tables 6 and 7 show the influence of inequalities $ab < c$ and $ab > c$ in the resulting time cost. Here, item (iii) of Remark 13 is also clear.

k	m_k	$\mathrm{d}(m_k, T_{3,k})$	P	L	BCS	AL
1	9709	4854	0.004208	0.002923	0.015249	0.000422
2	87082148	43541073	0.188668	0.133961	1.012403	0.000332
3	808930875251	404465437624	19.829875	9.667596	69.573355	0.000392

Table 6: $T_{3,k} = \langle 7^k, 11^k, 7^k + 11^{2k}\rangle$, $m_k = P_k - S_k - k$

k	m_k	$\mathrm{d}(m_k, T_{4,k})$	P	L	BCS	AL
1	1349	674	0.001330	0.002596	0.009378	0.000342
2	1007588	503793	0.006507	0.155632	0.489768	0.000453
3	764232891	382116444	0.045628	9.864927	35.554438	0.000263

Table 7: $T_{4,k} = \langle 7^k, 11^k, 7^k + 11^k\rangle$, $m_k = P_k - S_k - k$

7.3 $a = 1$

Let us take the semigroups $T_{5,k} = \langle 1, 7^k, 11^k\rangle$. Table 8 confirms that Algorithm P is slower than algorithms L and BCS. This rule is not noticeable with respect to Algorithm BCS for small values of k. However, it turns apparent from the value $k = 6$.

k	m_k	$\mathrm{d}(m_k, T_{5,k})$	P	L	BCS	AL
1	57	29	0.001047	0.000917	0.002000	0.000381
2	5756	2878	0.003230	0.002486	0.006424	0.000301
3	454855	227427	0.024524	0.009076	0.056042	0.000251
4	35135994	17567996	0.212520	0.048399	0.383123	0.000379
5	2706606293	1353303145	2.070613	0.350236	2.447490	0.001594
6	208420490872	104210245434	20.946929	2.581581	16.702903	0.001885
7	16048502956131	8024251478063	225.232675	16.902235	116.667887	0.009188

Table 8: $T_{5,k} = \langle 1, 7^k, 11^k\rangle$, $m_k = P_k - S_k - k$

Consider now the semigroups $T_{6,k} = \langle 1, 7^k, 7^k + 1\rangle$, where $c \approx b$. According to Remark 13-(ii), the algorithms P, L and BCS have similar time cost. Table 9 shows this behaviour in algorithms P and L. Algorithm BCS runs between three and four times slower.

k	m_k	$d(m_k, T_{6,k})$	P	L	BCS	AL
1	39	20	0.000963	0.000689	0.001357	0.000307
2	2348	1174	0.002826	0.001878	0.006299	0.000307
3	117301	58650	0.013340	0.008977	0.033656	0.000405
4	5762394	2881196	0.065825	0.061175	0.203000	0.000253
5	282458435	141229216	0.382746	0.397065	1.322981	0.000200
6	13841169544	6920584770	2.668743	2.589575	9.066692	0.000212
7	678222249297	339111124646	17.936306	17.299389	61.854215	0.000212
8	33232924804790	16616462402392	127.708571	120.464809	439.603427	0.000302

Table 9: $T_{6,k} = \langle 1, 7^k, 7^k + 1 \rangle$, $m_k = P_k - S_k - k$

7.4 Almost medium and large input data

Now we take larger input values for the Algorithm AL. Usually, our algorithm can manage almost middle input values at acceptable time output. However, when the involved $S^{\pm}$ sums take some proper parameters, the time cost can be almost constant. These cases allow the Algorithm AL to take large input values.

We consider the same semigroups of the previous sections to see these behaviours. When the output values m_k and $d(m_k, T)$ turn to be large, tables will show $\ell(m_k)$ and $\ell(d(m_k, T))$.

Table 10 and Table 11 belong to the case $a > 1$ with $c \notin \langle a, b \rangle$. The case $a > 1$ with $c \in \langle a, b \rangle$, is represented by Table 12 when $ab < c$ and Table13 when $ab > c$. Finally, the case $a = 1$ is represented by tables 14 and 15.

k	$\ell(m_k)$	$\ell(d(m_k, T_{1,k}))$	AL
10	38	38	0.000424
100	378	377	0.000966
1000	3773	3773	0.002669
10000	37730	37730	0.051687
100000	377299	377298	1.350448
1000000	3772982	3772982	25.325951

Table 10: $T_{1,k} = \langle 7^k, 11^k, \mathfrak{f}(7^k, 11^k) \rangle$, $m_k = P_k - S_k - k$

k	m_k	$d(m_k, T_{2,k})$	AL
4	514710794634	257355397315	0.000495
5	435933001714249	217966500857122	0.000889
6	369233168511568240	184616584255784117	0.000775
7	312740333247126511823	156370166623563255908	0.003953
8	264891049902986514370070	132445524951493257185031	0.010668
9	224362718316312996430224405	112181359158156498215112198	0.007589
10	190035222340650307226923642236	95017611170325153613461821113	0.013501
11	160959833316889266300917603625499	80479916658444633150458801812744	0.155148
12	136332978818970809672136276502054306	68166489409485404836068138251027147	5.684347
13	115474033059634827078142773316758235361	57737016529817413539071386658379117674	0.888997
14	97806506001508122984196519581781213521032	48903253000754061492098259790890606760509	31.661762
15	82842110583277181850188186745272781621519975	41421055291638590925094093372636390810759980	80.498716

Table 11: $T_{2,k} = \langle 7^k, 11^k, 11^k + 1 \rangle$, $m_k = P_k - S_k - k$

Algorithm AL allows almost middle length inputs, above a hundred digits. Several instances of this inputs at reasonable time output are given in tables 11, 12 and 14. The nature of the involved $S^{\pm}$ sums has an interesting property. Some parameters taken by these sums make almost constant the time cost of the denumerant's calculation. In these cases, the algorithm can handle large inputs (million digits) at a small time cost. Tables 10, 13 and 15 show some instances of this good behaviour.

k	m_k	$\mathrm{d}(m_k, T_{3,k})$	AL
4	7535450720580234	3767725360290115	0.000684
5	702070554503525553785	351035277251762776890	0.009376
6	654118736532593706215344	327059368266296853107669	0.003036
7	6094424053060467191130813247	3047212026530233595565406620	0.308053
8	56781748786352120105926189224470	28390874393176060052963094612231	0.497334
9	529035553379910639083032546370845061	264517776689955319541516273185422526	5.588338
10	4929024250806922465243407641240300023740	2464512125403461232621703820620150011865	52.150354
11	45923718944749929614349668788645626831480843	22961859472374964807174834394322813415740416	248.336705

Table 12: $T_{3,k} = \langle 7^k, 11^k, 7^k + 11^{2k} \rangle$, $m_k = P_k - S_k - k$

k	$\ell(m_k)$	$\ell(\mathrm{d}(m_k, T_{4,k}))$	AL
10	30	29	0.000494
100	293	293	0.000435
1000	2928	2928	0.002186
10000	29279	29279	0.046576
100000	292789	292789	1.200240
1000000	2927884	2927884	23.496686

Table 13: $T_{4,k} = \langle 7^k, 11^k, 7^k + 11^k \rangle$, $m_k = P_k - S_k - k$

Now, we briefly comment this almost constant time cost behaviour of Algorithm AL in tables 10, 13 and 15. In fact, almost all the time is spent in the computation of the related L-shape, that is $O(\log c)$.

The semigroup $T_{1,n} = \langle 7^n, 11^n, \mathfrak{f}(7^n, 11^n) \rangle$, from Theorem 3, has related the L-shape $\mathcal{H}_{1,n} = \mathrm{L}(11^n - 1, 7^n - 1, 1, 1)$ with $\delta = \theta = 1$. Then, this is the case $a > 1$ with $c \notin \langle a, b \rangle$. We have to calculate some of the sums $\sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{z_{0,1}} - k\widehat{\theta}}{\delta} \right\rfloor$, $\sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{z_{0,2}} - k\widehat{\delta}}{\theta} \right\rfloor$, $\sum_{k=0}^{k_1-1} \left\lfloor \frac{\widehat{y_0} + k\widehat{h}}{y} \right\rfloor$, $\sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{y_0} + k\widehat{h}}{y} \right\rfloor$, $\sum_{k=0}^{k_0-1} \left\lfloor \frac{\widehat{x_0} + k\widehat{l}}{w} \right\rfloor$ and $\sum_{k=0}^{k_0-1} \left\lfloor \frac{\widehat{x_0} + k\widehat{l}}{w} \right\rfloor$. From $\delta = \theta = w = y = 1$, all these sums are calculated at constant time. Therefore, the fast computation of denumerants in Table 10 is clear now.

k	m_k	$\mathrm{d}(m_k, T_{5,k})$	AL
8	1235736071423990	617868035711992	0.465418
9	95151692050870129	47575846025435061	5.482949
10	7326680446366300788	3663340223183150390	18.740051
11	564154396101848452607	282077198050924226299	241.083486

Table 14: $T_{5,k} = \langle 1, 7^k, 11^k \rangle$, $m_k = P_k - S_k - k$

The semigroup $T_{4,n} = \langle 7^n, 11^n, 7^n + 11^n \rangle$ has related the L-shape $\mathcal{H}_{4,n} = \mathrm{L}(11^n, 7^n + 1, 11^n - 1, 7^n)$ with $\delta = 0$ and $\theta = 1$. This is the case $a > 1$ with $c \in \langle a, b \rangle$ and parameters $\lambda = \mu = 1$ and $\lambda < b = 11^n$. Thus, following this case at page 24, we have two possibilities:

k	$\ell(m_k)$	$\ell(\mathrm{d}(m_k, T_{6,k}))$	AL
10	17	17	0.000623
100	170	169	0.000652
1000	1691	1690	0.000622
10000	16902	16902	0.003141
100000	169020	169020	0.061677
1000000	1690197	1690196	0.742673
10000000	16901961	16901961	10.886006

Table 15: $T_{6,k} = \langle 1, 7^k, 7^k + 1\rangle$, $m_k = P_k - S_k - k$

- When $k_0 = 0$, it has to be computed the sum $\sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{y_0}+k\widehat{h}}{y} \right\rfloor$ with $h = 7^n + 1$ and $y = 7^n$. From $h = y + 1$, it follows that $\widehat{h} = 1$ and the sum can be computed at constant cost from Theorem 8.
- Otherwise, when $1 \leq k_0 \leq A_m$, the algorithm calculates the sum $\sum_{k=0}^{k_0-1} \left\lfloor \frac{\widehat{x_0}+k\widehat{l}}{w} \right\rfloor$. Then, $l = w + 1$ holds and, by the previous argument, the sum can be computed at constant time cost.

Therefore, the fast behaviour of the algorithm in Table 13 is now clear.

Finally, let us consider the semigroups $T_{6,n} = \langle 1, 7^n, 7^n + 1\rangle$. A related L-shape is $\mathcal{H}_{6,n} = \mathrm{L}(7^n, 2, 7^n - 1, 1)$ with $\delta = 0$ and $\theta = 1$. This is the case $a = 1$ with parameters $\lambda = \mu = 1$ and $c < 2b$. Here we also have two possible cases:

- When $k_0 = 0$, there is only one sum to be computed, $S_1^+ = \sum_{k=0}^{A_m} \left\lfloor \frac{\widehat{y_0}+k\widehat{h}}{y} \right\rfloor$. Here we have $y = 1$. Thus, this sum is calculated at constant time.
- If $1 \leq k_0 \leq A_m$, we have to compute S_1^+ of the previous case and $S_2^+ = \sum_{k=0}^{k_0-1} \left\lfloor \frac{\widehat{x_0}+k\widehat{l}}{w} \right\rfloor$. Again, from $l = w + 1$ and Theorem 8, the sum S_2^+ can also be calculated at constant time cost.

Thus, the speed of the algorithm in Table 15 is now clear.

Remark 14 *Many semigroups have related an L-shape* $\mathrm{L}(l, h, w, y)$ *with* $\delta = 1$ *and/or* $\theta = 1$, $w = 1$ *and/or* $y = 1$. *Additionally, many elements of the semigroup* $m \in T$ *have null coefficient multiplying* k *in the* $S^{\pm}$ *sums. So, the fast behaviour of this algorithm eventually can be habitual.*

8 Conclusion

Algorithm AL accepts almost medium input data to calculate denumerants of numerical 3-semigroups at acceptable speed using an ordinary computer (tables 11, 12 and 14). As far as we know, this algorithm is faster than usual known implemented algorithms for embedding dimension three numerical semigroups. This is the behaviour in the worst case. Eventually, this algorithm accepts large input data (tables 10, 13 and 15).

The main tool of this algorithm is the hS-type set of ordered indices of intervals. As the computation techniques for obtaining these sets become faster, the time cost of this algorithm turns to be smaller.

It is difficult to generalize the algorithm to larger embedding dimensions because of the related minimum distance diagrams. Less is known about these diagrams related to numerical n-semigroups for $n \geq 4$, mainly a generic geometrical description.

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