DECOMPOSITION SPACES, INCIDENCE ALGEBRAS AND MÖBIUS INVERSION I: BASIC THEORY

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Abstract. This is the first in a series of papers devoted to the theory of decomposition spaces, a general framework for incidence algebras and Möbius inversion, where algebraic identities are realised by taking homotopy cardinality of equivalences of ∞-groupoids. A decomposition space is a simplicial ∞-groupoid satisfying an exactness condition, weaker than the Segal condition, expressed in terms of active and inert maps in $\mathbb{\Delta}$. Just as the Segal condition expresses composition, the new exactness condition expresses decomposition, and there is an abundance of examples in combinatorics.

After establishing some basic properties of decomposition spaces, the main result of this first paper shows that to any decomposition space there is an associated incidence coalgebra, spanned by the space of 1-simplices, and with coefficients in ∞-groupoids. We take a functorial viewpoint throughout, emphasising conservative ULF functors; these induce coalgebra homomorphisms. Reduction procedures in the classical theory of incidence coalgebras are examples of this notion, and many are examples of decalage of decomposition spaces. An interesting class of examples of decomposition spaces beyond Segal spaces is provided by Hall algebras: the Waldhausen $S_\bullet$-construction of an abelian (or stable infinity) category is shown to be a decomposition space.

In the second paper in this series we impose further conditions on decomposition spaces, to obtain a general Möbius inversion principle, and to ensure that the various constructions and results admit a homotopy cardinality. In the third paper we show that the Lawvere–Menni Hopf algebra of Möbius intervals is the homotopy cardinality of a certain universal decomposition space. Two further sequel papers deal with numerous examples from combinatorics.

Note: The notion of decomposition space was arrived at independently by Dyckerhoff and Kapranov [17] who call them unital 2-Segal spaces. Our theory is quite orthogonal to theirs: the definitions are different in spirit and appearance, and the theories differ in terms of motivation, examples, and directions.

Contents

Note. This paper originally formed Sections 1 and 2 of the manuscript [21], which has now been split into six papers.

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0. Introduction

The notion of incidence algebra of a locally finite poset is an important construction in algebraic combinatorics, with applications to many fields of mathematics. In this work we generalise this construction in three directions: (1) we replace posets by categories and ∞-categories; (2) we replace scalar coefficients in a field by ∞-groupoids, working at the objective level, ensuring natively bijective proofs [22]; and most importantly: (3) we replace the Segal condition, which essentially characterises ∞-categories among simplicial ∞-groupoids, by a weaker condition that still allows the construction of incidence algebras. Simplicial ∞-groupoids satisfying this axiom are called decomposition spaces, seen as a systematic framework for decomposing structures, whereas categories constitute the systematic framework for composing structures.

In the present work we focus on incidence *coalgebras*; incidence algebras are just the convolution algebras given by their linear duals. The fundamental role played by coalgebras was established by Rota and his collaborators, the work with Joni [31] being a milestone.

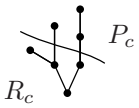
We briefly preface the historically motivated introduction below with a preview of one of the examples that motivated us, and which will serve as a running example.

0.1. Running example: the Hopf algebra of rooted trees. (We return to this example in 3.3, 5.2, 9.5, 10.4.) The Butcher–Connes–Kreimer Hopf algebra of rooted trees [8, 14, 47] is the free commutative algebra on the set of iso-classes of rooted trees T, with the comultiplication defined by summing over certain admissible cuts c:

$$\Delta(T) \;=\; \sum_{c\in \mathrm{adm.cuts}(T)} P_c \otimes R_c \tag{1}$$

An admissible cut c partitions the nodes of T into two subsets or 'layers'

(2)



One layer must form a rooted subtree R_c (or be empty), and its complement forms the 'crown', a subforest P_c regarded as a monomial of trees.

We can formalise this construction as follows. Let $\mathbf{H}_k$ denote the groupoid of forests with $k-1$ compatible admissible cuts, partitioning the forest into k layers (which may be empty). These form a simplicial groupoid $\mathbf{H}$, where simplicial degeneracy maps repeat a cut, inserting an empty layer, and face maps forget a cut, joining adjacent layers, or discard the top or bottom layer.

The comultiplication (1) arises from this simplicial groupoid by a pull-push formula (see 5.1, 5.2 below): for a tree $T \in \mathbf{H}_1$, take the homotopy sum over the homotopy fibre $d_1^{-1}(T) \subset \mathbf{H}_2$, and for each element c in the fibre return the pair (d_2c, d_0c) consisting of the two layers. Finally take homotopy cardinality to arrive at $P_c \otimes R_c$.

We note three things about this construction. Firstly, it is essential to work with simplicial groupoids rather than simplicial sets: had we passed to sets of iso-classes (of forests with cuts), crucial information would be lost, essentially because trees with a cut admit isomorphisms that do not fix the underlying tree — see [26] for detailed explanation of this point. Secondly, we took homotopy cardinality at the last step, but in fact the whole construction is so formal and natural that it works on the 'objective level' of groupoid slices. Refraining from taking cardinality yields a natively 'bijective' version. Finally, and most importantly, this simplicial groupoid is not a Segal object: that is, is not the (fat) nerve of a category. Indeed, the Segal condition would imply that any tree with an

admissible cut could be reconstructed uniquely knowing just the layers above and below the cut. But this is manifestly false: there are *many* trees with cuts that have the same layers as (2).

The main discovery is that there is a weaker condition than the Segal condition that allows the construction of a coassociative incidence coalgebra: this is the decomposition-space axiom that we introduce in §3. It has a clear combinatorial interpretation (see the pictures in 3.3), has a clean categorical description as an exactness condition 3.1, and is a general condition satisfied also by examples from other areas of mathematics, such as the Waldhausen $S_\bullet$-construction 10.7. See Dyckerhoff and Kapranov [17] for further outlook.

Background and motivation

Leroux's notion of Möbius category [52] generalises at the same time locally finite posets (Rota [63]) and Cartier–Foata finite-decomposition monoids [10], the two classical settings for incidence (co)algebras and Möbius inversion. The finiteness conditions in the definition of Möbius category ensure that the comultiplication law

$$\Delta(f) \;=\; \sum_{b\circ a=f} a\otimes b \tag{3}$$

is well defined on the vector space spanned by the set of arrows. This defines the classical incidence coalgebra.

An important advantage of having the classical settings of posets and monoids on the same footing is they may then be connected by an appropriate class of functors, the *CULF* functors (standing for 'conservative' and 'ULF' = 'unique lifting of factorisations'; see §4). In particular it gives a nice explanation of the important process of reduction, to get the most interesting algebras out of posets, a process that was sometimes rather ad hoc. For the most classical example of this process, consider the divisibility poset $(\mathbb{N}^\times, |)$ as a category. It admits a CULF functor to the multiplicative monoid $(\mathbb{N}^\times, \times)$, considered as a category with only one object. This functor induces a homomorphism of incidence coalgebras which is precisely the reduction map from the 'raw' incidence coalgebra of the divisibility poset to its reduced incidence coalgebra, which is isomorphic to the Cartier–Foata incidence coalgebra of the multiplicative monoid.

Shortly after Leroux's work, Dür [14] studied more involved categorical structures to extract further examples of incidence algebras and study their Möbius functions. In particular he realised the Hopf algebra of rooted trees as the reduced incidence coalgebra of a certain category of root-preserving forest embeddings, modulo the equivalence relation that identifies two root-preserving forest embeddings if their complement crowns are isomorphic forests (see 10.4). Another prominent example fitting into Dür's formalism is the Faà di Bruno bialgebra, previously obtained in [32] from the category of surjections, which is however not a Möbius category.

Our work on Faà di Bruno formulae in bialgebras of trees [20] prompted us to look for a more general version of Leroux's theory, which would naturally realise the Faà di Bruno and Butcher–Connes–Kreimer bialgebras directly as incidence coalgebras. A sequence of generalisations and simplifications of the theory led to the notion of decomposition space which is the central notion of the present work.

Abstraction steps: from numbers to sets, and from sets to ∞-groupoids

The first abstraction step is to follow the objective method, pioneered in this context by Lawvere and Menni [50], working directly with the combinatorial objects, using linear algebra with coefficients in ***Set*** rather than working with numbers and functions on the vector spaces spanned by the objects.

To illustrate the objective method, observe that a vector in the free vector space on a set S is just a collection of scalars indexed by (a finite subset of) S. The objective counterpart is a family of sets indexed by S, i.e. an object in the slice category $\boldsymbol{Set}_{/S}$, and linear maps at this level are given by spans $S \leftarrow M \rightarrow T$. The Möbius inversion principle states an equality between certain linear maps (elements in the incidence algebra). At the objective level, such an equality can be expressed as a bijection between sets in the spans representing those linear functors (see the second paper in this series [23]). In this way, algebraic identities are revealed to be just the cardinality of bijections of sets, which carry much more information.

In the present work we take coefficients in $\mathcal{S}$, the ∞-category of ∞-groupoids. The role of vector spaces is then played by slice ∞-categories $\mathcal{S}_{/S}$. In [22] we have developed the necessary 'homotopy linear algebra' and the notion of homotopy cardinality, extending many results of Baez–Hoffnung–Walker [3] who worked with 1-groupoids. In order to be able to recover numerical or algebraic results by taking cardinality, suitable finiteness conditions must be imposed, but as long as we work at the objective level, where all results and proofs are naturally bijective, these finiteness conditions do not play an essential role. Outside of this introduction we are not concerned with finiteness conditions and cardinality in the present paper, but will return to them in the second and third papers in this series [23, 24].

The price to pay for working at the objective level is the absence of additive inverses: in particular, Möbius functions cannot exist in the usual form of an alternating sum indexed by chains of different lengths. However, we can prove the following explicit equivalence of ∞-groupoids (cf. [23]):

$$\zeta * \Phi_{\mathrm{even}} \quad \simeq \quad \varepsilon \quad + \quad \zeta * \Phi_{\mathrm{odd}}.$$

We shall not here go into the definition of these ∞-groupoids. The point we wish to make is that upon taking homotopy cardinality, under the appropriate finiteness assumptions, and putting $\mu = |\Phi_{\mathrm{even}}| - |\Phi_{\mathrm{odd}}|$, one recovers the usual Möbius inversion formula $\zeta * \mu = \varepsilon$, which has thus been given a 'bijective' interpretation.

There are two levels of finiteness conditions needed in order to take cardinality and arrive at algebraic (numerical) results [23]: namely, just in order to obtain a numerical coalgebra, for each arrow f and for each $n \in \mathbb{N}$, there should be only finitely many decompositions of f into a chain of n arrows. Second, in order to obtain also Möbius inversion, the following additional finiteness condition is needed: for each arrow f, there is an upper bound on the number of non-identity arrows in a chain of arrows composing to f. The latter condition is important in its own right, as it is the condition for the existence of a length filtration (cf. [23, §6], useful in many applications [20, 43, 45]).

The importance of chains of arrows naturally suggests a simplicial viewpoint, regarding a category $\mathcal{C}$ as a simplicial set via its nerve $N\mathcal{C}$. Leroux's theory can be formulated in terms of simplicial sets, and many of the arguments then rely on certain simple pullback conditions, the first being the Segal condition which characterises categories among simplicial sets. Most importantly, in our exploitation of this simplicial viewpoint the comultiplication (3) can be written in terms of $N\mathcal{C}$ as a push-pull formula, $\Delta = (d_2, d_0)_! \circ d_1^*$.

The fact that combinatorial objects typically have symmetries prompted the upgrade from sets to groupoids, in fact a substantial conceptual simplification [20]. This upgrade is essentially straightforward, as long as the notions involved are taken in a correct homotopy sense: bijections of sets are replaced by equivalences of groupoids; the slices playing the role of vector spaces are homotopy slices, the pullbacks and fibres involved in the functors are homotopy pullbacks and homotopy fibres, and the sums are homotopy sums (i.e. colimits indexed by groupoids, just as ordinary sums are colimits indexed by sets).

In this setting one may abandon also the strict notion of simplicial object in favour of a pseudo-functorial analogue. For example, the monoidal nerve of $(\mathbb{B}, +, 0)$, the monoidal groupoid of finite sets and bijections under disjoint union, is actually only a pseudofunctor $\mathbf{B} : \mathbb{\Delta}^{\mathrm{op}} \to \boldsymbol{grpd}$ (see 2.17–2.18). This level of abstraction allows us to state for example that the incidence algebra of $\mathbf{B}$ is the category of species with the Cauchy product [26, §2.1] (suggested as an exercise by Lawvere and Menni [50]).

While it is doable to handle the 2-category theory involved to deal with groupoids, pseudo-functors, pseudo-natural isomorphisms, and so on, much conceptual clarity is obtained by passing immediately to the ∞-category $\mathcal{S}$ of ∞-groupoids: thanks to the monumental effort of Joyal [34], [35], Lurie [55] and others, ∞-groupoids can now be handled efficiently. At the elementary level where we work, all that is needed is some basic knowledge about (homotopy) pullbacks and (homotopy) sums, and everything looks very much like the category of sets. So we work throughout with certain simplicial ∞-groupoids. The appropriate notion of weak category in ∞-groupoids is that of Rezk complete Segal space [61]. Our theory at this level says that for any Rezk complete Segal space there is a natural incidence coalgebra defined with coefficients in ∞-groupoids (this is a special case of Theorem 7.4) and that the objective sign-free Möbius inversion principle holds [23].

The idea of decomposition spaces

The final abstraction step, which becomes the starting point for the paper, is to notice that in fact neither the Segal condition nor the Rezk condition is needed in full in order to get a (co)associative (co)algebra and a Möbius inversion principle. Coassociativity follows from (in fact is essentially equivalent to) the *decomposition-space axiom* (see §3 for the axiom, and the discussion at the beginning of §5 for its derivation from coassociativity): a decomposition space is a simplicial ∞-groupoid $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$ sending active-inert pushout squares in $\mathbb{\Delta}$ to pullbacks. Whereas the Segal condition is the expression of the ability to compose morphisms, the new condition is about the ability to decompose, which of course in general is easier to achieve than composability.

It is likely that all incidence (co)algebras can be realised directly (without imposing a reduction) as incidence (co)algebras of decomposition spaces. If a reduced incidence algebra construction is known, the decomposition space can be found by analysing the reduction step. For example, Dür [14] realises the q-binomial coalgebra as the reduced incidence coalgebra of the category of finite-dimensional vector spaces over a finite field and linear injections, by imposing the equivalence relation identifying two linear injections if their quotients are isomorphic. Trying to realise the reduced incidence coalgebra directly as a decomposition space immediately leads us to the Waldhausen $S_\bullet$-construction, which is a general class of examples: we show that for any abelian category or stable ∞-category, the Waldhausen $S_\bullet$-construction is a decomposition space (which is not Segal), cf. 10.7. Under the appropriate finiteness conditions, the resulting incidence algebras include the Hall algebras, as well as the derived Hall algebras first constructed by Toën [70].

Other examples of coalgebras that can be realised as incidence coalgebras of decomposition spaces but not of categories are Schmitt's Hopf algebra of graphs [66] and the Butcher–Connes–Kreimer Hopf algebra of rooted trees [11]. In a sequel paper [25], these examples are subsumed as examples of decomposition spaces induced from restriction species and directed restriction species.

For our present purposes, the appropriate notion of morphism between decomposition spaces is that of CULF functor, as these induce coalgebra homomorphisms (8.2). Many relationships between incidence coalgebras, and in particular most of the reductions that play a central role in the classical theory (from Rota [63] and Dür [14] to Schmitt [66]),

are induced from CULF functors. The simplicial viewpoint taken in this work reveals furthermore that many of these CULF functors are actually instances of the notion of *decalage* 4.7, which goes back to Lawvere [48] and Illusie [29]. Decalage is in fact an important ingredient in the theory to relate decomposition spaces to Segal spaces: we observe that the decalage of a decomposition space is a Segal space 4.10.

Throughout we have strived for deriving all results from elementary principles, such as pullbacks, factorisation systems and other universal constructions. It is also characteristic for our approach that we are able to reduce many technical arguments to simplicial combinatorics. The main notions are formulated in terms of the active-inert factorisation system in $\mathbb{\Delta}$ (see 2.2). To establish coassociativity we explore also $\underline{\mathbb{\Delta}}$ (the algebraist's Delta, including the empty ordinal) and establish and exploit a universal property of its twisted arrow category (§6). Sequels to this paper further vindicate this philosophy: In [24], in order to construct the universal decomposition space of intervals, we study the category Ξ of finite strict intervals, yet another variation of the simplex category, related to it by an adjunction. In [25], as a general method for establishing functoriality in inert maps, we study a certain category $\mathbb{\nabla}$ of convex correspondences in $\underline{\mathbb{\Delta}}$. These 'simplicial preliminaries' are likely to have applications also outside the theory of decomposition spaces.

Related work: 2-Segal spaces of Dyckerhoff and Kapranov

The notion of decomposition space was arrived at independently by Dyckerhoff and Kapranov [17]: a decomposition space is essentially the same thing as what they call a unital 2-Segal space. We hasten to give them full credit for having arrived at the notion first. Unaware of their work, we arrived at the same notion from a very different path, and the theory we have developed for it is mostly orthogonal to theirs.

The definitions are different in appearance: the definition of decomposition space refers to preservation of certain pullbacks, whereas the definition of 2-Segal space (reproduced in 3.2 below) refers to triangulations of convex polygons. The coincidence of the notions was noticed by Mathieu Anel because two of the basic results are the same: specifically, the characterisation in terms of decalage and Segal spaces (our Theorem 4.10) and the result that the Waldhausen $S_\bullet$-construction of a stable ∞-category is a decomposition space (our Theorem 10.15) were obtained independently (and first) in [17].

We were motivated by rather elementary aspects of combinatorics and quantum field theory, and our examples are all drawn from incidence algebras and Möbius inversion, whereas Dyckerhoff and Kapranov were motivated by representation theory, geometry, and homological algebra, and develop a theory with a much vaster range of examples in mind: in addition to Hall algebras and Hecke algebras they find cyclic bar construction, mapping class groups and surface geometry (see also [18] and [16]), construct a Quillen model structure and relate to topics of interest in higher category theory such as $(\infty, 2)$-categories and operads.

In the end we think our contribution is just a little corner of a vast theory, but an important little corner, and we hope that our viewpoints and insights will prove useful also for the rest of the theory.

Related work on Möbius categories

Where incidence algebras and Möbius inversion are concerned, our work descends from Leroux et al. [12, 52, 53], Dür [14], and Lawvere–Menni [50].

There is a different notion of Möbius category, due to Haigh [28]. The two notions have been compared, and to some extent unified, by Leinster [51], who calls Leroux's Möbius inversion *fine* and Haigh's *coarse*, as it only depends on the underlying graph of

the category. We should mention also the K-theoretic Möbius inversion for quasi-finite EI categories of Lück and collaborators [54], [19].

The classical theory of incidence algebras of posets reached a culmination with Schmitt's paper [66]. Subsequently, Ray and Schmitt [60] introduced a certain category of *cell-sets* consisting of sets equipped with an equivalence relation (and a compatible dimension function), whose morphisms are given by a clever multiset construction. They showed that coalgebraic structures could be defined at this level, prior to taking free vector spaces, and in this sense their theory can be seen as a precursor to the fullblown objective method of slices (indeed, their multiset morphisms are subsumed in the natural notion of linear functor between slices). Although cell-sets have equivalence relations built in, they do not account for symmetries in the same automatic way as groupoids do, and essentially their examples are still poset based. It seems likely that the main structures and constructions of [60] can be subsumed in the theory of decomposition spaces, and we hope to get the opportunity to take up this issue on a later occasion.

Outline of the present paper, section by section

We begin in Section 1 with a review of some elementary notions from the theory of ∞-categories, to render the paper accessible also to readers without prior experience with these notions. Section 2 contains a few preliminaries on simplicial objects and Segal spaces, and in Section 3 we introduce the main notion of this work, decomposition spaces:

Definition. A simplicial space $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$ is called a *decomposition space* when it takes active-inert pushouts in $\mathbb{\Delta}$ to pullbacks of ∞-groupoids.

We give some equivalent pullback characterisations, and observe that every Segal space is a decomposition space.

In Section 4 we turn to the relevant notion of morphism, that of CULF functor (meaning 'conservative' and 'ULF' = 'unique lifting of factorisations'):

Definition. A simplicial map is called *CULF* if it is cartesian on all active maps.

After some variations, we come to decalage, which features in the following important relationship between Segal and decomposition spaces:

Theorem 4.10. *A simplicial space X is a decomposition space if and only if both $\mathrm{Dec}_{\top}X$ and $\mathrm{Dec}_{\bot}X$ are Segal spaces, and the two comparison maps back to X are CULF.*

In Section 5 we introduce the incidence coalgebra associated to a decomposition space X. It is the slice ∞-category $\mathcal{S}_{/X_1}$, with the comultiplication map given by the span

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1.$$

We explain how a naive view of coassociativity provided the motivation for the decomposition-space axioms, but to formally establish the coassociativity result we first require more simplicial preliminaries, introduced in Section 6. In particular we introduce the twisted arrow category $\mathcal{D}$ of the category of finite ordinals, which is monoidal under *external sum*. We show that simplicial objects in a cartesian monoidal category can be characterised as monoidal functors on $\mathcal{D}$, and characterise decomposition spaces as those simplicial spaces whose extension to $\mathcal{D}$ preserves certain pullback squares.

In Section 7 the homotopy coassociativity of the incidence coalgebra is established exploiting the monoidal structure on $\mathcal{D}$:

Theorem 7.4. *For X a decomposition space, the slice ∞-category $\mathcal{S}_{/X_1}$ has the structure of a strong homotopy comonoid in the symmetric monoidal ∞-category **LIN**, with the*

comultiplication defined by the span

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1.$$

In Section 8 we show that CULF functors induce coalgebra homomorphisms. We also comment on a certain contravariant functoriality holding for simplicial maps that are equivalences in degree zero and are relatively Segal.

In Section 9 we introduce the notion of *monoidal decomposition space*, as a monoid object in the monoidal ∞-category of decomposition spaces and CULF maps. The incidence coalgebra of a monoidal decomposition space is naturally a *bialgebra*.

In Section 10 we give some basic examples to provide a taste of the breadth of applications. Further examples are expounded in detail in [25] and [26]. We begin with the example of finite sets and injections (which leads to the binomial coalgebra), to illustrate how decalage formalises reduction processes, and how the convolution product at the objective level is the Cauchy product of species. Coming to examples of decomposition spaces which are not Segal, we take a short look at Schmitt's Hopf algebra of graphs, and revisit the running example of the Butcher–Connes–Kreimer Hopf algebra, comparing with the construction of Dür [14]. Finally we consider the example of finite vector spaces, which leads to the general case of Waldhausen's $S_\bullet$-construction and Hall algebras. We establish that the $S_\bullet$ construction of an abelian category or a stable ∞-category is a decomposition space. This result was first proved by Dyckerhoff and Kapranov and constitutes a cornerstone in their work [17], [18], [16], [15], to which we refer for the remarkable richness of this class of examples.

Brief summary of the four sequels to this paper

The present paper originally formed the first two sections of a large manuscript [21] which has been split into altogether six papers of more manageable size. We briefly comment on the contents of the sequels.

The long appendix of [21] has become an independent paper [22] developing the necessary background on homotopy linear algebra.

In paper [23], the second in the trilogy, we introduce the notion of *complete* decomposition space in order to provide a notion of nondegeneracy necessary for the theory of Möbius inversion. In this context we can consider the linear functors Φ_n defined by spans $X_1 \leftarrow \vec{X}_n \to 1$, where $\vec{X}_n \subset X_n$ is the subspace of nondegenerate n-simplices, and prove the general Möbius inversion principle on the objective level:

$$\zeta * \Phi_{\text{even}} \;\simeq\; \varepsilon \;+\; \zeta * \Phi_{\text{odd}}.$$

Having established this, we analyse the finiteness conditions necessary to take cardinality and obtain numerical incidence algebras, and for the Möbius inversion principle to descend to these $\mathbb{Q}$-algebras. We identify two conditions on complete decomposition spaces: having *locally finite length* and being *locally finite*. Complete decomposition spaces that satisfy both finiteness conditions are called *Möbius decomposition spaces*. The first finiteness condition is equivalent to the existence of a certain *length filtration*, which is useful in applications. Although many examples coming from combinatorics do satisfy this condition, it is actually a rather strong condition, as witnessed by the following result:

Every decomposition space with length filtration is the left Kan extension of a semi-simplicial space.

This result holds for more general simplicial spaces that we term *stiff*, and we digress to establish this. We also consider an even weaker notion of *split* simplicial space, in which all face maps preserve nondegenerate simplices. This condition is the analogue of the condition for categories that identities are indecomposable, enjoyed in particular by Möbius categories in the sense of Leroux.

In paper [24] we come to what is perhaps the deepest theorem so far in our work. Lawvere showed in the 1980s that there is a Hopf algebra of Möbius intervals which contains the universal Möbius function [50]. This Hopf algebra, obtained from the collection of all iso-classes of Möbius intervals, is universal for incidence coalgebras of Möbius categories X, by virtue of the canonical coalgebra homomorphism from the incidence coalgebra of X sending an arrow in X to its factorisation interval. The universal Hopf algebra is not, however, the incidence coalgebra of a Möbius category.

We show that it *is* a decomposition space. We construct a (large) complete decomposition space U of all 'subdivided intervals', together with a canonical CULF classifying functor $X \to U$ for any complete decomposition space X. We prove that the space of CULF maps from X to U is connected, and conjecture that it is contractible.

If we also impose the relevant finiteness conditions, we obtain the result that the space of all Möbius intervals is a Möbius decomposition space. It follows that it admits a Möbius inversion formula with coefficients in finite ∞-groupoids or in $\mathbb{Q}$, and since every Möbius decomposition space admits a canonical CULF functor to it, we conclude that Möbius inversion in every incidence algebra is induced from this master formula.

In paper [25] we show that Schmitt coalgebras of restriction species [65] (such as graphs, matroids, posets, etc.) naturally define decomposition spaces. We also introduce a new notion of *directed restriction species*: whereas ordinary restriction species are presheaves of the category of finite sets and injections, directed restriction species are presheaves on the category of finite posets and convex maps. Examples covered by this notion are the Butcher–Connes–Kreimer bialgebra and the Manchon–Manin bialgebra of directed graphs. Both ordinary and directed restriction species are shown to be examples of a construction of decomposition spaces from what we call sesquicartesian fibrations, certain cocartesian fibrations over the category of finite ordinals that are also cartesian over convex maps.

In paper [26] we give examples from classical (and less classical) combinatorics. The first batch of examples, similar to the binomial posets of Doubilet–Rota–Stanley [13], are straightforward but serve to illustrate two key points: (1) the incidence algebra in question is realised directly from a decomposition space, without a reduction step, and reductions are typically given by CULF functors; (2) at the objective level, the convolution algebra is a monoidal structure of species (specifically: the usual Cauchy product of species, the shuffle product of $\mathbb{L}$-species, the Dirichlet product of arithmetic species, the Joyal–Street external product of q-species, and the Morrison 'Cauchy' product of q-species). In each of these cases, a power series representation results from taking cardinality. The next class of examples includes the Faà di Bruno bialgebra, the Butcher–Connes–Kreimer bialgebra of trees, with several variations, and similar structures on directed graphs (cf. Manchon [58] and Manin [59]). Another important class of examples is given by Hall algebras, cf. also 10.7 below. We conclude the paper by computing the Möbius function in a few cases, and commenting on certain cancellations that occur in the process of taking cardinality, substantiating that these cancellations are not possible at the objective level. This is related to the distinction between bijections and natural bijections.

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1. Preliminaries on ∞-groupoids and ∞-categories

1.1. Groupoids and ∞-groupoids. Although most of our motivating examples can be naturally formulated in the 2-category ***Grpd*** of 1-groupoids, we have chosen to work in the ∞-category $\mathcal{S}$ of ∞-groupoids. This is on one hand the natural generality of the theory, and on the other hand a considerable conceptual simplification: thanks to the monumental effort of Joyal [34], [35] and Lurie [55], the theory of ∞-categories has now reached a stage where it is just as workable as the theory of 1-groupoids — if not more! It also contains ordinary category theory: one regards a category $\mathcal{C}$ as an ∞-category via its nerve $N\mathcal{C}$. Some details can be found in 1.3 below. The philosophy is that, modulo a few homotopy *caveats*, one is allowed to think as if working in the category of sets. A recent forceful vindication of this philosophy is Homotopy Type Theory [69], in which a syntax that resembles set theory is shown to be a powerful language for general homotopy types.

A recurrent theme in the present work is to upgrade combinatorial constructions from sets to ∞-groupoids. To this end the first step consists in understanding the construction in abstract terms, often in terms of pullbacks and sums, and then the second step consists in copying over the construction to the ∞-setting. The ∞-category theory needed will be accordingly elementary, and it is our contention that it should be feasible to read this work without prior experience with ∞-groupoids or ∞-categories, simply by substituting the word 'set' for the word '∞-groupoid'. Even at the set level, our theory contributes interesting insight, revealing many constructions in the classical theory to be governed by very general principles proven useful also in other areas of mathematics.

The following short review of some basic aspects of ∞-categories should suffice for reading this paper and its sequels.

1.2. From posets to Rezk categories. A few remarks may be in order to relate the homotopy viewpoint with classical combinatorics. A 1-groupoid is the same as an ordinary groupoid, and a 0-groupoid is the same as a set. A (-1)-groupoid is the same as a truth value: up to equivalence there exist only two (-1)-groupoids, namely the contractible groupoid (a point) and the empty groupoid. A poset is essentially the same as a category in which all the mapping spaces are (-1)-groupoids. An ordinary category is a category in which all the mapping spaces are 0-groupoids. Hence the theory of incidence algebras of posets of Rota and collaborators can be seen as the (-1)-level of the theory. Cartier–Foata theory and Leroux theory take place at the 0-level. We shall see that in a sense the natural setting for combinatorics is the 1-level, since this level naturally takes into account that combinatorial structures can have symmetries. (From this viewpoint, it looks as if the classical theory compensates for working one level below the natural one by introducing reductions.) It is convenient to follow this ladder to infinity: the good notion of category with ∞-groupoids as mapping spaces is that of Rezk complete Segal space, also called Rezk category; this is the level of generality of the present work.

1.3. ∞-categories. By ∞-category we mean quasi-category [34]. These are simplicial sets satisfying the weak Kan condition: inner horns admit a filler. (An ordinary category

is a simplicial set in which every inner horn admits a *unique* filler.) This theory has already been developed by Joyal [34, 35] and Lurie [55]. The main point, Joyal's great insight, is that category theory can be generalised to quasi-categories, and that the results look the same, although to bootstrap the theory very different techniques are required. There are other implementations of ∞-categories, such as complete Segal spaces; see Bergner [7] for a survey. We will only use results that hold in all implementations, and for this reason we say ∞-category instead of referring explicitly to quasi-categories. Put another way, we shall only ever distinguish quasi-categories up to (categorical) equivalence, and most of the constructions rely on universal properties such as pullback, which in any case only determine the objects up to equivalence. Every 1-category is also a quasi-category via its nerve. In particular we have, for each $n \geq 0$, the ∞-category $\Delta[n]$ which is the nerve of the linearly ordered set $\{0 \leq 1 \leq \cdots \leq n\}$.

1.4. ∞-groupoids. An ∞-groupoid is an ∞-category in which all morphisms are invertible. We often say 'space' instead of ∞-groupoid, as they are a combinatorial substitute for topological spaces up to homotopy; for example, to each object x in an ∞-groupoid X, there are associated homotopy groups $\pi_n(X, x)$ for $n > 0$. In terms of quasi-categories, ∞-groupoids are precisely Kan complexes, i.e. simplicial sets in which every horn, not just the inner ones, admits a filler.

∞-groupoids play the role analogous to sets in classical category theory. In particular, for any two objects x, y in an ∞-category $\mathcal{C}$ there is (instead of a hom set) a *mapping space* $\mathrm{Map}_{\mathcal{C}}(x, y)$ which is an ∞-groupoid. ∞-categories form a (large) ∞-category denoted $\boldsymbol{Cat}_\infty$. ∞-groupoids form a (large) ∞-category denoted $\mathcal{S}$; it can be described explicitly as the coherent nerve of the (simplicially enriched) category of Kan complexes. Given two ∞-categories $\mathcal{D}$, $\mathcal{C}$, there is a *functor ∞-category* $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$. In terms of quasi-categories, the functor ∞-category is just the internal hom of simplicial sets. As an important example of a functor ∞-category, for a given ∞-category I we have the ∞-category of presheaves $\mathrm{Fun}(I^{\mathrm{op}}, \mathcal{S})$, and there is a Yoneda lemma that works as in the case of ordinary categories [55, Lemmas 5.1.5.2, 5.5.2.1]. Since $\mathcal{D}$ and $\mathcal{C}$ are objects in the ∞-category $\boldsymbol{Cat}_\infty$ we also have the ∞-groupoid $\mathrm{Map}_{\boldsymbol{Cat}_\infty}(\mathcal{D}, \mathcal{C})$, which can also be described as the maximal sub-∞-groupoid inside $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$.

1.5. Defining ∞-categories and sub-∞-categories. While in ordinary category theory one can define a category by saying what the objects and the arrows are (and how they compose), this from-scratch approach is more difficult for ∞-categories, as one would have to specify the simplices in all dimensions and verify the filler condition (that is, describe the ∞-category as a quasi-category). In practice, ∞-categories are constructed from existing ones by general constructions that automatically guarantee that the result is again an ∞-category, although the construction typically uses universal properties in such a way that the resulting ∞-category is only defined up to equivalence. To specify a sub-∞-category of an ∞-category $\mathcal{C}$, it suffices to specify a subcategory of the homotopy category of $\mathcal{C}$ (i.e. the category whose hom sets are π_0 of the mapping spaces of $\mathcal{C}$), and then pull back along the components functor. What this amounts to in practice is to specify the objects (closed under equivalences) and specifying for each pair of objects x, y a full sub-∞-groupoid of the mapping space $\mathrm{Map}_{\mathcal{C}}(x, y)$, also closed under equivalences, and closed under composition.

1.6. Diagrams. Since arrows in an ∞-category do not compose on the nose (one can talk about 'a' composite, not 'the' composite), the 1-categorical notion of commutative diagram in the naive sense is not appropriate here. Commutative triangle in an ∞-category $\mathcal{C}$ means instead 'object in the functor ∞-category $\mathrm{Fun}(\Delta[2], \mathcal{C})$': the 2-dimensional face of $\Delta[2]$ is mapped to a 2-cell in $\mathcal{C}$ mediating between the composite of the 01 and 12

edges and the long edge 02. Similarly, 'commutative square' means object in the functor ∞-category $\mathrm{Fun}(\Delta[1]\times\Delta[1],\mathcal{C})$. In general, 'commutative diagram of shape I' means object in $\mathrm{Fun}(I,\mathcal{C})$.

1.7. Adjoints, limits and colimits. There are notions of adjoint functors, limits and colimits, which behave essentially in the same way as these notions in ordinary category theory, and are characterised by universal properties up to equivalence — although to set up the theory and prove the theorems, much more technical proofs are required. Most importantly, the limit over an empty diagram defines *terminal object*, which may or may not exist. It does exist in $\mathcal{S}$ where it is the singleton set, or any contractible ∞-groupoid, in any case denoted 1.

1.8. Pullbacks and fibres. Central to this work is the notion of pullback: given two morphisms of ∞-groupoids $X\to B\leftarrow Y$, there is a commutative square

$$\begin{array}{ccc} X\times_B Y & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & B \end{array}$$

called the pullback, an example of a limit. It is defined via a universal property, as a terminal object in a certain auxiliary ∞-category consisting of commutative squares with sides $X\to B\leftarrow Y$. All formal properties of pullbacks of sets carry over to ∞-groupoids.

For an object b in an ∞-groupoid B, we denote by $\ulcorner b\urcorner : 1\to B$ the morphism that picks out b. Given a morphism of ∞-groupoids $p: X\to B$ and an object $b\in B$, the fibre of p over b is by definition the pullback

$$\begin{array}{ccc} X_b & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ 1 & \xrightarrow[\ulcorner b\urcorner]{} & B. \end{array}$$

1.9. Monomorphisms. A map of ∞-groupoids $f: X\to Y$ is a *monomorphism* when its fibres are (-1)-groupoids, that is, are either empty or contractible. In some respects, this notion behaves like for sets: for example, if f is a monomorphism, then there is a complement $Z:=Y\smallsetminus X$ such that $Y\simeq X+Z$. Hence a monomorphism is essentially an equivalence from X onto some connected components of Y. On the other hand, a crucial difference between sets and ∞-groupoids is that diagonal maps of ∞-groupoids are not in general monomorphisms. In fact $X\to X\times X$ is a monomorphism if and only if X is discrete (i.e. equivalent to a set).

1.10. Working in the ∞-category of ∞-groupoids, versus working in the model category of simplicial sets. When working with ∞-categories in terms of quasi-categories, one often works in the Joyal model structure on simplicial sets (whose fibrant objects are precisely the quasi-categories). This is a very powerful technique, exploited masterfully by Joyal [35] and Lurie [55], and essential to bootstrap the whole theory. In the present work, we can benefit from their work, and since our constructions are generally elementary, we do not need to invoke model structure arguments, but can get away with synthetic arguments. To illustrate the difference, consider the following version of the Segal condition (see 2.10 for details): we shall formulate it and use it by simply saying

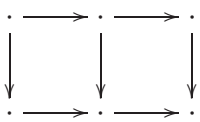
the natural square

$$\begin{array}{ccc} X_2 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_0 \end{array}$$

is a pullback. This is a statement taking place in the ∞-category of ∞-groupoids. A Joyal–Lurie style formulation would rather take place in the category of simplicial sets with the Joyal model structure and say something like *the natural map* $X_2 \to X_1 \times_{X_0} X_1$ *is an equivalence*. Here $X_1 \times_{X_0} X_1$ refers to the actual 1-categorical pullback in the category of simplicial sets, which does not coincide with X_2 on the nose, but is only naturally equivalent to it.

The following lemma extends a familiar result in 1-category theory and is used many times in our work.

Lemma 1.11. *If in a prism diagram of* ∞*-groupoids*



the outer rectangle and the right-hand square are pullbacks, then the left-hand square is a pullback.

A few remarks are in order. Note that we talk about a prism, i.e. a $\Delta[1] \times \Delta[2]$-diagram: although we have only drawn two of the squares of the prism, there is a third, whose horizontal sides are composites of the two indicated arrows. The triangles of the prism are not drawn either, because they are the fillers that exist by the axioms of quasi-categories. The proof follows the proof in the classical case, except that instead of saying 'given two arrows such and such, there exists a unique arrow making the diagram commute, etc.', one has to argue with equivalences of mapping spaces (or slice ∞-categories). See for example [55, Lemma 4.4.2.1]. for the dual case of pushouts.

1.12. Homotopy sums. In ordinary category theory, a colimit indexed by a discrete category (that is, a set) is the same thing as a sum (coproduct). For ∞-categories, the role of sets is played by ∞-groupoids. A colimit indexed by an ∞-groupoid is called a *homotopy sum*. In the case of 1-groupoids, these sums are ordinary sums weighted by inverses of symmetry factors. Their importance was stressed in [20]: by dealing with homotopy sums instead of ordinary sums, the formulae start to look very much like in the case of sets. For example, given a map of ∞-groupoids $X \to B$, we have that X is the homotopy sum of its fibres.

1.13. Slice categories. Maps of ∞-groupoids with codomain B form the objects of a slice ∞-category $\mathcal{S}_{/B}$, which behaves very much like a slice category in ordinary category theory. For example, for the terminal object 1 we have $\mathcal{S}_{/1} \simeq \mathcal{S}$. Again a word of warning is due: when we refer to the ∞-category $\mathcal{S}_{/B}$ we only refer to an object determined up to equivalence of ∞-categories by a certain universal property (Joyal's insight of defining slice categories as adjoint to a join operation [34]). In the Joyal model structure for quasi-categories, this category is represented by an explicit simplicial set. However, there is more than one possibility, depending on which explicit version of the join operator is employed (and of course these are canonically equivalent). In the works of Joyal and Lurie, these different versions are distinguished, and each has some technical advantages. In the present work we shall only need properties that hold for both, and we shall not distinguish them.

1.14. Families. A map of ∞-groupoids $X \to B$ can be interpreted as a family of ∞-groupoids parametrised by B, namely the fibres X_b. Just as for sets, the same family can also be interpreted as a presheaf $B \to \mathcal{S}$. Precisely, for each ∞-groupoid S, we have the fundamental equivalence

$$\mathcal{S}_{/B} \xrightarrow{\sim} \mathrm{Fun}(B, \mathcal{S}),$$

which takes a family $X \to B$ to the functor sending $b \mapsto X_b$. In the other direction, given a functor $F : B \to \mathcal{S}$, its colimit is the total space of a family $X \to B$.

1.15. Beck–Chevalley equivalence. Pullback along a map of ∞-groupoids $f : J \to I$ defines an ∞-functor $f^* : \mathcal{S}_{/I} \to \mathcal{S}_{/J}$. This functor is right adjoint to the functor $f_! : \mathcal{S}_{/J} \to \mathcal{S}_{/I}$ given by post-composing with f. (The latter construction requires some care: as composition is not canonically defined, one has to choose composites. One can check that different choices yield equivalent functors.) The following Beck–Chevalley rule (push-pull formula) [27] holds for ∞-groupoids: given a pullback square

$$\begin{array}{ccc} \cdot & \xrightarrow{f} & \cdot \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle q} \\ \cdot & \xrightarrow[g]{} & \cdot \end{array}$$

there is a canonical equivalence of functors

$$p_! \circ f^* \simeq g^* \circ q_!. \tag{4}$$

1.16. Symmetric monoidal ∞-categories. There is a notion of symmetric monoidal ∞-category, but it is technically more involved than the 1-category case, since in general higher coherence data has to be specified beyond the 1-categorical associator and MacLane pentagon condition. This theory has been developed in detail by Lurie [56, Ch.2], subsumed in the general theory of ∞-operads. In the present work, a few monoidal structures play an important role, but since they are directly induced by cartesian product, we have preferred to deal with them in an informal (and possibly not completely rigorous) way, with the same freedom as one deals with cartesian products in ordinary category theory. The following case is the most important for our theory. It is defined rigorously in [22], as a straightforward consequence of results of Lurie.

1.17. The symmetric monoidal ∞-category *LIN*. The ∞-categories of the form $\mathcal{S}_{/I}$ form the objects of a symmetric monoidal ∞-category ***LIN***, described in detail in [22]: the morphisms are the linear functors, meaning that they preserve homotopy sums, or equivalently indeed all colimits. Such functors are given by spans: the span

$$I \xleftarrow{p} M \xrightarrow{q} J$$

defines the linear functor

$$q_! \circ p^* : \mathcal{S}_{/I} \longrightarrow \mathcal{S}_{/J}.$$

The ∞-category ***LIN*** plays the role of the category of vector spaces (although to be strict about that interpretation, and in particular to entertain a notion of cardinality to embody the analogy, certain finiteness conditions should be imposed — these play no essential role in the present paper).

The symmetric monoidal structure on ***LIN*** is easy to describe on objects:

$$\mathcal{S}_{/I} \otimes \mathcal{S}_{/J} = \mathcal{S}_{/I \times J}$$

just as the tensor product of vector spaces with bases indexed by sets I and J is the vector space with basis indexed by $I \times J$. The neutral object is $\mathcal{S}$.

2. Simplicial preliminaries and Segal spaces

Our work relies heavily on simplicial machinery. We briefly review the notions needed, to establish conventions and notation.

2.1. The simplex category (the topologist's Delta). Recall that the 'simplex category' $\mathbb{\Delta}$ is the category whose objects are the nonempty finite ordinals

$$[k] := \{0, 1, 2, \dots, k\},$$

and whose morphisms are the monotone maps. These are generated by the coface maps $d^i : [n-1] \to [n]$, which are the monotone injective functions for which $i \in [n]$ is not in the image, and codegeneracy maps $s^i : [n+1] \to [n]$, which are monotone surjective functions for which $i \in [n]$ has a double preimage. We write $d^\perp := d^0$ and $d^\top := d^n$ for the outer coface maps.

2.2. Active and inert maps (generic and free maps). The category $\mathbb{\Delta}$ has an active-inert factorisation system. An arrow of $\mathbb{\Delta}$ is termed *active* (also called *generic*), and written $g : [m] \twoheadrightarrow [n]$, if it preserves end-points, $g(0) = 0$ and $g(m) = n$. An arrow is termed *inert* (also called *free*), and written $f : [m] \rightarrowtail [n]$, if it is distance preserving, $f(i+1) = f(i)+1$ for $0 \le i \le m-1$. The active maps are generated by the codegeneracy maps and the inner coface maps, and the inert maps are generated by the outer coface maps. Every arrow in $\mathbb{\Delta}$ factors uniquely as an active map followed by an inert map, as detailed below.

2.3. Background remarks. The notions of generic and free maps are general notions in category theory, introduced by Weber [72, 73], who extracted the notions from earlier work of Joyal [33]. A recommended entry point to the theory is Berger–Melliès–Weber [6]. The notions make sense for example whenever there is a cartesian monad on a presheaf category $\mathcal{C}$: in the Kleisli category, the free maps are those from $\mathcal{C}$, and the generic maps are those generated by the monad. In practice, this is restricted to a suitable subcategory of combinatorial nature. In the case at hand the monad is the free-category monad on the category of directed graphs, and $\mathbb{\Delta}$ arises as the restriction of the Kleisli category to the subcategory of non-empty linear graphs. Other important instances of generic-free factorisation systems are found in the category of rooted trees [40] (where the monad is the free-operad monad), the category of Feynman graphs [36] (where the monad is the free-modular-operad monad), the category of directed graphs [44] (where the monad is the free-properad monad), and Joyal's cellular category Θ [5] (where the monad is the free-omega-category monad). The more recent terminology 'active/inert' is due to Lurie [56], and is more suggestive for the role the two classes of maps play.

2.4. Amalgamated ordinal sum. The *amalgamated ordinal sum over* $[0]$ of two objects $[m]$ and $[n]$, denoted $[m] \vee [n]$, is given by the pushout of inert maps

$$\begin{array}{ccc} [0] & \xrightarrow{(d^\top)^n} & [n] \\ {\scriptstyle (d^\perp)^m}\downarrow & & \downarrow{\scriptstyle (d^\perp)^m} \\ [m] & \xrightarrow[(d^\top)^n]{} & [m] \vee [n] = [m+n] \end{array} \tag{5}$$

This operation is not functorial on all maps in $\mathbb{\Delta}$, but on the subcategory $\mathbb{\Delta}_{\mathrm{act}}$ of active maps it is functorial and defines a monoidal structure on $\mathbb{\Delta}_{\mathrm{act}}$ (dual to ordinal sum (cf. Lemma 6.2)).

The inert maps $f : [n] \rightarrowtail [m]$ are precisely the maps that can be written

$$f : [n] \rightarrowtail [a] \vee [n] \vee [b].$$

Every active map with source $[a] \vee [n] \vee [b]$ splits as

$$([a] \xrightarrow{g_1} [a']) \ \vee \ ([n] \xrightarrow{g} [k]) \ \vee \ ([b] \xrightarrow{g_2} [b'])$$

With these observations we can be explicit about the active-inert factorisation:

Lemma 2.5. *With notation as above, the active-inert factorisation of the composite of an inert map f followed by an active map $g_1 \vee g \vee g_2$ is given by*

$$\begin{array}{ccc} [n] & \xrightarrow{f} & [a] \vee [n] \vee [b] \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_1 \vee g \vee g_2} \\ [k] & \longrightarrow & [a'] \vee [k] \vee [b'] \end{array} \tag{6}$$

2.6. Identity-extension squares. A square (6) in which g_1 and g_2 are identity maps is called an *identity-extension square.*

Lemma 2.7. *Active and inert maps in $\mathbb{\Delta}$ admit pushouts along each other, and the resulting maps are again active and inert. In fact, active-inert pushouts are precisely the identity extension squares.*

$$\begin{array}{ccc} [n] & \longrightarrow & [a] \vee [n] \vee [b] \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \vee g \vee \mathrm{id}} \\ [k] & \longrightarrow & [a] \vee [k] \vee [b] \end{array}$$

These pushouts are fundamental to this work. We will define decomposition spaces to be simplicial spaces $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$ that send these pushouts to pullbacks.

The previous lemma has the following easy corollary.

Corollary 2.8. *Every codegeneracy map is a pushout (along an inert map) of $s^0 : [1] \to [0]$, and every active coface maps is a pushout (along an inert map) of $d^1 : [1] \to [2]$.*

2.9. Simplicial spaces and Segal spaces. Our main object of study will be simplicial ∞-groupoids subject to various exactness conditions, all formulated in terms of pullbacks. More precisely we work in the functor ∞-category

$$\mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{S}),$$

whose objects are functors $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$, from the ∞-category $\mathbb{\Delta}^{\mathrm{op}}$ to the ∞-category $\mathcal{S}$. We prefer to call these objects *simplicial spaces* rather than simplicial ∞-groupoids. As explained in 1.6 the simplicial identities for X are not strictly commutative squares but $\Delta[1] \times \Delta[1]$-diagrams in $\mathcal{S}$, hence come equipped with a homotopy between the two ways around in the square. But this is precisely the setting for pullbacks.

Consider a simplicial space $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$. We recall the *Segal maps*

$$(\partial_{0,1}, \dots, \partial_{r-1,r}) : X_r \longrightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1, \qquad r \geq 0,$$

where $\partial_{k-1,k} : X_r \to X_1$ is induced by the map $[1] \rightarrowtail [r]$ sending 0,1 to $k-1, k$.

A *Segal space* is a simplicial space satisfying the Segal condition, namely that the Segal maps are equivalences. (This is automatic for $r = 0, 1$ as the Segal map is just the identity map $X_r \to X_r$, by convention).

Lemma 2.10. *The following conditions are equivalent, for any simplicial space X:*

(1) *X satisfies the Segal condition,*

$$X_r \xrightarrow{\simeq} X_1 \times_{X_0} \cdots \times_{X_0} X_1 \qquad \textit{for all } r \geq 0.$$

(2) *The following square is a pullback for all* $p, q \geq r \geq 0$

$$\begin{array}{ccc} X_{p-r+q} & \xrightarrow{{d_0}^{p-r}} & X_q \\ {\scriptstyle {d_{p+1}}^{q-r}}\downarrow & \lrcorner & \downarrow{\scriptstyle {d_{r+1}}^{q-r}} \\ X_p & \xrightarrow[{d_0}^{p-r}]{} & X_r\,. \end{array}$$

(3) *The following square is a pullback for all* $n > 0$

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{d_\perp} & X_n \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_n & \xrightarrow[d_\perp]{} & X_{n-1}\,. \end{array}$$

(4) *The following square is a pullback for all* $p, q \geq 0$

$$\begin{array}{ccc} X_{p+q} & \xrightarrow{{d_0}^{p}} & X_q \\ {\scriptstyle {d_{p+1}}^{q}}\downarrow & \lrcorner & \downarrow{\scriptstyle {d_1}^{q}} \\ X_p & \xrightarrow[{d_0}^{p}]{} & X_0\,. \end{array}$$

Proof. It is straightforward to show that the Segal condition implies (2). Now (3) and (4) are special cases of (2). Also (3) implies (2): the pullback in (2) is a composite of pullbacks of the type given in (3). Finally one shows inductively that (4) implies the Segal condition (1). □

A simplicial map $F : Y \to X$ is *cartesian* on an arrow $[n] \to [k]$ in $\mathbb{\Delta}$ if the naturality square for F with respect to this arrow is a pullback.

Lemma 2.11. *If a simplicial map* $F : Y \to X$ *is cartesian on outer coface maps, and if* X *is a Segal space, then* Y *is a Segal space too.*

2.12. Rezk completeness. Let J denote the (ordinary) nerve of the groupoid generated by one isomorphism $0 \to 1$. A Segal space X is *Rezk complete* when the natural map

$$\mathrm{Map}(1, X) \to \mathrm{Map}(J, X)$$

(obtained by precomposing with $J \to 1$) is an equivalence of ∞-groupoids. It means that the space of identity arrows is equivalent to the space of equivalences. (See [61, Thm.6.2], [7] and [38].) A Rezk complete Segal space is also called a *Rezk category*.

2.13. Ordinary nerve. Let $\mathcal{C}$ be a small 1-category. The *nerve* of $\mathcal{C}$ is the simplicial set

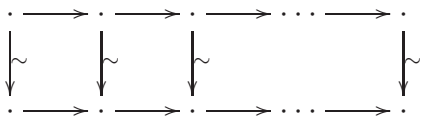
$$\begin{array}{rcl} N\mathcal{C} : \mathbb{\Delta}^{\mathrm{op}} & \longrightarrow & \boldsymbol{Set} \\ [n] & \longmapsto & \mathrm{Fun}([n], \mathcal{C}), \end{array}$$

where $\mathrm{Fun}([n], \mathcal{C})$ is the *set* of strings of n composable arrows. Subexamples of this are given by any poset or any monoid. The simplicial sets that arise like this are precisely those satisfying the Segal condition (which is strict in this context). If each set is regarded as a discrete ∞-groupoid, $N\mathcal{C}$ is thus a Segal space. In general it is not Rezk complete, since some object may have a nontrivial automorphism. As an example, if $\mathcal{C}$ is a one-object groupoid (i.e. a group), then inside $(N\mathcal{C})_1$ the space of equivalences is the whole set $(N\mathcal{C})_1$, but the degeneracy map $s_0 : (N\mathcal{C})_0 \to (N\mathcal{C})_1$ is not an equivalence (unless the group is trivial).

2.14. The fat nerve of an essentially small 1-category. In most cases it is more interesting to consider the *fat nerve*. Given a 1-category $\mathcal{C}$, the fat nerve of $\mathcal{C}$ is the simplicial 1-groupoid

$$\begin{array}{rcl} \mathbf{N}\mathcal{C} : \mathbb{\Delta}^{\mathrm{op}} & \longrightarrow & \boldsymbol{Grpd} \\ [n] & \longmapsto & \mathrm{Map}([n], \mathcal{C}), \end{array}$$

where $\mathrm{Map}([n], \mathcal{C})$ is the mapping space, defined as the maximal subgroupoid of the functor 1-category $\mathrm{Fun}([n], \mathcal{C})$. In other words, $(\mathbf{N}\mathcal{C})_n$ is the 1-groupoid whose objects are strings of n composable arrows in $\mathcal{C}$ and whose morphisms are isomorphisms of such strings:



It is straightforward to check the Segal condition, remembering that the pullbacks involved are homotopy pullbacks. For instance, the pullback $X_1 \times_{X_0} X_1$ has as objects strings of 'weakly composable' arrows, in the sense that the target of the first arrow is isomorphic to the source of the second, and a comparison isomorphism is specified. The Segal map $X_2 \to X_1 \times_{X_0} X_1$ is the inclusion of the subgroupoid consisting of strictly composable pairs. But any weakly composable pair is isomorphic to a strictly composable pair, and the comparison isomorphism is unique, hence the inclusion $X_2 \hookrightarrow X_1 \times_{X_0} X_1$ is an equivalence. Furthermore, the fat nerve is Rezk complete. Indeed, it is easy to see that inside X_1, the equivalences are the invertible arrows of $\mathcal{C}$. But any invertible arrow is equivalent to an identity arrow.

Note that if $\mathcal{C}$ is a category with no non-trivial isomorphisms (e.g. any Möbius category in the sense of Leroux) then the fat nerve coincides with the ordinary nerve, and if $\mathcal{C}$ is just equivalent to such a category then the fat nerve is level-wise equivalent to the ordinary nerve of any skeleton of $\mathcal{C}$.

2.15. Joyal–Tierney $t^!$ — the fat nerve of an ∞-category. The fat nerve construction is just a special case of the general construction $t^!$ of Joyal and Tierney [38], which is a functor from quasi-categories to complete Segal spaces, meaning specifically certain simplicial objects in the category of Kan complexes: given a quasi-category $\mathcal{C}$, the complete Segal space $t^!\mathcal{C}$ is given by

$$\begin{array}{rcl} \mathbb{\Delta}^{\mathrm{op}} & \longrightarrow & \boldsymbol{Kan} \\ [n] & \longmapsto & \big[[k] \mapsto \boldsymbol{sSet}(\Delta[n] \times \Delta'[k], \mathcal{C})\big] \end{array}$$

where $\Delta'[k]$ denotes (the ordinary nerve of) the groupoid freely generated by a string of k invertible arrows. They show that $t^!$ constitutes in fact a (right) Quillen equivalence between the simplicial sets with the Joyal model structure, and bisimplicial sets with the Rezk model structure.

Taking a more invariant viewpoint, talking about ∞-groupoids abstractly, the Joyal–Tierney $t^!$ functor associates to an ∞-category $\mathcal{C}$ the Rezk complete Segal space

$$\begin{array}{rcl} \mathbf{N}\mathcal{C} : \mathbb{\Delta}^{\mathrm{op}} & \longrightarrow & \mathcal{S} \\ [n] & \longmapsto & \mathrm{Map}(\Delta[n], \mathcal{C}). \end{array}$$

If $\mathcal{C}$ is a 1-category regarded as an ∞-category (via its ordinary nerve) this agrees with the fat nerve 2.14 regarded as a simplicial ∞-groupoid.

2.16. Fat nerve of bicategories with only invertible 2-cells. From a bicategory $\mathcal{C}$ with only invertible 2-cells one can get a simplicial bigroupoid by a construction analogous

to the fat nerve. (In fact, this can be viewed as the $t^!$ construction applied to the Duskin nerve of $\mathcal{C}$.) The *fat nerve* of a bicategory $\mathcal{C}$ is the complete Segal bigroupoid

$$\begin{aligned} \mathbf{N}\mathcal{C} : \mathbb{\Delta}^{\mathrm{op}} &\longrightarrow \boldsymbol{2Grpd} \\ [n] &\longmapsto \boldsymbol{PsFun}([n], \mathcal{C}), \end{aligned}$$

the bigroupoid of normalised pseudofunctors.

2.17. Monoidal groupoids. Important examples of the previous situation come from monoidal groupoids $(\mathcal{M}, \otimes, I)$. Consider $\mathcal{M}$ as a one-object bicategory $B\mathcal{M}$ with composition $\otimes$. This is often termed the classifying space of $\mathcal{M}$. Applying the fat nerve yields a Segal bigroupoid $\mathbf{N}(B\mathcal{M})$, as above, whose zeroth space is the classifying space of the full subgroupoid $\mathcal{M}^{\mathrm{eq}}$ spanned by the tensor-invertible objects.

The fat nerve construction can be simplified considerably in the case that $\mathcal{M}^{\mathrm{eq}}$ is contractible. This happens precisely when every tensor-invertible object is isomorphic to the unit object I and I admits no non-trivial automorphisms.

Proposition 2.18. *If $(\mathcal{M}, \otimes, I)$ is a monoidal groupoid such that $\mathcal{M}^{\mathrm{eq}}$ is contractible, then the Segal bigroupoid $\mathbf{N}B\mathcal{M}$ is equivalent to the monoidal nerve: the simplicial 1-groupoid*

$$\begin{aligned} \mathbb{\Delta}^{\mathrm{op}} &\longrightarrow \boldsymbol{Grpd} \\ [n] &\longmapsto \mathcal{M} \times \mathcal{M} \times \cdots \times \mathcal{M} \end{aligned} \tag{7}$$

where the outer face maps project away an outer factor, the inner face maps tensor together two adjacent factors, and the degeneracy maps insert a neutral object. This weakly simplicial 1-groupoid is strictly simplicial if and only if the monoidal structure of $\mathcal{M}$ is strict.

We have omitted the proof, to avoid going into 2-category theory.

Examples of monoidal groupoids satisfying the conditions of the proposition are the monoidal groupoid $(\mathbb{B}, +, 0)$ of finite sets and bijections, or the monoidal groupoid of vector spaces and linear isomorphisms under direct sum. In contrast, the monoidal groupoid of vector spaces and linear isomorphisms under tensor product is not of this kind, as the unit object $\Bbbk$ has many automorphisms. In this case the monoidal nerve (7) gives a Segal 1-groupoid that is not Rezk complete.

3. Decomposition spaces

Recall from Lemma 2.7 that active and inert maps in $\mathbb{\Delta}$ admit pushouts along each other.

3.1. Definition. A *decomposition space* is a simplicial space

$$X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$$

such that the image of any pushout diagram in $\mathbb{\Delta}$ of an active map g along an inert map f is a pullback of ∞-groupoids,

$$X\left(\begin{array}{ccc} [p] & \xleftarrow{g'} & [m] \\ {\scriptstyle f'}\uparrow & \lrcorner & \uparrow{\scriptstyle f} \\ [q] & \xleftarrow[g]{} & [n] \end{array}\right) \quad = \quad \begin{array}{ccc} X_p & \xrightarrow{g'^*} & X_m \\ {\scriptstyle f'^*}\downarrow & \lrcorner & \downarrow{\scriptstyle f^*} \\ X_q & \xrightarrow[g^*]{} & X_n. \end{array}$$

3.2. Remark. The notion of decomposition space can be seen as an abstraction of coalgebra, cf. §5 below: it is precisely the condition required to obtain a counital coassociative comultiplication on $\mathcal{S}_{/X_1}$.

The notion is equivalent to the notion of unital (combinatorial) 2-Segal space introduced by Dyckerhoff and Kapranov [17] (their Definition 2.3.1, Definition 2.5.2, Definition 5.2.2, Remark 5.2.4). Briefly, their definition goes as follows. For any triangulation T of a convex polygon with n vertices, there is induced a simplicial subset $\Delta^T \subset \Delta[n]$. A simplicial space X is called 2-Segal if, for every triangulation T of every convex n-gon, the induced map $\mathrm{Map}(\Delta[n], X) \to \mathrm{Map}(\Delta^T, X)$ is a weak homotopy equivalence. Unitality is defined separately in terms of pullback conditions involving degeneracy maps, similar to our (8) below. The equivalence between decomposition spaces and unital 2-Segal spaces follows from Proposition 2.3.2 of [17] which gives a pullback criterion for the 2-Segal condition.

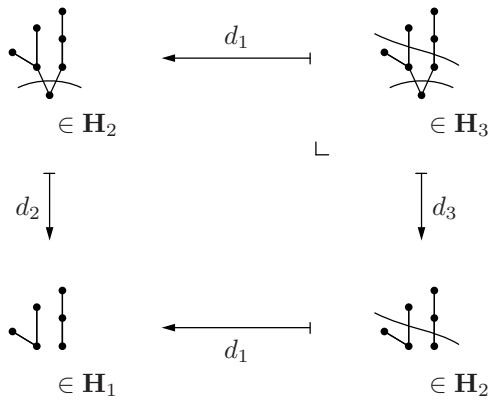
3.3. Running example: the decomposition space of rooted trees. We give an example, briefly previewed in 0.1, of a decomposition space which is not a Segal space, to illustrate the combinatorial meaning of the pullback condition: it is about structures that can be decomposed but not always composed. This example corresponds to the Butcher–Connes–Kreimer Hopf algebra of trees, as will shall see when we return to it in 5.2.

We define a simplicial groupoid $\mathbf{H}$: take $\mathbf{H}_1$ to be the groupoid of forests and, more generally, let $\mathbf{H}_k$ be the groupoid of forests with $k-1$ compatible admissible cuts, partitioning the forest into k layers (which may be empty), numbered from leaves to the root. Thus $\mathbf{H}_0$ is the trivial groupoid with one object: the empty forest.

These groupoids form a simplicial object: the outer face maps delete the bottom or the top layer, and the inner face maps join adjacent layers. The degeneracy maps insert an empty layer (i.e. duplicate an admissible cut). The simplicial identities are obviously verified, and one can easily check that $\mathbf{H}$ is in fact a decomposition space. Having the pullback

$$\begin{array}{ccc} \mathbf{H}_2 & \xleftarrow{d_1} & \mathbf{H}_3 \\ {\scriptstyle d_2}\downarrow & \llcorner & \downarrow{\scriptstyle d_3} \\ \mathbf{H}_1 & \xleftarrow[d_1]{} & \mathbf{H}_2 \end{array}$$

means any tree with two compatible admissible cuts ($\in \mathbf{H}_3$) is uniquely determined by a pair of elements in $\mathbf{H}_2$ with common image in $\mathbf{H}_1$ (under the indicated face maps). For example, the following picture represents elements corresponding to each other in the four groupoids.



The horizontal maps join layers one and two (i.e. forget the first admissible cut). The vertical maps discard the last layer. Clearly the diagram commutes. To reconstruct the tree with two admissible cuts (upper right-hand corner), most of the information is already available in the upper left-hand corner, namely the underlying tree and one of

the cuts. But the remaining cut is precisely available in the lower right-hand corner, and their common image in $\mathbf{H}_1$ says precisely how this missing piece of information is to be implanted.

Note that $\mathbf{H}$ is not a Segal space: in the diagram above there is a forest with a cut where the two layers do *not* determine the forest. Thus the square

$$\begin{array}{ccc} \mathbf{H}_2 & \xrightarrow{d_0} & \mathbf{H}_1 \\ {\scriptstyle d_2}\downarrow & & \downarrow{\scriptstyle d_1} \\ \mathbf{H}_1 & \xrightarrow[d_0]{} & \mathbf{H}_0 = 1 \end{array}$$

is not a pullback square as required by the Segal condition 2.10 (4) (with $p = q = 1$).

3.4. Alternative formulations of the pullback condition. To verify the conditions of the definition, it will in fact be sufficient to check a smaller collection of squares. On the other hand, the definition will imply that many other squares of interest are pullbacks too. The formulation in terms of active and inert maps is preferred both for practical reasons and for its conceptual simplicity compared to the smaller or larger collections of squares.

Recall from Lemma 2.7 that the active-inert pushouts used in the definition are just the identity extension squares,

$$\begin{array}{ccc} [n] & \xrightarrow{\quad g \quad} & [k] \\ \downarrow & & \downarrow \\ [a] \vee [n] \vee [b] & \xrightarrow[\mathrm{id} \vee g \vee \mathrm{id}]{} & [a] \vee [k] \vee [b]\,. \end{array}$$

Such a square can be written as a vertical composite of squares in which either $a = 1$ and $b = 0$, or vice-versa. In turn, since the active map g is a composite of inner coface maps $d^i : [m-1] \to [m]$ $(0 < i < m)$ and codegeneracy maps $s^j : [m+1] \to [m]$, these squares are horizontal composites of pushouts of a single active d^i or s^j along $d^\perp$ or $d^\top$. Thus, to check that X is a decomposition space, it is sufficient to check the following special cases are pullbacks, for $0 < i < n$ and $0 \le j \le n$:

$$\begin{array}{ccc} X_{1+n} & \xrightarrow{d_{1+i}} & X_n \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[d_i]{} & X_{n-1}, \end{array} \qquad \begin{array}{ccc} X_{n+1} & \xrightarrow{d_i} & X_n \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_n & \xrightarrow[d_i]{} & X_{n-1}, \end{array}$$

$$\begin{array}{ccc} X_{1+n} & \xrightarrow{s_{1+j}} & X_{1+n+1} \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[s_j]{} & X_{n+1}, \end{array} \qquad \begin{array}{ccc} X_{n+1} & \xrightarrow{s_j} & X_{n+1+1} \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_n & \xrightarrow[s_j]{} & X_{n+1}. \end{array} \tag{8}$$

The following proposition shows we can be more economic: instead of checking all $0 < i < n$ it is enough to check all $n \ge 2$ and *some* $0 < i < n$, and instead of checking all $0 \le j \le n$ it is enough to check the case $j = n = 0$.

Proposition 3.5. *A simplicial space X is a decomposition space if and only if the following diagrams are pullbacks*

$$\begin{array}{ccc} X_1 & \xrightarrow{s_1} & X_2 \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_0 & \xrightarrow[s_0]{} & X_1, \end{array} \qquad \begin{array}{ccc} X_1 & \xrightarrow{s_0} & X_2 \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_0 & \xrightarrow[s_0]{} & X_1, \end{array}$$

and the following diagrams are pullbacks for some choice of $i = i_n$, $0 < i < n$, for each $n \geq 2$:

$$\begin{array}{ccc} X_{1+n} & \xrightarrow{d_{1+i}} & X_n \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[d_i]{} & X_{n-1}, \end{array} \qquad \begin{array}{ccc} X_{n+1} & \xrightarrow{d_i} & X_n \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_n & \xrightarrow[d_i]{} & X_{n-1}. \end{array}$$

Proof. To see the non-necessity of the other degeneracy cases, observe that for $n > 0$, every degeneracy map $s_j : X_n \to X_{n+1}$ is the section of an *inner* face map d_i (where $i = j$ or $i = j + 1$). Now in the diagram

$$\begin{array}{ccccc} X_{1+n} & \xrightarrow{s_{1+j}} & X_{1+n+1} & \xrightarrow{d_{1+i}} & X_{1+n} \\ {\scriptstyle d_\perp}\downarrow & & \downarrow{\scriptstyle d_\perp} & & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[s_j]{} & X_{n+1} & \xrightarrow[d_i]{} & X_n, \end{array}$$

the horizontal composites are identities, so the outer rectangle is a pullback, and the right-hand square is a pullback since it is one of cases outer face with inner face. Hence the left-hand square, by Lemma 1.11, is a pullback too. The case $s_0 : X_0 \to X_1$ is the only degeneracy map that is not the section of an inner face map, so we cannot eliminate the two cases involving this map. The non-necessity of the other inner-face-map cases is the content of the following lemma. □

Lemma 3.6. *The following are equivalent for a simplicial space X.*

(1) *For each $n \geq 2$, the following diagram is a pullback for all $0 < i < n$:*

$$\begin{array}{ccc} X_{1+n} & \xrightarrow{d_{1+i}} & X_n \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[d_i]{} & X_{n-1} \end{array} \qquad \left(\text{resp.} \quad \begin{array}{ccc} X_{n+1} & \xrightarrow{d_i} & X_n \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_n & \xrightarrow[d_i]{} & X_{n-1} \end{array} \right).$$

(2) *For each $n \geq 2$, the above diagram is a pullback for some $0 < i < n$.*
(3) *For each $n \geq 2$, the following diagram is a pullback:*

$$\begin{array}{ccc} X_{1+n} & \xrightarrow{{d_2}^{n-1}} & X_2 \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[{d_1}^{n-1}]{} & X_1 \end{array} \qquad \left(\text{resp.} \quad \begin{array}{ccc} X_{n+1} & \xrightarrow{{d_1}^{n-1}} & X_2 \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_n & \xrightarrow[{d_1}^{n-1}]{} & X_1 \end{array} \right).$$

Proof. The hypothesised pullback in (2) is a special case of that in (1), and that in (3) is a horizontal composite of those in (2), since there is a unique active map $[1] \to [n]$ in $\mathbb{\Delta}$

for each n. The implication (3) $\Rightarrow$ (1) follows by Lemma 1.11 and the commutativity for $0 < i < n$ of the diagram

$$\begin{array}{ccccc} X_{1+n} & \xrightarrow{d_{1+i}} & X_n & \xrightarrow{{d_2}^{n-1}} & X_2 \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[d_i]{} & X_{n-1} & \xrightarrow[{d_1}^{n-1}]{} & X_1\,. \end{array}$$

Similarly for the 'resp.' case. □

Proposition 3.7. *Any Segal space is a decomposition space.*

Proof. Let X be Segal space. In the diagram ($n \geq 2$)

$$\begin{array}{ccccc} X_{n+1} & \xrightarrow{d_n} & X_n & \xrightarrow{d_\top} & X_{n-1} \\ {\scriptstyle d_\perp}\downarrow & & {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[d_{n-1}]{} & X_{n-1} & \xrightarrow[d_\top]{} & X_{n-2}, \end{array}$$

since the horizontal composites are equal to $d_\top \circ d_\top$, both the outer rectangle and the right-hand square are pullbacks by the Segal condition (2.10 (3)). Hence the left-hand square is a pullback. This establishes the third pullback condition in Proposition 3.5. In the diagram

$$\begin{array}{ccccc} X_1 & \xrightarrow{s_1} & X_2 & \xrightarrow{d_\top} & X_1 \\ {\scriptstyle d_\perp}\downarrow & & {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_0 & \xrightarrow[s_0]{} & X_1 & \xrightarrow[d_\top]{} & X_0, \end{array}$$

since the horizontal composites are identities, the outer rectangle is a pullback, and the right-hand square is a pullback by the Segal condition. Hence the left-hand square is a pullback, establishing the first of the pullback conditions in Proposition 3.5. The remaining two conditions of Proposition 3.5, those involving $d_\top$ instead of $d_\perp$, are obtained similarly by interchanging the roles of $\perp$ and $\top$. □

3.8. Remark. This result was also obtained by Dyckerhoff and Kapranov [17] (Propositions 2.3.3, 2.5.3, and 5.2.6).

Corollary 2.8 implies the following important property of decomposition spaces.

Lemma 3.9. *In a decomposition space X, every active face map is a pullback of $d_1 : X_2 \to X_1$, and every degeneracy map is a pullback of $s_0 : X_0 \to X_1$.*

Thus, even though the spaces in degree ≥ 2 are not fibre products of X_1 as in a Segal space, the higher active face maps and degeneracies are determined by 'unit' and 'composition',

$$X_0 \xrightarrow{s_0} X_1 \xleftarrow{d_1} X_2.$$

In $\mathbb{\Delta}^{\mathrm{op}}$ there are more pullbacks than those between active and inert. Diagram (5) in 2.4 is a pullback in $\mathbb{\Delta}^{\mathrm{op}}$ that is not preserved by all decomposition spaces, though it is preserved by all Segal spaces. On the other hand, certain other pullbacks in $\mathbb{\Delta}^{\mathrm{op}}$ are preserved by general decomposition spaces. We call them colloquially 'bonus pullbacks':

Lemma 3.10. *Let X be a decomposition space. For all $n \geq 3$ and all $0 < i < j < n$, the following squares of active face and degeneracy maps are pullbacks.*

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{d_i} & X_n \\ {\scriptstyle d_{j+1}}\downarrow & \lrcorner & \downarrow{\scriptstyle d_j} \\ X_n & \xrightarrow[d_i]{} & X_{n-1} \end{array} \qquad \begin{array}{ccc} X_{n-3} & \xrightarrow{s_{i-1}} & X_{n-2} \\ {\scriptstyle s_{j-2}}\downarrow & \lrcorner & \downarrow{\scriptstyle s_{j-1}} \\ X_{n-2} & \xrightarrow[s_{i-1}]{} & X_{n-1} \end{array}$$

$$\begin{array}{ccc} X_{n-1} & \xrightarrow{d_i} & X_{n-2} \\ {\scriptstyle s_j}\downarrow & \lrcorner & \downarrow{\scriptstyle s_{j-1}} \\ X_n & \xrightarrow[d_i]{} & X_{n-1} \end{array} \qquad \begin{array}{ccc} X_{n-1} & \xrightarrow{d_{j-1}} & X_{n-2} \\ {\scriptstyle s_{i-1}}\downarrow & \lrcorner & \downarrow{\scriptstyle s_{i-1}} \\ X_n & \xrightarrow[d_j]{} & X_{n-1}. \end{array}$$

Proof. We do the first square; the others are very similar. In the composite square

$$\begin{array}{ccccc} X_{n+1} & \xrightarrow{d_i} & X_n & \xrightarrow{{d_0}^i} & X_{n-i} \\ {\scriptstyle d_{j+1}}\downarrow & & \downarrow{\scriptstyle d_j} & & \downarrow{\scriptstyle d_{j-i}} \\ X_n & \xrightarrow[d_i]{} & X_{n-1} & \xrightarrow[{d_0}^i]{} & X_{n-1-i} \end{array}$$

the right-hand square is a pullback by the decomposition-space axiom. The composite horizontal maps are composites of bottom face maps, since ${d_0}^i \circ d_i = {d_0}^{i+1}$. Therefore also the composite square is a pullback, again by the decomposition-space axiom. But then the left-hand square is a pullback by the usual pullback argument of Lemma 1.11. □

3.11. Remark. Informally, the lemma states that a given degeneracy map s_i forms pullbacks against any other face or degeneracy map, except against d_{i+1} (and except against itself), and that a given active face map d_i forms pullbacks against any other face or degeneracy maps, except against s_{i-1} (and except against itself). The cases excluded will play a role to characterise important special classes of decomposition spaces: the pullback squares with s_i against itself characterise *complete* decomposition spaces [23, §2], while the pullback squares with s_i against d_{i+1} expresses the property of being *split* [23, §5].

3.12. Remark. In 1-category theory, all commuting squares of codegeneracy maps in $\mathbb{\Delta}$ are *absolute* pushouts (see Joyal–Tierney [39, Thm. 1.2.1]), hence in every simplicial *set* X the squares of Case 2 of Lemma 3.10 are pullbacks. However, those pushout squares are not absolute in the sense of ∞-categories, and not all simplicial *spaces* X satisfy this condition, which is a special feature of decomposition spaces.

4. CULF functors and decalage

A simplicial map $F : Y \to X$ is called *ULF (unique lifting of factorisations)* if it is a cartesian natural transformation on each active coface map of $\mathbb{\Delta}$. It is called *conservative* if it is cartesian on each codegeneracy map. It is called *CULF* if it is both conservative and ULF.

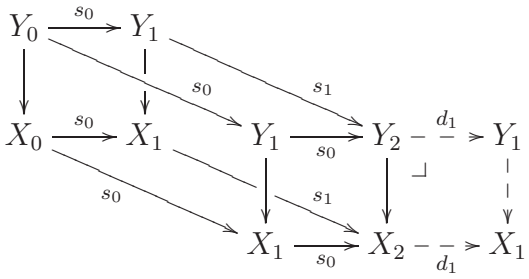
Lemma 4.1. *For a simplicial map $F : Y \to X$, the following are equivalent.*

1. *F is cartesian on each inner coface map and on each codegeneracy map (i.e. CULF).*
2. *F is cartesian on each active map of the form $[1] \to [n]$.*
3. *F is cartesian on all active maps.*

Proof. The implication (1) $\Rightarrow$ (2) is easy since the active map $[1] \to [n]$ factors as a sequence of inner coface maps (or is a codegeneracy map if $n = 0$). For the implication (2) $\Rightarrow$ (3), consider a general active map $[n] \to [m]$, and observe that if F is cartesian on the composite of active maps $[1] \to [n] \to [m]$ and also on the active map $[1] \to [n]$, then it is cartesian on $[n] \to [m]$ also, by Lemma 1.11. The implication (3) $\Rightarrow$ (1) is trivial. □

Proposition 4.2. *Any ULF map between decomposition spaces is conservative also.*

Proof. By Lemma 4.1(2) it is enough to prove that F is cartesian on active maps of the form $[1] \to [n]$. Since F is ULF, we already know it is cartesian on $[1] \to [n]$ for $n \geq 1$, so it remains to check the map $s^0 : [1] \to [0]$. In the diagram



the front square is a pullback since it is a section to the dashed square, which is a pullback since F is ULF. The top and bottom faces of the cube are pullbacks by Lemma 3.10, so the back face is a pullback square by the basic Lemma 1.11. □

Lemma 4.3. *A simplicial map between decomposition spaces is CULF if and only if it is cartesian on the active map* $[1] \to [2]$.

Proof. Suppose X, Y are decomposition spaces. By Lemma 3.9 all active face maps in X, Y are pullbacks of $d_1 : X_2 \to X_1$, $d_1 : Y_2 \to Y_1$. If $F : Y \to X$ is cartesian on the active map $[1] \to [2]$ it therefore follows by a basic pullback argument that it is cartesian on all active maps of $\underline{\Delta}$. The map F is thus ULF, and hence is CULF by Proposition 4.2. □

4.4. Remark. The notion of CULF can be seen as an abstraction of coalgebra homomorphism, cf. 8.2 below: 'conservative' corresponds to counit preservation, 'ULF' corresponds to comultiplicativity.

In the special case where X and Y are fat nerves of 1-categories, then the condition that the square

$$\begin{array}{ccc} Y_0 & \longrightarrow & Y_1 \\ \downarrow & \lrcorner & \downarrow \\ X_0 & \longrightarrow & X_1 \end{array}$$

be a pullback is precisely the classical notion of conservative functor (i.e. if $f(a)$ is invertible then already a is invertible).

Similarly, the condition that the square

$$\begin{array}{ccc} Y_1 & \longleftarrow & Y_2 \\ \downarrow & \llcorner & \downarrow \\ X_1 & \longleftarrow & X_2 \end{array}$$

be a pullback is an up-to-isomorphism version of the classical notion of ULF functor, implicit already in Content–Lemay–Leroux [12], and perhaps made explicit first by Lawvere [49]; it is equivalent to the notion of discrete Conduché fibration [30]. See Street [68] for the 2-categorical notion. In the case of the Möbius categories of Leroux, where there are no invertible arrows around, the two notions of ULF coincide.

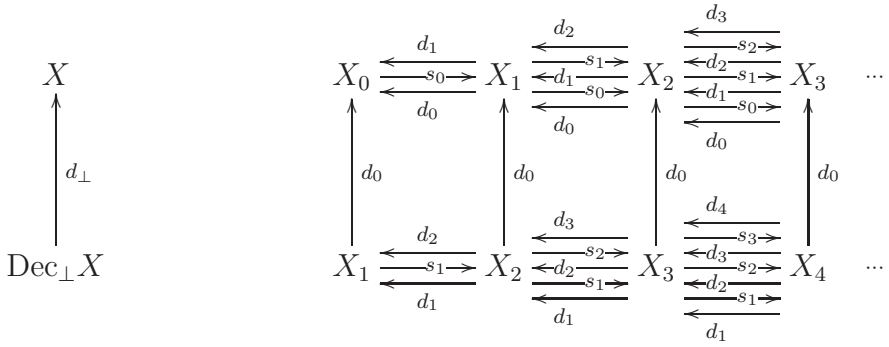
4.5. Example. Here is an example of a functor which is not CULF in Lawvere's sense (is not CULF on classical nerves), but which is CULF in the homotopical sense, on fat nerves. Namely, let $\boldsymbol{OI}$ denote the category of finite ordered sets and monotone injections. Let $\boldsymbol{I}$ denote the category of finite sets and injections. The forgetful functor $\boldsymbol{OI} \to \boldsymbol{I}$ is not CULF in the classical sense, because the identity monotone map $\underline{2} \to \underline{2}$ admits a factorisation in $\boldsymbol{I}$ that does not lift to $\boldsymbol{OI}$, namely the factorisation into two nontrivial transpositions. However, it is CULF in our sense, as can easily be verified by checking that the square

$$\begin{array}{ccc} \boldsymbol{OI}_1 & \longleftarrow & \boldsymbol{OI}_2 \\ \downarrow & \llcorner & \downarrow \\ \boldsymbol{I}_1 & \longleftarrow & \boldsymbol{I}_2 \end{array}$$

is a pullback of groupoids, by computing the fibres of the horizontal maps over a given monotone injection.

Lemma 4.6. *If X is a decomposition space and $F : Y \to X$ is CULF then also Y is a decomposition space.*

4.7. Decalage. (See Illusie [29, VI.1]). Given a simplicial space X (as in the top row of the following diagram) the *lower dec* $\mathrm{Dec}_\perp X$ is a new simplicial space (the bottom row of the diagram) obtained by deleting X_0 and shifting everything one place down, deleting also all d_0 face maps and all s_0 degeneracy maps. It comes equipped with a simplicial map, which we call the *dec map*, $d_\perp : \mathrm{Dec}_\perp X \to X$ given by the original d_0:



In fact $\mathrm{Dec}_\perp$ is a comonad on simplicial spaces, with the dec map $d_\perp$ as its counit.

Similarly the *upper dec* $\mathrm{Dec}_\top X$ is obtained by instead deleting, in each degree, the last face map $d_\top$ and the last degeneracy map $s_\top$. The deleted last face map becomes the *dec map* $d_\top \colon \mathrm{Dec}_\top X \to X$.

4.8. Slice interpretation. If $X = N\mathcal{C}$ is the strict nerve of a category $\mathcal{C}$ then there is a close relationship between the upper dec and the slice construction: $\mathrm{Dec}_\top X$ is the disjoint union of all (the nerves of) the slice categories of $\mathcal{C}$:

$$\mathrm{Dec}_\top X = \sum_{x \in X_0} N(\mathcal{C}_{/x}).$$

(In general it is a homotopy sum.)

Any individual slice category can be extracted from the upper dec, by exploiting that the upper dec comes with a canonical augmentation given by (iterating) the bottom face

map. The slices are the fibres of this augmentation:

$$\begin{array}{ccc} N\mathcal{C}_{/x} & \longrightarrow & \mathrm{Dec}_\top X \\ \downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ 1 & \xrightarrow[\ulcorner x \urcorner]{} & X_0\,. \end{array}$$

There is a similar relationship between the lower dec and the coslices.

Proposition 4.9. *If X is a decomposition space then $\mathrm{Dec}_\top X$ and $\mathrm{Dec}_\perp X$ are Segal spaces, and the dec maps $d_\top : \mathrm{Dec}_\top X \to X$ and $d_\perp : \mathrm{Dec}_\perp X \to X$ are CULF.*

Proof. We put $Y = \mathrm{Dec}_\top X$ and check the pullback condition 2.10 (3),

$$\begin{array}{ccc} Y_{n+1} & \xrightarrow{d_\perp} & Y_n \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ Y_n & \xrightarrow[d_\perp]{} & Y_{n-1}. \end{array}$$

This is the same as

$$\begin{array}{ccc} X_{n+2} & \xrightarrow{d_\perp} & X_{n+1} \\ {\scriptstyle d_{\top-1}}\downarrow & \lrcorner & \downarrow{\scriptstyle d_{\top-1}} \\ X_{n+1} & \xrightarrow[d_\perp]{} & X_n, \end{array}$$

and since here the vertically drawn maps (which with respect to Y are outer face maps) are inner face maps in X, this pullback square is one of the decomposition-space axioms. The CULF conditions say that the various $d_\top$ form pullbacks with all active maps in X. But this follows from the decomposition-space axiom for X. □

Theorem 4.10. *For a simplicial space $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$, the following are equivalent*

1. *X is a decomposition space*
2. *both $\mathrm{Dec}_\top X$ and $\mathrm{Dec}_\perp X$ are Segal spaces, and the respective dec maps back to X are CULF.*
3. *both $\mathrm{Dec}_\top X$ and $\mathrm{Dec}_\perp X$ are Segal spaces, and the respective dec maps back to X are conservative.*
4. *both $\mathrm{Dec}_\top X$ and $\mathrm{Dec}_\perp X$ are Segal spaces, and the following squares are pullbacks:*

$$\begin{array}{ccc} X_1 & \xrightarrow{s_1} & X_2 \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_0 & \xrightarrow[s_0]{} & X_1, \end{array} \qquad \begin{array}{ccc} X_1 & \xrightarrow{s_0} & X_2 \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_0 & \xrightarrow[s_0]{} & X_1. \end{array}$$

Proof. The implication (1) ⇒ (2) is just the preceding Proposition, and the implications (2) ⇒ (3) ⇒ (4) are specialisations. The implication (4) ⇒ (1) follows from Proposition 3.5. □

4.11. Remark. Dyckerhoff and Kapranov [17] (Theorem 6.3.2) obtain the result that a simplicial space is 2-Segal (i.e. a decomposition space except that there are no conditions imposed on degeneracy maps) if and only if both Decs are Segal spaces.

4.12. Right and left fibrations. A simplicial map $F : Y \to X$ is called a *right fibration* if it is cartesian on all bottom face maps $d_\perp$. This implies that it is also cartesian on all active maps (i.e. is CULF), as follows from an easy argument with the basic pullback

Lemma 1.11. The terminology is motivated by the case where Y and X are Segal spaces, in which case it corresponds to standard usage in the theory of ∞-categories. If X and Y are fat nerves of 1-categories, then 'right fibration' corresponds to groupoid fibration in the sense of Street [67]. Similarly, F is called a *left fibration* if it is cartesian on $d_\top$ (and consequently on all active maps also).

Proposition 4.13. *If $F : Y \to X$ is a CULF functor then $\mathrm{Dec}_\bot(F) : \mathrm{Dec}_\bot Y \to \mathrm{Dec}_\bot X$ is a right fibration. Similarly, $\mathrm{Dec}_\top(F) : \mathrm{Dec}_\top Y \to \mathrm{Dec}_\top X$ is a left fibration.*

Proof. It is clear that if F is CULF then so is $\mathrm{Dec}_\bot(F)$. The further claim is that $\mathrm{Dec}_\bot(F)$ is also cartesian on d_0. But d_0 was originally a d_1, and in particular was active, hence $\mathrm{Dec}_\bot(F)$ is cartesian on this map. □

5. The incidence coalgebra of a decomposition space

We now turn to the incidence coalgebra (with ∞-groupoid coefficients) associated to any decomposition space, and explain the origin of the decomposition-space axioms.

The *incidence coalgebra* of a decomposition space X will be a comonoid object in the symmetric monoidal ∞-category $\boldsymbol{LIN}$, and the underlying object is $\mathcal{S}_{/X_1}$. Since $\mathcal{S}_{/X_1} \otimes \mathcal{S}_{/X_1} = \mathcal{S}_{/X_1 \times X_1}$, and since linear functors are given by spans, to define a comultiplication functor is to give a span

$$X_1 \leftarrow M \to X_1 \times X_1 \,.$$

For any simplicial space X, we can consider the following structure maps on $\mathcal{S}_{/X_1}$.

5.1. Comultiplication and counit. The span

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1$$

defines a linear functor, the *comultiplication*

$$\begin{array}{rcl} \Delta : \mathcal{S}_{/X_1} & \longrightarrow & \mathcal{S}_{/(X_1 \times X_1)} \\ (A \xrightarrow{a} X_1) & \longmapsto & (d_2, d_0)_! \circ {d_1}^*(a). \end{array}$$

Likewise, the span

$$X_1 \xleftarrow{s_0} X_0 \xrightarrow{t} 1$$

defines a linear functor, the *counit*

$$\begin{array}{rcl} \varepsilon : \mathcal{S}_{/X_1} & \longrightarrow & \mathcal{S} \\ (A \xrightarrow{a} X_1) & \longmapsto & t_! \circ {s_0}^*(a). \end{array}$$

If X is the nerve of a category (for example, a poset) then X_2 is the set of all composable pairs of arrows. The comultiplication is just the formula (3) from the introduction

$$\Delta(f) \;=\; \sum_{b \circ a = f} a \otimes b,$$

and the counit is the classical counit, sending identity arrows to 1 and other arrows to 0.

5.2. Running example: the Hopf algebra of rooted trees. Recall from Example 3.3 that $\mathbf{H}$ is the decomposition space in which $\mathbf{H}_1$ is the groupoid of rooted forests and $\mathbf{H}_2$ is the groupoid of rooted forests with an admissible cut. Taking pullback along $d_1 : \mathbf{H}_2 \to \mathbf{H}_1$ is to consider all possible admissible cuts c of a given forest, and taking lowershriek along

$(d_2, d_0) : \mathbf{H}_2 \to \mathbf{H}_1 \times \mathbf{H}_1$ is to return the forests P_c and R_c found in the two layers on either side of the cut. We thus have a comultiplication functor

$$\begin{aligned} \mathcal{S}_{/\mathbf{H}_1} &\longrightarrow \mathcal{S}_{/\mathbf{H}_1} \otimes \mathcal{S}_{/\mathbf{H}_1} \\ (\ulcorner T \urcorner : 1 \to \mathbf{H}_1) &\longmapsto (\ulcorner P_c \urcorner : 1 \to \mathbf{H}_1) \otimes (\ulcorner R_c \urcorner : 1 \to \mathbf{H}_1) \end{aligned}$$

which is an objective version of the Butcher–Connes-Kreimer comultiplication, cf. 0.1(1).

5.3. Coassociativity of the comultiplication. The desired coassociativity diagram (which should commute up to equivalence)

$$\begin{array}{ccc} \mathcal{S}_{/X_1} & \xrightarrow{\Delta} & \mathcal{S}_{/X_1 \times X_1} \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta\otimes\mathrm{id}} \\ \mathcal{S}_{/X_1 \times X_1} & \xrightarrow[\mathrm{id}\otimes\Delta]{} & \mathcal{S}_{/X_1 \times X_1 \times X_1} \end{array}$$

is a diagram of linear functors defined by the spans in the *outline* of the following diagram.

$$\begin{array}{ccccc} X_1 & \xleftarrow{d_1} & X_2 & \xrightarrow{(d_2,d_0)} & X_1 \times X_1 \\ {\scriptstyle d_1}\uparrow & & {\scriptstyle d_1}\uparrow & \textcircled{1} & \uparrow{\scriptstyle d_1\times\mathrm{id}} \\ X_2 & \xleftarrow{d_2} & X_3 & \xrightarrow[(d_3,d_0d_0)]{} & X_2 \times X_1 \\ {\scriptstyle(d_2,d_0)}\downarrow & \textcircled{2} & \downarrow{\scriptstyle(d_2^2,d_0)} & & \downarrow{\scriptstyle(d_2,d_0)\times\mathrm{id}} \\ X_1 \times X_1 & \xleftarrow[\mathrm{id}\times d_1]{} & X_1 \times X_2 & \xrightarrow[\mathrm{id}\times(d_2,d_0)]{} & X_1 \times X_1 \times X_1 \end{array}$$

Coassociativity will follow from Beck–Chevalley equivalences if the *interior* part of the diagram can be established, with pullbacks ①, ② as indicated. Now the upper right-hand square ①, for example, will be a pullback if and only if its composite with the first projection is a pullback:

$$\begin{array}{ccccc} X_2 & \xrightarrow{(d_2,d_0)} & X_1 \times X_1 & \xrightarrow{\mathrm{pr}_1} & X_1 \\ {\scriptstyle d_1}\uparrow & \textcircled{1} & {\scriptstyle d_1\times\mathrm{id}}\uparrow & & \uparrow{\scriptstyle d_1} \\ X_3 & \xrightarrow[(d_3,d_0d_0)]{} & X_2 \times X_1 & \xrightarrow{\mathrm{pr}_1} & X_2 \end{array}$$

But demanding the outer rectangle to be a pullback is precisely one of the basic decomposition-space axioms. This argument is the origin of the decomposition-space axioms.

If one is just interested in coassociativity at the level of π_0 of the incidence coalgebra, this square and its twin are all that are needed. This was the case in the work of Toën [70] who dealt with the case where X is the Waldhausen $S_\bullet$-construction of a dg category, and in the work of Dyckerhoff and Kapranov [17] for exact ∞-categories.

5.4. Homotopy coherence of coassociativity. For coassociativity of the incidence coalgebra at the objective level, higher coherence has to be established, which will require the full decomposition-space axioms. To establish coassociativity in a strong homotopy sense we must deal on an equal footing with all 'reasonable' spans

$$\prod X_{n_j} \leftarrow \prod X_{m_j} \to \prod X_{k_i} \tag{9}$$

which could arise from composites of products of the comultiplication and counit. We therefore take a more abstract approach, relying on some more simplicial machinery.

6. Decomposition spaces as monoidal functors

In this section, in order to establish the homotopy coherent coassociativity of the incidence coalgebra, we study the twisted arrow category $\mathcal{D}$ of the category of finite ordinals, with a certain external tensor product $\oplus$. In Proposition 6.6 we show that simplicial objects in a cartesian monoidal category correspond to monoidal functors from $\mathcal{D}$, which enables us to characterise decomposition spaces as monoidal functors $X : (\mathcal{D}, \oplus) \to (\mathcal{S}, \times)$ satisfying an exactness condition. The purpose of this viewpoint is to deal with products of the form $\prod X_{k_i}$, as they appear in the 'reasonable spans' (9), to which we return in §7.

6.1. The category $\underline{\Delta}$ of finite ordinals (the algebraist's Delta). We denote by $\underline{\Delta}$ the category of all finite ordinals (including the empty ordinal) and monotone maps. Clearly $\Delta \subset \underline{\Delta}$, but this is not the most useful relationship between the two categories, and we will use a different notation for the objects of $\underline{\Delta}$, given by their cardinality, with an underline:

$$\underline{n} = \{1, 2, \dots, n\}.$$

The category $\underline{\Delta}$ is monoidal under ordinal sum

$$\underline{m} + \underline{n} := \underline{m+n},$$

with $\underline{0}$ as the neutral object.

Recall that Δ_{act} is the subcategory of Δ containing only the active maps, and that it is a monoidal category under amalgamated ordinal sum $\vee$ (cf. 2.4).

Lemma 6.2. *There is a canonical equivalence of monoidal categories (an isomorphism, if we consider the usual skeleta of these categories)*

$$\begin{array}{ccc} (\underline{\Delta}, +, \underline{0}) & \simeq & (\Delta_{\mathrm{act}}^{\mathrm{op}}, \vee, [0]) \\ \underline{k} & \leftrightarrow & [k] \end{array}$$

Proof. The map from left to right sends $\underline{k} \in \underline{\Delta}$ to

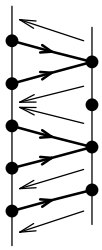
$$\mathrm{Hom}_{\underline{\Delta}}(\underline{k}, \underline{2}) \simeq [k] \in \Delta_{\mathrm{act}}^{\mathrm{op}}.$$

The map in the other direction sends $[k]$ to the ordinal

$$\mathrm{Hom}_{\Delta_{\mathrm{act}}}([k], [1]) \simeq \underline{k}.$$

In both cases, functoriality is given by precomposition. □

In both categories we can picture the objects as a line with some dots. The dots then represent the elements in $\underline{k}$, while the edges represent the elements in $[k]$; a map operates on the dots when considered a map in $\underline{\Delta}$ while it operates on the edges when considered a map in Δ_{act}. Here is a picture of a certain map $\underline{5} \to \underline{4}$ in $\underline{\Delta}$ and of the corresponding map $[5] \leftarrow [4]$ in Δ_{act}.



6.3. A twisted arrow category of $\underline{\Delta}$. Consider the category $\mathcal{D}$ whose objects are the arrows $\underline{n} \to \underline{k}$ of $\underline{\Delta}$ and whose morphisms (g, f) from $a : \underline{m} \to \underline{h}$ to $b : \underline{n} \to \underline{k}$ are

commutative squares

(10)
$$\begin{array}{ccc} \underline{m} & \xrightarrow{g} & \underline{n} \\ {\scriptstyle a}\downarrow & (g,f) & \downarrow{\scriptstyle b} \\ \underline{h} & \xleftarrow[f]{} & \underline{k}. \end{array}$$

That is, $\mathcal{D}^{\mathrm{op}}$ is the twisted arrow category [57, 4] of $\underline{\Delta}$.

6.4. Factorisation system on $\mathcal{D}$. There is a canonical factorisation system on $\mathcal{D}$: any morphism (10) factors uniquely as

$$\begin{array}{ccccc} \underline{m} & \xrightarrow{=} & \underline{m} & \xrightarrow{g} & \underline{n} \\ {\scriptstyle a=fbg}\downarrow & \varphi & \downarrow{\scriptstyle bg} & \gamma & \downarrow{\scriptstyle b} \\ \underline{h} & \xleftarrow[f]{} & \underline{k} & \xleftarrow[=]{} & \underline{k}. \end{array}$$

The maps $\varphi = (\mathrm{id}, f) : fb \to b$ in the left-hand class of the factorisation system are termed *segalic*,

(11)
$$\begin{array}{ccc} \underline{m} & \xrightarrow{=} & \underline{m} \\ {\scriptstyle fb}\downarrow & \varphi & \downarrow{\scriptstyle b} \\ \underline{h} & \xleftarrow[f]{} & \underline{k}. \end{array}$$

The maps $\gamma = (g, \mathrm{id}) : bg \to b$ in the right-hand class are termed *ordinalic* and may be identified with maps in various slice categories $\underline{\Delta}_{/\underline{k}}$

(12)
$$\begin{array}{ccc} \underline{m} & \xrightarrow{g} & \underline{n} \\ {\scriptstyle bg}\downarrow & \gamma & \downarrow{\scriptstyle b} \\ \underline{k} & \xleftarrow[=]{} & \underline{k}. \end{array}$$

6.5. External sum. Observe that $\underline{\Delta}$ is isomorphic to the subcategory of objects with target $\underline{k} = \underline{1}$, termed the *connected objects* of $\mathcal{D}$,

(13)
$$\underline{\Delta} \xrightarrow{=} \underline{\Delta}_{/\underline{1}} \xrightarrow{\subseteq} \mathcal{D}.$$

The ordinal sum operation in $\underline{\Delta}$ induces a monoidal operation in $\mathcal{D}$: the *external sum* $(\underline{n}{\to}\underline{k}) \oplus (\underline{n'}{\to}\underline{k'})$ of objects in $\mathcal{D}$ is their ordinal sum $\underline{n}+\underline{n'} \to \underline{k}+\underline{k'}$ as morphisms in $\underline{\Delta}$. The neutral object is $\underline{0} \to \underline{0}$. The inclusion functor (13) is not monoidal, but it is easily seen to be oplax monoidal by means of the codiagonal map $\underline{1}+\underline{1} \to \underline{1}$.

Each object $\underline{m} \xrightarrow{a} \underline{k}$ of $\mathcal{D}$ is an external sum of connected objects,

(14)
$$a \;=\; a_1 \oplus a_2 \oplus \cdots \oplus a_k \;=\; \bigoplus_{i \in \underline{k}} \left(\underline{m_i} \xrightarrow{a_i} \underline{1} \right),$$

where $\underline{m_i}$ is (the cardinality of) the fibre of a over $i \in \underline{k}$.

Any segalic map (11) and any ordinalic map (12) in $\mathcal{D}$ may be written uniquely as external sums

$$\varphi \;=\; \varphi_1 \oplus \varphi_2 \oplus \cdots \oplus \varphi_h \;=\; \bigoplus_{j\in\underline{h}} \begin{pmatrix} \underline{m}_j & \xrightarrow{=} & \underline{m}_j \\ \downarrow & \varphi_j & \downarrow b_j \\ \underline{1} & \longleftarrow & \underline{k}_j \end{pmatrix} \tag{15}$$

$$\gamma \;=\; \gamma_1 \oplus \gamma_2 \oplus \cdots \oplus \gamma_k \;=\; \bigoplus_{i\in\underline{k}} \left(\underline{m}_i \xrightarrow{\gamma_i} \underline{n}_i\right) \tag{16}$$

where each γ_i is a map in $\underline{\mathbb{\Delta}}_{/\underline{1}} = \underline{\mathbb{\Delta}}$.

In fact $\mathcal{D}$ is a universal monoidal category in the following sense.

Proposition 6.6. *For any cartesian category $(\mathcal{C}, \times, 1)$, there is an equivalence*

$$\mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C}) \;\simeq\; \mathrm{Fun}^{\otimes}((\mathcal{D}, \oplus, 0), (\mathcal{C}, \times, 1))$$

between the categories of simplicial objects X in $\mathcal{C}$ and of monoidal functors $\overline{X} : \mathcal{D} \to \mathcal{C}$. The correspondence between X and $\overline{X}$ is determined by following properties.

(a) The functors $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{C}$ and $\overline{X} : \mathcal{D} \to \mathcal{C}$ agree on the common subcategory $\mathbb{\Delta}^{\mathrm{op}}_{\mathrm{act}} \cong \underline{\mathbb{\Delta}}$,

$$\begin{array}{ccc} \mathbb{\Delta}^{\mathrm{op}}_{\mathrm{act}} & \hookrightarrow & \mathbb{\Delta}^{\mathrm{op}} \\ \cong\downarrow & \searrow^{X} & \\ \underline{\mathbb{\Delta}} & \hookrightarrow & \mathcal{D} \xrightarrow{\overline{X}} \mathcal{C}. \end{array}$$

(b) Let $(\underline{m} \xrightarrow{a} \underline{k}) = \bigoplus_i (\underline{m}_i \xrightarrow{a_i} \underline{1})$ be the external sum decomposition (14) of any object of $\mathcal{D}$, and denote by $f_i : [m_i] \rightarrowtail [m_1] \vee \ldots \vee [m_k] = [m]$ the canonical inert map in $\mathbb{\Delta}$, for $i \in \underline{k}$. Then

$$\overline{X}\begin{pmatrix} \underline{m} & \xrightarrow{=} & \underline{m} \\ \downarrow & \varphi & \downarrow a \\ \underline{1} & \longleftarrow & \underline{k} \end{pmatrix} \;=\; (X(f_1), \ldots, X(f_k)) \;:\; X_m \longrightarrow \prod_{i\in\underline{k}} X_{m_i}$$

and each $X(f_i)$ is the composite of $\overline{X}(\varphi)$ with the projection to X_i.

Proof. Given $\overline{X}$, property (a) says that there is a unique way to define X on objects and active maps. Conversely, given X, then for any object $a : \underline{m} \to \underline{k}$ in $\mathcal{D}$ we have

$$\overline{X}_a \;=\; \prod_{i\in\underline{k}} \overline{X}_{a_i} \;=\; \prod_{i\in\underline{k}} X_{m_i}$$

using (14), and for any ordinalic map γ we have

$$\overline{X}(\gamma) \;=\; \prod_{i\in\underline{k}} \overline{X}(\gamma_i) \;=\; \prod_{i\in\underline{k}} X(g_i)$$

using (16), where $g_i \in \mathbb{\Delta}^{\mathrm{op}}_{\mathrm{act}}$ corresponds to $\gamma_i \in \underline{\mathbb{\Delta}}$.

Thus we have a bijection between functors X defined on $\mathbb{\Delta}^{\mathrm{op}}_{\mathrm{act}}$ and monoidal functors $\overline{X}$ defined on the ordinalic subcategory of $\mathcal{D}$. Now we consider the inert and segalic maps. Given $\overline{X}$, property (b) says that for any inert map $f_r : [m_r] \to [m]$ we may define

$$X(f_r) = \left(X_m \xrightarrow{\overline{X}(\varphi)} \prod_{i\in\underline{k}} X_{m_i} \twoheadrightarrow X_{m_r}\right)$$

We may assume $k = 3$: given the factorisation

$$\varphi = \left(\begin{array}{ccccccc} \underline{m} & \xrightarrow{=} & \underline{m}_{<r} + \underline{m}_r + \underline{m}_{>r} & \xrightarrow{=} & \sum_{i\in\underline{k}} m_i \\ \downarrow & \varphi_2 & \downarrow & \varphi_1\oplus\mathrm{id}\oplus\varphi_3 & \downarrow \\ \underline{1} & \longleftarrow & \underline{3} & \longleftarrow & \underline{k} \end{array} \right)$$

one sees the value $X(f_r)$ is well defined from the following diagram

$$\begin{array}{ccccc} X_m & \xrightarrow{\overline{X}(\varphi_2)} & X_{m_{<r}} \times X_{m_r} \times X_{m_{>r}} & \xrightarrow{\overline{X}(\varphi_1)\times\mathrm{id}\times\overline{X}(\varphi_3)} & \prod_{i\in\underline{k}} X_{m_i} \\ & \searrow^{X(f_r)} & & & \downarrow \\ & & & & X_{m_r}\,. \end{array}$$

Functoriality of X on a composite of inert maps, say $[m_3] \rightarrowtail [\sum_2^4 m_i] \rightarrowtail [\sum_1^5 m_i]$, now follows from the diagram

$$\begin{array}{ccccc} X_{\sum_1^5 m_i} & & \longrightarrow & & \prod_1^5 X_{m_i} & \longrightarrow\!\!\!\!\rightarrow & X_{m_3} \\ & \searrow & & \nearrow & & \searrow & \uparrow \\ & & X_{m_1} \times X_{\sum_2^4 m_i} \times X_{m_5} & & & & \prod_2^4 X_{m_i} \\ & & & \searrow & & \nearrow & \\ & & & & X_{\sum_2^4 m_i} & & \end{array}$$

in which the first triangle commutes by functoriality of $\overline{X}$.

Conversely, given X, property (b) says how to define $\overline{X}$ on segalic maps with connected domain and hence, by (15), on all segalic maps. Functoriality of $\overline{X}$ on a composite of segalic maps, say $(\mathrm{id}, \underline{1} \leftarrow \underline{h} \leftarrow \underline{k})$, follows from functoriality of X:

$$\begin{array}{ccccc} X_m & \xrightarrow{(X([m_j]\rightarrowtail[m]))_{j\in\underline{h}}} & \prod_{j\in\underline{h}} X_{m_j} & \xrightarrow{\prod_{j\in\underline{h}}(X([m_i]\rightarrowtail[m_j]))_{i\in\underline{k}_j}} & \prod_{j\in\underline{h}}\prod_{i\in\underline{k}_j} X_{m_i} \\ & \searrow_{(X([m_i]\rightarrowtail[m]))_{i\in\underline{k}}} & & & \downarrow \\ & & & & \end{array}$$

It remains only to check that the construction of $\overline{X}$ from X (and of X from $\overline{X}$) is well defined on composites of ordinalic followed by segalic (inert followed by active) maps. One then has the mutually inverse equivalences required. Consider the factorisations in $\mathcal{D}$,

$$\begin{array}{ccccccc} \underline{m} & \xrightarrow{=} & \underline{m} & \xrightarrow{g} & \underline{n} \\ \downarrow & \varphi & \downarrow & \gamma & \downarrow \\ \underline{1} & \longleftarrow & \underline{k} & \xleftarrow{=} & \underline{k} \end{array} \quad = \quad \begin{array}{ccccccc} \underline{m} & \xrightarrow{g} & \underline{n} & \xrightarrow{=} & \underline{n} \\ \downarrow & \gamma' & \downarrow & \varphi' & \downarrow \\ \underline{1} & \xleftarrow{=} & \underline{1} & \longleftarrow & \underline{k}. \end{array}$$

To show that $\overline{X}$ is well defined, we must show that the diagrams

$$\begin{array}{ccccc} X_m & \xrightarrow{\overline{X}(\varphi)=(X(f_1),\ldots,X(f_k))} & \prod X_{m_i} & \longrightarrow\!\!\!\!\rightarrow & X_{m_r} \\ \downarrow{\scriptstyle \overline{X}(\gamma')=X(\tilde{g})} & & \downarrow{\scriptstyle \overline{X}(\gamma')=\prod X(\tilde{g}_i)} & & \downarrow{\scriptstyle X(\tilde{g}_r)} \\ X_n & \xrightarrow[\overline{X}(\varphi')=(X(f'_1),\ldots,X(f'_k))]{} & \prod X_{n_i} & \longrightarrow\!\!\!\!\rightarrow & X_{n_r}, \end{array}$$

commute for each r, where $\tilde{g}$, $\tilde{g}_i$ in $\mathbb{\Delta}_{\mathrm{act}}$ correspond to g, g_i in $\underline{\mathbb{\Delta}}$. This follows by functoriality of X, since $\tilde{g}$ restricted to n_r is the corestriction of $\tilde{g}_r$. Finally we observe that this

diagram, with $k = 3$ and $r = 2$, also serves to show that the construction of X from $\overline{X}$ is well defined on

$$\begin{array}{ccc} [m_1{+}m_2{+}m_3] & \xleftarrow{\quad f_2 \quad} & [m_2] \\ {\scriptstyle\tilde{g}}\uparrow & & \uparrow{\scriptstyle\tilde{g}_2} \\ [n_1{+}n_2{+}n_3] & \xleftarrow[\quad f_2' \quad]{} & [n_2] \end{array}$$

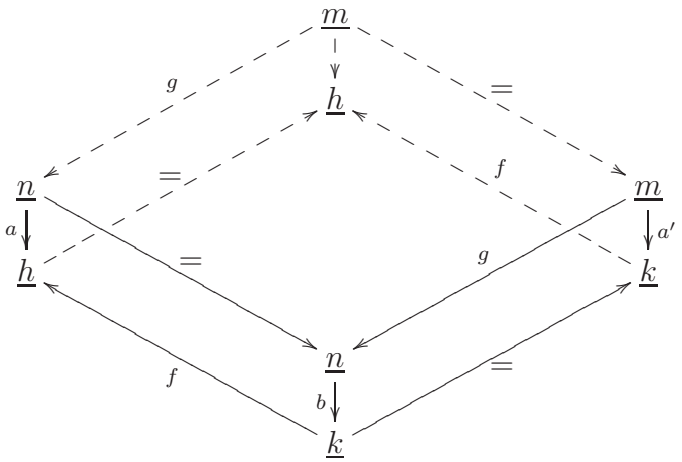
□

Lemma 6.7. *In the category $\mathcal{D}$, ordinalic and segalic maps admit pullback along each other, and the results are again maps of the same type.*

(This is a general fact about opposites of twisted arrow categories.)

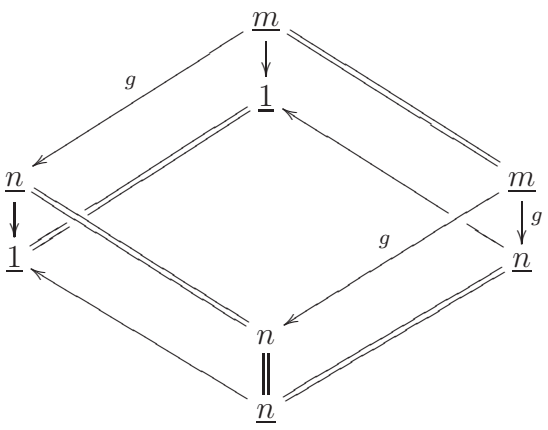
Proof. In the diagram below, the map from a to b is segalic (given essentially by the bottom map f) and the map from a' to b is ordinalic (given essentially by the top map g):

(17)



To construct the pullback, we are forced to repeat f and g, completing the squares with the corresponding identity maps. The connecting map in the resulting object is $fbg : \underline{m} \to \underline{h}$. It is clear from the presence of the four identity maps that this is a pullback. □

6.8. Examples. Every segalic-ordinalic pullback is the external sum of *connected* pullbacks, that is, those segalic-ordinalic pullbacks as above where $h = 1$. A segalic-ordinalic pullback over $b = \mathrm{id}$ is termed a *special* pullback. Any map $g : \underline{m} \to \underline{n}$ in $\underline{\Delta}$ defines canonically a *connected special* pullback:



We now have the following important characterisation of decomposition spaces.

Proposition 6.9. *For $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$ a simplicial space, the following are equivalent.*

(1) *X is a decomposition space.*
(2) *The corresponding monoidal functor $\overline{X} : \mathcal{D} \to \mathcal{S}$ preserves pullbacks of the kind described in Lemma 6.7.*
(3) *For every active map $g : [n] \twoheadrightarrow [m]$ the following square is a pullback*

$$\begin{array}{ccc} X_m & \longrightarrow & X_{m_1} \times \cdots \times X_{m_n} \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle g_1^* \times \cdots \times g_n^*} \\ X_n & \longrightarrow & X_1 \times \cdots \times X_1, \end{array}$$

where $g = g_1 \vee \cdots \vee g_n$ with $g_i : [1] \twoheadrightarrow [m_i]$, and the horizontal maps are induced by the inert maps $[m_i] \rightarrowtail [m_1] \vee \cdots \vee [m_n] = [m]$ and $[1] \rightarrowtail [1] \vee \cdots \vee [1] = [n]$.

Proof. Since $\overline{X}$ is monoidal, condition (2) is equivalent to the condition that the connected pullbacks are preserved. Now the $\overline{X}$-image of a connected pullback is a diagram

$$\begin{array}{ccc} X_m & \longrightarrow & X_{m_1} \times \cdots \times X_{m_k} \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & X_{n_1} \times \cdots \times X_{n_k}. \end{array}$$

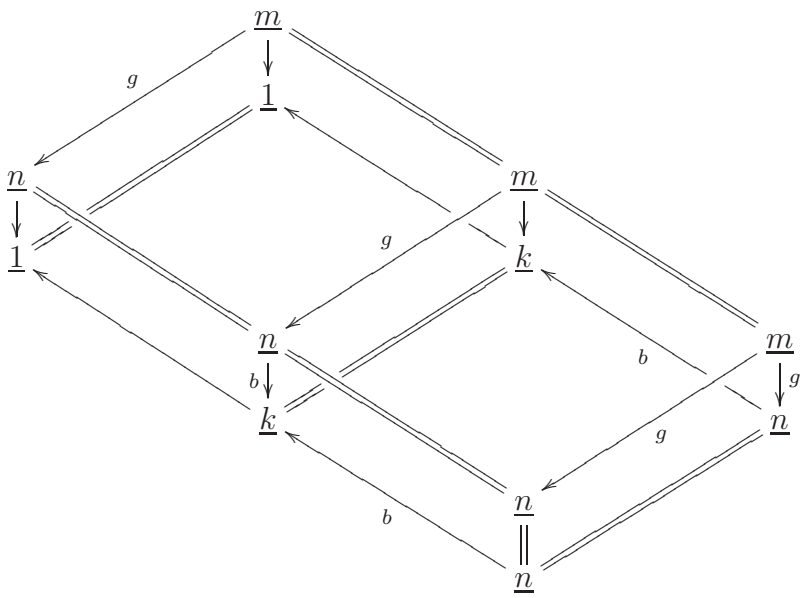
We can factor this into a vertical composite of such diagrams in which the map on the left is a single face or degeneracy map. Then the map on the right is a product of maps, one of which, say the ith factor, is again a single face or degeneracy map, and the rest are identities. To check if each of these new simpler squares are pullbacks we consider the projections onto the non-trivial factor:

$$\begin{array}{ccccc} X_m & \longrightarrow & X_{m_1} \times \cdots \times X_{m_k} & \longrightarrow & X_{m_i} \\ \downarrow & & \downarrow & & \downarrow \\ X_n & \longrightarrow & X_{n_1} \times \cdots \times X_{n_k} & \longrightarrow & X_{n_i}. \end{array}$$

But by construction of $\overline{X}$, the composite horizontal maps are precisely inert maps in the sense of the simplicial space X, and the vertical maps are precisely active maps in the sense that they are arbitrary maps in $\underline{\mathbb{\Delta}}$ and hence (in the other direction) active maps in $\mathbb{\Delta}$, under the duality in Lemma 6.2. Since the right-hand square is always a pullback, the standard pullback argument of Lemma 1.11 shows that the total square is a pullback (i.e. we have a decomposition space) if and only if the left-hand square is a pullback (i.e. the pullback condition on $\overline{X}$ is satisfied). This proves (1) $\Leftrightarrow$ (2).

The diagram in condition (3) is the image of a connected special pullback, as in Example 6.8, so (2) $\Rightarrow$ (3). Finally we show that (3) $\Rightarrow$ (2). As $\overline{X}$ is monoidal, (3) is equivalent to preservation of all special pullbacks, just as (2) is equivalent to preservation of just the connected pullbacks. Now any connected pullback (as in the northwest half of the following diagram) can be composed in a canonical way with a special pullback (the

southeast half of the diagram) to form a special connected pullback:



Hence, by the pullback Lemma 1.11, if special pullbacks are preserved then so are connected pullbacks. Note that (1) $\Leftrightarrow$ (3) can also be proved directly, without reference to $\mathcal{D}$. □

6.10. Example. If g is the bottom degeneracy map $\underline{3} \to \underline{2}$ in $\underline{\Delta}$, corresponding to the active map $d^1 : [2] \twoheadrightarrow [3]$ in Δ_{act}, the special connected pullback square in Example 6.8 is sent to

$$\begin{array}{ccc} X_3 & \xrightarrow{(d_3, d_0 d_0)} & X_2 \times X_1 \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle d_1 \times \mathrm{id}} \\ X_2 & \xrightarrow[(d_2, d_0)]{} & X_1 \times X_1, \end{array}$$

as in item (3) of the proposition. This is precisely square ① of the basic coassociativity argument in 5.3.

7. Proof of strong homotopy coassociativity of the incidence coalgebra

We proceed to establish that, if X is a decomposition space, then the comultiplication and counit defined in 5.1 make $\mathcal{S}_{/X_1}$ a coassociative and counital coalgebra in a strong homotopy sense.

We have more generally, for any $n \geq 0$, the generalised comultiplication maps

$$\Delta_n : \mathcal{S}_{/X_1} \longrightarrow \mathcal{S}_{/X_1 \times \cdots \times X_1} \tag{18}$$

defined by the spans

$$X_1 \leftarrow X_n \to X_1 \times \cdots \times X_1. \tag{19}$$

The case $n = 0$ is the counit map, $n = 1$ gives the identity, and $n = 2$ is the comultiplication Δ we considered above. The coassociativity result will follow from Lemma 7.2 and Proposition 7.3 that say that all combinations (composites and tensor products) of these generalised comultiplication maps are canonically equivalent whenever they have the same source and target. For this we exploit the category $\mathcal{D}$ introduced in §6, designed exactly to encode also products of the various spaces X_k.

7.1. Reasonable spans and reasonable linear functors. A *reasonable span* in $\mathcal{D}$ is a span $a \xleftarrow{g} m \xrightarrow{f} b$ in which g is ordinalic and f is segalic. Clearly the external sum of two reasonable spans is reasonable, and the composite of two reasonable spans is reasonable (by Lemma 6.7).

Let $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$ be a fixed decomposition space, and interpret it also as a monoidal functor $\overline{X} : \mathcal{D} \to \mathcal{S}$, via Proposition 6.6. A span in $\mathcal{S}$ of the form

$$\overline{X}_a \leftarrow \overline{X}_m \to \overline{X}_b$$

is called *reasonable* if it is induced by a reasonable span in $\mathcal{D}$.

Recall from 1.17 that a functor between slices of $\mathcal{S}$ is linear if it is defined by a span in $\mathcal{S}$. A linear functor is called *reasonable* if the defining span is reasonable. That is, a reasonable linear functor is a functor that is defined by a pullback along an ordinalic map followed by a lowershriek along a segalic map.

Lemma 7.2. *For X a decomposition space, tensor products and composites of reasonable linear functors are again reasonable linear functors.*

Proof. Cartesian products of reasonable spans in $\mathcal{S}$ are again reasonable since $\overline{X}$ is monoidal. Hence tensor products of reasonable linear functors are again reasonable. A composite of reasonable linear functors is induced by the composite reasonable span in $\mathcal{D}$, using Proposition 6.9. Hence reasonable linear functors are closed under composition. □

The interest in these notions is of course that the generalised comultiplication maps Δ_n of (18) are reasonable linear functors. In fact they are the 'only' reasonable linear functors:

Proposition 7.3. *Any reasonable linear functor*

$$\mathcal{S}_{/X_1} \longrightarrow \mathcal{S}_{/X_1 \times \cdots \times X_1}, \quad n \geq 0$$

is canonically equivalent to the nth comultiplication map Δ_n.

Proof. We have to show that the only reasonable span of the form $X_1 \leftarrow \coprod X_{m_i} \to X_1 \times \cdots \times X_1$ is (19). Indeed, the left leg must come from an ordinalic map, so since X_1 has only one factor, the middle object has also only one factor, i.e. is the image of $\underline{m} \to \underline{1}$. On the other hand, the right leg must be segalic, which forces $m = n$. □

Thus we have:

Theorem 7.4. *For X a decomposition space, the slice ∞-category $\mathcal{S}_{/X_1}$ has the structure of a strong homotopy comonoid in the symmetric monoidal ∞-category **LIN**, with the comultiplication defined by the span*

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1.$$

8. Functoriality of the incidence coalgebra construction

We have associated to any decomposition space X its incidence coalgebra with underlying slice ∞-category $\mathcal{S}_{/X_1}$. We now investigate the functoriality of this construction. Given a simplicial map $F : X \to Y$ between decomposition spaces there are induced linear functors

$$F_! : \mathcal{S}_{/X_1} \to \mathcal{S}_{/Y_1}, \qquad F^* : \mathcal{S}_{/Y_1} \to \mathcal{S}_{/X_1},$$

defined by postcomposition with, and pullback along, $F_1 : X_1 \to Y_1$. In this section we will give conditions on the simplicial map F for these linear functors to be coalgebra homomorphisms.

8.1. Covariant functoriality. It is an important feature of CULF maps that they induce coalgebra homomorphisms:

Lemma 8.2. *If $F : X \to Y$ is a CULF map between decomposition spaces then $F_! : \mathcal{S}_{/X_1} \to \mathcal{S}_{/Y_1}$ is a coalgebra homomorphism.*

Proof. In the diagram

$$\begin{array}{ccccc} X_1 & \xleftarrow{g} & X_n & \xrightarrow{f} & X_1^n \\ {\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_n} & & \downarrow{\scriptstyle F_1^n} \\ Y_1 & \xleftarrow[g']{} & Y_n & \xrightarrow[f']{} & Y_1^n \end{array}$$

the left-hand square is a pullback since F is conservative (case $n = 0$) and ULF (cases $n > 1$). Hence by the Beck–Chevalley condition we have an equivalence of functors $g'^* \circ F_{1!} \simeq F_{n!} \circ g^*$, and by postcomposing with $f'_!$ we arrive at the coalgebra homomorphism condition $\Delta'_n F_{1!} \cong F_{1!}^n \Delta_n$. □

8.3. Remark. If Y is a Segal space, then the statement can be improved to an if-and-only-if statement.

8.4. Example. An important class of CULF maps are the dec maps (Proposition 4.9):

$$d_\perp : \mathrm{Dec}_\perp X \to X \qquad \text{and} \qquad d_\top : \mathrm{Dec}_\top X \to X.$$

Many coalgebra maps in the classical theory of incidence coalgebras, notably reduction maps, are induced from decalage in this way, as we shall see in Section 10, and as further amplified in [26].

8.5. Contravariant functoriality. There is also a contravariant functoriality for certain simplicial maps, which we briefly explain, although it will not be needed elsewhere in this paper.

We will say that a functor between decomposition spaces $F : X \to Y$ is *relatively Segal* when for any 'spine' (i.e. an inclusion of the string of principal edges into a simplex)

$$\Delta[1] \coprod_{\Delta[0]} \cdots \coprod_{\Delta[0]} \Delta[1] \longrightarrow \Delta[n],$$

the space of fillers in the diagram

$$\begin{array}{ccc} \Delta[1] \coprod_{\Delta[0]} \cdots \coprod_{\Delta[0]} \Delta[1] & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \Delta[n] & \longrightarrow & Y \end{array}$$

is contractible. Note that the precise condition is that the following square is a pullback:

$$\begin{array}{ccc} \mathrm{Map}(\Delta[n], X) & \longrightarrow & \mathrm{Map}(\Delta[1] \coprod_{\Delta[0]} \cdots \coprod_{\Delta[0]} \Delta[1], X) \\ \downarrow & & \downarrow \\ \mathrm{Map}(\Delta[n], Y) & \longrightarrow & \mathrm{Map}(\Delta[1] \coprod_{\Delta[0]} \cdots \coprod_{\Delta[0]} \Delta[1], Y). \end{array}$$

This can be rewritten

$$\begin{array}{ccc} X_n & \longrightarrow & X_1 \times_{X_0} \cdots \times_{X_0} X_1 \\ \downarrow & & \downarrow \\ Y_n & \longrightarrow & Y_1 \times_{Y_0} \cdots \times_{Y_0} Y_1. \end{array} \tag{20}$$

(Hence the ordinary Segal condition for a simplicial space X is the case where Y is a point.)

Proposition 8.6. *If $F : X \to Y$ is relatively Segal and $F_0 : X_0 \to Y_0$ is an equivalence, then*

$$F^* : \mathcal{S}_{/Y_1} \to \mathcal{S}_{/X_1}$$

is naturally a coalgebra homomorphism, that is, there is a canonical equivalence of functors

$$\Delta_n F_1{}^* \simeq F_1{}^{*n} \Delta'_n.$$

where Δ_n and Δ'_n are the comultiplication maps for X and Y.

Proof. In the diagram

$$\begin{array}{ccccc} X_1 & \xleftarrow{g} & X_n & \xrightarrow{f} & X_1^n \\ {\scriptstyle F_1}\downarrow & & {\scriptstyle F_n}\downarrow & & \downarrow{\scriptstyle F_1^n} \\ Y_1 & \xleftarrow[g']{} & Y_n & \xrightarrow[f']{} & Y_1^n \end{array}$$

we claim that the right-hand square is a pullback for all n. In this case, by the Beck–Chevalley condition, we would have an equivalence of functors $f_! \circ F_n{}^* \simeq F_1^n{}^* \circ f'_!$, and by precomposing with g'^* we would arrive at the required coalgebra homomorphism condition.

The claim for $n = 0$ amounts to

$$\begin{array}{ccc} X_0 & \xrightarrow{f} & 1 \\ {\scriptstyle F_0}\downarrow & \lrcorner & \downarrow \\ Y_0 & \xrightarrow[f']{} & 1 \end{array}$$

which is precisely to say that F_0 is an equivalence. For $n > 1$ we can factor the square as

$$\begin{array}{ccccc} X_n & \xrightarrow{f} & X_1 \times_{X_0} \cdots \times_{X_0} X_1 & \longrightarrow & X_1 \times \cdots \times X_1 \\ {\scriptstyle F_n}\downarrow & \lrcorner & \downarrow{\scriptstyle F_1^n} & & \downarrow{\scriptstyle F_1^n} \\ Y_n & \xrightarrow[f']{} & Y_1 \times_{Y_0} \cdots \times_{Y_0} Y_1 & \longrightarrow & Y_1 \times \cdots \times Y_1 \end{array}$$

Here the left-hand square is a pullback since F is relatively Segal. It remains to prove that the right-hand square is a pullback. For the case $n = 2$, this whole square is the pullback of the square

$$\begin{array}{ccc} X_0 & \longrightarrow & X_0 \times X_0 \\ \downarrow & \lrcorner & \downarrow \\ Y_0 & \longrightarrow & Y_0 \times Y_0 \end{array}$$

which is a pullback precisely when F_0 is mono. But we have assumed it is even an equivalence. The general case $n > 2$ is easily obtained from the $n=2$ case by an iterative argument. □

8.7. Remarks. It should be mentioned that in order for contravariant functoriality to preserve finiteness as in [23], and hence restrict to coefficients in homotopy-finite ∞-groupoids, it is necessary furthermore to require that F is finite, cf. [22].

When both X and Y are Segal spaces, then the relative Segal condition is automatically satisfied, because the horizontal maps in (20) are then equivalences. In this case, we recover the classical results on contravariant functoriality by Content–Lemay–Leroux [12,

Prop. 5.6] and Leinster [51], where the only condition is that the functor be bijective on objects (in addition to requiring F finite, necessary since they work on the level of vector spaces).

9. Monoidal decomposition spaces

The ∞-category of decomposition spaces (as a full subcategory of simplicial spaces), has finite products. Hence there is a symmetric monoidal structure on the ∞-category $\boldsymbol{Dcmp}^{\mathrm{culf}}$ of decomposition spaces and CULF maps. We still denote this product as $\times$, although of course it is not the categorical product in $\boldsymbol{Dcmp}^{\mathrm{culf}}$.

9.1. Definition. A *monoidal decomposition space* is a monoid object (X, m, e) in $(\boldsymbol{Dcmp}^{\mathrm{culf}}, \times, 1)$. A *monoidal functor* between monoidal decomposition spaces is a monoid homomorphism in $(\boldsymbol{Dcmp}^{\mathrm{culf}}, \times, 1)$.

By this we mean a monoid in the homotopy sense, that is, an algebra in the sense of Lurie [56, §4.1]. We do not wish at this point to go into the technicalities of this notion, since in our examples, the algebra structure will be given simply by sums (or products).

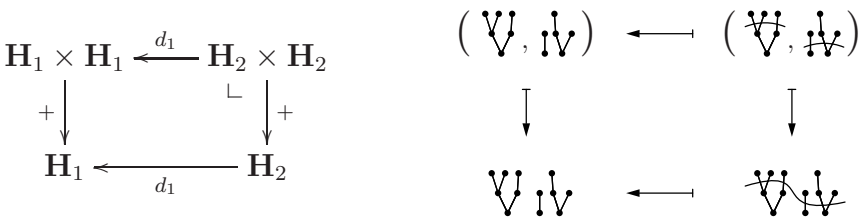
9.2. Example. Recall that a category $\mathcal{E}$ with finite sums is *extensive* [9] when the natural functor $\mathcal{E}_{/A} \times \mathcal{E}_{/B} \to \mathcal{E}_{/A+B}$ is an equivalence. The fat nerve of an extensive 1-category is a monoidal decomposition space. The multiplication is given by taking sum, the neutral object by the initial object, and the extensive property ensures precisely that, given a factorisation of a sum of maps, each of the maps splits into a sum of maps in a unique way.

A key example is the category of sets, or of finite sets. Certain subcategories, such as the category of finite sets and surjections, or the category of finite sets and bijections, inherit the crucial property $\mathcal{E}_{/A} \times \mathcal{E}_{/B} \simeq \mathcal{E}_{/A+B}$. They fail, however, to be extensive in the strict sense, since the monoidal structure $+$ in these cases is not the categorical sum. Instead they are examples of *monoidal extensive* categories, meaning a monoidal category $(\mathcal{E}, \boxplus, 0)$ for which $\mathcal{E}_{/A} \times \mathcal{E}_{/B} \to \mathcal{E}_{/A \boxplus B}$ is an equivalence (and it should then be required separately that also $\mathcal{E}_{/0} \simeq 1$). The fat nerve of a monoidal extensive 1-category is a monoidal decomposition space.

Lemma 9.3. *The lower dec of a monoidal decomposition space has again a natural monoidal structure, and the dec map is a monoidal functor. The same is true for the upper dec.*

9.4. Bialgebras. For a monoidal decomposition space the resulting coalgebra is also a bialgebra. Indeed, the fact that the monoid multiplication is CULF means that it induces a coalgebra homomorphism, and similarly with the unit. Note that in the bialgebras arising like this, the algebra and coalgebra are not on entirely equal footing: while the comultiplication is induced from internal, simplicial data in X, the multiplication is induced by extra structure (the monoidal structure), and is given by spans with trivial pullback component. In examples coming from combinatorics, the monoid structure will typically be given by categorical sum.

9.5. Running example: the Hopf algebra of rooted trees. The decomposition space $\mathbf{H}$ of Examples 3.3 and 5.2 has a canonical monoidal structure given by disjoint union. Recall that $\mathbf{H}_k$ is the groupoid of forests with $k-1$ compatible admissible cuts. The disjoint union of two such structures is given by taking the disjoint union of the underlying forests, with the cuts concatenated. This clearly defines a simplicial map from $\mathbf{H} \times \mathbf{H}$ to $\mathbf{H}$. To say that it is CULF is to establish that squares like this are pullbacks:



But this is clear: a pair of forests each with an admissible cut can be uniquely reconstructed if we know what the two underlying forests are (an element in $\mathbf{H}_1 \times \mathbf{H}_1$) and we know how the disjoint union is cut (an element in $\mathbf{H}_2$) — provided of course that we can identify the disjoint union of those two underlying forests with the underlying forest of the disjoint union (which is to say that the data agree down in $\mathbf{H}_1$). It follows that the resulting incidence coalgebra is also a bialgebra, in fact Hopf algebra.

Proposition 9.6. *If $F : X \to Y$ is a monoidal CULF functor between monoidal decomposition spaces, then $F_! : \mathcal{S}_{/X_1} \to \mathcal{S}_{/Y_1}$ is a bialgebra homomorphism.*

10. Examples

10.1. Injections and the monoidal groupoid of sets under sum. Let $\mathbf{I}$ be the fat nerve of the category of finite sets and injections, and let $\mathbf{B}$ be the monoidal nerve of the monoidal groupoid $(\mathbb{B}, +, 0)$ of finite sets and bijections (see 2.18). If we construct the incidence coalgebra of the decomposition space $\mathbf{I}$ and impose the equivalence relation 'having isomorphic complements' then, as observed by Dür [14], we obtain the binomial coalgebra. The binomial coalgebra also arises directly as the incidence coalgebra of $\mathbf{B}$, and Dür's reduction arises as a CULF functor from a decalage:

Lemma 10.2. *There is a levelwise equivalence of simplicial groupoids*

$$\mathrm{Dec}_\perp \mathbf{B} \xrightarrow{\simeq} \mathbf{I}$$

given in degree k by

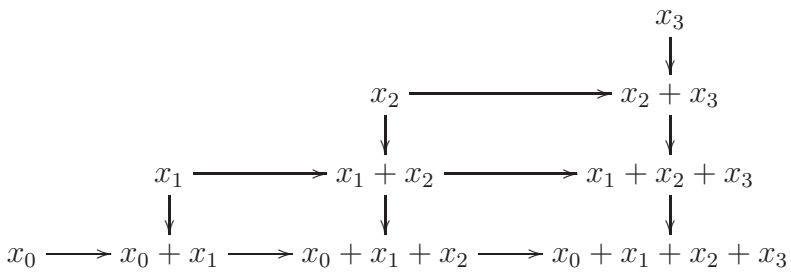
$$(x_0, \dots, x_k) \longmapsto [x_0 \subseteq x_0 + x_1 \subseteq \cdots \subseteq x_0 + \cdots + x_k],$$
$$(y_0, y_1 \setminus y_0, \dots, y_k \setminus y_{k-1}) \longmapsfrom [y_0 \subseteq y_1 \subseteq \cdots \subseteq y_k].$$

The canonical CULF functor

$$d_\perp : \mathrm{Dec}_\perp \mathbf{B} \to \mathbf{B}, \qquad (x_0, \dots, x_k) \mapsto (x_1, \dots, x_k)$$

defines the reduction map $r : \mathbf{I} \to \mathbf{B}$.

The equivalence may also be represented using diagrams reminiscent of those in Waldhausen's $S_\bullet$-construction, cf. 10.7 below. As an example, both groupoids $\mathbf{I}_3$ and $(\mathrm{Dec}_\perp \mathbf{B})_3 = \mathbf{B}_4$ are equivalent to the groupoid of diagrams



For $i < 3$, the face maps $d_i : \mathbf{I}_3 \to \mathbf{I}_2$ and $d_i : (\mathrm{Dec}_\perp \mathbf{B})_3 \to (\mathrm{Dec}_\perp \mathbf{B})_2$ act by erasing the column beginning x_i and the row beginning x_{i+1}. The top face map d_3 erases the last column. The face map $d_0 : \mathbf{B}_4 \to \mathbf{B}_3$ erases the bottom row.

Both $\mathbf{I}$ and $\mathbf{B}$ are monoidal decomposition spaces under disjoint union, and $\mathbf{I} \simeq \mathrm{Dec}_\perp \mathbf{B} \to \mathbf{B}$ is a monoidal functor by Lemma 9.3, inducing a homomorphism of bialgebras $\mathcal{S}_{/\mathbf{I}_1} \to \mathcal{S}_{/\mathbf{B}_1}$ by Proposition 9.6, which is the reduction map described by Dür [14].

Recall from [22, 2.3] that the functors $\ulcorner S \urcorner : 1 \to \mathbf{B}_1$, $1 \mapsto S$, play the role of a basis of $\mathcal{S}_{/\mathbf{B}_1}$ as S ranges over $\pi_0\mathbf{B}_1$. The comultiplication on $\mathcal{S}_{/\mathbf{B}_1}$ is

$$\Delta(\ulcorner S \urcorner) \;=\; \sum_{A+B=S} \ulcorner A \urcorner \otimes \ulcorner B \urcorner$$

(where the sum is more specifically over all $A, B \subset S$, $A \cup B = S$, $A \cap B = \varnothing$). The decomposition space $\mathbf{B}$ is *locally finite* (see [23, §7]), that is, $\mathbf{B}_1$ has finite automorphism groups and the maps $s_0 : \mathbf{B}_0 \to \mathbf{B}_1$ and $d_1 : \mathbf{B}_2 \to \mathbf{B}_1$ are finite. Therefore we can take cardinality (as in [22]), giving the classical binomial coalgebra spanned by symbols δ_n (the cardinality of $\ulcorner n \urcorner : 1 \to \mathbf{B}_1$) with

$$\Delta(\delta_n) = \sum_{a+b=n} \frac{n!}{a!\,b!}\, \delta_a \otimes \delta_b.$$

As a bialgebra we have $(\delta_1)^n = \delta_n$ and one recovers the comultiplication from $\Delta(\delta_n) = \big(\delta_0 \otimes \delta_1 + \delta_1 \otimes \delta_0\big)^n$.

The objective level is much richer. The linear dual [23] of $\mathcal{S}_{/\mathbf{B}_1}$ is $\mathcal{S}^{\mathbf{B}_1}$, the category of groupoid-valued species [2], [41], and its multiplication is the monoidal structure given by the convolution formula

$$(F * G)[S] = \sum_{A+B=S} F[A] \times G[B],$$

which is precisely the Cauchy product of species (see [1]). The cardinality of this monoidal category is the profinite-dimensional vector space $\mathbb{Q}^{\pi_0\mathbf{B}_1}$ with pro-basis given by the symbols δ^n (dual to δ_n), with convolution product

$$\delta^a * \delta^b = \frac{n!}{a!\,b!}\, \delta^{a+b}.$$

This is isomorphic to the algebra $\mathbb{Q}[[z]]$, where δ^n corresponds to $z^n/n!$ and the cardinality of a species F corresponds precisely to its exponential generating series [32].

10.3. Graphs. The following coalgebra of graphs is due to Schmitt [66, §12]. For a graph G with vertex set V (admitting multiple edges and loops), and a subset $U \subset V$, define $G|U$ to be the graph whose vertex set is U, and whose edges are those edges of G both of whose incident vertices belong to U. On the vector space spanned by iso-classes of graphs, define a comultiplication by the rule

$$\Delta(G) = \sum_{A+B=V} G|A \otimes G|B.$$

This coalgebra is the cardinality of the coalgebra of a decomposition space (cf. [26]), but not directly of a category. Indeed, define a simplicial groupoid with $\mathbf{G}_1$ the groupoid of graphs, and more generally let $\mathbf{G}_k$ be the groupoid of graphs with an ordered partition of the vertex set into k (possibly empty) parts. In particular, $\mathbf{G}_0 = 1$ is the contractible groupoid consisting only of the empty graph. The outer face maps delete the first or last part of the graph, and the inner face maps join adjacent parts. The degeneracy maps insert an empty part. It is clear that this is not a Segal space: a graph structure on a given set cannot be reconstructed from knowledge of the graph structure of the parts of the set, since chopping up the graph and restricting to the parts throws away all information about edges going from one part to another. One can easily check that it is a decomposition space. It is clear that the cardinality of the resulting coalgebra is Schmitt's coalgebra of

graphs. Note that disjoint union of graphs makes this into a bialgebra, in fact a Hopf algebra.

10.4. Running example: the Hopf algebra of rooted trees. Dür [14, IV §3] gave an incidence-coalgebra construction of the Butcher–Connes–Kreimer Hopf algebra by starting with the category $\mathcal{C}$ of rooted forests and root-preserving inclusions, generating a coalgebra (in our language the incidence coalgebra of the fat nerve of $\mathcal{C}$), and imposing the equivalence relation that identifies two root-preserving forest inclusions if their complement crowns are isomorphic forests. To be precise, this yields the opposite of the Butcher–Connes–Kreimer coalgebra, in the sense that the factors P_c and R_c are interchanged. To remedy this, one should use $\mathcal{C}^{\mathrm{op}}$ instead of $\mathcal{C}$.

As we have seen (in our running example 0.1, 3.3, 5.2, 9.5), we can obtain the Butcher–Connes–Kreimer Hopf algebra directly from the (monoidal) decomposition space (see [25] for more details) $\mathbf{H}$ where $\mathbf{H}_1$ denotes the groupoid of forests, and $\mathbf{H}_2$ is the groupoid of forests with an admissible cut, and so on. The relationship with Dür's construction is this (cf. [25]): the 'raw' decomposition space $\mathbf{N}(\mathcal{C}^{\mathrm{op}})$ is the decalage of $\mathbf{H}$, in close analogy with Lemma 10.2:

$$\mathrm{Dec}_{\top}\mathbf{H} \simeq \mathbf{N}(\mathcal{C}^{\mathrm{op}}).$$

Furthermore, under this identification, the dec map $\mathrm{Dec}_{\top}\mathbf{H} \to \mathbf{H}$, always a (monoidal) CULF functor, realises precisely Dür's reduction: on $(\mathbf{N}\mathcal{C}^{\mathrm{op}})_1 \to \mathbf{H}_1$ it sends a root-preserving forest inclusion to its crown, that is, its complement. More generally, on $(\mathbf{N}\mathcal{C}^{\mathrm{op}})_k \to \mathbf{H}_k$ it sends a sequence of forest inclusions $F_0 \subset F_1 \subset \cdots \subset F_k$ to

$$F_1 \smallsetminus F_0 \subset \cdots \subset F_k \smallsetminus F_0.$$

10.5. Restriction species, directed restriction species, and operads. The graph example $\mathbf{G}$ is an example of a decomposition space coming from a restriction species in the sense of Schmitt [65] (see also [1]). The tree example $\mathbf{H}$ is an example of a decomposition space coming from a *directed restriction species*, a notion introduced in [25], formalising the idea of considering only decompositions compatible with an underlying poset structure, as exemplified clearly by the notion of admissible cut.

While the decomposition space $\mathbf{H}$ is not a Segal space, it admits important variations which *are* Segal spaces, namely by replacing the combinatorial trees above by various kinds of *operadic* trees. These yield only bialgebras instead of Hopf algebras, but the Segal property has been exploited to good effect in various contexts [42], [20], [43], [45]. These examples are subsumed in the general notion of incidence bialgebra of an operad, cf. [26] and [46].

10.6. q-binomials: $\mathbb{F}_q$-vector spaces. Consider the finite field $\mathbb{F}_q$ with q elements. The q-binomial coalgebra (see Dür [14, 1.54]) may be obtained as a reduction of the incidence coalgebra of the category ***vect***, of finite-dimensional $\mathbb{F}_q$-vector spaces and $\mathbb{F}_q$-linear injections, by identifying two injections if their cokernels are isomorphic.

The same coalgebra can be obtained without reduction as follows. Put $\mathbf{V}_0 = 1$ (the contractible groupoid of 0-dimensional vector spaces), let $\mathbf{V}_1$ be the maximal subgroupoid of ***vect***, and let $\mathbf{V}_2$ be the groupoid of short exact sequences. The span

$$\begin{array}{ccccc} \mathbf{V}_1 & \longleftarrow & \mathbf{V}_2 & \longrightarrow & \mathbf{V}_1 \times \mathbf{V}_1 \\ E & \longleftarrow\!\shortmid & [E' \to E \to E''] & \longmapsto & (E', E'') \end{array}$$

(together with the span $\mathbf{V}_1 \leftarrow \mathbf{V}_0 \to 1$) defines a coalgebra on $\mathcal{S}_{/\mathbf{V}_1}$ which (after taking cardinality) is the q-binomial coalgebra, without further reduction. The groupoids and maps involved are part of a simplicial groupoid $\mathbf{V} : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$, namely the Waldhausen $S_\bullet$-construction of ***vect***, which is a decomposition space but not a Segal space (cf. 10.7

below). The lower dec of $\mathbf{V}$ is naturally equivalent to the fat nerve of ***vect***, and the comparison map d_0 is the reduction map of Dür.

Although we have postponed the notion of the dual incidence *algebra* to [23], we wish to mention that in this case the incidence algebra is $\mathcal{S}^{\mathbf{V}_1}$, which is the category of groupoid-valued q-species, and the convolution tensor product resulting from our constructions is the *external product of q-species* of Joyal–Street [37] (except that they work with vector-space valued q-species). A main contribution of [37] is to show that this monoidal structure carries a non-trivial braiding. This is a very interesting structure, which cannot be seen after taking cardinality.

One can compute explicitly (see [26]) the section coefficients of the comultiplication (or the convolution product) to find the Hall numbers

$$\frac{|\mathrm{SES}_{k,n,n-k}|}{\big|\mathrm{Aut}(\mathbb{F}_q^k)\big|\,\big|\mathrm{Aut}(\mathbb{F}_q^{n-k})\big|} = \binom{n}{k}_q,$$

where $\mathrm{SES}_{k,n,n-k}$ denotes the groupoid of short exact sequences of fixed vector spaces of dimensions k, n, and $n-k$.

This example is a special case of the following general construction with wide-ranging ramifications and consequences.

10.7. Waldhausen $S_\bullet$-construction of an abelian category [71]**.** We follow Lurie [56, Subsection 1.2.2] for the account of Waldhausen's $S_\bullet$-construction. For I a linearly ordered set, let $\mathrm{Ar}(I)$ denote the category of arrows in I: the objects are pairs of elements $i \le j$ in I, and the morphisms are relations $(i,j) \le (i',j')$ whenever $i \le i'$ and $j \le j'$. A *gap complex* in an abelian category $\mathcal{A}$ is a functor $F : \mathrm{Ar}(I) \to \mathcal{A}$ such that

(1) For each $i \in I$, the object $F(i,i)$ is a zero object.
(2) For every $i \le j \le k$, the associated diagram

$$\begin{array}{ccc} 0 \simeq F(j,j) & \rightarrowtail & F(j,k) \\ \uparrow & & \uparrow\!\!\uparrow \\ F(i,j) & \rightarrowtail & F(i,k) \end{array}$$

is a pushout (or equivalently a pullback).

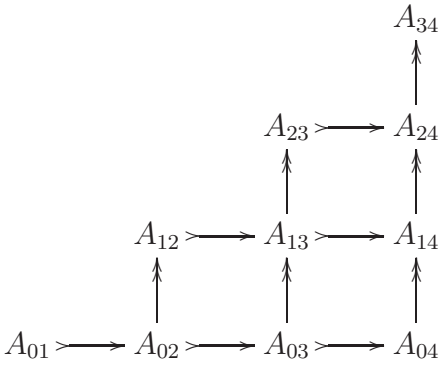
Since the pullback of a monomorphism is always a monomorphism, and the pushout of an epimorphism is always an epimorphism, it follows that automatically the horizontal maps are monomorphisms and the vertical maps are epimorphisms, as already indicated with the arrow typography. Altogether, it is just a convenient way of saying 'short exact sequence' or '(co)fibration sequence'.

Let $\mathrm{Gap}(I,\mathcal{A})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Ar}(I),\mathcal{A})$ consisting of the gap complexes, and $\mathrm{Gap}(I,\mathcal{A})^{\mathrm{eq}}$ its maximal subgroupoid. The assignment

$$[n] \mapsto \mathrm{Gap}([n],\mathcal{A})^{\mathrm{eq}}$$

defines a simplicial space $S_\bullet\mathcal{A} : \Delta^{\mathrm{op}} \to \mathcal{S}$, which by definition is the Waldhausen $S_\bullet$-construction of $\mathcal{A}$. Intuitively (or essentially), the groupoid $\mathrm{Gap}([n],\mathcal{A})^{\mathrm{eq}}$ has as objects

staircase diagrams like the following (picturing $n = 4$).



Informally, the face map d_i 'erases' all objects containing an i index. The degeneracy map s_i repeats the ith row and the ith column.

A string of composable monomorphisms $(A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n)$ determines, up to canonical isomorphism, short exact sequences $A_{ij} \rightarrowtail A_{ik} \twoheadrightarrow A_{jk} = A_{ij}/A_{ik}$ with $A_{0i} = A_i$. Hence the whole diagram can be reconstructed up to isomorphism from the bottom row. (Similarly, since epimorphisms have uniquely determined kernels, the whole diagram can also be reconstructed from the last column.)

We have

$$d_0(A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n) = (A_2/A_1 \rightarrowtail \cdots \rightarrowtail A_n/A_1)$$
$$s_0(A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n) = (0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n)$$

The simplicial maps d_i, s_i for $i \geq 1$ are more straightforward: the simplicial set $\mathrm{Dec}_\perp(S_\bullet\mathcal{A})$ is just the fat nerve of $\mathrm{mono}(\mathcal{A})$.

Lemma 10.8. *The projection $S_{n+1}\mathcal{A} \to \mathrm{Map}([n], \mathrm{mono}(\mathcal{A}))$ is an equivalence. Similarly the projection $S_{n+1}\mathcal{A} \to \mathrm{Map}([n], \mathrm{epi}(\mathcal{A}))$ is an equivalence.*

More precisely:

Proposition 10.9. *These equivalences assemble into levelwise simplicial equivalences*

$$\mathrm{Dec}_\perp(S_\bullet\mathcal{A}) \simeq \mathbf{N}(\mathrm{mono}(\mathcal{A}))$$
$$\mathrm{Dec}_\top(S_\bullet\mathcal{A}) \simeq \mathbf{N}(\mathrm{epi}(\mathcal{A})).$$

Theorem 10.10. *The Waldhausen $S_\bullet$-construction of an abelian category $\mathcal{A}$ is a decomposition space.*

Proof. For convenience we write $S_\bullet\mathcal{A}$ simply as $S_\bullet$. The previous proposition already implies that the two Decs of $S_\bullet$ are Segal spaces. By Theorem 4.10, it is therefore enough to establish that the squares

$$\begin{array}{ccc} S_1 & \xrightarrow{s_1} & S_2 \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ S_0 & \xrightarrow[s_0]{} & S_1 \end{array} \qquad \begin{array}{ccc} S_1 & \xrightarrow{s_0} & S_2 \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle d_2} \\ S_0 & \xrightarrow[s_0]{} & S_1 \end{array}$$

are pullbacks. Since a zero object has no nontrivial automorphisms, $s_0 : S_0 \to S_1$ is a monomorphism of groupoids, given by the inclusion of the groupoid of zero objects into $S_1 = \mathcal{A}^{\mathrm{iso}}$. The map $d_0 : S_2 \to S_1$ sends a monomorphism to its cokernel, and its fibre over a zero object is the full subgroupoid of S_2 consisting of those monomorphisms whose cokernel is zero. Clearly these are precisely the isos, so the fibre is just $\mathcal{A}^{\mathrm{iso}} = S_1$. The other pullback square is established similarly, but arguing with epimorphisms instead of monomorphisms. □

10.11. Remark. Waldhausen's $S_\bullet$-construction was designed for more general categories than abelian categories, namely what are now called Waldhausen categories, where the cofibrations play the role of the monomorphisms, but where there is no stand-in for the epimorphisms. The theorem does not generalise to Waldhausen categories in general, since in that case $\mathrm{Dec}_\top(S_\bullet)$ is not necessarily a Segal space of any class of arrows.

10.12. Waldhausen $S_\bullet$ of a stable ∞-category. The same construction works in the ∞-setting, by considering stable ∞-categories instead of abelian categories. Let $\mathcal{A}$ be a stable ∞-category (see Lurie [56, §1.1.1]). Just as in the abelian case, the assignment

$$[n] \mapsto \mathrm{Gap}([n], \mathcal{A})^{\mathrm{eq}}$$

defines a simplicial space $S_\bullet\mathcal{A} : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$, which by definition is the Waldhausen $S_\bullet$-construction of $\mathcal{A}$. Note that in the case of a stable ∞-category, in contrast to the abelian case, every map can arise as either horizontal or vertical arrow in a gap complex. Hence the role of monomorphisms (cofibrations) is played by all maps, and the role of epimorphisms is also played by all maps.

Lemma 10.13. *Suppose $\mathcal{A}$ is a stable ∞-category. For each $k \in \mathbb{N}$, the two projection functors $S_{k+1}\mathcal{A} \to \mathrm{Map}(\Delta[k], \mathcal{A})$ are equivalences.*

From the description of the face and degeneracy maps, the following more precise result follows readily, comparing with the fat nerves in the sense of 2.15:

Proposition 10.14. *For $\mathcal{A}$ a stable ∞-category, we have natural (levelwise) simplicial equivalences*

$$\mathrm{Dec}_\bot(S_\bullet\mathcal{A}) \simeq \mathbf{N}\mathcal{A}$$

$$\mathrm{Dec}_\top(S_\bullet\mathcal{A}) \simeq \mathbf{N}\mathcal{A}.$$

Theorem 10.15. *Waldhausen's $S_\bullet$-construction of a stable ∞-category $\mathcal{A}$ is a decomposition space.*

Proof. The proof is exactly the same as in the abelian case, relying on the following three facts:

(1) The Decs are Segal spaces.
(2) $s_0 : S_0 \to S_1$ is a monomorphism of ∞-groupoids.
(3) A map (playing the role of monomorphisms) is an equivalence if and only if its cofibre is the zero object, and a map (playing the role of epimorphism) is an equivalence if and only if its fibre is the zero object.

□

10.16. Remark. This theorem was proved independently (and first) by Dyckerhoff and Kapranov [17], Theorem 7.3.3. They prove it more generally for exact ∞-categories, a notion they introduce. Their proof that Waldhausen's $S_\bullet$-construction of an exact ∞-category is a decomposition space is somewhat more complicated than ours above. In particular their proof of unitality (the pullback condition on degeneracy maps) is technical and involves Quillen model structures on certain marked simplicial sets à la Lurie [55]. We do not wish to go into exact ∞-categories here, and refer instead the reader to [17], but we wish to point out that our simple proof above works as well for exact ∞-categories. This follows since the three points in the proof hold also for exact ∞-categories, which in turn is a consequence of the definitions and basic results provided in [17, Sections 7.2 and 7.3].

10.17. Hall algebras. The finite-support incidence algebra of a decomposition space X is defined in [23, 7.15]; see also [17]. In order for it to admit a cardinality, the required assumptions are, in addition to X_1 being locally finite, that X_0 be finite and that $X_2 \to X_1 \times X_1$ be a finite map. In the case of $X = S_\bullet\mathcal{A}$ for an abelian category $\mathcal{A}$, this translates into the condition that Ext^0 and Ext^1 be finite (which in practice means 'finite dimension over a finite field'). The finite-support incidence algebra in this case is the *Hall algebra* of $\mathcal{A}$ (cf. Ringel [62]; see also [64], although these sources twist the multiplication by the so-called Euler form).

For a stable ∞-category $\mathcal{A}$, with mapping spaces assumed to be locally finite ([22, 3.1]), the finite-support incidence algebra of $S_\bullet\mathcal{A}$ is the *derived Hall algebra*. These were introduced by Toën [70] in the setting of dg-categories.

Hall algebras were one of the main motivations for Dyckerhoff and Kapranov [17] to introduce 2-Segal spaces. We refer to their work for development of this important topic, recommending as entry point the lecture notes of Dyckerhoff [15].

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