A Formula for the Kirchhoff Index

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Abstract

We show here that the Kirchhoff index of a network is the average of the Wiener capacities of its vertices. Moreover, we obtain a closed-form formula for the effective resistance between any pair of vertices when the considered network has some symmetries which allows us to give the corresponding formulas for the Kirchhoff index. In addition, we find the expression for the Foster's n-th Formula.

1 Introduction and preliminaries

The computation of the effective resistance between any pair of vertices of a network as well as the computation of the Kirchhoff index has interest in electric circuit and probabilistic theory. In the last years, it has been proved the utility of the Kirchhoff index in Chemistry as a better alternative to other parameters used for discriminating among different molecules with similar shapes and structures, see for instance [1, 2]. The effective resistance and the Kirchhoff index have been computed for some class of graphs with symmetries, see [1, 3, 4]. In particular, J. L. Palacios in [5] gave a closed-form formula for the Kirchhoff index for *distance-regular graphs* and a class of graphs of diameter two. His approach is based on the first and second Foster's Formula. Later, in [6] he extended these techniques to a class of graphs with diameter three by proving the so-called third Foster's Formula.

In this paper we use a different approach based on discrete Potential Theory in order to compute the effective resistances. Specifically, we consider the so-called equilibrium measures of the network associated with the combinatorial Laplacian kernel and the corresponding Wiener capacities, see [7, 8, 9]. In particular, we prove that the Kirchhoff index is nothing else but the average of the Wiener capacities of the vertices of the network. Moreover, when the network has symmetries the equilibrium measures can be computed by hand and hence we can obtain explicit formulas for the effective resistances and the Kirchhoff index. This is the case of distance-regular graphs, the so-called *weighted barbell networks* and the *wagon wheel network* that we analyze at the end of the paper.

Although we do not use the Foster's formulas in our proofs, a full generalization of those formulas can be easily obtained from the expression of the effective resistance in terms of equilibrium measures. Of course, following the Palacios' technique the Foster's formulas are of potential application in the computation of Kirchhoff index for graphs or networks with diameter greater than three.

In this paper Γ denotes a *network*; that is, a simple and finite connected graph, with vertex set $V = \{1, 2, \dots, n\}$ and edge set E, in which each edge (i, j) has been assigned a *conductance* $c_{ij} > 0$. In addition, when $(i, j) \notin E$ we define $c_{ij} = 0$ and in particular $c_{ii} = 0$ for any i. We define the *(weighted) degree of* i as $\delta_i = \sum\limits_{j=1}^{n} c_{ij}$ and the value $q_i = \max\limits_{1 \le j \le n} \{c_{ij}\}$.

The matrix $P = (p_{ij})$, where $p_{ij} = \dfrac{c_{ij}}{\delta_i}$ is usually called *transition probability matrix* of the reversible Markov chain associated with the network. More generally, for any $k \ge 1$, the k-th power of P, $(p_{ij}^{(k)})$, is called the *k-step transition probability matrix*. Its ij entry is the probability that after k steps the Markov chain attains vertex j when starting from vertex i. Moreover, for $k \ge 2$ this value is given by the identity $p_{ij}^{(k)} = \sum\limits_{l_1,\dots,l_{k-1}=1}^{n} \dfrac{c_{il_1} c_{l_1 l_2} \cdots c_{l_{k-1} j}}{\delta_i \delta_{l_1} \cdots \delta_{l_{k-1}}}$, which implies that $\sum\limits_{j=1}^{n} p_{ij}^{(k)} = 1$ and also that $\sum\limits_{i=1}^{n} \delta_i p_{ij}^{(k)} = \delta_j$, for any $k \ge 1$. The trace of the k-step transition probability matrix is denoted by $\mathsf{tr}(P^k)$.

The *combinatorial Laplacian* of Γ is the matrix L whose entries are $L_{ij} = -c_{ij}$ for all

$i \neq j$ and $L_{ii} = \delta_i$. Therefore, for each vector $u \in \mathbb{R}^n$ and for each $i = 1, \dots, n$

$$(Lu)_i = \delta_i u_i - \sum_{j=1}^{n} c_{ij} u_j = \sum_{j=1}^{n} c_{ij}(u_i - u_j). \tag{1}$$

It is well-known that $Lu = 0$ iff $u = a\mathsf{e}$, $a \in \mathbb{R}$, where e is the vector whose entries equal one. Therefore, given $f \in \mathbb{R}^n$, the linear system $Lu = f$ has solution iff $\sum\limits_{i=1}^{n} f_i = 0$ and in this case there exists a unique solution up to a constant. In addition, the combinatorial Laplacian verifies the *minimum principle*, see [9]. In particular, this properties implies that if $u \in \mathbb{R}^n$ verifies $u_i \geq 0$ and $(Lu)_j \geq 0$, for any $j \neq i$, then $u_j \geq 0$ for all $j = 1, \dots, n$.

If for each $i = 1, \dots, n$, e^i denotes the ith unit vector, with 1 in the ith position, and 0 elsewhere, the linear system $Lu = \mathsf{e} - n\mathsf{e}^i$ has a unique solution denoted by ν^i such that $\nu^i_i = 0$. This solution was called by some of the authors *equilibrium measure* of the set $V \setminus \{i\}$, see [9, 10]. In these references, the authors proved that any equilibrium measure can be obtained as the solution of a linear programming problem and also as the solution of a convex quadratic programming problem. The value $\mathsf{cap}(i) = \sum\limits_{j=1}^{n} \nu^i_j$ is called the *Wiener capacity of vertex* i.

Lemma 1.1 *It is verified that* $\nu^i_j \geq \dfrac{1}{q_i}$ *for any* $j \neq i$. *In addition,* $\mathsf{cap}(i) \geq \dfrac{n-1}{q_i}$ *and the equality holds iff* $c_{ij} = q_i$ *for* $j \neq i$.

Proof. Consider $u \in \mathbb{R}^n$ given by $u_i = 0$ and $u_j = \dfrac{1}{q_i}$ for $j \neq i$. Then, $(Lu)_j = \dfrac{c_{ij}}{q_i} \leq 1$ for any $j \neq i$ and the equality holds iff $c_{ij} = q_i$. Applying the minimum principle we obtain that $\nu^i_j \geq u_j$ for any $j \neq i$ and hence $\mathsf{cap}(i) \geq \sum\limits_{j=1}^{n} u_j = \dfrac{n-1}{q_i}$. Moreover, the equality holds iff $\nu^i_j = u_j$ for any $j \neq i$; that is, iff $c_{ij} = q_i$ for any $j \neq i$. ∎

2 An explicit formula for the Kirchhoff index

One of the main problems in Network Theory is to calculate the effective resistance between any pair of vertices. If $i, j \in V$, the *effective resistance* between i and j is defined as $R_{ij} = u_i - u_j$, where $u \in \mathbb{R}^n$ is any solution of the linear system $Lu = \mathsf{e}^i - \mathsf{e}^j$. Note that R_{ij} does not depend on the chosen solution. Therefore, $R_{ij} = R_{ji}$ and $R_{ii} = 0$. The *Total resistance* or *Kirchhoff index* of the network is defined as

$$R(\Gamma) = \frac{1}{2} \sum_{i,j=1}^{n} R_{ij}. \tag{2}$$

The following result express the effective resistance in terms of equilibrium measures and it was proved in [9, Corollary 4.2]. We include its proof here for the sake of completeness and because it allows us to obtain directly a closed-form formula for the Kirchhoff index of Γ.

Proposition 2.1 *For any $i,j=1,\dots,n$ it is verified that $R_{ij}=\dfrac{1}{n}(\nu_j^i+\nu_i^j)$ and hence*

$$R(\Gamma)=\frac{1}{n}\sum_{i=1}^{n}\mathsf{cap}(i).$$

Proof. If we consider $u=\dfrac{1}{n}(\nu^j-\nu^i)$, then $Lu=\mathsf{e}^i-\mathsf{e}^j$ and hence

$$R_{ij}=u_i-u_j=\frac{1}{n}(\nu_i^j+\nu_j^i).$$

Therefore,

$$R(\Gamma)=\frac{1}{2n}\sum_{i,j=1}^{n}(\nu_i^j+\nu_j^i)=\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\nu_j^i=\frac{1}{n}\sum_{i=1}^{n}\mathsf{cap}(i).\qquad\blacksquare$$

Taking into account the lower bounds for equilibrium measures and its corresponding Wiener capacities established in Lemma 1.1, we obtain the following lower bounds for the effective resistances and the Kirchhoff index.

Corollary 2.2 *For any $1\le i<j\le n$ it is verified that $R_{ij}\ge\dfrac{q_i+q_j}{nq_iq_j}$. Moreover,*

$$R(\Gamma)\ge\frac{(n-1)}{n}\sum_{i=1}^{n}\frac{1}{q_i}$$

and the equality holds iff there exists $c>0$ such that $c_{ij}=c$ for any $i,j=1,\dots,n$, $i\ne j$; that is, iff Γ is a complete network with constant conductances.

Observe that when Γ is a graph then $R_{ij}\ge\dfrac{2}{n}$ for any $1\le i<j\le n$ and $R(\Gamma)\ge n-1$ with equality iff Γ is the complete graph, a well-known property, see for instance [1].

As a by-product of the expression of the effective resistance given in Proposition 2.1, we can derived a full generalization of the so-called Foster's Identities, see [6]. We remark that the case $k=1$ is the most popular Foster's formula and the case $k=3$ is, in fact, due to J.L. Palacios.

Proposition 2.3 *For any $k \geq 1$ it is verified that*

$$\frac{1}{2}\sum_{i,j=1}^{n} \delta_i R_{ij} p_{ij}^{(k)} = n - k + \sum_{j=1}^{k-1} \mathsf{tr}(P^j).$$

Proof. First note that

$$\sum_{i,j=1}^{n} \delta_i R_{ij} p_{ij}^{(k)} = \frac{1}{n}\sum_{i,j=1}^{n} \delta_i \nu_j^i p_{ij}^{(k)} + \frac{1}{n}\sum_{i,j=1}^{n} \delta_i \nu_i^j p_{ij}^{(k)} = \frac{2}{n}\sum_{i,j=1}^{n} \delta_i \nu_j^i p_{ij}^{(k)},$$

since $\delta_i p_{ij}^{(k)} = \delta_j p_{ji}^{(k)}$. So, it suffices to prove that $\frac{1}{n}\sum_{i,j=1}^{n} \delta_i \nu_j^i p_{ij}^{(k)} = n - k + \sum_{j=1}^{k-1} \mathsf{tr}(P^j)$. Applying that $L\nu^i = \mathsf{e} - n\mathsf{e}^i$ we get that

$$\begin{aligned}
\frac{1}{n}\sum_{i,j=1}^{n} \delta_i \nu_j^i p_{ij}^{(k+1)} &= \frac{1}{n}\sum_{i,l=1}^{n} \delta_i p_{il}^{(k)} \sum_{j=1}^{n} \frac{c_{lj}}{\delta_l}\nu_j^i = \frac{1}{n}\sum_{i,l=1}^{n} \frac{\delta_i}{\delta_l} p_{il}^{(k)} (\delta_l \nu_l^i + n\mathsf{e}_l^i - 1) \\
&= \frac{1}{n}\sum_{i,l=1}^{n} \delta_i p_{il}^{(k)} \nu_l^i + \sum_{i=1}^{n} p_{ii}^{(k)} - \frac{1}{n}\sum_{i,l=1}^{n} \frac{\delta_i}{\delta_l} p_{il}^{(k)} \\
&= \frac{1}{n}\sum_{i,l=1}^{n} \delta_i p_{il}^{(k)} \nu_l^i + \mathsf{tr}(P^k) - 1,
\end{aligned}$$

since $\sum_{i,l=1}^{n} \frac{\delta_i}{\delta_l} p_{il}^{(k)} = \sum_{l=1}^{n} \frac{1}{\delta_l} \sum_{i=1}^{n} \delta_i p_{il}^{(k)} = n$. The result follows keeping in mind that

$$\frac{1}{n}\sum_{i,l=1}^{n} \delta_i p_{il} \nu_l^i = \frac{1}{n}\sum_{i=1}^{n} (\delta_i \nu_i^i + n\mathsf{e}_i^i - 1) = n - 1. \quad \blacksquare$$

Let us point out that to compute the effective resistance between any pair of vertices and hence the Kirchhoff index it suffices to solve n equilibrium problems. However, it is clear that the number of problems that we have to solve, could be drastically reduced if we have additional information about the network structure. The most striking cases appear when Γ has some type of symmetries that allow us to obtain by hand the equilibrium measures. One of the main examples of this situation is the case of *distance-regular graphs*. This kind of graphs have been studied by N. Biggs [11], J.L. Palacios [5] and by the authors in [7, 8]. A connected graph Γ is called *distance-regular* if there are integers b_i, c_i, $i = 0, \dots, d$ such that for any two vertices $i, j \in V$ at distance $\ell = d(i, j)$, there are exactly c_ℓ neighbours of j in $\Gamma_{\ell-1}(i)$ and b_ℓ neighbours of j in $\Gamma_{\ell+1}(i)$, where $\Gamma_\ell(i)$ is the set of vertices at distance ℓ from i. In particular, Γ is regular of degree $\delta = b_0$. Moreover,

$a_i = \delta - c_i - b_i$ is the number of neighbours of j in $\Gamma_\ell(i)$ and clearly, $b_d = c_0 = 0$, $c_1 = 1$ and the diameter of Γ is d. The sequence

$$\iota(\Gamma) = \{b_0, b_1, \dots, b_{d-1}; c_1, \dots, c_d\},$$

is called the *intersection array* of Γ. In addition, the number of vertices in $\Gamma_\ell(i)$ is independent of the choice of i and will be denoted by k_ℓ. Then, $k_0 = 1$, $k_1 = \delta$ and the following equalities hold:

$$k_\ell = \frac{b_0 \cdots b_{\ell-1}}{c_1 \cdots c_\ell},\ \ell = 2, \dots, d \ \text{ or equivalently } \ k_{\ell+1} c_{\ell+1} = k_\ell b_\ell,\ \ell = 2, \dots, d-1. \qquad (3)$$

In [7] we proved that the equilibrium measure ν_j^i depend only on the distance between vertices i and j. Specifically, $\nu_j^i = \displaystyle\sum_{l=0}^{d(i,j)-1} \frac{1}{k_l b_l} \sum_{m=l+1}^{d} k_m$ and hence we get the following result, that was previously obtained in [5] by using a different approach.

Proposition 2.4 *For any $i, j = 1, \dots, n$ it is verified that* $R_{ij} = \dfrac{2}{n} \displaystyle\sum_{l=0}^{d(i,j)-1} \frac{1}{k_l b_l} \sum_{m=l+1}^{d} k_m$ *and hence*

$$R(\Gamma) = \sum_{l=0}^{d-1} \frac{1}{k_l b_l} \Big(\sum_{m=l+1}^{d} k_m \Big)^2.$$

Some particular cases of the above formula are also important. For instance, we can consider the *complete graph, K_n*; that is the distance regular graph of diameter $d = 1$, whose intersection array is $\iota(K_n) = \{n-1; 1\}$. Therefore, $R_{ij} = \dfrac{2}{n}$ for any $i \neq j$ and $R(K_n) = n - 1$, equality that in this case is nothing but that the so-called first Foster's identity. We can also consider the cycle C_n on n vertices. In this case, $d = \lfloor \frac{n}{2} \rfloor$ and the intersection array is given by $\iota(\Gamma) = \{2, 1, \dots, 1; 1, \dots, 1, c_d\}$, where $c_d = 2$ for even n and $c_d = 1$ for odd n. Then, $R_{ij} = \dfrac{d(i,j)}{n} \Big(n - d(i,j) \Big)$ and hence $R(C_n) = \dfrac{n}{12} (n^2 - 1)$.

Another interesting family of this type of graphs is formed by the so-called *strongly regular graphs*; that is, distance-regular graphs of diameter $d = 2$. Therefore if Γ is an strongly regular graph, then its intersection array is $\iota(\Gamma) = \{\delta, b_1; 1, c_2\}$ and hence it is characterized by three parameters. Then, $R_{ij} = \dfrac{2(b_1 + c_2)}{n\, c_2}$ if $d(i,j) = 1$, $R_{ij} = \dfrac{2(1 + b_1 + c_2)}{n\, c_2}$ if $d(i,j) = 2$ and hence $R(\Gamma) = \dfrac{\delta}{c_2^2} (b_1 + (c_2 + b_1)^2)$.

We finish this paper by calculating the Kirchhoff index for two types of networks that have some symmetries but that are not distance-regular graphs; in fact, they are not even regular.

The first example is the so-called *weighted barbell graph* on $n = k+m+r$ vertices, where $m \ge 2$ and $k, r \ge 1$: start with a weighted path on m vertices, labeled as $x_{k+1}, ..., x_{k+m}$ and attach a complete network of order $k+1$ at vertex x_{k+1} and a complete network of order $r+1$ at vertex x_{k+m}. Denote by $\{x_1, \dots, x_k\}$ the set of new vertices of the complete network attached to x_{k+1} and by $\{x_{k+m+1}, \dots, x_{k+m+r}\}$ the set of new vertices of the complete network attached to x_{k+m}. Moreover, the conductances are given by $c_{ij} = a$, $1 \le i < j \le k$; $c_{i,k+1} = c_0$, $1 \le i \le k$; $c_{k+i,k+i+1} = c_i$, $1 \le i \le m-1$; $c_{k+m,k+m+i} = c_m$, $1 \le i \le r$, and $c_{k+m+i,k+m+j} = b$, $1 \le i < j \le r$, where $c_0, \dots, c_m > 0$ and $a, b \ge 0$. Observe that when $a = 0$, respectively $b = 0$, then the attached network at vertex x_{k+1}, respectively at vertex x_{k+m}, is a weighted star. On the other hand, when $k = r = 1$, then Γ is nothing else than a weighted path on $m+2$ vertices whose conductances are $c_0, \dots, c_m$.

Because the symmetries in Γ, it suffices to calculate the equilibrium measures ν^i, $i = k, \dots, k+m+1$. Then, the following identities are easy to verify:

$$\begin{aligned}
\nu_j^k &= \frac{n}{ka + c_0}, \quad 1 \le j \le k-1, \\
\nu_{k+j}^k &= \frac{(k-1)}{c_0} + \frac{n(1-k)a}{c_0(ka+c_0)} + \sum_{l=0}^{j-1} \frac{r+m-l}{c_l}, \quad 1 \le j \le m, \\
\nu_{k+m+j}^k &= \frac{(k-1)}{c_0} + \frac{n(1-k)a}{c_0(ka+c_0)} + \frac{(1-r)}{c_m} + \sum_{l=0}^{m} \frac{r+m-l}{c_l}, \quad 1 \le j \le r,
\end{aligned}$$

for any $i = 1, \dots, m$,

$$\begin{aligned}
\nu_j^{k+i} &= \frac{(1-k)}{c_0} + \sum_{l=0}^{i-1} \frac{k+l}{c_l}, \quad 1 \le j \le k, \quad \nu_{k+j}^{k+i} = \sum_{l=j}^{i-1} \frac{k+l}{c_l}, \quad 1 \le j \le i-1 \\
\nu_{k+j}^{k+i} &= \sum_{l=i}^{j-1} \frac{r+m-l}{c_l}, \quad i+1 \le j \le m, \quad \nu_{k+m+j}^{k+i} = \frac{(1-r)}{c_m} + \sum_{l=i}^{m} \frac{r+m-l}{c_l}, \quad 1 \le j \le r,
\end{aligned}$$

and finally,

$$\begin{aligned}
\nu_j^{k+m+1} &= \frac{(r-1)}{c_m} + \frac{n(1-r)b}{c_m(rb+c_m)} + \frac{(1-k)}{c_0} + \sum_{l=0}^{m} \frac{k+l}{c_l}, \quad 1 \le j \le k, \\
\nu_{k+j}^{k+m+1} &= \frac{(r-1)}{c_m} + \frac{n(1-r)b}{c_m(rb+c_m)} + \sum_{l=j}^{m} \frac{k+l}{c_l}, \quad 1 \le j \le m, \\
\nu_{k+m+j}^{k+m+1} &= \frac{n}{rb + c_m}, \quad 2 \le j \le r.
\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}
\mathsf{cap}(i) &= \frac{n^2(k-1)}{k[ka+c_0]} + \frac{(m+r)^2}{kc_0} + \frac{r}{c_m} + \sum_{l=1}^{m-1} \frac{(m+r-l)^2}{c_l}, \quad i=1,\dots,k,\\
\mathsf{cap}(k+i) &= \frac{k(1-k)}{c_0} + \frac{r(1-r)}{c_m} + \sum_{l=0}^{i-1} \frac{(k+l)^2}{c_l} + \sum_{l=i}^{m} \frac{(r+m-l)^2}{c_l}, \quad i=1,\dots,m,\\
\mathsf{cap}(k+m+i) &= \frac{n^2(r-1)}{r[rb+c_m]} + \frac{k}{c_0} + \frac{(m+k)^2}{rc_m} + \sum_{l=1}^{m-1} \frac{(k+l)^2}{c_l}, \quad i=1,\dots,r.
\end{aligned}$$

Consequently, it results that

$$R(\Gamma) = \frac{n(k-1)}{ka+c_0} + \frac{n(r-1)}{rb+c_m} + \frac{(m+r)}{c_0} + \frac{(m+k)}{c_m} + \sum_{l=1}^{m-1} \frac{(k+l)(m+r-l)}{c_l}.$$

Moreover, the formulas for the equilibrium measures imply that

$$\begin{aligned}
R_{ij} &= \frac{2}{ka+c_0}, \quad 1 \le i < j \le k,\\
R_{i,k+j} &= \frac{(1-k)a}{c_0(ka+c_0)} + \sum_{l=0}^{j-1} \frac{1}{c_l}, \quad 1 \le i \le k, \quad 1 \le j \le m,\\
R_{i,k+m+j} &= \frac{(1-k)a}{c_0(ka+c_0)} + \frac{(1-r)b}{c_m(rb+c_m)} + \sum_{l=0}^{m} \frac{1}{c_l}, \quad 1 \le i \le k, \quad 1 \le j \le r,\\
R_{k+i,k+j} &= \sum_{l=i}^{j-1} \frac{1}{c_l}, \quad 1 \le i < j \le m,\\
R_{k+i,k+m+j} &= \frac{(1-r)b}{c_m(rb+c_m)} + \sum_{l=i}^{m} \frac{1}{c_l}, \quad 1 \le i \le m, \quad 1 \le j \le r,\\
R_{k+m+i,k+m+j} &= \frac{2}{rb+c_m}, \quad 1 \le i < j \le r.
\end{aligned}$$

We conclude the analysis of this example by specifying the above formulas for some particular cases that have its own interest and that have been considered in the literature. When $r = k = 2\ell - 1$ and $m = 2\ell + 1$ where $\ell \ge 2$ and, in addition, all conductances equal 1, then Γ is called *barbell graph on $n = 6\ell - 1$ vertices*, see for instance [12]. In this case, we obtain that

$$R(\Gamma) = \frac{2}{3\ell}\Big(26\ell^4 - 9\ell^3 + 31\ell^2 - 21\ell + 3\Big).$$

When $m = 2$ and $r = k = \ell - 1$, $\ell \ge 2$, the corresponding weighted barbell network is sometimes called *weighted dumbbell network*, see [12]. Then, $n = 2\ell$ and

$$R(\Gamma) = \frac{2\ell(\ell-2)}{(\ell-1)a+c_0} + \frac{2\ell(\ell-2)}{(\ell-1)b+c_2} + \frac{(\ell+1)}{c_0} + \frac{\ell^2}{c_1} + \frac{(\ell+1)}{c_2}.$$

If, in addition, $a = b$ and $c_2 = c_0$, then $R(\Gamma) = \dfrac{4\ell(\ell-2)}{(\ell-1)a + c_0} + \dfrac{2(\ell+1)}{c_0} + \dfrac{\ell^2}{c_1}$. This identity was obtained in [13, Formula 39] by using a different approach based on the eigenvalues of the combinatorial Laplacian.

When $k = r = 1$; that is, when Γ is the weighted path on $n = m + 2$ vertices with conductances $c_0, \dots, c_m$, we get the well-known identity $R(\Gamma) = \sum\limits_{l=0}^{m} \dfrac{(l+1)(m+1-l)}{c_l}$, that for unitary weights becomes $R(\Gamma) = \dfrac{n}{6}\,(n^2 - 1)$.

Let us now consider the so-called *wagon wheel network* with $n \geq 3$ vertices. It consists in attaching a vertex, say n, to a weighted cycle on $n - 1$ vertices, $\{1, \dots, n-1\}$, with uniform conductance $a > 0$. Moreover, the conductances of the spoke edges are $c_{i,n} = c > 0$ for any $i = 1, \dots, n-1$. Then, we can verify straightforwardly that

$$\nu_j^n = \frac{1}{c}, \quad 1 \leq j \leq n-1$$

and hence $\mathsf{cap}(n) = \dfrac{n-1}{c}$. To calculate the equilibrium measures ν^i, $1 \leq i \leq n-1$, we need to remember some properties of the *First and Second order Chebyshev Polynomials*, that are respectively defined by the following recurrences:

$$\begin{array}{llll} T_0(x) = 1, & T_1(x) = x, & T_{m+2}(x) = 2\,x\,T_{m+1}(x) - T_m(x), & m \geq 0, \\ U_{-2}(x) = -1, & U_{-1}(x) = 0, & U_m(x) = 2\,x\,U_{m-1}(x) - U_{m-2}(x), & m \geq 0. \end{array} \tag{4}$$

Moreover, for any $m \geq 0$ we have that $T_m(x) = xU_{m-1}(x) - U_{m-2}(x)$ and also that $2(x-1)\sum\limits_{l=0}^{m} U_l(x) = U_{m+1}(x) - U_m(x) - 1$, for any $x \in \mathbb{R}$.

Tacking into account the above properties, it is easy to verify that if $q = 1 + \dfrac{c}{2a}$, then for any $1 \leq i \leq n-1$ the values of the equilibrium measure ν^i are given by

$$\begin{aligned} \nu_j^i &= \frac{n}{2a\Big[T_{n-1}(q) - 1\Big]}\Big[U_{n-2}(q) - U_{|i-j|-1}(q) - U_{n-2-|i-j|}(q)\Big], \quad 1 \leq j \leq n-1, \\ \nu_n^i &= \frac{nU_{n-2}(q)}{2a\Big[T_{n-1}(q) - 1\Big]} - \frac{1}{c}, \end{aligned}$$

which implies that $\mathsf{cap}(i) = \nu_n^i + \sum\limits_{l=1}^{n-1} \nu_l^i = n\nu_n^i - \dfrac{1}{c}$, since $(n-1)c\nu_n^i - c\sum\limits_{l=1}^{n-1} \nu_l^i = 1$. Therefore, $R(\Gamma) = \dfrac{1}{n}\Big[\mathsf{cap}(n) + (n-1)\mathsf{cap}(1)\Big] = (n-1)\nu_n^1$ and hence

$$R(\Gamma) = \frac{n(n-1)U_{n-2}(q)}{2a\Big[T_{n-1}(q) - 1\Big]} - \frac{(n-1)}{c}.$$

We can obtain an alternative expression for the Kirchhoff index taking into account that $T'_m(x) = mU_{m-1}(x)$ for any $m \geq 0$ and that

$$\frac{T'_{n-1}(q)}{2a\Big[T_{n-1}(q)-1\Big]} = \frac{1}{2a}\sum_{l=0}^{n-2}\frac{1}{\Big[q-\cos\Big(\frac{2l\pi}{n-1}\Big)\Big]} = \frac{1}{c}+\sum_{l=1}^{n-2}\frac{1}{c+2a\Big[1-\cos\Big(\frac{2l\pi}{n-1}\Big)\Big]}$$

since $\Big\{\cos\Big(\frac{2l\pi}{n-1}\Big)\Big\}_{l=0}^{n-2}$ are the roots of the polynomial $T_{n-1}(x)-1$. Therefore,

$$R(\Gamma) = \frac{1}{c}+\sum_{l=1}^{n-2}\frac{n}{c+2a\Big[1-\cos\Big(\frac{2l\pi}{n-1}\Big)\Big]},$$

an identity that was obtained in [13, Formula 38] by using newly the approach based on the combinatorial Laplacian's eigenvalues.

Finally, the formulas for the equilibrium measures imply that

$$R_{ij} = \frac{1}{a\Big[T_{n-1}(q)-1\Big]}\Big[U_{n-2}(q)-U_{|i-j|-1}(q)-U_{n-2-|i-j|}(q)\Big], \quad 1 \leq i,j \leq n-1,$$
$$R_{i,n} = \frac{U_{n-2}(q)}{2a\Big[T_{n-1}(q)-1\Big]}, \quad 1 \leq i \leq n.$$

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