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Exploring nonlocal correlations in many-body systems

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Abstract. We address the question of how to evaluate the Bell correlations depth, i.e. the number of parties sharing genuinely nonlocal correlations in a multipartite system. Previous work [1] has shown that it is possible to construct inequalities based on two-body symmetric correlation functions that can reveal k-nonlocality, but only for $k \leq 6$. We extend this result up to $k \leq 24$ by means of a new algorithm whose complexity is poly(k). Furthermore, we provide a hierarchical approximation that in principle could be used to extend such procedure to arbitrarily large values of k. Since the inequalities that we consider only require assessing two-body symmetric correlation functions, they are suitable for experiments involving a large number of particles. Hence, our results open the way for the study of the multipartite nonlocality depth of many-body systems, such as Bose-Einstein condensates or thermal ensembles.

1. Introduction

In 1964, Bell proved his celebrated theorem: no theory of local hidden variables can ever explain the predictions of quantum mechanics [2]. The discovery that some quantum states exhibit nonlocal properties not only has had a profound impact in the foundations of physics, but also it is the key resource in many protocols for quantum information communications and processing [3, 4].

Recently, the study of nonlocality in the context of many-body systems has attracted a lot of interest. Appart from its potential applications in other device independent protocols [5], the observation of nonlocal correlations could provide new insights into condensed matter physics. A question of special relevance for that matter is how to determine the *nonlocality depth*, that is, the number of particles sharing genuinely nonlocal correlations. In [1] it was shown that it is possible to construct inequalities that can reveal the nonlocality depth k of a quantum system for $k < 7$, in terms of two-body symmetric correlation functions. As the two-body symmetric correlation functions can be estimated from collective measurements [6], such inequalities are of great interest for experiments in many-body systems. Here, we explore how to extend such results to arbitrarily large values of k.

2. Multipartite nonlocality

2.1. Preliminaries about polytopes

In the study of multipartite nonlocality, we make an extensive use of the theory of polytopes. A *polytope* in $\mathbf{R}^N$ can be defined as the (bounded) set resulting from the intersection of a finite number of half-spaces (a *half-space* is the set of all points in $\mathbf{R}^N$ that satisfy a given linear inequality, also referred as *facet*). Alternatively, one can define a polytope as the convex hull (the set of all convex combinations) of a set of points in $\mathbf{R}^N$, also referred as *vertices.* In fact, both definitions coincide, as the Minkowski-Weyl Theorem shows. In order to pass from one representation to the other, we make use of the C++ library *lrslib* [7].

2.2. The multipartite Bell scenario

Let us consider a Bell experiment in which we have a system with N parties, such that in each party it is possible to perform m distinct measurements, and each measurement can have d possible outcomes. This is called an (N, m, d) scenario. As an example of an $(N, 2, 2)$ scenario, we can think of an experiment with a system composed of N atoms with spin $S = 1/2$, where we can measure the observables s_x, s_z in each atom, having two possible measurement outcomes per site ($\pm\frac{1}{2}$). In what follows, we only consider $(N, 2, 2)$ experiments.

We denote the measurements as $\mathcal{M}^{(i)}_{x_i}$, where $i \in \{1, ..., N\}$ labels each party and $x_i \in \{0, 1\}$ labels each of the two possible measurement settings, while the outcomes are denoted as $a_i \in \{0, 1\}$. The *correlations* are then the conditional probabilities of obtaining a given set of outcomes upon the measurement of some set of observables

$$P(a_1, ..., a_N|x_1, ..., x_N), \quad x_i \in \{0, 1\}, \quad a_i \in \{-1, 1\}. \tag{1}$$

Since they are probabilities, the correlations have to be positive and normalized.

It is convenient to define a vector $\mathbf{p}$ of dimension 4^N whose entries $p_{\mathbf{ax}}$ are the correlations corresponding to each pair $(\mathbf{a}, \mathbf{x}) \equiv (a_1, ..., a_N|x_1, ..., x_N)$.

Information cannot be sent faster than light, and in consequence it is impossible for an observer to instantaneously send information to another one just by changing his measurements settings. This constraint is introduced with the no-signaling conditions.

Definition 1. The N-partite no-signaling polytope $\mathcal{NS}_N$ is the set of all positive and normalized correlations $\mathbf{p} \in \mathbf{R}^{4^N}$ satisfying the no-signaling conditions

$$\sum_{a_i=0}^{1} p(a_1, ..., a_i, ..., a_N|x_0, ..., x_i, ..., x_N) = \sum_{a_i=0}^{1} p(a_1, ..., a_i, ..., a_N|x_0, ..., x'_i, ..., x_N). \tag{2}$$

Most of the works in the field of nonlocality are concerned with the question of how to detect nonlocal correlations. However, we want to address a more general problem:

how to evaluate the nonlocality depth, that is, the number of particles sharing genuine nonlocal correlations in a multipartite system. We use the definition of nonlocality depth given in [1].

Definition 2. Let $L_k = \{\mathcal{A}_i\}_{i=1}^{L}$ be a partition of the set $I = \{1, ..., N\}$ into L subsets such that the size of each subset $\mathcal{A}_i$ is at most k. We say that the correlations $\mathbf{p} \in \mathbb{R}^{4^N}$ are k-producible with respect to L_k if they admit the following decomposition

$$p(\mathbf{a}|\mathbf{x}) = \int_\Lambda d\lambda\, p_i(\lambda) \prod_{i=1}^{L} p(\mathbf{a}_{\mathcal{A}_i}|\mathbf{x}_{\mathcal{A}_i}, \lambda), \quad \int_\Lambda d\lambda\, p(\lambda) = 1, \quad p(\lambda) \geq 0, \tag{3}$$

where $\mathbf{a}_{\mathcal{A}_i} \equiv (a_{j_1}, ..., a_{j_{|\mathcal{A}_i|}})$, with $j_1, ..., j_{|\mathcal{A}_i|} \in \mathcal{A}_i$, and $p_i(\mathbf{a}_{\mathcal{A}_i}|\mathbf{x}_{\mathcal{A}_i}, \lambda) \in \mathcal{NS}_{|\mathcal{A}_i|}$

Definition 3. We say that the correlations $\mathbf{p} \in \mathbb{R}^{4^N}$ are k-producible if they admit the following decomposition

$$p(\mathbf{a}|\mathbf{x}) = \sum_{S \in S_k} q_S p_S(\mathbf{a}_S|\mathbf{x}_S), \quad \sum_{S \in S_k} q_S = 1, \quad q_S \geq 0, \tag{4}$$

where S_k is the set of all L_k partitions. The set of all k-producible correlations with N parties is the $\mathcal{PN}_{N,k}$ polytope. If some correlations are not k-producible, then they are called k-nonlocal. The minimal k for which some correlations are k-producible is called the nonlocality depth.

Remark. The $\mathcal{PN}_{N,k}$ set is indeed a convex polytope, whose vertices are given by $\prod_{i=1}^{L} p_i(\mathbf{a}_{\mathcal{A}_i}|\mathbf{x}_{\mathcal{A}_i})$. Each factor $p_i(\mathbf{a}_{\mathcal{A}_i}|\mathbf{x}_{\mathcal{A}_i})$ is a vertex of the $|\mathcal{A}_i|$-partite no-signaling polytope.

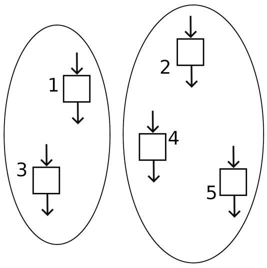
We assume that the probabilities $p(\mathbf{a}_{\mathcal{A}_i}|\mathbf{x}_{\mathcal{A}_i})$ that appear in the factorization given in Definition 2 are no-signalling. We could have considered a factorization in terms of quantum correlations, but the quantum set is harder to characterize. In addition, the no-signaling conditions are the weaker constraints compatible with a reasonable physical model (in particular, they contain the quantum correlations as a subset).

Heuristically, we can think of this definition of k-producible correlations as a generalization of the standard definition of local correlations, but replacing local strategies with no-signaling strategies involving k parties at most. In particular, the 1-producible correlations are the local correlations, and the $\mathcal{PN}_{N,1}$ polytope is the local polytope, denoted as $\mathcal{L}_N$. It should be clear that

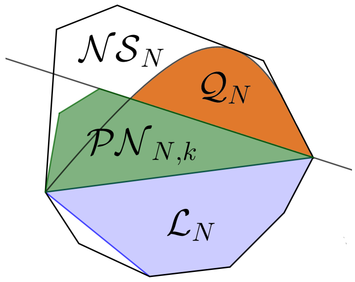
$$\mathcal{L}_N \subset \mathcal{PN}_{N,2} \subset ... \subset \mathcal{PN}_{N,N-1} \subset \mathcal{NS}_N. \tag{5}$$

The correlations with nonlocality depth $k+1$ lie outside the $\mathcal{PN}_{N,k}$ set, so our task is to obtain inequalities of the form

$$\sum_{\mathbf{a},\mathbf{x}} c_{\mathbf{ax}} p(\mathbf{a}|\mathbf{x}) \geq -\beta_N^k, \tag{6}$$



(a)



(b)

Figure 1: (a) Example of a 5-partite 3-producible state, corresponding to the partition $L_3 = \{\{1,3\},\{2,4,5\}\}$ (the ellipses represent nonlocal correlations). (b) Sketch of the different sets of correlations: $\mathcal{L}_N \subset \mathcal{PN}_{N,k} \subset \mathcal{NS}_N$ and $\mathcal{Q}_N \subset \mathcal{NS}_N$. Our goal is to find the hyperplanes that delimit $\mathcal{PN}_{N,k}$.

where $\beta_N^k \equiv -\min\left\{\sum_{p(\mathbf{a}|\mathbf{x})\in\mathcal{PN}_{N,k}} c_{\mathbf{ax}}p(\mathbf{a}|\mathbf{x})\right\}$.

If a quantum system exhibits correlations violating any inequality of the form (6), we can use such inequality to prove that it is k-nonlocal. For this reason, it is important to characterize the quantum set, i.e. the set of correlations that can be obtained from quantum systems.

Definition 4. The N-partite quantum set $\mathcal{Q}_N$ is the set of all correlations $\mathbf{p} \in \mathbb{R}^{4^N}$ that can be represented with the Born's rule for some measurements $\{\Pi_{a_i}^{(x_i)}\}$ and an N-partite density matrix $\rho_{1\ldots N}$

$$p(\mathbf{a}|\mathbf{x}) = \mathrm{Tr}\left[\rho_{1\ldots N}\bigotimes_i \Pi_{a_i}^{(x_i)}\right]. \tag{7}$$

$\mathcal{Q}_N$ is a convex set like $\mathcal{NS}_N$ and $\mathcal{PN}_{N,k}$, but it is not a polytope. It can be shown easily that $\mathcal{Q}_N \subsetneq \mathcal{NS}_N$. The relation of $\mathcal{PN}_{N,k}$ and $\mathcal{Q}_N$ is more complicated, but it has been proved that in many cases $\mathcal{Q}_N$ is not contained in $\mathcal{PN}_{N,k}$: *there are quantum correlations outside* $\mathcal{PN}_{N,k}$ [1]. Consider some function $I : \mathcal{PN}_{N,k} \to \mathbb{R}$ such that $I(\mathbf{p}) \geq -\beta_k \quad \forall \mathbf{p} \in \mathcal{PN}_{N,k}$. If there are correlations $\mathbf{q} \in \mathcal{Q}_N$ for which $I(\mathbf{q}) = -\beta_Q < -\beta_k$, we say that the inequality $I(\mathbf{p}) \geq -\beta_k$ is violated by quantum correlations, with a quantum violation given by $\beta_Q - \beta_k$. Our final goal is to obtain inequalities violated by quantum correlations.

2.3. Two-body symmetric correlators

Instead of describing a system in terms of the probabilities $p(\mathbf{a}|\mathbf{x})$, we could have used the expectation values of the measurements $\langle \mathcal{M}_{x_{i_1}}^{(i_1)} \ldots \mathcal{M}_{x_{i_k}}^{(i_k)} \rangle$, also called correlation functions or *correlators*. For an $(N,2,2)$ scenario and assuming no-signaling conditions, probabilities and correlators are related through the formula

$$p(\mathbf{a}|\mathbf{x})=\frac{1}{2^N}\left[1+\sum_{k=1}^{N}\sum_{1\leq i_1<i_2...<i_k\leq N}a_{i_i}...a_{i_k}\langle\mathcal{M}_{x_{i_1}}^{(i_1)}...\mathcal{M}_{x_{i_k}}^{(i_k)}\rangle\right]. \tag{8}$$

Calculating $\mathcal{PN}_{N,k}$ for large values of N is impossible in practice as the number of vertices grows very fast. Furthermore, even if we could calculate all the facets of $\mathcal{PN}_{N,k}$, a general inequality of the form (6) requires the measurement of $exp(N)$ different correlations, what makes it impractical for experimental purposes. In order to overcome this difficulty, we focus on the inequalities involving only *two-body symmetric correlators*

$$S_l\equiv\sum_{i=1}^{N}\langle\mathcal{M}_l^{(i)}\rangle,\quad S_{lm}\equiv\sum_{i,j=1,i\neq j}^{N}\langle\mathcal{M}_l^{(i)}\mathcal{M}_m^{(j)}.\rangle \tag{9}$$

Two-body symmetric correlators can be estimated from collective measurements in condensed matter experiments [6]. In addition, while the dimension of the whole space of correlations is 4^N, the space of two-body symmetric correlators is only 5-dimensional, what substantially reduces the computational complexity of the problem of obtaining valid Bell-like inequalities for k-nonlocality.

Let $\mathcal{PN}_{N,k}^{2,S}$ and $\mathcal{NS}_k^{2,S}$ be the (orthogonal) projections of $\mathcal{PN}_{N,k}$ and $\mathcal{NS}_k$ in the space of two-body symmetric correlators, respectively (projection in the following sense: $(S_0,S_1,S_{00},S_{01},S_{11},S_{000},S_{001},...)\rightarrow(S_0,S_1,S_{00},S_{01},S_{11})$). A fundamental result is that the vertices of $\mathcal{PN}_{N,k}^{2,S}$ can be obtained from the vertices of all the polytopes $\mathcal{NS}_p^{2,S}$, with $p\in\{1,...,k\}$. Here we outline how to do it. First, any vertex of $\mathcal{PN}_{N,k}^{2,S}$ is a projected vertex of $\mathcal{PN}_{N,k}$. Second, since the two-body symmetric correlators are permutationally invariant, the vertices of $\mathcal{PN}_{N,k}^{2,S}$ do not depend on which no-signaling strategy adopts each party, but only on the number of parties that are assigned to each vertex of the p-partite no-signaling polytope. Therefore, we can parametrize the vertices of $\mathcal{PN}_{N,k}^{2,S}$ with the populations $\xi_{p,i}$ of each vertex of the p-partite no-signaling polytopes, with $p\in\{1,...,k\}$ and $i\in\{1,...,n_p\}$ (n_p is the number of vertices of $\mathcal{NS}_p^{2,S}$). Let $\mathbf{S}(p,i)=(S_0(p,i),S_1(p,i),S_{00}(p,i),S_{01}(p,i),S_{11}(p,i))$ be the i-th vertex of $\mathcal{NS}_p^{2,S}$. Taking into account that expectation values belonging to different strategies can be factorized, we can arrive to a closed expression for the two-body correlators of the vertices of $\mathcal{PN}_{N,k}^{2,S}$

$$S_m(\boldsymbol{\xi})=\sum_{p=1}^{k}\sum_{i=1}^{n_p}\xi_{pi}S_m(p,i) \tag{10}$$

$$\begin{aligned}S_{mn}(\boldsymbol{\xi})=\sum_{p=1}^{k}\sum_{i=1}^{n_p}\xi_{pi}S_{m,n}(p,i)+\sum_{p=1}^{k}\sum_{i=1}^{n_p}\xi_{pi}(\xi_{pi}-1)S_m(p,i)S_n(p,i)\\+\sum_{(p,i)\neq(q,j)}\xi_{pi}\xi_{qj}S_m(p,i)S_n(q,j),\end{aligned} \tag{11}$$

with the populations satisfying that $\sum_{p,i}p\xi_{p,i}=N$ (the total number of parties is N).

3. Approximating the $\mathcal{NS}_k$ polytope from outside

As we have argued, the problem of characterizing $\mathcal{PN}^{2,S}_{N,k}$ reduces to the calculation of all the polytopes $\mathcal{NS}^{2,S}_p$ with $p \in \{1,...,k\}$. Yet, we are only interested in finding inequalities that enclose $\mathcal{PN}^{2,S}_{N,k}$, so an approximation can be sufficient for our purposes, as long as it is an approximation from outside. The strategy that we employ is as follows: first, we compute the vertices of a polytope that approximates $\mathcal{NS}^{2,S}_k$ from outside; second, we will use such vertices and Eqs. 10, 11 to obtain inequalities enclosing $\mathcal{PN}^{2,S}_{N,k}$, and third, we prove that there are quantum correlations that violate our inequalities.
Consider the space V^S_k spanned by the full-body symmetric correlators, which are defined as $S_{m_1...m_l} \equiv \sum_{i_1\neq i_2\neq ...\neq i_l}\langle M^{(i_1)}_{m_1}M^{(i_2)}_{m_2}...M^{(i_1)}_{m_l}\rangle, \quad l \leq k$. The outer approximation of $\mathcal{NS}^{2,S}_k$ is calculated in two steps. In the first place we calculate the facets of $\mathcal{NS}^S_k$, which is the projection of $\mathcal{NS}_k$ in the space V^S_k.
We can also define the correlators $T^{j_1...j_l}_{m_1...m_l} \equiv \sum_{i_1\neq i_2\neq ...\neq i_l}\langle M^{(j_1)}_{m_1}M^{(j_2)}_{m_2}...M^{(j_1)}_{m_l}\rangle - S_{m_1...m_l}$, which span the complementary subspace of V^S_k.
In the correlator representation, the no-signaling and normalization conditions are automatically fulfilled, so we only need to impose positivity in order to find the facets of $\mathcal{NS}_k$. Starting from expression (8) (using Lemma 1, given in the Annex in Appendix A) we find that the facets of $\mathcal{NS}^S_k$ are given by

$$1+\sum_{p=1}^{k}\sum_{1\leq i_1<i_2...<i_p\leq k}\frac{k!}{(k-p)!}a_{i_i}...a_{i_p}S_{x_{i_1}...x_{i_p}} \geq 0 \quad \forall \mathbf{a},\mathbf{x}\in\mathbb{B}^k. \tag{12}$$

It would seem that the number of facets grows exponentially as the number of pairs $\mathbf{a},\mathbf{x}$ is 4^k, but in fact most of the facets given in Eq. (12) are redundant. Indeed, as we show in Theorem 1, the number of facets of $\mathcal{NS}^S_k$ grows as $\mathcal{O}(k^3)$.
Let us introduce the notation $S_{(q-j)(j)} = S_{\underbrace{0...0}_{\text{q-j}}\underbrace{1...1}_{\text{j}}}$.
The following theorem constitutes the first new result of this master thesis. The proof is given in the Annex, in Appendix A.

Theorem 1. *The half-space representation of* $\mathcal{NS}^S_k$ *is given by*

$$\mathcal{NS}^S_k = \left\{\mathbf{S}\in V^k_S : 1+\sum_{j=1}^{k}\sum_{i=1}^{j}d_{ji}(l,u,v)S_{(j-i)(i)} \geq 0, \quad \forall (l,u,v)\in\mathbb{P}_k\right\}, \tag{13}$$

where $\mathbb{P}_k \equiv \{(a,b,c)\in\mathbb{N}^3 | a<k, b<x-a, c<a\}$ *and the coefficients are given by*

$$d_{ji}(l,u,v) = \frac{k!}{(k-j)!}\sum_{r=0}^{j-i}\sum_{s=0}^{i}(-1)^{j-r-s}\binom{u}{r}\binom{k-l-u}{j-i-r}\binom{v}{s}\binom{l-v}{i-s}. \tag{14}$$

Although we have an explicit expression for the facets of $\mathcal{NS}^S_k$, the problem of projecting them in the space of two-body correlators is highly non-trivial. Computing all the vertices of $\mathcal{NS}^S_k$ and then projecting them is not possible: for a polytope with n facets

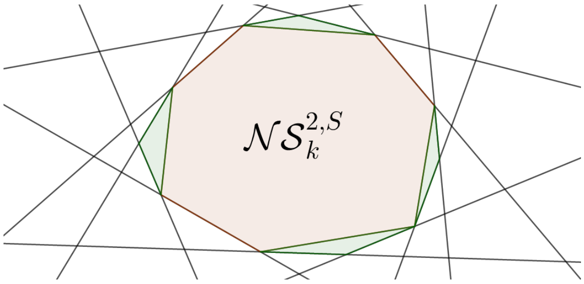


Figure 2: The algorithm finds a set of hyperplanes in $\mathbb{R}^5$ enclosing $\mathcal{NS}_k^{2,S}$. The resulting set of hyperplanes define an outer approximation (green) of $\mathcal{NS}_k^{2,S}$ (orange).

and v vertices in d dimensions, the complexity of the vertex enumeration problem is $\mathcal{O}(nvd)$, but in the worst case $v \sim n^{\lfloor d/2 \rfloor}$ (and in our case $d \sim k^2$, $n \sim k^3$). The standard approach would be using the Fourier-Motzkin algorithm to project the set of inequalities given by (13) in the space of two-body correlators, as it was done in [1]. Nevertheless, the time complexity of the Fourier-Motzkin algorithm is exponential, so with this approach it is not possible to go much further than $k = 6$. Instead, we make use of a novel algorithm designed to give outer approximations of the projection of a given set of inequalities in a lower dimensional space.
The idea behind the algorithm is quite simple: we pick a ray in the space of two body correlators, and using linear programming we find the hyperplane perpendicular to this ray that is closest to $\mathcal{NS}_k^S$ while enclosing it; doing that for a large number of rays, the algorithm finds a good approximation for the projection (figure 2). In each iteration, the set of rays is updated in order to improve the approximation.

Algorithm for the projection of $\mathcal{NS}_k^S$:

Input: the inequalities that define $\mathcal{NS}_k^S$ as given by Eq. (13)
1) (Initialization). Set $t = 1$. Generate n_d rays distributed uniformly over the unit 4-sphere: $X = \{\mathbf{x}_i\}_{i=1}^{n_d}$. Solve the following minimization problem:

$$\mathbf{S}_i^* = \arg \max_{\mathbf{S} \in \mathcal{NS}_k^S, \mathbf{x}_i \cdot \mathbf{S} \geq 0} \mathbf{x}_i \cdot \mathbf{S} \tag{15}$$

for all $\mathbf{x}_i \in X$. Store the vectors $\mathbf{v}_i = (S_{i,0}^*, S_{i,0}^*, S_{i,00}^*, S_{i,01}^*, S_{i,11}^*)$ in $\mathcal{V}^{(0)} = \{\mathbf{v}_i\}_i$.
2) (Calculation of the facets). Eliminate all the redundant vertices in the convex hull of $\mathcal{V}^{(n)}$. Calculate the facets corresponding to the convex hull of $\mathcal{V}^{(n)}$, and store them in $\mathcal{F}^{(n)}$.
3) (Update) Update $X = \{\mathbf{x}_i\}_i$, where now $\mathbf{x}_i$ is the normal ray of each facet of the set $\mathcal{F}^{(n)}$. Solve the maximization problem 15 for each $\mathbf{x}_i$. Store the vectors $\mathbf{v}_i = (S_{i,0}^*, S_{i,0}^*, S_{i,00}^*, S_{i,01}^*, S_{i,11}^*)$ in $\mathcal{V}^{(n+1)}$. Increase the level counter: $t \to t + 1$.
4) Stop if all the facets in $\mathcal{F}^{(n)}$ are already optimal. If not, go to step (5).
5) Stop if $t > t_c$. If not, return to step 2)
Output: $\mathcal{A} = \bigcap_i \{\mathbf{S} \in V_k^{2,S} \,|\, \mathbf{x}_i \cdot \mathbf{S} \leq \mathbf{x}_i \cdot \mathbf{v}_i\}$ is an outer approximation of $\mathcal{NS}_k^{2,S}$.

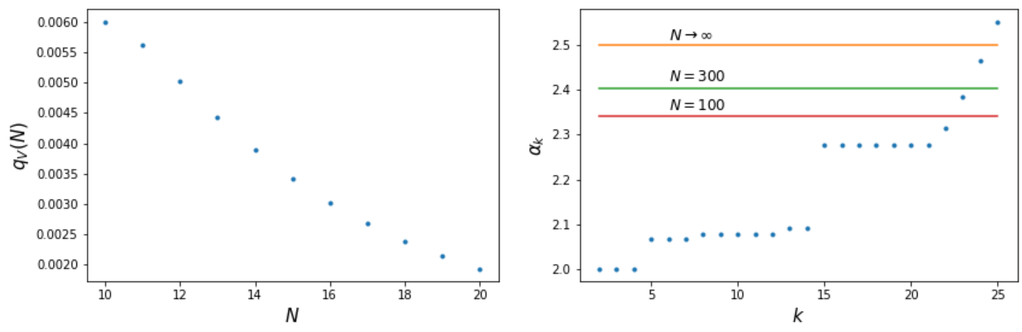
The main advantage of the algorithm with respect to Fourier-Motzkin is that the maximization problem (15) can be phrased as a linear program, as explained in the Annex in Appendix B, so the complexity of each iteration of the algorithm is poly(k). Note that if condition 4) is fulfilled, the output is the exact projection (the convex hull of $\mathcal{V}^{(n)}$ is contained in $\mathcal{NS}_k^{2,S}$, which in turn is contained in $\mathcal{A}$. If $\mathcal{V}^{(n)}$ and $\mathcal{A}$ coincide, they have to be equal to $\mathcal{NS}_k^{2,S}$). Nevertheless, the algorithm is not guaranteed to converge in polynomial time, although for our purposes an outer approximation is enough.

4. Nonlocality depth inequalities

Using the algorithm that we discussed above, we were able to calculate the vertices of $\mathcal{NS}_k^{2,S}$ for $k \leq 20$ (the algorithm converged after three iterations in all cases). We could obtain inequalities for k-nonlocality in two different limits: $k = N - 1$ and $k \ll N$.
The situation in which the nonlocality depth of an N-partite system is N (all parties nonlocally correlated) is referred as *genuine multipartite nonlocality*. In order to provide inequalities capable of revealing genuine multipartite nonlocality, we calculated all the vertices of the $\mathcal{PN}_{N,N-1}^{2,S}$ polytope using Eqs. (10) and (11). As in this particular case $N = k - 1$, all these vertices can be calculated from the partitions of the form $L_k = \{\mathcal{A}_1, \mathcal{A}_2\}$ composed of two sets with sizes $|\mathcal{A}_1| = k, k-1, ..., \lceil \frac{k}{2} \rceil$, $|\mathcal{A}_2| = k - |\mathcal{A}_1|$ (note that any other possible partition is included in one of them). After calculating the lower bounds $\beta_{N-1} = \min_{\mathbf{S} \in \mathcal{PN}_{N,N-1}^{2,S}} I(\mathbf{S})$, the final step is to search the inequalities of the form $I(\mathbf{S}) \geq -\beta_{N-1}$ for which β_{N-1} is smaller than the quantum bound $\beta_N^Q = \min_{\mathbf{S} \in \mathcal{Q}_N^{2,S}} I(\mathbf{S})$, i.e. the inequalities that can be violated by quantum systems. We used the procedure to calculate the quantum bound introduced in [8].
After calculating the facets of the $\mathcal{PN}_{N,N-1}^{2,S}$ for $10 \leq N \leq 14$, we realized that the following inequality for genuine multipartite nonlocality

$$106N(S_0 + S_1) + 21S_{00} + 64S_{01} + 21S_{11} \geq -106N^2 + 946N - 2520 \tag{16}$$

is valid in the range $10 \leq N \leq 21$. As the calculation of the facets is too costly for $N > 14$, for $N = 15, 16, 17, 18, 19, 20, 21$ we minimized the linear function given by $I(\mathbf{S}) = 106N(S_0 + S_1) + 21S_{00} + 64S_{01} + 21S_{11}$ over the set of vertices of $\mathcal{PN}_{N,N-1}^{2,S}$, showing that in this range the minimum coincides with the value given above. The maximal relative quantum violation is presented in Figure 3. We consider that the inequality (16) is a promising candidate for a witness of genuine multipartite nonlocality valid for any for any number of parties $N \geq 10$, but further work is need in order to prove or falsify this conjecture.
The problem now is that for this method we need to calculate first all the vertices of the $\mathcal{NS}_k^{2,S}$ polytopes with $k < N$. Although the complexity of our algorithm is polynomial, it still grows very fast, making difficult to compute the projection of no-signaling polytopes involving a very large number of parties. With Theorem 2, also a new result, we provide a tool by which, in principle, we could overcome this difficulty.



(a) (b)

Figure 3: (a) The maximum relative quantum violation of inequality (16), defined as $q_v(N) \equiv \frac{\beta_N^Q - \beta_{N-1}}{\beta_{N-1}}$, as a function of N. (b) $\alpha_k = \frac{\beta_k}{N} - \frac{1}{2N}$ as a function of k for the inequality (18). The solid lines represent the quantum bound $\frac{\beta_N^Q}{N}$ for $N = 100, 300, \infty$.

Theorem 2. *Let us consider the map* $\phi_k : \mathbb{R}^5 \to \mathbb{R}^5$ *given by*

$$\phi_k(\mathbf{S}) = \left(\frac{S_0}{k}, \frac{S_1}{k}, \frac{S_{00}}{k(k-1)}, \frac{S_{01}}{k(k-1)}, \frac{S_{11}}{k(k-1)} \right). \tag{17}$$

If $k \leq l$, *then* $\phi_l(\mathcal{NS}_l^{2,S}) \subset \phi_k(\mathcal{NS}_k^{2,S})$.

The proof is given in the Annex, in Appendix A. Theorem 2 gives us a hierarchical approximation of $\mathcal{NS}_l^{2,S}$: we can approximate $\mathcal{NS}_l^{2,S}$ by $\phi_l^{-1}(\phi_k(\mathcal{NS}_k^{2,S}))$ for any $l > k$. The question now is whether this approximation is good enough to find inequalities that are violated by quantum correlations. We will show that it is, at least for moderate values of l.

In the case of $k < N$, we studied the following inequality

$$I(\mathbf{S}) \equiv 2S_0 + \frac{1}{2}S_{00} + S_{01} + \frac{1}{2}S_{11} \geq -\beta_k. \tag{18}$$

Substituting Eqs. (10) and (11) in the inequality (18), we see that

$$\begin{aligned} I(\mathbf{S}) = \sum_{p,i} \xi_{pi} \left[I(\mathbf{S}(p,i)) - \frac{1}{2}\left(S_0(p,i) + S_1(p,i)\right)^2 - S_0(p,i) - S_1(p,i) \right] - \frac{1}{2} \\ + \frac{1}{2}\left[\sum_{p,i} \xi_{pi} \left(S_0(p,i) + S_1(p,i)\right) \right]^2. \end{aligned} \tag{19}$$

The last term is positive semidefinite, so we can forget about it, as we are interested in a lower bound. Then the minimum is just given by

$$\beta_k = \min_{p \in \{1,\ldots,k\}} \min_{i \in \{1,\ldots,n_p\}} \frac{N}{p} \left[I(\mathbf{S}(p,i)) - \frac{1}{2}\left(S_0(p,i) + S_1(p,i)\right)^2 - S_0(p,i) - S_1(p,i) \right] + \frac{1}{2}. \tag{20}$$

The quantity $\alpha_k \equiv \frac{\beta_k}{N} - \frac{1}{2N}$ is shown in Figure 3 for several values of k (note that α_k is constant with respect to N). In [8] it was proved that the quantum bound of the inequality (18) has the limit $\lim_{N\to\infty} \frac{\beta_N^Q}{N} = \frac{5}{2}$. For $k \leq 24$ we observe that $\alpha_k < \frac{5}{2}$. Therefore, for a sufficiently large value of N, our inequality can reveal a nonlocality depth $k \leq 24$ in a quantum system. The bounds for $k = 21, 22, 23, 24, 25$ were obtained by means of the hierarchical approximation given by Theorem 2.

5. Conclusion

The main result of this master thesis is a new algorithm for the approximation of the projection of the k-partite no-signaling polytopes in the space of two-body symmetric correlators, with poly(k) complexity. With this algorithm we were able to obtain inequalities that can reveal the nonlocality depth k of a quantum system for $k \leq 24$. Although it is an improvement with respect to previous work, we still lack a method to construct inequalities for arbitrarily large values of the nonlocality depth. To solve this issue, we proposed a hierarchical approximation of $\mathcal{NS}_k^{2S}$, and we showed that at least it can be used to extend a bit further the capability of the inequality (18) to detect nonlocality depth. Even though this hierarchy does not seem to be appropriate for the detection of genuine multipartite nonlocality, we think that it is worth trying to study if it can be employed to obtain inequalities for k-nonlocality depth in the limit $k \ll N$. In addition, we found a new inequality (Eq. (16)) that can detect genuine multipartite nonlocality in the range $10 \leq N \leq 21$. Further work is needed to determine if it is still valid for any number of parties larger than 21.

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