A CARTAN-EILENBERG APPROACH TO HOMOTOPICAL ALGEBRA

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ABSTRACT. In this paper we propose an approach to homotopical algebra where the basic ingredient is a category with two classes of distinguished morphisms: strong and weak equivalences. These data determine the cofibrant objects by an extension property analogous to the classical lifting property of projective modules. We define a Cartan-Eilenberg category as a category with strong and weak equivalences such that there is an equivalence between its localization with respect to weak equivalences and the localised category of cofibrant objects with respect to strong equivalences. This equivalence allows us to extend the classical theory of derived additive functors to this non additive setting. The main examples include Quillen model categories and categories of functors defined on a category endowed with a cotriple (comonad) and taking values on a category of complexes of an abelian category. In the latter case there are examples in which the class of strong equivalences is not determined by a homotopy relation. Among other applications of our theory, we establish a very general acyclic models theorem.

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In their pioneering work [CE], H. Cartan and S. Eilenberg defined the notion of derived functors of additive functors between categories of modules. Their approach is based on the characterization of projective modules over a ring $R$ in terms of the notions of homotopy between morphisms of complexes of $R$-modules and of quasi-isomorphisms of complexes. Projective modules can be characterized from them: an $R$-module $P$ is projective if for every solid diagram

\[
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\downarrow{w} & & \downarrow{w} \\
P & \xrightarrow{f} & Y
\end{array}
\]

where $w$ is a quasi-isomorphism of complexes, there is a lifting $g$ such that the resulting diagram is homotopy commutative. The lifting is unique up to homotopy.

A. Grothendieck, in his Tohoku paper [Gr], introduced abelian categories and extended Cartan-Eilenberg methods to derive additive functors between them. Later on, Grothendieck stressed the importance of complexes, rather than modules, and promoted the introduction of derived categories by J.L. Verdier.

In modern language the homotopy properties of projective complexes can be summarized in the following manner. If $\mathcal{A}$ is an abelian category with enough projective objects, then there is an equivalence of categories

\[
\mathbf{K}_+(\text{Proj}(\mathcal{A})) \xrightarrow{\sim} \mathbf{D}_+(\mathcal{A})
\]

(0.1)

where $\mathbf{K}_+(\text{Proj}(\mathcal{A}))$ is the category of positive chain complexes of projective objects modulo homotopy and $\mathbf{D}_+(\mathcal{A})$ is the corresponding derived category. Additive functors can therefore be derived as follows. If $F: \mathcal{A} \to \mathcal{B}$ is an additive functor, it induces a functor $\widetilde{F}: \mathbf{K}_+(\text{Proj}(\mathcal{A})) \to \mathbf{K}_+(\mathcal{B})$ and by the equivalence (0.1), we obtain the derived functor $\mathbb{L}F: \mathbf{D}_+(\mathcal{A}) \to \mathbf{D}_+(\mathcal{B})$.

In order to derive non additive functors, D. Quillen, inspired by topological methods, introduced model categories in his notes on Homotopical Algebra [Q]. Since then, Homotopical Algebra has grown considerably as can be seen, for example, in the books [DHKS], [Ho], [Hi]. Quillen’s approach applies to classical homotopy theory as well as to rational homotopy, Bousfield localisation, or more recently to simplicial sheaves or motivic homotopy theory.
In a Quillen model category $\mathcal{C}$, a homotopy relation for morphisms is defined from the axioms and one of the main results of [Q] is the equivalence

$$\pi_{\mathcal{C}_{cf}} \sim \mathcal{C}[W^{-1}], \quad (0.2)$$

where $\pi_{\mathcal{C}_{cf}}$ is the homotopy category of the full subcategory $\mathcal{C}_{cf}$ of fibrant-cofibrant objects, and $\mathcal{C}[W^{-1}]$ is the localised category with respect to weak equivalences. The equivalence (0.2) extends the one for projective complexes (0.1) and allows derivation of functors in this setting.

The set of axioms of model categories is, in some sense, somewhat strong because there are interesting categories in which to do homotopy theory that do not satisfy all of them. Several authors (see [Br], [Ba] and others) have developed simpler alternatives, all of them focused on laterality, asking only for a left- (or right-) handed version of Quillen’s set of axioms. All these alternatives are very close to Quillen’s formulation.

Here we propose another approach which is closer to the original development by Cartan-Eilenberg. The initial data are two classes of morphisms $\mathcal{S}$ and $\mathcal{W}$ in a category $\mathcal{C}$, with $\mathcal{S} \subseteq \mathcal{W}$, which we call strong and weak equivalences, respectively. We define an object $M$ of $\mathcal{C}$ to be cofibrant if for every solid diagram

$$\begin{array}{ccc}
X & \xrightarrow{w} & Y \\
\downarrow{g} & & \downarrow{w} \\
M & \xrightarrow{f} & Y
\end{array}$$

with $w$ a weak equivalence, there is a unique lifting $g$ in $\mathcal{C}[\mathcal{S}^{-1}]$ such that the diagram is commutative. We say that $\mathcal{C}$ is a Cartan-Eilenberg category if it has enough cofibrant objects, that is, if for each object $X$ in $\mathcal{C}$ there is a weak equivalence $M_X \rightarrow X$ where the source is cofibrant. In that case the functor

$$\mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}] \sim \mathcal{C}[W^{-1}] \quad (0.3)$$

is an equivalence of categories, where $\mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}]$ is the full subcategory of $\mathcal{C}[\mathcal{S}^{-1}]$ generated by the cofibrant objects of $\mathcal{C}$.

In a Cartan-Eilenberg category we can derive functors exactly in the same way as Cartan-Eilenberg. If $\mathcal{C}$ is a Cartan-Eilenberg category and $F : \mathcal{C} \rightarrow \mathcal{D}$ is a functor which sends strong equivalences to isomorphisms, $F$ induces a functor $\tilde{F} : \mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}] \rightarrow \mathcal{D}$ and by the equivalence (0.3), we obtain the derived functor $LF : \mathcal{C}[W^{-1}] \rightarrow \mathcal{D}$.

Each Quillen model category produces a Cartan-Eilenberg category: the category of its fibrant objects, with $\mathcal{S}$ the class of homotopy equivalences and $\mathcal{W}$ the class of weak equivalences. Nevertheless, note the following differences with Quillen’s theory. First, in the Quillen context the class $\mathcal{S}$ appears as a consequence of the axioms while fibrant/cofibrant objects are part of them. Second, cofibrant objects in the Cartan-Eilenberg sense are homotopy invariant, in contrast with cofibrant objects in Quillen model categories. Actually, in a Quillen category a fibrant object is Cartan-Eilenberg cofibrant if and only if it is homotopy equivalent to a Quillen cofibrant one.

Another example covered by our presentation is that of Sullivan’s minimal models. We define minimal objects in a Cartan-Eilenberg category and call it a Sullivan category if any object
has a minimal model. As an example, we interpret some results of [GNPR1] as saying that the category of modular operads over a field of characteristic zero is a Sullivan category.

In closing this introduction, we want to highlight the definition of Cartan-Eilenberg structures coming from a cotriple. If $\mathcal{X}$ is a category with a cotriple $G$, $\mathcal{A}$ is an abelian category and $\mathbb{C}_+(\mathcal{A})$ denotes the category of positive chain complexes of $\mathcal{A}$, we define a structure of Cartan-Eilenberg category on the functor category $\text{Cat}(\mathcal{X}, \mathbb{C}_+(\mathcal{A}))$ (see theorem 6.1.4). We apply this result to obtain theorems of the acyclic models kind, extending results in [B] and [GNPR2]. We stress that in these examples the class of strong equivalences $S$ does not come from a homotopy relation. We also prove a cubical version of acyclic models used in [GN] without proof.

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1. Localisation of Categories

In this section we collect for further reference some mostly well-known facts about localisation of categories, and we introduce the notion of relative localisation, which plays an important role in the sequel.

1.1. Categories with weak equivalences.

1.1.1. By a category with weak equivalences we understand a pair $(\mathcal{C}, \mathcal{W})$ where $\mathcal{C}$ is a category and $\mathcal{W}$ is a class of morphisms of $\mathcal{C}$. Morphisms in $\mathcal{W}$ will be called weak equivalences.

We always assume that $\mathcal{W}$ is stable by composition and contains all the isomorphisms of $\mathcal{C}$, so that we can identify $\mathcal{W}$ with a subcategory of $\mathcal{C}$.

1.1.2. Recall that the category of fractions, or localisation, of $\mathcal{C}$ with respect to $\mathcal{W}$ is a category $\mathcal{C}[\mathcal{W}^{-1}]$ together with a functor $\gamma : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ such that:

(i) For all $w \in \mathcal{W}$, $\gamma(w)$ is an isomorphism.

(ii) For any category $\mathcal{D}$ and any functor $F : \mathcal{C} \to \mathcal{D}$ that transforms morphisms $w \in \mathcal{W}$ to isomorphisms, there exists a unique functor $F' : \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ such that $F' \circ \gamma = F$.

The uniqueness condition on $F'$ implies immediately that, when it exists, the localisation is uniquely defined up to isomorphism. The localisation exists if $\mathcal{W}$ is small, and, in general, the localisation always exists in a higher universe.

We denote by $\text{Cat}_W(\mathcal{C}, \mathcal{D})$ the category of functors from $\mathcal{C}$ to $\mathcal{D}$ that send morphisms in $\mathcal{W}$ to isomorphisms. The definition of the category of fractions means that for any category $\mathcal{D}$, the functor

$$\gamma^* : \text{Cat}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \to \text{Cat}_W(\mathcal{C}, \mathcal{D}), \quad G \mapsto G \circ \gamma$$

induces a bijection on the class of objects. This implies easily that $\gamma^*$ is an isomorphism of categories.

1.1.3. We say that the class of weak equivalences $\mathcal{W}$ is saturated if a morphism $f$ of $\mathcal{C}$ is in $\mathcal{W}$ when $\gamma f$ is an isomorphism. The saturation $\overline{\mathcal{W}}$ of $\mathcal{W}$ is the preimage by $\gamma$ of the isomorphisms of $\mathcal{C}[\mathcal{W}^{-1}]$. It is the largest saturated class of morphisms of $\mathcal{C}$ which contains $\mathcal{W}$. 
1.1.4. Let \((C, W)\) and \((C', W')\) be two categories with weak equivalences. A functor \(F : C \rightarrow C'\) such that \(F(W) \subseteq W'\) obviously induces a functor between the localised categories \(F' : C[W^{-1}] \rightarrow C'[W'^{-1}]\). Because of its potential applications, there has been interest in giving sufficient conditions that assure that \(F'\) is an equivalence of categories. Kahn and Sujatha ([KS]) have given a solution in the style of Quillen’s theorem A. In this paper we propose a different approach.

1.2. Hammocks. In this section we describe the localisation of categories by using hammocks as introduced by Dwyer-Kan ([DK]). Given a category with weak equivalences \((C, W)\) and two objects \(X\) and \(Y\) in \(C\), a \(W\)-zigzag \(f\) from \(X\) to \(Y\) is a finite sequence of morphisms of \(C\), going in either direction, between \(X\) and \(Y\)

\[
f : X \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow Y
\]

where the morphisms going from right to left are in \(W\). Because each \(W\)-zig is a diagram, it has a type, its index category. A morphism from a \(W\)-zigzag \(f\) to a \(W\)-zigzag \(g\) of the same type is a commutative diagram in \(C\)

![Hammock diagram](image)

A hammock between two \(W\)-zigzags \(f\) and \(g\) from \(X\) to \(Y\) of the same type is a finite sequence of morphisms of zigzags going in either direction. More precisely, it is a commutative diagram \(H\) in \(C\)

![Diagram of hammock between two zigzags](image)

such that

(i) in each column of arrows, all (horizontal) maps go in the same direction, and if they go to the left they are in \(W\) (in particular, any row is a \(W\)-zigzag),
(ii) in each row of arrows, all (vertical) maps go in the same direction, and they are arbitrary maps in \( C \),

(iii) the top \( W \)-zigzag is \( f \) and the bottom is \( g \).

If there is a hammock \( H \) between \( f \) and \( g \), and \( f' \) is a \( W \)-zigzag obtained from \( f \) adding identities, then adding the same identities in the hammock \( H \) and in the \( W \)-zigzag \( g \) we obtain a new hammock \( H' \) and a new \( W \)-zigzag \( g' \) such that \( H' \) is a hammock between \( f' \) and \( g' \).

We say that two \( W \)-zigzags \( f, g \) between \( X \) and \( Y \) are related if there exist \( W \)-zigzags \( f' \) and \( g' \) of the same type, obtained from \( f \) and \( g \) by adding identities, and a hammock \( H \) between \( f' \) and \( g' \). We consider the equivalence relation generated by related \( W \)-zigzags.

Let \( C_W \) be the category whose objects are the objects of \( C \) and, for any two objects \( X, Y \), the morphisms from \( X \) to \( Y \) are the equivalence classes of \( W \)-zigzags from \( X \) to \( Y \), and composition is juxtaposition of \( W \)-zigzags.

**Theorem 1.2.1.** ([DHKS], 33.10). The category \( C_W \), together with the obvious functor \( C \rightarrow C_W \) is a solution to the universal problem of the category of fractions \( C[W^{-1}] \).

In the cited reference there is a general hypothesis which concerns the class \( W \), which is not necessary for this result.

### 1.3. Categories with a congruence.

There are some situations where it is possible to give an easiest presentation of morphisms of the category \( C[W^{-1}] \), for example, when there is a calculus of fractions (see [GZ]). In this section we present an even simpler situation which will occur later, the localisation provided by some quotient categories.

1.3.1. Let \( C \) be a category and \( \sim \) a congruence on \( C \), that is, an equivalence relation between morphisms of \( C \) which is compatible with composition ([ML], page 51). We denote by \( C/\sim \) the quotient category, and by \( \pi : C \rightarrow C/\sim \) the universal canonical functor. We denote by \( S \) the class of morphisms \( f : X \rightarrow Y \) such that there exist \( g, g' : Y \rightarrow X \) such that \( fg \sim 1_Y \) and \( g'f \sim 1_X \). We will call \( S \) the class of equivalences associated to \( \sim \).

1.3.2. If \( \sim \) is a congruence one can also obtain the localised category \( \delta : C \rightarrow C[S^{-1}] \) of \( C \) with respect to the class \( S \) of equivalences defined by this congruence.

It follows easily from the definitions that one has:

**Proposition 1.3.3.** Let \( \sim \) be a congruence and \( S \) the associated class of equivalences. If \( S \) and \( \sim \) are compatible, that is, if \( f \sim g \) implies \( \delta f = \delta g \), then the categories \( C/\sim \) and \( C[S^{-1}] \) are canonically isomorphic.

**Example 1.3.4.** The congruence \( \sim \) is compatible with its class \( S \) of equivalences when it may be expressed by a cylinder object, or dually by a path object.

Recall that, if \( X \in \text{Ob} \ C \) a cylinder object over \( X \) is an object \( \text{Cyl}(X) \) in \( C \) together with morphisms \( i_0, i_1 : X \rightarrow \text{Cyl}(X) \) and \( p : \text{Cyl}(X) \rightarrow X \) such that \( p \in S \) and \( pi_0 = 1_X = pi_1 \).

Now, suppose that the congruence is determined by cylinder objects in the following way: \( f \sim g : X \rightarrow Y \) if and only if there exists a morphism \( H : \text{Cyl}(X) \rightarrow Y \) with \( Ht_0 = f \).
and $Hi_1 = g$”. Then $\sim$ and $S$ are compatible. In fact, if $f \sim g : X \to Y$, then we have the $S$-hammock

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
Cyl(X) & \xrightarrow{H} & Y \\
\downarrow & & \downarrow \\
X & \xleftarrow{i_0} & X
\end{array}
$$

between $f$ and $g$, which shows that $\delta(f) = \delta(g)$ in $C[S^{-1}]$.

More generally, $\sim$ and $S$ are compatible if $\sim$ is the equivalence relation transitively generated by a cylinder object.

1.4. **Relative localisation.** Let $\sim$ be a congruence on a category $C$. If $i : M \to C$ is a full subcategory, there is an induced congruence on $M$ and the quotient category $M/\sim$ is a full subcategory of $C/\sim$. Nevertheless, if $E$ denotes the class of equivalences associated to $\sim$, and $E_M$ the morphisms in $M$ which are in $E$, the functor $\tilde{i} : M[E_M^{-1}] \to C[E^{-1}]$ is not faithful, in general. More generally, if $E$ is an arbitrary class of morphisms in $C$, the functor $\tilde{i} : M[E_M^{-1}] \to C[E^{-1}]$ is neither faithful nor full.

**Definition 1.4.1.** Let $(C, E)$ be a category with weak equivalences and $M$ a full subcategory. The **relative localisation** of $M$ with respect to $(C, E)$, denoted by $M[E^{-1}, C]$, is the full subcategory of $C[E^{-1}]$ whose objects are those of $M$.

To simplify notations, in the situation above we write $M[E^{-1}]$ for $M[E_M^{-1}]$.

This relative localisation is necessary in order to express the main results of this paper (e.g. Theorem 2.3.2). See Remark 4.2.3 for an interesting example where the relative localisation $M[E^{-1}, C]$ is not equivalent to the localisation $M[E^{-1}]$. However, in some common situations there is no distinction between them, as for example in the proposition below, which is an abstract generalised version of theorem III.2.10 in [GMa].

**Proposition 1.4.2.** Let $(C, E)$ be a category with weak equivalences. Suppose that $E$ has a right calculus of fractions and that for every morphism $w : X \to M$ in $\overline{E}$, with $M \in \text{Ob } M$, there exists a morphism $N \to X$ in $\overline{E}$, where $N$ is in $M$. Then $\tilde{i} : M[E^{-1}, C]$ is an equivalence of categories.

**Proof.** Let’s prove that $\tilde{i}$ is full: if $f = \sigma^{-1} \circ g : M_1 \to M_2$ is a morphism in $C[E^{-1}]$ between objects of $M$, where $\sigma \in E$, take a weak equivalence $\rho : N \to X$ with $N \in \text{Ob } M$, whose existence is guaranteed by hypothesis. Then $f = (\sigma \circ \rho)^{-1} \circ (\rho \circ g)$ is a morphism of $M[E^{-1}]$. The faithfulness is proved in a similar way. \qed
2. CARTAN-EILENBERG CATEGORIES

In this section we define cofibrant objects in a relative setting given by two classes of morphisms, as a generalisation of projective complexes in an abelian category. We then introduce Cartan-Eilenberg categories and give some criteria to prove that a given category is Cartan-Eilenberg. We also relate these notions with Adams’ study of localisation in homotopy theory, [A].

2.1. Models in a category with strong and weak equivalences. In this section we introduce models of objects and diagrams in categories with two distinguished classes of morphisms.

Let $C$ be a category and $S, W$ two classes of morphisms of $C$. Recall that our classes of morphisms are closed under composition and contain all isomorphisms.

Definition 2.1.1. We say that $(C, S, W)$ is a category with strong and weak equivalences if $S \subseteq W$. Morphisms in $S$ are called strong equivalences and those in $W$ are called weak equivalences.

The basic example of a category with strong and weak equivalences is the category of bounded below chain complexes of $R$-modules $C_+^R$, for a ring $R$, with $S$ the class of homotopy equivalences and $W$ the class of quasi-isomorphisms.

Notation 2.1.2. It is convenient to fix some notation for the rest of the paper. Let $(C, S, W)$ be a category with strong and weak equivalences. We denote by $\delta : C \rightarrow C[S^{-1}]$ and $\gamma : C \rightarrow C[W^{-1}]$ the canonical functors. Since $S \subseteq W$, the functor $\gamma$ factors through $\delta$ in the form

\[
\begin{array}{ccc}
C & \xrightarrow{\gamma} & C[W^{-1}] \\
\delta \downarrow & & \downarrow \gamma' \\
C[S^{-1}] & \xrightarrow{\gamma} & \end{array}
\]

The functor $\gamma'$ will always stand for the functor defined by this factorisation.

Definition 2.1.3. Let $(C, S, W)$ be a category with strong and weak equivalences, $M$ a full subcategory of $C$ and $X$ an object of $C$. A left $S$-model of $X$ (with respect to $W$) in $M$ is an object $M$ in $M$ together with a morphism $\delta(M) \rightarrow \delta(X)$ which is in $\overline{\delta(W)}$.

We say that there are enough left $S$-models in $M$, or that $M$ is a subcategory of left models of $(C, S, W)$, if each object of $C$ has a left $S$-model in $M$ with respect to $W$.

2.2. Cofibrant objects.

Definition 2.2.1. Let $(C, S, W)$ be a category with strong and weak equivalences. An object $M$ of $C$ is called $(S, W)$-cofibrant, or simply cofibrant, if for each $w : X \rightarrow Y \in W$, the map

\[ w_* : C[S^{-1}](M, X) \rightarrow C[S^{-1}](M, Y), \quad g \mapsto w \circ g \]

is bijective.
That is to say, cofibrant objects are defined by a lifting property, in \( C[S^{-1}] \), with respect to weak equivalences: for any solid diagram as

\[
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\downarrow^{w} & & \downarrow^{w} \\
M & \xleftarrow{f} & Y
\end{array}
\]

with \( w \in W \) and \( f \) a morphism of \( C[S^{-1}] \), there exists a unique morphism \( g \) of \( C[S^{-1}] \) making the triangle commutative.

Cofibrant objects are characterised by a formal Whitehead type theorem, as follows from the next result.

**Proposition 2.2.2.** Let \((C, S, W)\) be a category with strong and weak equivalences, and \( M \) an object of \( C \). The following conditions are equivalent:

(i) \( M \) is cofibrant.
(ii) For all \( w : X \rightarrow Y \in \delta(W) \) the map \( w_* : C[S^{-1}](M, X) \rightarrow C[S^{-1}](M, Y) \) is bijective.
(iii) For all \( X \in \text{Ob} \ C \) the map \( C[S^{-1}](M, X) \rightarrow C[W^{-1}](\gamma'M, \gamma'X) \) is bijective.

**Proof.** Firstly, (i) is equivalent to (ii). Indeed, if (i) is satisfied, the functor \( C[S^{-1}](M, -) : C[S^{-1}] \rightarrow \text{Sets} \) sends the morphisms in \( \delta(W) \) to isomorphisms, hence it sends the morphisms in \( \delta(W) \) to isomorphisms and (ii) is satisfied. The converse is obvious.

Secondly, (i) follows immediately from (iii) because each \( w \in W \) induces an isomorphism in \( C[W^{-1}] \). In fact, if \( w : X \rightarrow Y \in W \) we have a commutative diagram

\[
\begin{array}{ccc}
C[S^{-1}](M, X) & \xrightarrow{\cong} & C[W^{-1}](\gamma'M, \gamma'X) \\
\downarrow^{w_*} & & \downarrow^{w_*} \\
C[S^{-1}](M, Y) & \xrightarrow{\cong} & C[W^{-1}](\gamma'M, \gamma'Y)
\end{array}
\]

Finally, let us see that (i) implies (iii). To see the surjectivity, for \( f \) a \( W \)-zigzag from \( M \) to \( X \), we look for a morphism from \( M \) to \( X \) in \( C[S^{-1}] \) equivalent to \( f \). If \( m \) is the length of \( f \), we proceed by induction on \( m \). Suppose \( m = 1 \). If \( f \in C \) the result is obvious. If \( f = w^{-1} \), where \( w : X \rightarrow M \in W \), since \( M \) is a cofibrant object, there exists \( s \in C[S^{-1}](M, X) \) such that \( w_* s = 1_M \) in \( C[S^{-1}](M, M) \), that is \( ws = 1_M \), hence \( f = w^{-1} = s \) in \( C[W^{-1}] \).

\[
\begin{array}{ccc}
X & \xrightarrow{s} & M \\
\downarrow^{w} & & \downarrow^{w} \\
M & \xleftarrow{1_M} & M
\end{array}
\]

Now, we suppose that \( m > 1 \). There are two cases:

a) Suppose that we can write \( f = w^{-1} f_1 \), where \( w : X \rightarrow X_1 \in W \) and \( f_1 : M \rightarrow X_1 \) is a \( W \)-zigzag in \( C \) which has length \( m - 1 \). By the induction hypothesis there exists \( f'_1 \in C[S^{-1}](M, X_1) \)
such that $f'_1 = f_1$ en $C[W^{-1}]$. Since $M$ is cofibrant there exists $f''_1 \in C[S^{-1}](M, X)$ such that $wf''_1 = f'_1$ in $C[S^{-1}](M, X_1)$. Therefore $f = f''_1$ in $C[W^{-1}]$.

\[ \begin{array}{c} X \\ \downarrow ^w \\ M \xrightarrow{f_1} X_1 \end{array} \]

b) Suppose we can write $f = gf_1$, where $g : X_1 \rightarrow X \in C$, and $f_1 : M \rightarrow X_1$ is a $W$-zigzag in $C$ which has length $m - 1$. By the induction hypothesis there exists $f'_1 \in C[S^{-1}](M, X_1)$ such that $f'_1 = f_1$ en $C$, hence $gf'_1 \in C[S^{-1}](M, X)$ satisfies $gf'_1 = f$ in $C[W^{-1}]$.

To see the injectivity, considerer $f, g \in C[S^{-1}](M, X)$ such that $\gamma'(f) = \gamma'(g)$. Let $f', g'$ be two $\delta(W)$-zigzags of $C[S^{-1}]$ which are obtained from $f, g$ adding some identities, and $H$ be a hammock in $C[S^{-1}]$ between the two $\delta(W)$-zigzags $f'$ and $g'$. We proceed by induction on the minimum number $n$ of columns of the hammock $H$ in which the morphisms go to the left. If $n = 0$, then $f = g$ trivially. If $n > 0$, we write, for example,

$H = (M \xrightarrow{f_{i0}} X_{i1} \xleftarrow{f_{i1}} X_{i2} \xrightarrow{f_{i2}} X_{i3} \xrightarrow{\cdots} X_{ip} \xrightarrow{f_{ip}} X )$

where

\[
X_{\bullet i} = \begin{pmatrix}
X_{1i} \\
a_{1i} \\
X_{2i} \\
a_{2i} \\
\vdots \\
a_{m-1,i} \\
X_{mi}
\end{pmatrix}
\]

are the columns of the hammock $H$. The top and bottom rows of $H$ are $f'$ and $g'$ respectively, hence their composition are $f$ and $g$ respectively. Since $M$ is cofibrant and $f_{i1} \in \delta(W)$, there exists a unique $h_i : M \rightarrow X_{i2}$ in $C[S^{-1}]$ such that $f_{i1} \circ h_i = f_{i0}$ for each $i$, that is $f_{\bullet 1} \circ h_{\bullet} = f_{\bullet 0}$. 

\[ \begin{array}{c} X_{i1} \\ \downarrow ^{f_{i1}} \\ X_{i2} \\
\vdots \\
X_{i+1,1} \xrightarrow{f_{i+1,1}} X_{i+1,2} \\
\vdots \\
M \xrightarrow{a_{11}} X_{i+1,0} \\
\vdots \\
X_{i+1,1} \xrightarrow{a_{i2}} X_{i+1,2} \\
\vdots \\
X_{i+1,1} \xrightarrow{a_{m-1,i}} X_{i+1,2} \end{array} \]
Moreover, for each \( i \), if \( a_{i1} \) goes down we have \( a_{i1} \circ f_{i0} = f_{i+1,0} \) and \( a_{i1} \circ f_{i1} = f_{i+1,1} \circ a_{i2} \), therefore we have

\[
f_{i+1,1} \circ a_{i2} \circ h_i = a_{i1} \circ f_{i1} \circ h_i = a_{i1} \circ f_{i0} = f_{i+1,0} = f_{i+1,1} \circ h_{i+1}
\]

and, since \( f_{i+1,1} \in \delta(W) \) and \( M \) is cofibrant, it follows that \( a_{i2} \circ h_i = h_{i+1} \). If \( a_{i1} \) goes up we can prove in the same way that \( a_{12} \circ h_{i+1} = h_i \). Hence we have a morphism \( h_\bullet : M \to X_{\bullet 2} \), and a hammock in \( C[S^{-1}] \) between two \( \delta(W) \)-zigzags from \( M \) to \( X \)

\[
K = (M \xrightarrow{f_{\circ \text{coh} \bullet}} X_{\bullet 3} \xrightarrow{f_{\bullet 4}} X_{\bullet 4} \xrightarrow{f_{\bullet 2}} X_{\bullet 3} \cdots X_{\bullet p} \xrightarrow{f_{\bullet p}} X)
\]

The composition of the top and bottom rows of \( K \) are \( f \), and \( g \) respectively, and \( K \) is shorter than \( H \). By the induction hypothesis we obtain \( f = g \).

\[
\square
\]

2.2.3. Now we can establish a basic fact of our theory which includes a formal version of the Whitehead theorem in the homotopy theory of topological spaces.

We denote by \( C_{\text{cof}} \) the full subcategory of \( C \) whose objects are the cofibrant objects of \( C \), by \( i : C_{\text{cof}}[S^{-1},C] \to C[S^{-1}] \) the inclusion functor, and by \( j : C_{\text{cof}}[S^{-1},C] \to C[W^{-1}] \) the composition \( \gamma' \circ i \).

**Theorem 2.2.4.** Let \( (C,S,W) \) be a category with strong and weak equivalences and \( M \) be a full subcategory of \( C_{\text{cof}} \). The functor

\[
j : M[S^{-1},C] \to C[W^{-1}]
\]

is full and faithful. In particular, \( j \) reflects isomorphisms, that is to say, if \( w : M \to N \in \delta(W) \), where \( M \) and \( N \) are in \( M \), then \( w \) is an isomorphism in \( C[S^{-1}] \).

\[
\square
\]

2.3. Cartan-Eilenberg categories. For a category \( C \) with strong and weak equivalences the general problem is to know if there are enough cofibrant objects. This problem is equivalent to the orthogonal category problem for \( (C[S^{-1}],\delta(W)) \) (see [Bo](I.5.4)), which has been studied by Casacuberta and Chorny in the context of homotopy theory (see [CCh]). If the subcategory of cofibrant objects is a left model subcategory of \( C \), the category \( C \) will be called a left Cartan-Eilenberg category. It is a non additive generalisation for the category of complexes of an abelian category with enough projective objects.

**Definition 2.3.1.** A category with strong and weak equivalences \( (C,S,W) \) is called a left Cartan-Eilenberg category if each object of \( C \) has a cofibrant left \( S \)-model.

A pair \( (C,W) \) is called a left Cartan-Eilenberg category when the triple \( (C,S,W) \), with \( S \) the class of isomorphisms of \( C \), is a left Cartan-Eilenberg category.

The next result follows directly from Theorem 2.2.4.

**Theorem 2.3.2.** A category with strong and weak equivalences \( (C,S,W) \) is a left Cartan-Eilenberg category if and only if

\[
j : C_{\text{cof}}[S^{-1},C] \to C[W^{-1}]
\]

is an equivalence of categories.

\[
\square
\]
When proving that a category with strong and weak equivalences is a Cartan-Eilenberg category, recognizing cofibrant objects may prove difficult, as the definition is given in terms of a lifting property in $\mathcal{C}[S^{-1}]$. The sufficient conditions we state in the next result are the basic properties of the category of graded-commutative differential $k$-algebras, as treated in [GM], in order to obtain Sullivan minimal models. This is also the course we followed to study the homotopy theory of modular operads in [GNPR1].

**Theorem 2.3.3.** Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a category with strong and weak equivalences and $\mathcal{M}$ a full subcategory of $\mathcal{C}$. Suppose that each object $M$ in $\mathcal{M}$ satisfies the following properties:

1. For any $w : Y' \to Y \in \mathcal{W}$ and any $f : M \to Y \in \text{Mor} \mathcal{C}$ there exists a morphism $f' : M \to Y' \in \text{Mor} \mathcal{C}[S^{-1}]$ such that $w \circ f' = f$ in $\mathcal{C}[S^{-1}]$.
2. For any $w : Y' \to Y \in \mathcal{W}$, the map $w_* : \mathcal{C}[S^{-1}](M, Y') \to \mathcal{C}[S^{-1}](M, Y)$ is injective.

and assume also that any object $X$ of $\mathcal{C}$ has a left model in $\mathcal{M}$ with respect to $\mathcal{W}$. Then,

1. Every object in $\mathcal{M}$ is cofibrant and $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category.
2. The functors $\mathcal{M}[S^{-1}, \mathcal{C}] \to \mathcal{C}_{cof}[S^{-1}, \mathcal{C}] \to \mathcal{C} [\mathcal{W}^{-1}]$ are equivalences of categories.

**Proof.** We have to prove that, for each $M$ in $\mathcal{M}$, given $w \in \mathcal{W}$ and $f \in \mathcal{C}[S^{-1}](M, Y)$, there exists $f' \in \mathcal{C}[S^{-1}](M, Y')$ such that $w f' = f$.

\[
\begin{array}{c}
M \xrightarrow{f} \xrightarrow{w} Y' \\
M \xrightarrow{f'} Y
\end{array}
\]

Suppose that $f \in \mathcal{C}[S^{-1}](M, Y)$ can be written as a $\mathcal{S}$-zigzag of length $m$. We proceed by induction on $m$. The case $m = 1$ is just the hypothesis.

Let $m > 1$. a) Suppose that $f = f_1 s^{-1} g$, where $g : M \to X_1 \in \mathcal{C}$, $s : X_2 \to X_1 \in \mathcal{S}$ and $f_1 : X_2 \to Y$ is a $\mathcal{S}$-zigzag of $\mathcal{C}$ which has length $m - 2$. Let $\epsilon : M_2 \to X_2$ be a left model of $X_2$ in $\mathcal{M}$. Then, there exists $g' \in \mathcal{C}[S^{-1}]$ such that $g = seg'$, by the induction hypothesis, and there exists $f_1' \in \mathcal{C}[S^{-1}]$ such that $f_1 \epsilon = w f_1'$, since $f_1 \epsilon$ has length $m - 1$. Let $f' := f_1' g'$. Then $w f' = w f_1' g' = f_1 \epsilon g' = f_1 s^{-1} s g' = f_1 s^{-1} g = f$.

b) Suppose that $f = f_1 g s^{-1}$, where $s : X_1 \to M \in \mathcal{S}$, $g : X_1 \to X_2 \in \mathcal{C}$ and $f_1 : X_2 \to Y$ is a $\mathcal{S}$-zigzag of $\mathcal{C}$ which has length $m - 2$. Let $\epsilon_i : M_i \to X_i$ be a left model of $X_i$ in $\mathcal{M}$, for $i = 1, 2$. Then, there exists $f_1' \in \mathcal{C}[S^{-1}]$ such that $f_1 \epsilon_2 = w f_1'$ by the induction hypothesis, and there exist $g' \in \mathcal{C}[S^{-1}]$ such that $g \epsilon_1 = \epsilon_2 g'$ and $h \in \mathcal{C}[S^{-1}]$ such that $Id_M = s \epsilon_1 h$ by (i). Let
2.4. Relation with reflective subcategories. In some cases, localisation of categories may be realised through reflective subcategories (see [Bo](I.3.5.2)) or, equivalently, by Adams idempotent functors (see [A], section 2). The following result relates Cartan-Eilenberg categories with these concepts. Observe that in order to be consistent with our laterality assumptions we consider the dual notions of coreflective subcategories and coidempotent functor.

Recall that a full and replete subcategory \( \mathcal{A} \) of a category \( \mathcal{B} \) is called coreflexive if the inclusion functor \( i : \mathcal{A} \longrightarrow \mathcal{B} \) has a right adjoint.

**Theorem 2.4.1.** Let \( (\mathcal{C}, \mathcal{S}, \mathcal{W}) \) be a category with strong and weak equivalences. Then conditions (i) and (ii) are equivalent:

(i) \( (\mathcal{C}, \mathcal{S}, \mathcal{W}) \) is a left Cartan-Eilenberg category.

(ii) The inclusion functor \( i : \mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}] \longrightarrow \mathcal{C}[\mathcal{S}^{-1}] \) admits a right adjoint \( r : \mathcal{C}[\mathcal{S}^{-1}] \longrightarrow \mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}] \) such that, if \( \epsilon' : i \circ r \Rightarrow 1 \) denotes the counit of the adjunction, then \( \epsilon'_X \in \delta(\mathcal{W}) \), for each \( X \in \mathcal{C} \). Moreover, \( \delta(\mathcal{W}) \) is the pre-image by \( r \) of the isomorphisms in \( \mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}] \).

Assume now that these conditions are satisfied and take \( R' = i \circ r : \mathcal{C}[\mathcal{S}^{-1}] \longrightarrow \mathcal{C}[\mathcal{S}^{-1}] \). Then, for every object \( X \) of \( \mathcal{C} \), \( R'(X) \) is cofibrant, and each cofibrant object is isomorphic in \( \mathcal{C}[\mathcal{S}^{-1}] \) to an object of the form \( R'(X) \). Furthermore, \( \epsilon'_{R'X} \) and \( R'(\epsilon'_X) \) are isomorphisms in \( \mathcal{C}[\mathcal{S}^{-1}] \).

**Proof.** Firstly, we assume that (ii) is satisfied. Then, for each object \( X \) of \( \mathcal{C} \), we have a morphism \( \epsilon'_X : R'(X) \longrightarrow X \) in \( \delta(\mathcal{W}) \), where \( R'(X) \) is cofibrant, hence \( (\mathcal{C}, \mathcal{S}, \mathcal{W}) \) has enough cofibrant objects and it is a left Cartan-Eilenberg category.

Next, let us prove that (i) implies (ii). By theorem 2.3.2, the functor \( j : \mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}] \longrightarrow \mathcal{C}[\mathcal{W}^{-1}] \) is an equivalence of categories. Let \( \overline{r} : \mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}] \) be a quasi-inverse functor of \( j \), and \( \alpha : j \overline{r} \Rightarrow 1 \) an isomorphism of functors. Put \( r := \overline{r} \circ \gamma' \). Since \( \gamma' i r = j \overline{r} \gamma' \), by proposition 2.2.2 the maps

\[
\mathcal{C}[\mathcal{S}^{-1}](i r X, X) \longrightarrow \mathcal{C}[\mathcal{W}^{-1}](\gamma' i r X, \gamma' X) \longrightarrow \mathcal{C}[\mathcal{W}^{-1}](\gamma' X, \gamma' X)
\]

are bijective. Let \( \epsilon'_X : i r (X) \longrightarrow X \) such that \( \gamma'(\epsilon'_X) = \alpha_{\gamma'X} \).
Then \( r \) is a right adjoint for \( i \), with counit \( \epsilon' \). Namely, for each cofibrant object \( M \) and each \( X \in C \), we have a commutative diagram

\[
\begin{array}{ccc}
C_{cof}[S^{-1}, C](M, r(X)) & \xrightarrow{(\epsilon'_X)_*} & C[S^{-1}](iM, X) \\
\downarrow \gamma'_i & & \downarrow \gamma' \\
C[W^{-1}](\gamma'iM, \gamma'iM, \gamma'(X)) & \xrightarrow{(\alpha_{\gamma', \gamma'}X)_*} & C[W^{-1}](\gamma'iM, \gamma'(X)),
\end{array}
\]

where the vertical arrows are bijective by proposition 2.2.2, and \((\alpha_{\gamma', \gamma'}X)_*\) is also a bijective map. Hence, the top arrow is bijective and \( r \) is a right adjoint of \( i \). Besides, \( \gamma'(\epsilon'_X) = \alpha_{\gamma', \gamma'} \) is an isomorphism in \( C[W^{-1}] \), hence \( \epsilon'_X \in \delta(W) \).

Furthermore, if \( w \in \delta(W) \), \( r(w) = \tau(\gamma'w) \) and \( \gamma'w \) is an isomorphism, therefore \( r(w) \) is an isomorphism. Conversely, if \( r(w) \) is an isomorphism, then \( j(r(w)) = (j \circ \tau)(\gamma'w) \) is also an isomorphism. Since \( j \circ \tau \) is an equivalence of categories, we conclude that \( \gamma'w \) is an isomorphism, that is, \( w \in \delta(W) \). Therefore we have proved (ii).

Assume now that (ii) is satisfied, and define the functor \( R' = i \circ r : C[S^{-1}] \rightarrow C[S^{-1}] \). For each object \( X \) in \( C \), \( \epsilon'_X \in \delta(W) \), hence \( r(\epsilon'_X) : rR'(X) \rightarrow r(X) \) is an isomorphism, and so it is \( R'\epsilon'_X \).

Since \( \epsilon'_{R'X} : (R')^2(X) \rightarrow R'X \) is in \( \delta(W) \), and \( R'(X) \), \( (R')^2(X) \) are cofibrant, by theorem 2.2.4 \( \epsilon'_{R'X} \) is an isomorphism. Finally, if \( M \) is cofibrant, \( \epsilon_M : R'(M) \rightarrow M \) is an isomorphism. \( \square \)

Following Adams [A], a coidempotent functor is a pair \((R, \epsilon)\) where \( R : B \rightarrow B \) is an endofunctor of a category \( B \) and \( \epsilon \) is a morphism \( \epsilon : R \Rightarrow 1_B \) such that \( R\epsilon = \epsilon R \) and

\[ \epsilon R : R^2 \Rightarrow R \]

is an isomorphism.

**Proposition 2.4.2.** Let \( B \) be a category together with an endofunctor \( R : B \rightarrow B \) and a morphism \( \epsilon : R \Rightarrow 1_B \) such that \( \epsilon R, R\epsilon : R^2 \Rightarrow R \) are isomorphisms. Then \((R, \epsilon)\) is a coidempotent functor.

**Proof.** It is enough to prove that \( R\epsilon = \epsilon R \). By naturality, we have the following commutative diagram
Indeed, the right square is commutative by naturality, and applying $R$ to this square we obtain the commutativity of the bottom square. Other squares are commutative by naturality. Since $R\epsilon$ and $\epsilon R$ are isomorphisms, we obtain from the middle squares $R^2\epsilon = R\epsilon R = \epsilon R^2$, and finally $\epsilon R = R\epsilon$ from the left square. 

**Corollary 2.4.3.** Let $\mathcal{C}$ be a category, $\mathcal{S}$ a class of morphisms in $\mathcal{C}$, $R' : \mathcal{C}[S^{-1}] \rightarrow \mathcal{C}[S^{-1}]$ a functor and $\epsilon' : R' \Rightarrow 1$ a morphism of functors. Take $\mathcal{W} = (R' \circ \delta)^{-1}(\mathcal{S})$ and assume that $R'\epsilon'$, and $\epsilon'R'$ are isomorphisms, then $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category.

**Proof.** By Proposition 2.4.2, $R'$ is a coidempotent functor. Denote by $\mathcal{D}$ the replete subcategory of $\mathcal{C}[S^{-1}]$ of objects isomorphic to some object of the form $R'(X)$, $i : \mathcal{D} \rightarrow \mathcal{C}[S^{-1}]$ the inclusion functor, and $r : \mathcal{C}[S^{-1}] \rightarrow \mathcal{D}$ the unique functor such that $R' = i \circ r$. Since $\mathcal{D}$ is a full subcategory of $\mathcal{C}[S^{-1}]$, to define $\eta : 1 \Rightarrow r \circ i$ it is enough to define a morphism $i\eta : i \Rightarrow i \circ r \circ i$. If $M = r(X)$ is an object of $\mathcal{D}$ we define

$$i\eta_{R'(X)} : R'(X) = ir(X) \longrightarrow (R')^2(X) = (ir)^2(X)$$

to be $\epsilon_{R'(X)}^{-1}$. It is easily to check that $\eta$ and $\epsilon$ are the unit and the counit, respectively, of an adjunction $i \vdash r$. So the result follows from theorem 2.4.1. 

**Remark 2.4.4.** If $\mathcal{S}$ is just the class of isomorphisms, then $\mathcal{C}_{cof}$ is the class of objects which are left orthogonal (see [Bo](I.5.4)) to $\mathcal{W}$, therefore $(\mathcal{C}, \mathcal{W})$ is a left Cartan-Eilenberg category if and only if $\mathcal{C}_{cof}$ is a coreflective subcategory of $\mathcal{C}$.

### 2.5. Resolvent functors.

Sometimes the coidempotent functor $R' : \mathcal{C}[S^{-1}] \rightarrow \mathcal{C}[S^{-1}]$ in theorem 2.4.1 comes from an endofunctor of $\mathcal{C}$ itself. We formalise this situation in the following definition.

**Definition 2.5.1.** Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a category with strong and weak equivalences. A left resolvent functor for $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a pair $(R, \epsilon)$ such that

(i) $R : \mathcal{C} \rightarrow \mathcal{C}$ is a functor and $R(X)$ is a cofibrant object for each $X \in Ob \mathcal{C}$,

(ii) $\epsilon : R \Rightarrow Id_{\mathcal{C}}$ is a morphism such that $\epsilon_X : R(X) \rightarrow X \in \mathcal{W}$ for each $X \in Ob \mathcal{C}$.

Sometimes a left resolvent functor is called a functorial cofibrant replacement.

Obviously, a left resolvent functor for $\mathcal{C}$ guarantees the existence of enough cofibrants objects, so $\mathcal{C}$ is a left Cartan-Eilenberg category. The Cartan-Eilenberg categories obtained in this way enjoy some other properties.

**Proposition 2.5.2.** Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a category with strong and weak equivalences and $(R, \epsilon)$ a left resolvent functor for $(\mathcal{C}, \mathcal{S}, \mathcal{W})$. Then $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category such that

1. The functor $\mathcal{C}_{cof}[S^{-1}] \rightarrow \mathcal{C}[W^{-1}]$ is an equivalence of categories.
2. We have $\mathcal{W} = R^{-1}(\mathcal{S})$, in particular $R(\mathcal{S}) \subseteq \mathcal{S}$.
3. For each $X \in Ob \mathcal{C}$, $R(\epsilon_X) \in \mathcal{S}$, and $\epsilon_{R(X)} \in \mathcal{S}$.
4. The cofibrant objects of $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ are the objects isomorphic, in $\mathcal{C}[S^{-1}]$, to $R(X)$ for some $X$. 

Proof. The pair \((R, \epsilon)\) induces a functor \(R' : \mathcal{C}[S^{-1}] \rightarrow \mathcal{C}[S^{-1}]\) and a morphism \(\epsilon' : R' \Rightarrow 1\) satisfying hypothesis of Corollary 2.4.3, so \((\mathcal{C}, \mathcal{S}, \mathcal{W})\) is a left Cartan-Eilenberg category.

The equivalence between \(\mathcal{C}_{cof}[S^{-1}]\) and \(\mathcal{C}_{cof}[S^{-1}, \mathcal{C}]\) is an easy consequence of the existence of the resolvent functor, since to any \(\mathcal{W}\)-zigzag in \(\mathcal{C}\) between two objects of \(\mathcal{C}_{cof}\) or any hammock between two such zigzags we can apply the functor \(R\) to obtain \(S\)-zigzags and hammocks in \(\mathcal{C}_{cof}\). Therefore, \(\mathcal{C}_{cof}[S^{-1}] \rightarrow \mathcal{C}[W^{-1}]\) is an equivalence of categories. This proves (1).

On the other hand, from theorem 2.4.1 it follows \(\mathcal{W} = \{w \in \text{Mor} \mathcal{C}; (R' \circ \delta)(w)\text{ is an isomorphism}\} = \{w \in \text{Mor} \mathcal{C}; R(w) \in \mathcal{S}\}\).

In particular \(R(S) \subseteq R(W) \subseteq \mathcal{S}\), hence we have (2).

Next, by (2), \(R\epsilon_X \in \mathcal{S}\), since \(\epsilon_X \in \mathcal{W}\), and by theorem 2.2.4 \(\epsilon_{RX} \in \mathcal{S}\) since \(\epsilon_{RX} \in \mathcal{W}\). This proves (3). Finally (4) follows from theorem 2.4.1.

The following result gives a useful criterion in order to obtain left resolvent functors, as we will see in section 6.

**Theorem 2.5.3.** Let \(\mathcal{C}\) be a category, \(\mathcal{S}\) a class of morphisms in \(\mathcal{C}\), \(R : \mathcal{C} \rightarrow \mathcal{C}\) a functor and \(\epsilon : R \Rightarrow 1\) a morphism such that

\[
R(S) \subseteq S, \quad R(\epsilon_X) \in S, \quad \epsilon_{RX} \in S,
\]

for all \(X \in \text{Ob} \mathcal{C}\). If we take \(W = R^{-1}(S)\), then \(S \subseteq W\) and \((R, \epsilon)\) is a left resolvent functor for \((\mathcal{C}, \mathcal{S}, \mathcal{W})\).

**Proof.** The functor \(R\) induces a functor \(R' : \mathcal{C}[S^{-1}] \rightarrow \mathcal{C}[S^{-1}]\) which satisfies the hypothesis of corollary 2.4.3.

**Remark 2.5.4.** Dualising the definitions of cofibrant object and left Cartan-Eilenberg category, we obtain the notions of **fibrant object** and **right Cartan-Eilenberg category**. All the preceding results have their corresponding dual. For example, for every right Cartan-Eilenberg category \(\mathcal{C}\), the relative localisation \(\mathcal{C}_{fib}[S^{-1}, \mathcal{C}]\) is a reflective subcategory of \(\mathcal{C}[S^{-1}]\), that is to say, the inclusion functor \(\mathcal{C}_{fib}[S^{-1}, \mathcal{C}] \rightarrow \mathcal{C}[S^{-1}]\) admits a left adjoint \(l : \mathcal{C}[S^{-1}] \rightarrow \mathcal{C}_{fib}[S^{-1}, \mathcal{C}]\).

Finally, a **right resolvent functor** is a pair \((R, \eta)\) where \(R : \mathcal{C} \rightarrow \mathcal{C}\) is a functor such that \(R(X)\) is a fibrant object and \(\eta_X : X \rightarrow R(X) \in \mathcal{W}\), for each \(X \in \text{Ob} \mathcal{C}\).

### 3. Models of Functors and Derived Functors

In this section we study functors defined on a Cartan-Eilenberg category \(\mathcal{C}\) and taking values in a category \(\mathcal{D}\) with a class of weak equivalences. We prove that, subject to a few hypotheses, certain categories of functors are also Cartan-Eilenberg categories. In this context we can realise derived functors, when they exist, as cofibrant models in the functor category. The classical example is the category of additive functors defined on a category of complexes of an abelian category with enough projective objects.
3.1. Derived functors. To begin with, we recall the definition of derived functor as set up by Quillen ([Q]).

**Definition 3.1.1.** Let \((C, W)\) be a category with weak equivalences. If \(F : C \to D\) is a functor, a right Kan extension \(\theta_F : \text{Ran}_\gamma F \to F\) of \(F\) along the localisation functor \(\gamma : C \to C[W^{-1}]\) is called a *left derived functor* of \(F\) with respect to \(W\), and denoted by \(\theta_F : L_W F \to F\).

If \(W\) has a right calculus of fractions, this definition agrees with the definition given by Deligne in [D2].

Recall that the category \(\text{Cat}(C[W^{-1}], D)\) is identified, by means of the functor \(\gamma^* : \text{Cat}(C[W^{-1}], D) \to \text{Cat}(C, D)\), with the full subcategory \(\text{Cat}_W(C, D)\) of \(\text{Cat}(C, D)\) whose objects are the functors which send morphisms in \(W\) to isomorphisms in \(D\). This is the class of tautologically derivable functors as ensues from the following easy lemma.

**Lemma 3.1.2.** Let \((C, W)\) be a category with weak equivalences. If \(F : C \to D\) is a functor which takes \(W\) into isomorphisms, then \(\text{Ran}_\gamma F\) and \(\text{Lan}_\gamma F\) exist and they agree with the functor \(F' : C[W^{-1}] \to D\) induced by \(F\).

If \(\text{Cat}'((C, W), D)\) denotes the category of left derivable functors from \((C, W)\) to \(D\), the category \(\text{Cat}_W(C, D)\) is a full subcategory of \(\text{Cat}'((C, W), D)\). Moreover, \(\mathbb{L}_W : \text{Cat}'((C, W), D) \to \text{Cat}_W(C, D)\) defines a functor and the canonical morphism \(\theta_F : \mathbb{L}_W F \to F\) gives a morphism \(\theta : \mathbb{L}_W \Rightarrow 1\). From lemma 3.1.2 the morphism \(\theta_L F\) is an isomorphism. On the other hand, the naturalness of \(\theta\) implies that the following diagram is commutative

\[
\begin{array}{ccc}
\mathbb{L}^2 F & \xrightarrow{\theta_L F} & \mathbb{L} F \\
\downarrow \theta_F & & \downarrow \theta_F \\
\mathbb{L} F & \xrightarrow{\theta F} & F
\end{array}
\]

Since the composition \(\theta_F \circ \mathbb{L} \theta_F = \theta_F \circ \theta_L F : \mathbb{L}^2 F \to F\) factors in a unique way through \(\theta_F\) we obtain \(\mathbb{L} \theta_F = \theta_L F\). Hence \((\mathbb{L}, \theta)\) is a coidempotent functor on \(\text{Cat}'((C, W), D)\). If we take the weak equivalences \(\widetilde{W}\) to be the class of morphisms of \(\text{Cat}'((C, W), D)\) whose image by \(\mathbb{L}_W\) is an isomorphism, and the strong equivalences the class of isomorphisms, from theorems 2.4.3 and 2.4.1 we obtain the following result.

**Theorem 3.1.3.** The pair \((\text{Cat}'((C, W), D), \widetilde{W})\) is a left Cartan-Eilenberg category, \(\text{Cat}_W(C, D)\) is the subcategory of cofibrant objects and \(\mathbb{L}_W : \text{Cat}'((C, W), D) \to \text{Cat}_W(C, D)\) is the corresponding coreflection, with counit morphism \(\theta : \mathbb{L}_W \Rightarrow 1\).

3.2. Functors on Cartan-Eilenberg categories. We give a derivability criterion for functors defined on a left Cartan-Eilenberg category.
3.2.1. Let \((C, S, W)\) be a left Cartan-Eilenberg category. We first summarize the different functors and adjunctions we have between the categories associated to \(C\).

There is a diagram

\[
\begin{array}{ccc}
C[S^{-1}] & \xrightarrow{\gamma} & C[W^{-1}] \\
\downarrow r & & \downarrow \lambda \\
C_{cof}[S^{-1}, C] & \xleftarrow{i} & C[W^{-1}]
\end{array}
\]

in which \(i\) is the inclusion functor and \(r\) the coreflection given by theorem 2.4.1, so that \(i \dashv r\). The functor \(\gamma\) is the localisation in \(\delta(W)\), see 2.1.2, and \(\tau\) is the unique functor such that \(r = \tau \circ \gamma'\). The other two functors in the diagram are defined by composition, \(j := \gamma' \circ i\) and \(\lambda := i \circ \tau\). Functors \(\tau\) and \(j\) are inverse equivalences.

**Lemma 3.2.2.** With the notation above, \(\lambda\) is left adjoint to \(\gamma'\).

**Proof.** By 2.4.1, the counit morphism \(\epsilon' : i \circ r \Rightarrow 1\) is in \(\overline{\delta(W)}\), and therefore, \(\gamma' \epsilon : j \circ r \Rightarrow \gamma'\) is an isomorphism. Since \(i \dashv r\) and \(\tau \dashv j\), one has \(\lambda = i \circ \tau \dashv j \circ r \cong \gamma'\), hence \(\lambda \dashv \gamma'\).

**Theorem 3.2.3.** Let \((C, S, W)\) be a left Cartan-Eilenberg category and \(D\) a category. Then functors in \(\text{Cat}_S(C, D)\) are left derivable, that is, \(\text{Cat}_S(C, D)\) is a full subcategory of \((\text{Cat}'((C, W), D))\). Moreover, if \(F \in \text{Ob} \text{Cat}_S(C, D)\),

\[
\L F = F' \circ \lambda,
\]

where \(F' : C[S^{-1}] \longrightarrow D\) denotes the functor induced by \(F\), defines a left derived functor of \(F\).

**Proof.** By lemma 3.2.2, \(\lambda\) is left adjoint to \(\gamma'\). Identifying as usual \(\text{Cat}_S(C, D)\) with \(\text{Cat}(C[S^{-1}], D)\), these functors induce a pair of functors

\[
\begin{array}{ccc}
\text{Cat}_S(C, D) & \xrightarrow{\gamma^*} & \text{Cat}_W(C, D) \\
\downarrow \lambda^* & & \downarrow \lambda^* \\
\text{Cat}_S(C, D) & \xleftarrow{\gamma^*} & \text{Cat}_W(C, D)
\end{array}
\]

which are also adjoint, \(\gamma^* \dashv \lambda^*\), as is easily seen.

By lemma (3.1.2), \(F' = \text{Ran}_\delta F\), and since a right adjoint to \(\gamma^*\) gives the right Kan extension along \(\gamma\) (see [ML](X.3)), \(\text{Ran}_\gamma F' = F' \circ \lambda\). Hence, by lemma (3.2.4) below, proof of which is an easy exercise, we have \(\text{Ran}_\gamma F = \text{Ran}_\gamma (\text{Ran}_\delta F) = F' \circ \lambda\).

**Lemma 3.2.4.** Let \(\gamma_1 : C_1 \longrightarrow C_2\) and \(\gamma_2 : C_2 \longrightarrow C_3\) be two composable functors, and \(\gamma = \gamma_2 \circ \gamma_1\). If \(F : C_1 \longrightarrow D\) is a functor such that \(\text{Ran}_{\gamma_2} (\text{Ran}_{\gamma_1} (F))\) exists, then \(\text{Ran}_\gamma F\) exists and \(\text{Ran}_\gamma F = \text{Ran}_{\gamma_2} (\text{Ran}_{\gamma_1} (F))\).

The next corollary is a derivability criterion for a functor that generalises to a non additive setting the standard derivability criterion for additive functors (see [GM], III.6 th. 8).

**Corollary 3.2.5.** Let \((C, W)\) be a category with weak equivalences, and \(F : C \longrightarrow D\) a functor. Denote by \(S\) the class of weak equivalences \(w \in W\) such that \(F(w)\) is an isomorphism. If \((C, S, W)\) is a left Cartan-Eilenberg category, the functor \(F\) has a left derived functor.
Taken together theorems 3.1.3 and 3.2.3 it follows immediately that \( \text{Cat}_S(C, D), \tilde{W} \) is a left Cartan-Eilenberg category. More specifically, we have:

**Theorem 3.2.6.** Let \((C, S, W)\) be a left Cartan-Eilenberg category and \(D\) a category. Let \(\tilde{W}\) be the class of morphisms of functors defined in 3.1.3. Then:

1. \(\tilde{W}\) induces in \(\text{Cat}_S(C, D)\) the class of morphisms of functors \(w : F \Rightarrow G : C \rightarrow D\) such that \(w_M\) is an isomorphism for all cofibrant objects \(M\) of \(C\).
2. \((\text{Cat}_S(C, D), \tilde{W})\) is a left Cartan-Eilenberg category and \(\text{Cat}_W(C, D)\) is the subcategory of cofibrant objects.
3. \((\lambda \circ \gamma')^*\) induces a left resolvent functor

\[
\tilde{R} : \text{Cat}_S(C, D) \rightarrow \text{Cat}_S(C, D), \quad F \mapsto F' \circ \lambda \circ \gamma',
\]

where \(F' : C[S^{-1}] \rightarrow D\) denotes the functor induced by \(F\).

**Proof.** (1) follows immediately from the definitions of \(\tilde{W}\) and \(\lambda\) and (2), (3) follow from theorems 3.1.3 and 3.2.3. \(\square\)

### 3.3. Models of functors.

When the target category \(D\) of a functor \(F : C \rightarrow D\) is endowed with a class of weak equivalences it is desirable to have cofibrant models for functors which send strong equivalences to weak equivalences. We prove that this is possible if the Cartan-Eilenberg category has a left resolvent functor.

So let \((C, S, W)\) be a Cartan-Eilenberg category with a left resolvent functor \((R, \epsilon)\) and \(D\) a category with a saturated class of weak equivalences \(W_D\). Denote by

\[
\text{Cat}((C, S), (D, W_D))
\]

the category of functors \(F \in \text{Cat}(C, D)\) which send \(S\) to \(W_D\).

**Definition 3.3.1.** Let \(F, G\) be objects of \(\text{Cat}((C, S), (D, W_D))\) and \(\phi : F \Rightarrow G\) a morphism.

(i) \(\phi\) is called a weak equivalence if \(\phi_M\) is in \(W_D\), for all \(M \in \text{Ob} C_{cof}\).

(ii) \(\phi\) is called a strong equivalence if \(\phi_X\) is in \(W_D\), for all \(X \in \text{Ob} C\).

We denote by \(\tilde{W}\) and \(\tilde{S}\) the weak and strong equivalences of \(\text{Cat}((C, S), (D, W_D))\), respectively.

The resolvent functor \(R\) induces the functor \(R^* : \text{Cat}((C, S), (D, W_D)) \rightarrow \text{Cat}((C, S), (D, W_D))\)

given by \(R^*(F) := F \circ R\), and the counit \(\epsilon : F \Rightarrow 1\) induces a counit \(\epsilon^* : R^* \Rightarrow 1\) by \(\epsilon^*_F := F \epsilon : F \circ R \Rightarrow F\).

It is easy to check that \(\tilde{W} = R^{-1}(\tilde{S})\) and \(R^* \epsilon^*_F, \epsilon^*_R(F) \in \tilde{S}\), for each \(F \in \text{Cat}((C, S), (D, W_D))\).

Hence, by theorem 2.5.3, we have the following result.

**Theorem 3.3.2.** Let \((C, S, W)\) be a category with a left resolvent functor \((R, \epsilon)\). With the previous notation we have

1. \((\text{Cat}((C, S), (D, W_D)), \tilde{S}, \tilde{W})\) is a left Cartan-Eilenberg category and \((R^*, \epsilon^*)\) is a left resolvent functor for \((\text{Cat}((C, S), (D, W_D)), \tilde{S}, \tilde{W})\).
(2) A functor $F \in \text{Cat} ((\mathcal{C}, \mathcal{S}), (\mathcal{D}, \mathcal{W}_D))$ is cofibrant if and only if $F(\mathcal{W}) \subseteq \mathcal{W}_D$.

Finally, by theorem 3.2.3, we obtain:

**Corollary 3.3.3.** Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a category with a left resolvent functor $(R, \epsilon)$. With the previous notation for each $F \in \text{Cat} ((\mathcal{C}, \mathcal{S}), (\mathcal{D}, \mathcal{W}_D))$, $F \epsilon : F \circ R \to F$ is a cofibrant model of $F$ and the left derived functor $L_F$ of $\gamma_D \circ F$ exists and is induced by $\gamma_D \circ F \circ R$.

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F \circ R} & \mathcal{D} \\
\gamma_C \downarrow & & \downarrow \gamma_D \\
\mathcal{C}[\mathcal{W}^{-1}] & \xrightarrow{L_F} & \mathcal{D}[\mathcal{W}^{-1}].
\end{array}
\]

4. **Quillen model categories and Sullivan minimal models**

We describe in this section how Cartan-Eilenberg categories relate to some other axiomatisations for homotopy theory.

**4.1. Quillen model categories.** Let $\mathcal{C}$ be a Quillen model category, that is, a category equipped with three classes of morphisms: weak equivalences $\mathcal{W}$, cofibrations $\text{cofib}$, and fibrations $\text{fib}$, satisfying Quillen’s axioms for a closed model category (see [Q], [Hi]).

Denote by $\mathcal{C}_c$, $\mathcal{C}_f$ and $\mathcal{C}_{c\text{f}}$ the full subcategories of cofibrant, fibrant and cofibrant-fibrant objects of $\mathcal{C}$, respectively. By [Q], theorem 1, $\mathcal{C}_f$ and $\mathcal{C}_c$ are subcategories of models of $(\mathcal{C}, \mathcal{W})$ in our sense (see Definition (2.1.3)). Moreover, the left homotopy relation of morphisms in $\mathcal{C}_f$ is a congruence (see [Q], p. 1.11). If $\mathcal{S}$ denotes the left homotopy equivalences, we have:

**Proposition 4.1.1.** Let $\mathcal{C}$ be a Quillen model category. A Quillen fibrant-cofibrant object is Cartan-Eilenberg cofibrant in $(\mathcal{C}_f, \mathcal{S}, \mathcal{W})$.

**Proof.** It follows from [Q], theorem 1’, by the characterisation of Cartan-Eilenberg cofibrant objects given in prop. 2.2.2. □

**Remark 4.1.2.** Observe that in a Quillen model category $\mathcal{C}$ the definition of Quillen cofibrant object is not homotopy invariant, while the subcategory of Cartan-Eilenberg cofibrant objects of $\mathcal{C}_f$ is stable by homotopy equivalences. In fact, the Cartan-Eilenberg cofibrants are the objects homotopy equivalent to Quillen cofibrant objects.

For instance, let $\mathcal{A}$ be an abelian category with enough projectives and $\mathcal{C} = C_+(\mathcal{A})$ the category of bounded below chain complexes. It is classical that taking quasi-isomorphisms as weak equivalences, epimorphisms as fibrations, and monomorphisms whose cokernel is a degree wise projective complex as cofibrations, $\mathcal{C} = C_+(\mathcal{A})$ is a Quillen model category with all objects fibrant. A contractible complex is Cartan-Eilenberg cofibrant, but it is not Quillen cofibrant unless it is projective, (see also [C]).

Since in a Quillen model category any object has a cofibrant model, from 4.1.1 we deduce:
Theorem 4.1.3. Let \( \mathcal{C} \) be a Quillen model category. Then \((\mathcal{C}_f, \mathcal{S}, \mathcal{W})\) is a left Cartan-Eilenberg category and \( \mathcal{C}_{cf} \) is a subcategory of left models of \( \mathcal{C}_f \).

4.1.4. Let \( \mathcal{C} \) be a Baues fibration category (see [Ba]), and \( \mathcal{W} \) its class of weak equivalences. Denote by \( \mathcal{C}_f \) and \( \mathcal{C}_{cf} \) the full subcategories of fibrant and cofibrant-fibrant objects of \( \mathcal{C} \), respectively. We take \( \mathcal{S} \) the class of morphisms of \( \mathcal{C}_f \) given by the isomorphisms and the homotopy equivalences of \( \mathcal{C}_{cf} \).

Proposition 4.1.5. Let \( \mathcal{C} \) be a Baues fibration category. With notation as above, \((\mathcal{C}_f, \mathcal{S}, \mathcal{W})\) is a left Cartan-Eilenberg category and \( \mathcal{C}_{cf} \) is a subcategory of left models of \( \mathcal{C}_f \).

Proof. By [Ba] proposition II.3.6, the inclusion \( \mathcal{C}_{cf} \to \mathcal{C} \) induces an equivalence of categories \( \mathcal{C}_{cf}[\mathcal{S}^{-1}, \mathcal{C}] \to \mathcal{C}[\mathcal{W}^{-1}] \). Moreover, by the dual of [Ba], proposition II.2.11, the Baues cofibrant objects are cofibrant objects in the sense of our definition, thus \((\mathcal{C}_f, \mathcal{S}, \mathcal{W})\) is a left Cartan-Eilenberg category and \( \mathcal{C}_{cf} \) is a subcategory of left models of \( \mathcal{C}_f \). □

4.2. Sullivan minimal models. In some Cartan-Eilenberg categories there is a subcategory \( \mathcal{M} \) of \( \mathcal{C}_{cof} \) which serves as a subcategory of left models. A typical situation is that of Sullivan minimal models. Let us give an abstract version. In all this paragraph \((\mathcal{C}, \mathcal{S}, \mathcal{W})\) will be a category with strong and weak equivalences.

Definition 4.2.1. Let \((\mathcal{C}, \mathcal{S}, \mathcal{W})\) be a category with strong and weak equivalences. We say that a cofibrant object \( M \) of \( \mathcal{C} \) is minimal if

\[
\text{End}_\mathcal{C}(M) \cap \mathcal{W} = \text{Aut}_\mathcal{C}(M).
\]

That is, any weak equivalence \( w : M \to M \) of \( \mathcal{C} \) is an isomorphism.

We denote by \( \mathcal{C}_{min} \) the full subcategory of \( \mathcal{C} \) whose objects are minimal in \((\mathcal{C}, \mathcal{W}, \mathcal{S})\).

Definition 4.2.2. We say that \((\mathcal{C}, \mathcal{W}, \mathcal{S})\) is a left Sullivan category if there are enough minimal left \( \mathcal{S} \)-models.

Remark 4.2.3. Observe that by the uniqueness property of the extension in definition 2.2.1, any cofibrant object of \( \mathcal{C} \) is minimal in the localized category \((\mathcal{C}[\mathcal{S}^{-1}], \delta(\mathcal{W}))\).

As a consequence of the definition, a left Sullivan category is a special kind of a left Cartan-Eilenberg category, one for which the canonical functor

\[
\mathcal{C}_{min}[\mathcal{S}^{-1}, \mathcal{C}] \to \mathcal{C}[\mathcal{W}^{-1}]
\]

is an equivalence of categories. Observe that by definition, if \( X \) is a minimal object and \( s : X \to X \) is in \( \mathcal{S} \), then \( s \) is an isomorphism, hence \( \mathcal{C}_{min}[\mathcal{S}^{-1}] = \mathcal{C}_{min} \), so that in this case the inclusion functor \( \mathcal{C}_{min}[\mathcal{S}^{-1}] \to \mathcal{C}[\mathcal{S}^{-1}, \mathcal{C}] \) is not, generally speaking, an equivalence of categories.

Examples 4.2.4. Let \( \mathbf{k} \) be a field of characteristic zero. From theorem 2.3.3 and Sullivan’s results on the existence of minimal models and their properties in the category \( \text{Adgc}(\mathbf{k})_0 \) of cohomologically connected commutative differential graded algebras over \( \mathbf{k} \), as stated in [GM], it follows that \( \text{Adgc}(\mathbf{k})_0 \) is a Sullivan category.
Analogously, there are enough minimal objects in the category $\mathbf{Op}(k)_1$ of differential operads over $k$, $P$, such that $H^*P(1) = 0$, (see [MSS]). From theorem 2.3.3 again it follows that $\mathbf{Op}(k)_1$ is a Sullivan category.

4.2.5. To illustrate the reasoning behind the above examples we consider more explicitly the case of modular operads over a field of characteristic zero $k$, (refer to [GK] and [GNPR1]).

In [GNPR1], 8.6.1, we defined minimal modular operads as modular operads obtained from the trivial operad 0 by a sequence of principal extensions. Indeed these modular operads are minimal in the sense of definition 4.2.1, as we prove in the next result.

**Theorem 4.2.6.** Let $\mathbf{MOp}(k)$ be the category of minimal operads over a field of characteristic zero $k$. Let $\mathcal{S}$ be the class of homotopy equivalences and $\mathcal{W}$ the class of homology equivalences of $\mathbf{MOp}(k)$. Then $(\mathbf{MOp}(k), \mathcal{S}, \mathcal{W})$ is a left Sullivan category and the subcategory of minimal modular operads defined in [GNPR1] is a subcategory of left models of $\mathbf{MOp}(k)$.

**Proof.** Let $\mathcal{M}$ be the category of minimal modular operads as defined in [GNPR1]. We check the hypothesis of corollary 2.3.3. The existence of enough left models in $\mathcal{M}$ is the content of theorem 8.6.3 of loc. cit. The lifting property (i) is theorem 8.7.2 of [GNPR1], which also gives uniqueness of the lifting modulo homotopy. Finally (ii) follows also from an iterated application of 8.7.2 [GNPR1].

□

**Corollary 4.2.7.** With the notation as above, the canonical functor

$$\mathbf{MOp}(k)_\text{min}/\sim \xrightarrow{\sim} \mathbf{MOp}(k)[\mathcal{W}^{-1}],$$

where $\sim$ denotes the homotopy relation, is an equivalence of categories.

5. CARTAN-EILENBERG CATEGORIES OF FILTERED OBJECTS

In this section we prove that some categories of filtered complexes and of filtered graded differential commutative algebras are Cartan-Eilenberg or Sullivan categories. In some cases one could prove that they are Quillen model categories, so for them the results would follow from theorem 4.1.3. We present instead a direct simpler proof.

5.1. Filtered complexes of an abelian category. Let $\mathcal{A}$ be an abelian category with enough projective objects. By a filtered complex of $\mathcal{A}$ we understand a pair $(X, W)$ where $X$ is a chain complex of $\mathcal{A}$ and $W$ is an increasing filtration of subcomplexes of $X$. We denote by $Gr^W_pX$ the complex $W_pX/W_{p-1}X$.

5.1.1. We denote by $\mathbf{FC}_+(\mathcal{A})$ the full subcategory category of filtered complexes $(X, W)$ such that

(i) the complex $X$ is bounded below: $X_p = 0$ for $p \ll 0$.
(ii) the filtration $W$ is bounded below, $W_p = 0$ if $p \ll 0$, and biregular, that is, finite on each $X_n$. 

5.1.2. As usual, we say that two morphisms \( f, g : (X, W) \to (Y, W) \) between filtered chain complexes are filtered homotopic if they are homotopic through a homotopy \( h \) such that \( h(W_pX) \subseteq W_pY \), for all \( p \). We denote by \( S \) the class of filtered homotopy equivalences.

Denote by \( W \) the class of filtered quasi-isomorphisms in \( \text{FC}_+(A) \): a filtered morphism \( f \) is in \( W \) if \( \text{Gr}^W_p(f) \) is a quasi-isomorphism for each \( p \) (equivalently, since filtrations are bounded below and biregular, \( W_p(f) \) is a quasi-isomorphism for each \( p \)). It is clear that \( S \subseteq W \).

**Theorem 5.1.3.** Let \( A \) be an abelian category with enough projective objects. Then the category \( (\text{FC}_+(A), S, W) \) is a left Cartan-Eilenberg category. The full subcategory of filtered complexes \( P \) such that, for all \( p \), \( \text{Gr}^W_pP \) is projective in each degree is a left model subcategory of cofibrant objects.

The proof will be a consequence of propositions 5.1.4, 5.1.6, which ensure that complexes \( P \) with \( \text{Gr}^W_P \) projective in each degree are cofibrant objects and that any filtered complex has a left model of such type.

**Proposition 5.1.4.** Let \( (P, W) \) be a filtered complex of \( \text{FC}_+(A) \), such that, for all \( p \), \( \text{Gr}^W_P \) is projective in each degree. Then, \( (P, W) \) is Cartan-Eilenberg cofibrant.

**Proof.** Let

\[
\begin{array}{ccc}
P & \xrightarrow{f} & X \\
\downarrow w & & \\
Y & & \\
\end{array}
\]

be a diagram with \( w \) a filtered quasi-isomorphism. To prove the existence of a unique lifting, up to homotopy, we define inductively on \( p \) a morphism \( g_p : W_pP \to W_pY \) and a homotopy \( h_p : W_pP \to W_pX \) such that \( h_p : wg_p \sim f \) as follows. For \( p \ll 0 \), \( W_pP = 0 \) so we take \( g_p = 0 \) and \( h_p = 0 \). Assume now that \( g | W_{p-1}P \) and \( h | W_{p-1}P \) have been defined, and consider the diagram

\[
\begin{array}{ccc}
W_{p-1}P & \xrightarrow{g | W_{p-1}P} & W_pY \\
\downarrow j & & \downarrow w \\
W_pP & \xrightarrow{f | W_p(P)} & W_pX
\end{array}
\]

The cokernel of \( j \) is projective in each degree and bounded below, hence lemma 5.1.5 applies, so there are morphisms \( g | W_p(P) \) and \( h | W_p(P) \), which are extensions of the previous data. As the filtration \( W \) is biregular, the \( g_p, h_p \) define a morphism \( g : P \to Y \) and a homotopy \( h : P \to X \) such that \( h : wg \sim f \). The uniqueness property also follows from 5.1.5. \( \square \)

**Lemma 5.1.5.** Let

\[
\begin{array}{ccc}
Q & \xrightarrow{\phi} & Y \\
\downarrow j & & \downarrow w \\
R & \xrightarrow{F} & X
\end{array}
\]

be a diagram of complexes such that \( w \) is a quasi-isomorphism, \( j \) is a monomorphism whose cokernel \( P \) is bounded below and projective in each degree, and \( \lambda : Q \to X \) a homotopy
\( \lambda : w \phi \simeq Fj \). Then, there is a morphism \( G : R \longrightarrow Y \) such that \( G \circ j = \phi \), and a homotopy \( H : w \circ G \simeq F \), such that \( Hj = \lambda \). Moreover, \( G \) is unique up to a homotopy which is trivial on \( Q \).

**Proof.** As \( P \) is projective in each degree, we may assume that \( R_i = P_i \oplus Q_i \), \( j = (0 \; \text{Id}_Q) \), and that the differential of \( R \) is given by a matrix

\[
D_i^R = \begin{pmatrix} \partial_i^P & 0 \\ \gamma_i & \partial_i^Q \end{pmatrix},
\]

where \( \gamma : P \longrightarrow Q \) satisfies \( \gamma \circ \partial + \partial \circ \gamma = 0 \). Then, \( F \) is of the form \( F = (f \; \psi) \), where \( \psi, f \) satisfy: \( \psi \partial = \partial \psi, \partial f - f \partial = \psi \gamma \), and \( \partial \lambda + \lambda \partial = \psi - w \circ \phi \).

We are looking for a chain morphism \( G = (g \; \phi) : R \longrightarrow Y \) and a homotopy \( H = (h \; \lambda) : R \longrightarrow X \), \( H : wG \simeq F \). That is, we look for homogeneous maps \( g : P \longrightarrow Y \) of degree \( 0 \) and \( h : P \longrightarrow X \) of degree \( 1 \), such that \( g \partial + \phi \gamma = \partial g, \lambda \gamma + h \partial + \partial h = f - wg \).

As \( P \) is bounded below, we may assume that we have defined such \( g_j, h_j \) for \( j < i \). Now, let \( C(w) \) be the cone of \( w \), and take

\[
\beta_i = \begin{pmatrix} f_i - h_{i-1} \partial_i - \lambda_i \gamma_i \\ -g_{i-1} \partial_i - \phi_i \gamma_i \end{pmatrix} : P_i \longrightarrow C_i(w).
\]

By induction we have \( D_i \beta_i = 0 \). But, \( w \) is a quasi-isomorphism, so \( C(w) \) is acyclic and as \( P_i \) is projective, there exists

\[
\alpha_i : P_i \longrightarrow C_{i+1}(w), \quad \text{such that} \quad D_{i+1} \alpha_i = \beta_i.
\]

Hence \( \alpha_i = \begin{pmatrix} h_i \\ g_i \end{pmatrix} \) satisfies the recurrence.

For uniqueness, observe that taking as \( G \) the difference between two solutions, we may assume that \( \phi = 0, F = 0, \lambda = 0 \). So we have a morphism of complexes \( G : R \longrightarrow Y \) such that \( Gj = 0 \) and a homotopy \( H : P \longrightarrow X \), \( H : 0 \simeq wg \), such that \( Hj = 0 \). We look for a homotopy \( K : 0 \simeq G \) such that \( Kj = 0 \) and a second homotopy \( \Theta : wK \simeq H \) such that \( \Theta j = 0 \). If \( G = (g \; 0), H = (h \; 0) \), with \( g \partial = \partial g, \) and \( h \partial + \partial h = wg \), the problem is equivalent to defining \( K = (k \; 0) \) and \( \Theta = (\theta \; 0) \) such that \( \partial k + k \partial = g \) and \( \partial \theta - \theta \partial = h - wk \). But this problem corresponds to the absolute case. \( \square \)

**Proposition 5.1.6.** Let \( (X, W) \) be a filtered complex of \( \text{FC}_+ (A) \). Then, there is a filtered complex \( (P, W) \) in \( \text{FC}_+ (A) \) such that, for all \( p \), \( Gr_p^W P \) is projective in each degree and a filtered quasi-isomorphism \( \rho : (P, W) \longrightarrow (X, W) \).

**Proof.** We use induction on \( p \) to prove that the filtered complexes \( (W_p X, W) \) have models as in the statement of the proposition, so the result follows from the regularity of the filtration \( W \).

As \( W \) is bounded below, we can take \( P_p = 0 \) for \( p < 0 \). Assume that there is a filtered quasi-isomorphism \( \rho_{p-1} : (P_{p-1}, W) \longrightarrow (W_{p-1} X, W) \), such that \( Gr_q^W P_{p-1} \) is projective for all \( q \). We want to extend this model to a model of \( W_p X \).
By composing \( \rho_{p-1} \) with the inclusion \( \iota_p : W_{p-1}X \to W_pX \), we get a filtered morphism \( \rho^p_{p-1} : (P_{p-1}, W) \to (W_pX, W) \). Let \( C\rho^p_{p-1} \) be its mapping cone with the induced filtration. Recall that, if \( f : A \to B \) is a chain map, its mapping cone \( Cf \) satisfies \( Cf_i = B_i \oplus A_{i-1} = (B \oplus A[1])_i \).

By the induction hypothesis and since \( \text{Gr}_q C\rho^p_{p-1} = C\text{Gr}_q \rho^p_{p-1} \), for all \( q \), we deduce that the filtered complex \( C\rho^p_{p-1} \) is filtered quasi-isomorphic to \( \text{Gr}_p W_pX \), where this complex has pure weight \( p \).

Let \( G_p \) be a projective model of the complex \( \text{Gr}_p W_pX \), that is, a projective complex together with a quasi-isomorphism \( G_p \to \text{Gr}_p W_pX \), which exists since \( A \) has enough projective objects. This quasi-isomorphism lifts, through the quasi-isomorphisms \( C\rho^p_{p-1} \to C\iota_p \to \text{Gr}_p W_pX \), to a quasi-isomorphism \( s : G_p \to C\rho^p_{p-1} \), which is a filtered quasi-isomorphism if we consider \( G_p \) as a pure complex of weight \( p \).

Consider the following commutative diagram of complexes

\[
\begin{array}{c}
G_p[-1] \xrightarrow{\xi} P_{p-1} \\
\downarrow s[-1] \quad \downarrow \text{id} \\
C\rho^p_{p-1}[-1] \xrightarrow{-p} P_{p-1} \to W_pX
\end{array}
\]

By the lemma below, there is a chain map \( \nu : C\xi \to W_pX \) which completes the previous diagram

\[
\begin{array}{c}
G_p[-1] \xrightarrow{\xi} P_{p-1} \to C\xi \to G_p \\
\downarrow s[-1] \quad \downarrow \text{id} \quad \downarrow \nu \quad \downarrow s \\
C\rho^p_{p-1}[-1] \xrightarrow{-p} P_{p-1} \xrightarrow{\rho^p_{p-1}} W_pX \to C\rho^p_{p-1}
\end{array}
\]

where the rows are distinguished triangles in the category of filtered complexes, the central square is commutative and the vertical morphisms define a morphism of triangles in the homotopy category. As \( s \) is a filtered quasi-isomorphism, so is \( \nu \), hence we may take \( P_p = C\xi \) with the induced filtration. \( \square \)

**Lemma 5.1.7.** Let

\[
\begin{array}{c}
B \xrightarrow{\eta} A \\
\downarrow \lambda \quad \downarrow \mu \\
C\xi[-1] \xrightarrow{-p\nu} Y \xrightarrow{\xi} X
\end{array}
\]

be a commutative diagram of filtered complexes of an abelian category; then there exists a filtered chain map \( \nu : C\eta \to X \) such that in the diagram

\[
\begin{array}{c}
B \xrightarrow{\eta} A \to C\eta \to B[1] \\
\downarrow \lambda \quad \downarrow \mu \quad \downarrow \nu \quad \downarrow \lambda[1] \\
C\xi[-1] \xrightarrow{-p\nu} Y \xrightarrow{\xi} X \to C\eta
\end{array}
\]

the central square is commutative and the right-hand square is filtered homotopy commutative. Moreover, its rows are distinguished triangles in the derived category of filtered complexes and the vertical maps define a morphism of distinguished triangles.
Proof. This lemma and its proof are the filtered version of lemma 8.6.4 of [GNPR1]. □

Remark 5.1.8. The induced equivalence of categories $K_+^\ast \mathcal{P}(\mathcal{A}) \to DF_\ast^{+}(\mathcal{A})$ has been obtained by Illusie (see [I] Cor. (V.1.4.7)) for complexes with a finite filtration.

5.2. Filtered complexes of vector spaces. As a special case of the results above, take $k$ a field and $\mathcal{A}$ the category of $k$-vector spaces. We write $\mathbf{FC}_+(k)$ for $\mathbf{FC}_+(\mathcal{A})$. In this case, all objects of $\mathbf{FC}_+(k)$ are cofibrant, so the Cartan-Eilenberg structure is useless. However, we can characterise the minimal complexes:

Proposition 5.2.1. Let $(X, W)$ be a filtered complex of $\mathbf{FC}_+(k)$ such that the induced differential on $\text{Gr}^W_p X$ is zero, for all $p$. Then $(X, W)$ is minimal.

Proof. Let $f : (X, W) \to (X, W)$ be a filtered quasi-isomorphism. As the differential on $\text{Gr}^W_p X$ is zero, $f$ induces an isomorphism $Grf : Gr^W_p X \to Gr^W_p X$. The filtration being regular, it follows that $f$ is an isomorphism. □

Theorem 5.2.2. Let $k$ be a field. Then, the category $(\mathbf{FC}_+(k), S, W)$ is a left Sullivan category. The full subcategory of filtered complexes $(M, W)$ such that the induced differential on $\text{Gr}^W_p M$ is zero, for all $p$, is the subcategory of minimal models.

Proof. By 5.2.1, it is enough to prove that any filtered complex $(X, W)$ of $\mathbf{FC}_+(k)$ has a model such that the differential on their associated graded complex is zero. As in 5.1.6, the proof is based on lemma 5.1.7.

We use induction to prove that $W_p X$ has a minimal model for any $p$. Assume we have a model $\rho_{p-1} : M_{p-1} \to W_{p-1}$ such that, for each $q$, the differential on the associated graded complex $\text{Gr}^W_q M_{p-1}$ vanishes, and consider the diagram

$$
\begin{array}{cccc}
N_p[-1] & \xrightarrow{\xi} & M_{p-1} & \\
s[-1] & & \downarrow{\text{id}} & \\
C\rho_{p-1}^p[-1] & \xrightarrow{-\rho} & M_{p-1} & \xrightarrow{\rho_{p-1}} W_p X
\end{array}
$$

where $N_p$ is a minimal model of the complex $Gr^W_p X$, that is, a model with zero differential, which exists since $k$ is a field. Reasoning as in the proof of 5.1.6, we apply 5.1.7 and deduce that there is a filtered quasi-isomorphism $\nu : C\xi \to W_p X$ which completes the previous diagram in the form

$$
\begin{array}{cccc}
N_p[-1] & \xrightarrow{\xi} & M_{p-1} & \to & C\xi & \to & N_p & \\
s[-1] & & \downarrow{\text{id}} & & \downarrow{\nu} & & \downarrow{s} & \\
C\rho_{p-1}^p[-1] & \xrightarrow{-\rho} & M_{p-1} & \xrightarrow{\rho_{p-1}} W_p X & \to & C\rho_{p-1}^p
\end{array}
$$

Observe that $C\xi$ has zero differential on the associated graded complex, hence we can take $M_p = C\xi$. □
Recall that a morphism of filtered modules $f : X \to Y$ is called strict if $f(W_p X) = W_p Y \cap f(X)$ for all $p$. By [D1], the differential $d$ is strict if and only if the spectral sequence $E_{pq}^r(W)$ degenerates in the $E_1$-term. As a consequence, if $(X, W)$ is a minimal filtered chain complex which is $d$-strict, by theorem 5.2.2 its differential vanishes. Hence we have the following result.

**Corollary 5.2.3.** If $(X, W)$ is a $d$-strict minimal filtered chain complex of $\text{FC}_+(k)$, then it is a minimal chain complex.

5.3. **Filtered Algebras.** In this section we review, using the formalism of Cartan-Eilenberg categories, the homotopy theory of filtered algebras, which Halperin and Tanré developed in [HT] by perturbation methods.

Let $R$ be a commutative ring such that $\mathbb{Q} \subset R$. We denote by $\text{FAlg}(R)$ the category of filtered differential graded-commutative (cdg for short) $R$-algebras in the sense of Halperin-Tanré [HT] (which correspond to the category of $(R,0)$-algebras in loc. cit.) Halperin-Tanré have developed part of the structure of a Quillen model category for $\text{FAlg}(R)$. In particular, they have defined models and a filtered homotopy relation between morphisms. We interpret their results in our setting as follows.

**Theorem 5.3.1.** Let $R$ be a commutative ring such that $\mathbb{Q} \subset R$. The category of filtered cdg $R$-algebras, with filtered homotopy equivalences as strong equivalences and filtered quasi-isomorphisms as weak equivalences is a left Cartan-Eilenberg category, and the subcategory of Halperin-Tanré models is a left model subcategory of cofibrant objects.

**Proof.** By a Halperin-Tanré model of a filtered cdg algebra $A$ we understand an $(R,0)$-extension $R \to M$ which is a model of the morphism $R \to A$ as defined in [HT], (4.1). By theorem (4.2) of loc. cit. there are enough Halperin-Tanré models. Moreover, in case $\mathbb{Q} \subset R$, their application (7.7) proves that they are Cartan-Eilenberg cofibrant objects of $\text{FAlg}(R)$, so the result follows. □

If $R = k$ is a field of characteristic zero, Halperin-Tanré also define a notion of minimal model for filtered cdg algebras concentrated in positive degrees, see definition (8.3) of loc. cit., and prove that, if the filtered cdg algebra $A$ satisfies $H^0(Gr^W_0 A) = k$, then $A$ has such a minimal model [HT] (8.11). Their theorem (8.4) proves that their minimal models are minimal objects in our sense, so we deduce the following result.

**Theorem 5.3.2.** Let $k$ be a field of characteristic zero. The category of filtered cdg $k$-algebras, concentrated in degrees $\geq 0$ and such that $H^0(Gr^W_0 A) = k$, with filtered homotopy equivalences as strong equivalences and filtered quasi-isomorphisms as weak equivalences, is a left Sullivan category, and the Halperin-Tanré minimal models are the minimal objets.

One can also state the previous results for $(R,r)$-algebras, $r > 0$, in the sense of Halperin-Tanré. We recall also that Neisendorfer-Taylor noticed the existence of cofibrant models for filtered cdg $k$-algebras under some finiteness condition ([NT], proof of Prop. 1).
In this section we give a Cartan-Eilenberg structure to some functor categories equipped with a cotriple. From the Cartan-Eilenberg structure we deduce the general version of the acyclic models theorem given by Barr, [B], and a filtered acyclic models theorem.

6.1. Functor categories and cotriples.

6.1.1. Let $\mathcal{A}$ be an abelian category and denote by $\mathcal{C}_+(\mathcal{A})$ the category of positive chain complexes of $\mathcal{A}$. Following Barr ([B]), we say that a class of morphisms $\mathcal{S}$ of $\mathcal{C}_+(\mathcal{A})$ is a class of acyclic morphisms if it satisfies the following five properties:

(i) The isomorphisms are in $\mathcal{S}$.
(ii) A morphism $f$ is in $\mathcal{S}$ if and only if $f[1]$ is in $\mathcal{S}$.
(iii) The class $\mathcal{S}$ is homotopy invariant, that is, if $f$ and $g$ are homotopic morphisms, then $f \in \mathcal{S}$ if and only if $g \in \mathcal{S}$.
(iv) All morphisms of $\mathcal{S}$ are quasi-isomorphisms.
(v) Let $f : C_{\ast\ast} \to D_{\ast\ast}$ be a morphism of double complexes. If $f_n : C_{\ast n} \to D_{\ast n}$ is in $\mathcal{S}$, for all $n \geq 0$, then $\text{Tot} f : \text{Tot} C_{\ast\ast} \to \text{Tot} D_{\ast\ast}$ is in $\mathcal{S}$.

For example, one can take for $\mathcal{S}$ the class of homotopy equivalences or that of quasi-isomorphisms, (cf. [B], Chap. 5). These are the two extreme cases, since any other class of acyclic morphisms lies between them.

6.1.2. Let $\mathcal{X}$ be a category and $\mathcal{A}$ an abelian category. The functor category $\text{Cat}(\mathcal{X}, \mathcal{A})$ is an abelian category. Besides the classes of homotopy equivalences or quasi-isomorphisms, we can consider the following class of acyclic morphisms $\mathcal{S}$ in $\mathcal{C}_+(\text{Cat}(\mathcal{X}, \mathcal{A})) = \text{Cat}(\mathcal{X}, \mathcal{C}_+(\mathcal{A}))$.

Take $\Sigma$ a class of acyclic morphisms in $\mathcal{C}_+(\mathcal{A})$ and define a class of morphisms $\mathcal{S}$ of $\text{Cat}(\mathcal{X}, \mathcal{C}_+(\mathcal{A}))$ by

$$\mathcal{S} = \{ f : K \to L \mid f(X) \in \Sigma, \forall X \in \text{Ob} \mathcal{X} \}.$$ 

Then $\mathcal{S}$ is a class of acyclic morphisms. We shall say that $\mathcal{S}$ is defined point-wise.

For example, if $\Sigma$ is the class of homotopy equivalences in $\mathcal{C}_+(\mathcal{A})$, we say that $\mathcal{S}$ is the class of point-wise homotopy equivalences. In the sequel this class will be denoted by $\mathcal{S}_{\text{ph}}$. Observe that in contrast to the case of homotopy equivalences in $\text{Cat}(\mathcal{X}, \mathcal{C}_+(\mathcal{A}))$, the point-wise homotopy equivalences have homotopy inverses over each object $X$ of $\mathcal{X}$, but these homotopy inverses are not required to be functorial.

6.1.3. Let $G = (G, \varepsilon, \delta)$ be a cotriple defined in the category $\text{Cat}(\mathcal{X}, \mathcal{C}_+(\mathcal{A}))$.

For any functor $K$ in $\text{Cat}(\mathcal{X}, \mathcal{C}_+(\mathcal{A}))$, let $B_{\ast}(K)$ be the value at $K$ of the simplicial standard construction associated to the cotriple $G$, so that $B_n(K) = G^{n+1}(K)$, see [ML]. The morphism $\varepsilon$ defines an augmentation $B_{\ast}(K) \to K$. Since $K$ is a chain complex, there is a naturally defined double complex associated to $B_{\ast}(K)$, with total complex denoted by $B(K)$, (cf. [GNPR2]). This construction defines a functor

$$B : \text{Cat}(\mathcal{X}, \mathcal{C}_+(\mathcal{A})) \to \text{Cat}(\mathcal{X}, \mathcal{C}_+(\mathcal{A})), \quad K \mapsto B(K) = \text{Tot} B_{\ast}(K),$$

and a natural transformation $\varepsilon : B \Rightarrow 1$. 
Theorem 6.1.4. Let $\mathcal{X}$ be a category, $\mathcal{A}$ an abelian category, $G$ a cotriple defined in $\text{Cat}(\mathcal{X}, \mathcal{C}_+(\mathcal{A}))$, and $S$ a class of acyclic morphisms of $\text{Cat}(\mathcal{X}, \mathcal{C}_+(\mathcal{A}))$ such that $G(S) \subseteq S$. If $W = B^{-1}(S)$, then the pair $(B, \varepsilon)$ is a left resolvent functor for $(\text{Cat}(\mathcal{X}, \mathcal{C}_+(\mathcal{A})), S, W)$.

In particular, $(\text{Cat}(\mathcal{X}, \mathcal{C}_+(\mathcal{A})), S, W)$ is a left Cartan-Eilenberg category and the cofibrant objects are the functors strongly equivalent to $B(K)$ for some $K \in \text{Ob} \text{Cat}(\mathcal{X}, \mathcal{C}_+(\mathcal{A}))$.

Proof. This is a consequence of theorem 2.5.3 as soon as we verify its hypothesis. To prove that $B(S) \subseteq S$, take $s \in S$. Then, $G^i(s) = s \circ G^i \in S$, for all $i \geq 1$, because $G(S) \subseteq S$, hence the result follows from property (v) of acyclic classes of morphisms.

Now, let $K$ be an object of $\text{Cat}(\mathcal{X}, \mathcal{C}_+(\mathcal{A}))$. We have to prove that $B(\varepsilon_K)$ and $\varepsilon_{BK}$ are in $S$. The morphism $\varepsilon_{BK}$ is in $S$ by property (v) of a class of acyclic morphisms. Indeed, it is the morphism associated to a contractible simplicial object $\varepsilon : B_\bullet(BK) \to BK$, so the result follows.

As for $B(\varepsilon_K)$, it is equal to the morphism

$$B^2(K) = \text{Tot}_i G^{i+1}(\text{Tot} K G^{i+1})$$

$$= \text{Tot}_i (\text{Tot} K G^{i+1}) G^{i+1}$$

$$= \text{Tot}_i (\text{Tot} K G^{i+1} G^{i+1})$$

$$\to \text{Tot}_i (G^{i+1}(K)) = B(K)$$

where the last morphism in the sequence is $\text{Tot}_i(\varepsilon_{G^{i+1}K})$, which is in $S$ by (6.1.1), (v) and proposition 6.1.5 below. □

This result is an application of our theory that is not covered by the classical homotopy theories since the class $S$ does not come necessarily from a homotopy equivalence, as we shall see in the next examples. The cofibrant model of a functor with respect to this Cartan-Eilenberg structure is the (non-additive) derived functor defined by Barr-Beck from the cotriple (see [BB]).

In order to recognize cofibrant objects in functor categories it is useful to apply the following criterion.

Proposition 6.1.5. With notation as in theorem above.

(1) Let $K$ be an object of $\text{Cat}(\mathcal{X}, \mathcal{C}_+(\mathcal{A}))$. Suppose that $\varepsilon_K : G(K) \to K$ splits, that is to say, there is a natural transformation $\theta : K \to G(K)$ such that $\varepsilon_K \theta = \text{id}$. Then, for any class of acyclic morphisms $S$ of $\text{Cat}(\mathcal{X}, \mathcal{C}_+(\mathcal{A}))$, $K$ is cofibrant.

(2) Suppose that $S$ is defined point-wise and that for any $X \in \text{Ob} \mathcal{X}$, the morphism $\varepsilon_K(X)$ splits, then $K$ is cofibrant.

Proof. We prove (1), as (2) is proved similarly. The splitting $\theta$ defines an extra degeneracy of the augmented simplicial object $B_\bullet(K) \to K$, so it is contractible and the result follows. □

Example 6.1.6. In this example we prove that the functor of singular chains is a cofibrant model for the functor $H_0(\cdot, \mathbb{Z})$ on the category of complex valued functors on topological spaces with a convenient Cartan-Eilenberg structure.
Let $\mathcal{X} = \mathbf{Top}$ be the category of topological spaces and consider the classical cotriple in $\mathbf{Top}$ defined by

$$G(X) = \bigoplus_{n, \sigma \in \mathbf{Top}(\Delta^n, X)} (\Delta^n, \sigma),$$

where $(\Delta^n, \sigma)$ is a copy of $\Delta^n$ indexed by $\sigma$, with convenient coaugmentation and comultiplication transformations (see [B], Chap. 8, (1.12)). It naturally defines a cotriple in $\mathbf{Cat}(\mathbf{Top}, C_+(\mathbb{Z}))$ by sending $K$ to $K \circ G$, with the evident extensions of the transformations $\varepsilon, \delta$. We denote also by $G$ the extended cotriple in $\mathbf{Cat}(\mathbf{Top}, C_+(\mathbb{Z}))$.

Take $\mathcal{S}$ the class of homotopy equivalences in $\mathbf{Cat}(\mathbf{Top}, C_+(\mathbb{Z}))$. The compatibility assumption $G(\mathcal{S}) \subseteq \mathcal{S}$ is clear, so we obtain, from theorem 6.1.4, a left Cartan-Eilenberg category structure on $\mathbf{Cat}(\mathbf{Top}, C_+(\mathbb{Z})), \mathcal{S}, B^{-1}(\mathcal{S})$.

Let $S_* : \mathbf{Top} \longrightarrow C_+(\mathbb{Z})$ be the functor of singular chains and $\tau : S_* \longrightarrow H_0(-, \mathbb{Z})$ the natural augmentation.

We apply proposition 6.1.5 to prove that $S_*$ is cofibrant. It suffices to define the splitting of $\varepsilon_{S_0}$ given by the natural transformation $\theta : S_n \longrightarrow S_n \circ G$ which, for any topological space $X$, sends a singular simplex $\sigma : \Delta^n \longrightarrow X$ to the simplex $\theta(\sigma) = \text{id} : \Delta^n \longrightarrow (\Delta^n, \sigma) \subseteq G(X)$.

The morphism $\tau : S_* \longrightarrow H_0$ is a weak equivalence. In fact, for any topological space $X$ the morphism

$$G\tau : \bigoplus_{\sigma : \Delta^n \longrightarrow X} S_*((\Delta^n, \sigma)) \longrightarrow \bigoplus_{\sigma : \Delta^n \longrightarrow X} H_0((\Delta^n, \sigma), \mathbb{Z})$$

is a homotopy equivalence, since the simplexes $\Delta^n$ are contractible. Hence, $S_*$ is a cofibrant model for $H_0(-, \mathbb{Z})$ in $\mathbf{Cat}(\mathbf{Top}, C_+(\mathbb{Z})), \mathcal{S}, B^{-1}(\mathcal{S})$.

Observe that if we take $\mathcal{S}_{ph}$ the class of point-wise homotopy equivalences in $\mathbf{Cat}(\mathbf{Top}, C_+(\mathbb{Z}))$, then it is easy to prove that $B^{-1}(\mathcal{S}_{ph}) = B^{-1}(\mathcal{S})$, so $S_*$ is also a cofibrant model for $H_0(-, \mathbb{Z})$ in the left Cartan-Eilenberg category $\mathbf{Cat}(\mathbf{Top}, C_+(\mathbb{Z})), \mathcal{S}_{ph}, B^{-1}(\mathcal{S})$.

**Example 6.1.7.** The next example is a variation for differentiable manifolds of the previous one.

Let $\mathcal{X} = \mathbf{Diff}$ be the category of differentiable manifolds with corners. Consider the cotriple $G^\infty$ defined on $\mathbf{Cat}(\mathbf{Diff}, C_+(\mathbb{Z}))$ by

$$G^\infty(K)(X) = \bigoplus_{n, \sigma \in \mathbf{Diff}(\Delta^n, X)} K(\Delta^n, \sigma),$$

with morphisms $\varepsilon, \delta$ defined as in the example above. Take $\mathcal{S}$ the class of homotopy equivalences in $\mathbf{Cat}(\mathbf{Diff}, C_+(\mathbb{Z}))$. By theorem 6.1.4, $\mathbf{Cat}(\mathbf{Diff}, C_+(\mathbb{Z})), \mathcal{S}, B^{-1}(\mathcal{S})$ is a left Cartan-Eilenberg category.

Denote by $S_*^\infty : \mathbf{Diff} \longrightarrow C_+(\mathbb{Z})$ the functor of differentiable singular chains. Reasoning as in the topological case, it follows that $S_*^\infty$ is a cofibrant model of $H_0(-, \mathbb{Z})$ in $\mathbf{Cat}(\mathbf{Diff}, C_+(\mathbb{Z})), \mathcal{S}, B^{-1}(\mathcal{S})$, and hence also in the left Cartan-Eilenberg structure $\mathbf{Cat}(\mathbf{Diff}, C_+(\mathbb{Z})), \mathcal{S}_{ph}, B^{-1}(\mathcal{S})$ coming from the point-wise homotopy equivalences.
As Eilenberg proved in [E], the natural transformation $S_\infty^* \to S_*$ is a point-wise homotopy equivalence (Eilenberg’s proof for differentiable manifolds can be easily extended to manifolds with corners, see also [Hu]), hence $S_*$ is a cofibrant model of $H_0(-,\mathbb{Z})$ in $(\textbf{Cat}(\textbf{Diff}, C_+(\mathbb{Z})), S_{ph}, B^{-1}(S))$. However, one can verify that $S_\infty^*$ and $S_*$ are not homotopy equivalent functors from $\textbf{Diff}$ to $C_+(\mathbb{Z})$, so $S_*$ is not a cofibrant model of $H_0(-,\mathbb{Z})$ in $(\textbf{Cat}(\textbf{Diff}, C_+(\mathbb{Z})), S, B^{-1}(S))$.

6.1.8. Reinterpreting theorem 6.1.4 in terms of acyclic models, we obtain the general acyclic models theorem proved by Barr, [B]. Let us recall the main definitions of [B], (see also [GNPR2]). We maintain the notation of theorem 6.1.4.

A morphism $\alpha : K \to L$ of $\textbf{Cat}(\mathcal{X}, C_+(\mathcal{A}))$ is a $G$-equivalence if $G(\alpha)$ is in $\mathcal{S}$. Since we are assuming that $G(\mathcal{S}) \subseteq \mathcal{S}$, it is easy to see that $\alpha$ is a $G$-equivalence if and only if $B(\alpha) \in \mathcal{S}$ (see [GNPR2], proposition 4.2.6), that is, the $G$-equivalences are the weak equivalences in theorem 6.1.4.

A functor $F : \mathcal{X} \to C_+(\mathcal{A})$ is $G$-acyclic if the augmentation $\tau : F \to H_0F$ is a $G$-equivalence. An object $K$ of $\textbf{Cat}(\mathcal{X}, C_+(\mathcal{A}))$ is $G$-presentable if the augmentation $\varepsilon : B(K) \to K$ is a morphism of $\mathcal{S}$. By (2) in theorem 6.1.4, $K$ is $G$-presentable if and only if it is cofibrant.

Now, we derive a variation of Barr’s acyclic models theorem, [B], Chap. 5, (3.1), as an immediate consequence of the Cartan-Eilenberg structure of $\textbf{Cat}(\mathcal{X}, C_+(\mathcal{A}))$.

**Theorem 6.1.9.** (Acyclic models theorem) Let $\mathcal{X}$ be a category, $\mathcal{A}$ an abelian category, $G$ a cotriple defined in $\textbf{Cat}(\mathcal{X}, C_+(\mathcal{A}))$, and $\mathcal{S}$ a class of acyclic morphisms of $\textbf{Cat}(\mathcal{X}, C_+(\mathcal{A}))$ such that $G(\mathcal{S}) \subseteq \mathcal{S}$. Let $K, L$ be objects of $\textbf{Cat}(\mathcal{X}, C_+(\mathcal{A}))$ such that $K$ is $G$-presentable and $L$ is $G$-acyclic. Then, any morphism $\phi : H_0(GK) \to H_0(GL)$ admits a unique extension to a morphism $K \to L$ in $\textbf{Cat}(\mathcal{X}, C_+(\mathcal{A}))[\mathcal{S}^{-1}]$.

**Proof.** The morphism $\phi : H_0(GK) \to H_0(GL)$ induces a morphism $\phi G^* : H_0^* B_* (K) \to H_0^* B_* (L)$, where $H_0^* B_* K$ denotes the graded complex of the 0-th homology of the rows of the double complex $B_* K$. On the other hand we have a morphism $H_0^* B_* (K) \to H_0^* K$ and a commutative solid diagram

$$
\begin{array}{c}
K \\
\downarrow \\
H_0 K \\
\downarrow \\
H_0^* B_* K \\
\downarrow \\
H_0^* B_* L \\
\downarrow \\
H_0^* L
\end{array}
$$

Since $B_* K$ is cofibrant for the left Cartan-Eilenberg structure of $\textbf{Cat}(\mathcal{X}, C_+(\mathcal{A}))$ given by theorem 6.1.4 and $L \to H_0 L$ is a weak equivalence, because $L$ is $G$-acyclic, there is a lifting $B_* K \to L$. Finally, since $K$ is $G$-presentable, the morphism $B K \to K$ is in $\mathcal{S}$, so the lifting above defines a morphism $K \to L$ in $\textbf{Cat}(\mathcal{X}, C_+(\mathcal{A}))[\mathcal{S}^{-1}]$ making the corresponding diagram commutative.

**Remark 6.1.10.** In [GNPR2] we have presented some variations of the acyclic models theorem in the monoidal and the symmetric monoidal settings. They also follow from a convenient Cartan-Eilenberg structure.
6.2. A filtered acyclic models theorem. We end this paper by proving a filtered acyclic models theorem in the context of cubical topological spaces, which was used in ([GN],(1.5.13)) without proof.

6.2.1. We begin by extending theorem 6.1.4 to functors taking values in the category of filtered complexes of an abelian category.

Let $\mathcal{A}$ be an abelian category and denote by $F^+C^+(\mathcal{A})$ the category of filtered chain complexes with non negative weights, that is to say, chain complexes $X$ with an increasing filtration $W$ such that $W_{-1} = 0$.

The total functor of a double complex extends to the category of filtered double complexes by

$$W_pTot(X_{ij}) = \sum_{i+j=n} W_pX_{ij},$$

for each $p$. Then the notion of class of acyclic morphisms (see 6.1.1) is easily extended to $F^+C^+(\mathcal{A})$.

In the same way, a cotriple $G = (G, \varepsilon, \delta)$ defined in $\text{Cat}(\mathcal{X}, F^+C^+(\mathcal{A}))$, induces a functor

$$B : \text{Cat}(\mathcal{X}, F^+C^+(\mathcal{A})) \longrightarrow \text{Cat}(\mathcal{X}, F^+C^+(\mathcal{A})), $$

and a natural transformation $\varepsilon : B \Rightarrow 1$.

**Theorem 6.2.2.** Let $\mathcal{X}$ be a category, $\mathcal{A}$ an abelian category. Let $G$ be a cotriple and $S$ a class of acyclic morphisms defined in $\text{Cat}(\mathcal{X}, F^+C^+(\mathcal{A}))$, such that $G(S) \subseteq S$. If $W = B^{-1}(S)$, then the pair $(B, \varepsilon)$ is a left resolvent functor for $(\text{Cat}(\mathcal{X}, F^+C^+(\mathcal{A})), S, W)$.

In particular, $(\text{Cat}(\mathcal{X}, F^+C^+(\mathcal{A})), S, W)$ is a left Cartan-Eilenberg category and the cofibrant objects are the functors strongly equivalent to $B(K)$ for some $K \in \text{Ob} \text{Cat}(\mathcal{X}, F^+C^+(\mathcal{A}))$.

6.2.3. Next, we define a cotriple in the category of cubical spaces, extending the classical cotriple $G$ on the category $\text{Top}$ recalled in 6.1.6.

Fix a non-negative integer $N \geq 0$. Let $\square := \square_N$ be the cubic category of dimension $N$, that is to say, the category associated to the ordered set of non empty subsets of $\{0, 1, \ldots, N\}$. Denote by $\square^{\text{op}}\text{Top}$ the category of $\square$-diagrams of topological spaces, that is to say, functors $\square^{\text{op}} \longrightarrow \text{Top}$.

For each topological space $T$ and each $\alpha \in \square$, we denote by $T \times \square_{\alpha}$, the $\square$-diagram defined by $(T \times \square_\alpha)_\beta = T$, if $\beta \in \square_\alpha$, and $= \emptyset$ otherwise. Note that to give a morphism $T \times \square \longrightarrow X_\bullet$ is equivalent to give the component $\sigma_\alpha : T \longrightarrow X_\alpha$ of $\sigma$.

Let $G$ be the cotriple in $\square^{\text{op}}\text{Top}$ associated to the set of objects $\{\Delta^n \times \square_\alpha; n \in \mathbb{N}, \alpha \in \square\}$, that is to say (see [B], Chap. 4, (2.2)), for each $X_\bullet$ in $\square^{\text{op}}\text{Top}$ we have

$$G(X_\bullet) = \bigsqcup_{n, \sigma: \Delta^n \longrightarrow X_\alpha} (\Delta^n \times \square_\alpha, \sigma),$$

with the induced coaugmentation and comultiplication.

6.2.4. Let $S$ be the class of filtered homotopy equivalences in $\text{Cat}(\square^{\text{op}}\text{Top}, F^+C^+(\mathcal{A}))$. Then $S$ is a class of acyclic morphisms of $\text{Cat}(\square^{\text{op}}\text{Top}, F^+C^+(\mathcal{A}))$, and $G(S) \subseteq S$. Hence we can
apply theorem 6.2.2, as was done in theorem 6.1.9, to obtain the following filtered acyclic models theorem.

**Theorem 6.2.5.** Let $K, L : \square^{op}\text{Top} \to F_+C_+(A)$ be functors such that $K$ is $G$-presentable and $L$ is $G$-acyclic. Then, each filtered morphism $H_0(GK) \to H_0(GL)$ admits an extension to a filtered morphism $K \to L$, unique up to filtered homotopy.

**6.2.6.** Next, we introduce a $G$-presentable functor. The functor of singular chains $S_* : \text{Top} \to C_+ (\mathbb{Z})$ induces a functor $K : \square^{op}\text{Top} \to F_+C_+ (\mathbb{Z})$ by

$$K(X_\bullet) = (\text{Tot} S_*(X_\bullet), W), \quad W_p = \sum_{|\alpha| \leq p} S_*(X_\alpha).$$

As in the classical case, $K$ is $G$-presentable. Indeed, the morphism

$$\theta_{X_\bullet} : \text{Tot} S_*(X_\bullet) \to \text{Tot} S_*(G(X_\bullet)),$$

$$\theta_{X_\bullet} (\sigma : \Delta^n \to X_\alpha) = \left( \Delta^n \times \square_\alpha \xrightarrow{id} (\Delta^n \times \square_\alpha, \sigma) \subset G(X_\bullet) \right)$$

is a functorial splitting of $\epsilon_{X_\bullet}$.

**6.2.7.** On the other hand the geometric realisation functor $| | : \square^{op}\text{Top} \to \text{Top}$ induces a functor $L : \square^{op}\text{Top} \to F_+C_+ (\mathbb{Z})$ by

$$L(X_\bullet) = (S_*|X_\bullet|, W), \quad W_p (S_*|X_\bullet|) = S_* (|sq_pX_\bullet|).$$

The functor $L$ is $G$-acyclic since $L_* (\Delta^n \times \square_\alpha) \to H_0 L_* (\Delta^n \times \square_\alpha)$ is a filtered homotopy equivalence.

Hence, we deduce from theorem 6.2.5 the following result (see [GN], p. 29).

**Corollary 6.2.8.** Any family of filtered morphisms

$$\phi_{n, \alpha} : H_0 (\text{Tot} S_*(\Delta^n \times \square_\alpha), W) \to H_0 (S_*(|\Delta^n \times \square_\alpha|), W)$$

admits an extension to a natural filtered morphism $\phi_X : (\text{Tot} S_*(X_\bullet), W) \to (S_* (|X_\bullet|), W)$, unique up to natural filtered homotopy.

**References**


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