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Rainbow connectivity of Moore cages of girth 6

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\textbf{A B S T R A C T}

Let $G$ be an edge-colored graph. A path $P$ of $G$ is said to be rainbow if no two edges of $P$ have the same color. An edge-coloring of $G$ is a rainbow $t$-coloring if for any two distinct vertices $u$ and $v$ of $G$ there are at least $t$ internally vertex-disjoint rainbow $(u, v)$-paths. The rainbow $t$-connectivity $rc_t(G)$ of a graph $G$ is the minimum integer $j$ such that there exists a rainbow $t$-coloring using $j$ colors. A $(k; g)$-cage is a $k$-regular graph of girth $g$ and minimum number of vertices denoted $n(k; g)$. In this paper we focus on $g = 6$. It is known that $n(k; 6) \geq 2(k^2 - k + 1)$ and when $n(k; 6) = 2(k^2 - k + 1)$ the $(k; 6)$-cage is called a Moore cage. In this paper we prove that the rainbow $k$-connectivity of a Moore $(k; 6)$-cage $G$ satisfies that $k \leq rc_k(G) \leq k^2 - k + 1$. It is also proved that the rainbow 3-connectivity of the Heawood graph is 6 or 7.

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1. Introduction

All graphs considered in this work are finite, simple and undirected. We follow the book of Bondy and Murty [1] for terminology and notations not defined here. Let $G$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. The distance between two vertices $u$ and $v$, denoted by $d_G(u, v)$, is the length of a shortest $(u, v)$-path. For each vertex $v \in V(G)$ we use $N_G(v)$ and $d_G(v)$ to denote the set of neighbors and the degree of $v$ in $G$. A graph $G$ is called $k$-regular if each of its vertices has degree $k$. The girth $g(G)$ of $G$ is the length of a shortest cycle in $G$.

An edge-coloring of a graph $G$ is a function $\rho : E(G) \rightarrow R$, where $R$ is a set of distinct colors. Throughout this paper we only consider edge-colorings. Let $G$ be an edge-colored graph. A path $P$ in $G$ is called rainbow if no two edges of $P$ are colored the same. Chartrand, Johns, McKeon and Zhang [3] defined the rainbow connecting colorings. An edge-colored graph $G$ is said to be rainbow connected if there exists a rainbow path between every two distinct vertices of $G$. Clearly, every connected graph $G$ has an edge-coloring that makes it rainbow connected (simply color the edges of $G$ with distinct colors). The rainbow connection number $rc(G)$ of a connected graph $G$ is the minimum number of colors that are needed to make $G$ rainbow connected.

Menger [14] proved that a graph $G$ is $t$-connected if and only if there are at least $t$ internally vertex-disjoint $(u, v)$-paths for every two distinct vertices $u$ and $v$. Schiermeyer studied rainbow $t$-connected graphs with a minimum number of edges [15], and very recently the rainbow connectivity of certain products of graphs has been studied in [12]. Similar to rainbow connecting colorings, an edge-coloring is called a rainbow $t$-coloring if for every pair of distinct vertices $u$ and $v$ there are at least $t$ internally disjoint rainbow $(u, v)$-paths. Clearly, coloring the edges of a $t$-connected graph $G$ with as many colors as edges, every two vertices of $G$ are connected by $t$ internally vertex-disjoint rainbow paths. Thus, the rainbow $t$-connectivity $rc_t(G)$ (defined by Chartrand et al. [4]) of a graph $G$ can be defined as the minimum integer $j$ such that there...
exists a rainbow $t$-coloring using $j$ colors. Moreover, \( rc(G) = rc_1(G) \) and \( rc_t(G) \leq rc_{2t}(G) \) for \( 1 \leq t_1 \leq t_2 \). The complexity of computing the \( rc \) has been studied in [7]. For 2-connected graphs it has been proved that \( rc(G) \leq \lceil |V(G)|/2 \rceil \), see [8]. Also for $t$-connected graphs with \( t \geq 5 \) and girth \( g(G) \geq 5 \) it has been proved that \( rc(G) \leq |V(G)|/t + 19 \), see [8]. Some \( rc_{2t}(G) \) has been computed when \( G \) is a complete graph or a complete bipartite graph in [4]. For more references on rainbow connectivity and rainbow k-connectivity see [9] and the book by Li and Sun [10] or the survey by Li, Shi, and Sun [10].

Given two integers \( k \geq 2 \) and \( g \geq 3 \) a \((k; g)\)-cage is a \( k \)-regular graph of girth \( g \) and minimum number of vertices, that is denoted by \( n(k; g) \). For more information on cages see the survey on cages [5]. In this paper we focus on the case \( g = 6 \). It is known that \( n(k; 6) \geq 2(k^2 - k + 1) \) and concerning the connectivity of any \((k; 6)\)-cage, it has been proved that they are \( k \)-connected [13]. When \( n(k; 6) = 2(k^2 - k + 1) \) the \((k; 6)\)-cage is called a Moore \((k; 6)\)-cage. It is known that the incidence graph of a projective plane of order \( k - 1 \) is a Moore \((k; 6)\)-cage [5,6].

**Definition 1.1.** A projective plane \((\mathcal{P}, \mathcal{L})\) is a non-empty set \( \mathcal{P} \) of points together with a set \( \mathcal{L} \) of non-empty subsets of \( \mathcal{P} \), called lines, satisfying the following axioms:

GP1. For any two distinct points \( p \) and \( p' \), there exists a unique line \( \ell \) connecting them.

GP2. For any two distinct lines \( \ell \) and \( \ell' \), there exists a unique point \( p \) in their intersection.

GP3. There exist at least four points such that no three of them are collinear.

From this definition it follows that each point \( p \in \mathcal{P} \) belongs to \( n + 1 \) lines and each \( \ell \in \mathcal{L} \) line contains \( n + 1 \) points yielding that \( |\mathcal{P}| = |\mathcal{L}| = n^2 + n + 1 \). Thus, the number \( n \) is said to be the order of the projective plane \((\mathcal{P}, \mathcal{L})\) which must be \( n \geq 2 \).

The incidence graph of a projective plane \((\mathcal{P}, \mathcal{L})\) of order \( n \) is a bipartite graph \( G \) with vertex set \( \mathcal{P} \cup \mathcal{L} \). A vertex \( p \in \mathcal{P} \) is adjacent to a vertex \( \ell \in \mathcal{L} \) if and only if \( p \) is incident with \( \ell \) in \((\mathcal{P}, \mathcal{L})\). Note that \( G \) is a Moore \((n + 1; 6)\)-cage, because it is a regular graph of degree \( n + 1 \) with \( 2(n^2 + n + 1) \) vertices and girth 6. Moreover, the diameter of \( G \) is three. A Moore \((n + 1; 6)\)-cage has been constructed for \( q \) where \( q \) is a prime power. In Fig. 2 is depicted the \((3; 6)\)-cage (Heawood graph), which is the incidence graph of the Fano plane.

Chartrand, Johns, McKeon and Zhang [2] showed that the rainbow 3-connectivity of the Petersen graph is 5, and the rainbow 3-connectivity of the Heawood graph is between 5 and 7 inclusive. In this paper we prove that if \( G \) is a Moore \((k; 6)\)-cage, then \( k \leq rc_6(G) \leq k^2 - k + 1 \). It is also proved that the rainbow 3-connectivity of the Heawood graph is 6 or 7.

2. Bounds on the rainbow connectivity of cages

In this section we give a lower bound and an upper bound for the rainbow \( k \)-connectivity of a \((k; 6)\)-Moore cage.

**Theorem 2.1.** Let \( G \) be the incidence graph of a projective plane of order \( n \geq 3 \) and let \( \rho : E(G) \to R \) be a coloring of \( G \). If every path of \( G \) of length at most 3 is rainbow, then \( \rho \) is a rainbow \((n + 1)\)-coloring.

**Proof.** Let \( G \) be the incidence graph of a projective plane \((\mathcal{P}, \mathcal{L})\). Since the diameter of \( G \) is three, we distinguish three different cases according to the distance between two vertices in \( G \).

Case 1. Let \( a \in \mathcal{P} \) and \( L \in \mathcal{L} \) be such that \( d_G(a, L) = 3 \). Then there is a geodesic \((a, L_a, b, L)\) in \( G \) which is rainbow by hypothesis. Let \( L_a^{(i)} \), \( i = 1, \ldots, n \), be the \( n \) lines adjacent to \( a \) different from \( L_a \). Observe that \( |N_G(L_a^{(i)}) \cap N_G(L)| = 1 \) because \( G \) is the incidence graph of a projective plane and let \( \{p^{(i)}\} = N_G(L_a^{(i)}) \cap N_G(L) \) for \( i = 1, \ldots, n \). Note that \( p^{(i)} \neq a, b \) and \( p^{(i)} \neq p^{(j)} \) for \( i \neq j \) because \( G \) has girth 6. The paths \( \{(a, L_a^{(i)}, p^{(i)}, L) : 1 \leq i \leq n\} \) are internally vertex-disjoint paths between \( a \) and \( L \), and they are rainbow by hypothesis.

Case 2. Let \( a, b \in \mathcal{P} \) be such that \( d_G(a, b) = 2 \) and let \( (a, L_ab, b) \) be the geodesic between \( a \) and \( b \) which is unique because the girth is 6, that is, \( N_G(a) \cap N_G(b) = \{L_ab\} \). This geodesic is rainbow by hypothesis. Let \( L_{ab}^{(i)} \), \( i = 1, \ldots, n \), be the \( n \) lines adjacent to \( a \) different from \( L_ab \) and let \( L_{ab}^{(i)} \), \( i = 1, \ldots, n \), be the \( n \) lines adjacent to \( b \) different from \( L_ab \). Let \( \{p^{(i)}\} = N_G(L_{ab}^{(i)}) \cap N_G(L) \), for \( i = 1, \ldots, n \), and observe that \( p^{(i)} \neq p^{(j)} \) for \( i \neq j \) because \( G \) has girth 6. Denote the color of the edge \( al^{(i)} \) by \( r_i = \rho(al^{(i)}) \), \( i = 1, \ldots, n \), and note that \( r_i \neq r_j \) for \( i \neq j \) because by hypothesis paths of length 2 are rainbow. Analogously, denote by \( r_i' = \rho(bl^{(i)}), t = 1, \ldots, n \), and observe that \( r_i' \neq r_j' \) for \( t \neq h \) by hypothesis. If there is no color in common among these sets of colors \( \{r_i\}, \{r_i'\} \), then the \( n \) paths \( \{(a, L_{ab}^{(i)}), p^{(i)}, L) : 1 \leq i \leq n\} \) are internally vertex-disjoint rainbow paths between \( a \) and \( b \). If there are \( k \) colors in common, without loss of generality we may assume that \( r_i = r_i' \) for \( i = 1, \ldots, k, k \leq n \), and \( r_i \neq r_j \) for \( t = k + 1, \ldots, n \). Then the \( n \) paths \( \{(a, L_{ab}^{(i)}), u^{(i)}, b) : 1 \leq i \leq n\} \), where \( u^{(i)} = N_G(L_{ab}^{(i)}) \cap N_G(L_{ab}^{(i'+1)}), i = 1, \ldots, n \), and the sum of superindex is taken modulo \( n \), are internally vertex-disjoint rainbow paths between \( a \) and \( b \) by hypothesis and the girth of \( G \) is 6. The case when \( L', L \in \mathcal{L} \) such that \( d_G(L, L') = 2 \) is solved analogously by duality.

Case 3. Let \( a \in \mathcal{P} \) and \( A \in \mathcal{L} \) be such that \( d_G(a, A) = 1 \). Let \( \{(L^{(i)}, L) : 1 \leq i \leq n\} = N_G(a) - A \) and \( \{(a^{(i)}, \ldots, a^{(i)} = N_G(A) - a \). Moreover, let \( \{M^{(i)} = N_G(a^{(i)}) - A \) and let \( \{b^{(i)}, \ldots, b^{(i)} = N_G(L^{(i)}) - a \) for \( i = 1, 2, \ldots, n \). Since there exists a perfect matching between the sets \( N_G(a) - A \) and \( N_G(L^{(i)}) - a \), for all \( i, j \), we may assume without loss of generality that \( b^{(i)} a^{(i)} \in E(G) \). Let \( r_1 = \rho(aL) \) and \( s_1 = \rho(Aa) \).
First, suppose that \( \rho(l^{(1)}b^{(1)}) = s_2 \neq s_1 \). If \( \rho(M^{(1)}a^{(1)}) = r_2 \neq r_1 \), then \( \rho(b^{(1)}M^{(1)}) \not\in \{r_1, r_2, s_1, s_2\} \), because by hypothesis paths of length 3 are rainbow. Therefore the path
\[
(a, L^{(1)}, b^{(1)}, M^{(1)}, a^{(1)}, A)
\]
is rainbow. Then we have to suppose that \( \rho(M^{(1)}a^{(1)}) = r_1 \), which implies that \( \rho(M^{(1)}a^{(1)}) = r_j \neq r_1 \) for all \( j \geq 2 \) since paths of length 2 are rainbow by hypothesis. Since \( n \geq 3 \), we can take \( j \in \{2, \ldots, n\} \) such that \( \rho(l^{(1)}b^{(1)}) = s_j \neq s_1 \). Then \( \rho(b^{(1)}M^{(1)}) \not\in \{r_1, r_j, s_1, s_j\} \), since paths of length 3 are rainbow by hypothesis, which implies that the path
\[
(a, L^{(1)}, b^{(1)}, M^{(1)}, a^{(1)}, A)
\]
is rainbow. Second, suppose that \( \rho(l^{(1)}b^{(1)}) = s_1 \). Then \( \rho(l^{(1)}b^{(1)}) = s_j \neq s_1 \) for all \( j = 2, \ldots, n \). Since \( n \geq 3 \), we can take \( j \in \{2, \ldots, n\} \) such that \( \rho(M^{(1)}a^{(1)}) = r_j \neq r_1 \). Then \( \rho(b^{(1)}M^{(1)}) \not\in \{r_1, r_j, s_1, s_j\} \), since paths of length 3 are rainbow by hypothesis, yielding that the path
\[
(a, L^{(1)}, b^{(1)}, M^{(1)}, a^{(1)}, A)
\]
is rainbow in either case we can find a rainbow path of length 5 between \( a \) and \( A \) through vertices \( l^{(1)}, a^{(1)} \) and vertices in \( N_G(l^{(1)}) - a \) and through vertices in \( N_G(a^{(1)}) - A \). Repeating this process for each \( i = 2, \ldots, n \), we find \( n \) internally vertex-disjoint rainbow paths between \( a \) and \( A \) which along with the edge \( aA \) give us \( n + 1 \) vertex-disjoint \((a, A)\)-paths.

**Definition 2.1.** Let \( (P, L) \) be a projective plane and \( G \) the corresponding incidence graph. For all \( L \in L \) let \( \sigma_L : L \to L \) be a permutation such that \( \sigma_L(a) \neq a \) for every \( a \in L \). For each edge \( e \) of \( G \) with \( e \in P \) and \( L \in L \), we color \( e \) with the color \( \sigma_L(e) \). This coloring over the edges of \( G \) is said to be a \( \sigma \)-coloring.

As an example of **Definition 2.1**, let us consider the following permutations of lines of Heawood graph defining a \( \sigma \)-coloring shown in Fig. 1.

\[
\sigma_{L_1} = (132); \sigma_{L_2} = (147); \sigma_{L_3} = (165); \sigma_{L_4} = (264); \sigma_{L_5} = (273); \sigma_{L_6} = (354); \sigma_{L_7} = (367).
\]

**Lemma 2.1.** Let \( G \) be the incidence graph of a projective plane of order \( n \geq 2 \) with a \( \sigma \)-coloring. Then every path of length at most three of \( G \) is rainbow.

**Proof.** If a path has length one, clearly it is rainbow. Let \( (a, L, b) \) be a path of length two of \( G \). Since \( \sigma_L \) is a permutation of the points of \( L \) and \( a, b \in L \) with \( a \neq b \), then \( \sigma_L(a) \neq \sigma_L(b) \). Let \( (L, a, L') \) be a path of length two of \( G \). In this case \( [a] = L \cap L' \), and \( \sigma_L, \sigma_L \) are permutations of the points of \( L \) and \( L' \), respectively. If \( \sigma_L(a) = \sigma_L(a) = [a] \), then \( p \in L \cap L' \), that is \( p = a \), which is a contradiction because \( \sigma_L(a) \neq a \) and \( \sigma_L(a) = a \) according to **Definition 2.1**.

Let \( (a, L_{ab}, b, L_b) \) be a path of length three of \( G \). Then \( \sigma_{L_{ab}}(a) \neq \sigma_{L_{ab}}(b) \neq \sigma_{L_b}(b) \). If \( \sigma_{L_{ab}}(a) = \sigma_{L_b}(b) = p \), then \( p \in L_{ab} \cap L_b = \{b\} \), yielding that \( p = b \), which is a contradiction because \( \sigma_{L_{ab}}(a) \neq b \) by **Definition 2.1**.

As an immediate consequence of **Theorem 2.1** and **Lemma 2.1** we can write the following result.

**Theorem 2.2.** Let \( G \) be the incidence graph of a projective plane of order \( n \geq 3 \) with a \( \sigma \)-coloring. Then \( G \) is rainbow \((n + 1)\)-connected and \( r_{\sigma(n+1)}(G) \leq n^2 + n + 1 \).

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The following assertions hold for $i \in \{1, 2\}$:

(i) If $[2i - 1, 2i]$, $[2i - 3, 2i - 4] \in [r]$, then $[2i + 1, 2i + 2]$, $[2i + 4, 2i + 5] \notin [r]$.

(ii) If $[2i - 1, 2i]$, $[2i - 7, 2i - 6] \in [r]$, then $[2i + 1, 2i - 8]$, $[2i + 3, 2i + 4] \notin [r]$.

(iii) If $[2i - 1, 2i]$, $[2i + 7, 2i + 6] \in [r]$, then $[2i + 5, 2i + 4]$, $[2i - 5, 2i - 6] \notin [r]$.

(iv) If $[2i - 1, 2i]$, $[2i + 3, 2i - 6] \in [r]$, then $[2i + 1, 2i + 2]$, $[2i - 2, 2i + 7] \notin [r]$.

(v) If $[2i - 1, 2i]$, $[2i + 3, 2i + 2] \in [r]$, then $[2i - 3, 2i - 2]$, $[2i - 6, 2i - 5] \notin [r]$.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
edge & color & edge & color \\
\hline
$1L_1$ & 2 & $6L_4$ & 4 \\
$2L_1$ & 3 & $2L_5$ & 7 \\
$3L_1$ & 1 & $5L_5$ & 2 \\
$1L_2$ & 4 & $7L_5$ & 5 \\
$4L_2$ & 7 & $3L_6$ & 4 \\
$7L_2$ & 1 & $4L_6$ & 5 \\
$1L_3$ & 6 & $5L_6$ & 3 \\
$5L_3$ & 1 & $3L_7$ & 6 \\
$6L_3$ & 5 & $6L_7$ & 7 \\
$2L_4$ & 6 & $7L_7$ & 3 \\
$4L_4$ & 2 & & \\
\hline
\end{tabular}
\end{table}

Fig. 2. Heawood graph with a $\sigma$-coloring which is not 3-rainbow.

**Remark 2.1.** In Theorem 2.2, the hypothesis $n \geq 3$ is necessary as shown for the $\sigma$-coloring depicted in Fig. 2 of Heawood graph. We can check that this $\sigma$-coloring satisfies the hypothesis of Lemma 2.1, but between 1 and $L_1$ there are no 3 internally rainbow vertex-disjoint paths.

However, the $\sigma$-coloring of Heawood graph shown in Fig. 1 does work.

### 3. Rainbow 3-connectivity of Heawood graph

In the previous section we have described a rainbow 3-coloring of the Heawood graph of 7 colors. We prove that the rainbow 3-connectivity of Heawood graph is at least 6.

**Lemma 3.1.** Let $G$ be a $k$-regular and $k$-connected graph, and let $\rho$ be a rainbow $k$-coloring of $G$. If $e_1$ and $e_2$ are two incident edges, then $\rho(e_1) \neq \rho(e_2)$.

**Proof.** Suppose by contradiction that there are two incident edges $e_1 = uv$, $e_2 = vw$ of $G$ such that $\rho(e_1) = \rho(e_2)$. Since $G$ is rainbow $k$-connected there are $k$ vertex disjoint rainbow paths between vertices $u$ and $w$. Since $d(u) = d(w) = k$, it follows that among these $k$ vertex disjoint rainbow paths there is one containing $e_1$ and that path cannot contain $e_2$, and there must be another path containing $e_2$ and this path cannot contain $e_1$. A contradiction, because these two paths are not vertex-disjoint.

Let $\rho$ be a coloring of a graph $G$. A **chromatic class** $[r]$ is the set of edges of $G$ with color $r$. By Lemma 3.1, the following corollary is immediate.

**Corollary 3.1.** Let $G$ be the incidence graph of a projective plane of order $n$ and let $\rho$ be a rainbow $(n + 1)$-coloring of $G$. Then every chromatic class is independent.

It is well known that the Heawood graph can be described as a bipartite graph with $V(G) = \mathbb{Z}_{14}$ and $E(G) = \{[2i, 2i + 1], [2i, 2i - 1], [2i + 1, 2i + 6] : i = 0, \ldots, 6\}$, see Fig. 3. In the rest of the paper we use this notation for the Heawood graph.

**Lemma 3.2.** Let $H$ be the Heawood graph, let $\rho : E(H) \to R$ be a rainbow 3-coloring of $H$ with $|R| = 5$, and let $[r]$ be a chromatic class. The following assertions hold for $i \in \{0, \ldots, 6\}$:

(i) If $[2i - 1, 2i]$, $[2i - 3, 2i - 4] \in [r]$, then $[2i + 1, 2i + 2]$, $[2i + 4, 2i + 5] \notin [r]$.

(ii) If $[2i - 1, 2i]$, $[2i - 7, 2i - 6] \in [r]$, then $[2i + 1, 2i - 8]$, $[2i + 3, 2i + 4] \notin [r]$.

(iii) If $[2i - 1, 2i]$, $[2i + 7, 2i + 6] \in [r]$, then $[2i + 5, 2i + 4]$, $[2i - 5, 2i - 6] \notin [r]$.

(iv) If $[2i - 1, 2i]$, $[2i + 3, 2i - 6] \in [r]$, then $[2i + 1, 2i + 2]$, $[2i - 2, 2i + 7] \notin [r]$.

(v) If $[2i - 1, 2i]$, $[2i + 3, 2i + 2] \in [r]$, then $[2i - 3, 2i - 2]$, $[2i - 6, 2i - 5] \notin [r]$.

**Proof.** Note that if $d_H(a,b) = 2$ for $a, b \in V(H)$, then the shortest $(a, b)$-path is unique because the girth of $H$ is 6. Let $N(a) = \{c, a', a''\}$ and $N(b) = \{c, b', b''\}$. Then $a, c, b$ is the shortest path between $a$ and $b$. Since $\rho$ is a rainbow 3-coloring, it follows that between $a$ and $b$ there are another two vertex disjoint rainbow paths which must have even length at least 4 because $H$ is bipartite. Moreover, since $|R| = 5$ these paths must have length exactly 4. If $aa', bb' \in [r]$, then there must be
unique paths of length 2 joining $a'$ with $b'$ and $b'$ with $a''$ without edges in $[r]$. To prove the lemma we use this fact and we only indicate the shortest path $(a, c, b)$ in most of the cases.

(i) Suppose that $\{2i - 1, 2i\}, \{2i - 3, 2i - 4\} \not\subseteq [r]$. Let us consider the path of length two $\{2i - 4, 2i - 5, 2i\}$. One vertex disjoint rainbow path between $2i - 4$ and $2i$ must join $2i - 3$ with $2i + 1 \in N(2i) \setminus \{2i - 5, 2i - 1\}$ since $\{2i - 1, 2i\} \subseteq [r]$, and having no edges in $[r]$. This path is $\{2i - 3, 2i + 2, 2i + 1\}$ and $\{2i + 1, 2i + 2\} \not\subseteq [r]$. And the other vertex disjoint rainbow path must join $2i - 1$ with $2i + 5 \in N(2i) \setminus \{2i - 5, 2i - 3\}$ since $\{2i - 4, 2i - 3\} \subseteq [r]$, and having no edges in $[r]$. This path is $\{2i - 1, 2i + 4, 2i + 5\}$ and $\{2i + 5, 2i + 4\} \not\subseteq [r]$.

(ii) Suppose that $\{2i - 1, 2i\}, \{2i - 7, 2i - 6\} \subseteq [r]$. The result follows by considering the path $\{2i - 1, 2i - 2, 2i - 7\}$.

(iii) Suppose that $\{2i - 1, 2i\}, \{2i + 7, 2i + 6\} \subseteq [r]$. The result follows by considering the path $\{2i - 1, 2i - 2, 2i + 7\}$.

(iv) Suppose that $\{2i - 1, 2i\}, \{2i + 3, 2i + 8\} \subseteq [r]$. The result follows by considering the path $\{2i - 1, 2i + 4, 2i + 3\}$.

(v) Suppose that $\{2i - 1, 2i\}, \{2i + 3, 2i + 2\} \subseteq [r]$. The result follows by considering the path $\{2i, 2i + 1, 2i + 2\}$.

\begin{theorem}
Let $H$ be the Heawood graph. Then $6 \leq rc_3(H) \leq 7$.
\end{theorem}

\begin{proof}
Let $\rho : E(H) \rightarrow R$ be a rainbow 3-coloring on the edges of $H$. We reason by contradiction assuming that $rc_3(H) = |R| = 5$, which implies that there is a chromatic class $[r]$ with $|r| \geq 5$ because $|E(H)| = 21 = \sum |r|$. Let $[r]$ be such a chromatic class. Observe that a matching of at least 5 edges in Heawood graph always contains two edges at distance 2. Without loss of generality suppose that $\{1, 2\} \subseteq [r]$. At distance two of $\{1, 2\}$ there are 8 edges which induce a cycle of length 8: $C = \{7, 12, 13, 4, 5, 10, 9, 8, 7\}$. Assume that $\{7, 8\} \subseteq [r]$. Since $\{7, 8\}, \{1, 2\} \subseteq [r]$, by item (ii) of Lemma 3.2 (taking $i = 4$), it follows that $\{9, 0\}, \{11, 12\} \not\subseteq [r]$.

We consider the following cases according to the edges in $E(C) \cap [r]$.

Suppose that there are four edges in $C$ with color $r$. In this case, the class $[r]$ must contain the edges $\{1, 2\}, \{7, 8\}, \{9, 10\}, \{5, 4\}$ and $\{13, 12\}$. Since $\{7, 8\}, \{5, 4\} \subseteq [r]$, by item (i) of Lemma 3.2 (taking $i = 4$), it follows that $\{9, 10\}, \{13, 12\} \not\subseteq [r]$, a contradiction. Hence $C$ contains at most 3 edges in $[r]$ including $\{7, 8\}$.

Suppose that $\{7, 8\}, \{13, 12\} \in E(C) \cap [r]$. Since $\{1, 2\}, \{13, 12\} \subseteq [r]$ it follows that $\{3, 4\} \not\subseteq [r]$ by item (i) of Lemma 3.2 (taking $i = 1$). If $\{5, 10\} \subseteq [r]$, by Lemma 3.1 and (1) there is no other edge belonging to $[r]$, see Fig. 3(a), and so $|r| = 4$, which is a contradiction. Hence, $\{5, 10\} \not\subseteq [r]$. If $\{5, 4\} \subseteq [r]$, then taking into account that $\{7, 8\} \subseteq [r]$, it follows by item (i) of Lemma 3.2 (taking $i = 4$) that $\{12, 13\} \not\subseteq [r]$, a contradiction. Thus, $\{5, 4\} \not\subseteq [r]$. If $\{9, 10\} \subseteq [r]$, using that $\{13, 12\} \subseteq [r]$, item (v) of Lemma 3.2 (taking $i = 5$) implies that $\{7, 8\} \not\subseteq [r]$ which is a contradiction; then $\{9, 10\} \not\subseteq [r]$. Furthermore, if $\{5, 6\} \subseteq [r]$ using that $\{12, 13\} \subseteq [r]$, item (iii) of Lemma 3.2 (taking $i = 3$) implies that $\{11, 10\} \not\subseteq [r]$ yielding that $|r| = 4$ which is a contradiction. Hence, if $\{7, 8\} \subseteq [r]$ then $\{12, 13\} \not\subseteq [r]$. By symmetry, if $\{7, 8\} \subseteq [r]$ then $\{9, 10\} \not\subseteq [r]$. Thus, if $[r]$ contains two edges of $C$ these two edges must be at distance at least 2 in $C$.

Suppose that $\{7, 8\}, \{5, 10\} \subseteq [r] \cap E(C)$. Observe that the only other edges that can be in $[r]$ are $\{3, 4\}, \{13, 0\}, \{13, 4\}$ (see Fig. 3(b)). By item (iv) of Lemma 3.2 (taking $i = 1$), $\{1, 2\}, \{5, 10\} \subseteq [r]$ implies that $\{3, 4\} \not\subseteq [r]$, yielding that $|r| \leq 4$, a contradiction. Thus, $\{5, 10\} \not\subseteq [r]$. By symmetry $\{4, 13\} \not\subseteq [r]$.

Suppose that $\{7, 8\}, \{5, 4\} \subseteq [r] \cap E(C)$. At this point the only edges that can be in $[r]$ are $\{3, 4\}, \{10, 11\}$ (see Fig. 3(c)). By item (v) of Lemma 3.2 (taking $i = 1$), $\{1, 2\}, \{5, 4\} \subseteq [r]$ implies that $\{10, 11\}, \{13, 0\} \not\subseteq [r]$, yielding that $|r| = 4$ which is a contradiction. Thus, we conclude that $[r] \cap E(C) = \{7, 8\}$.

Therefore, we have all the edges incident with $\{1, 2\}, \{7, 8\}$ (by Lemma 3.1) together with the edges of $C$ minus $\{7, 8\}$, and $\{11, 12\}, \{9, 0\}$ (by (1)) do not belong to $[r]$. Hence, the edges that can be in $[r]$ are $\{3, 4\}, \{5, 6\}, \{10, 11\}$ and $\{13, 0\}$. Suppose $\{13, 0\} \subseteq [r]$. Then $\{7, 8\}, \{13, 0\} \subseteq [r]$ implies that $\{3, 4\} \not\subseteq [r]$ by item (ii) of Lemma 3.2, and $\{10, 11\}, \{13, 0\} \subseteq [r]$. Therefore, we have all the edges incident with $\{1, 2\}, \{7, 8\}$ (by Lemma 3.1) together with the edges of $C$ minus $\{7, 8\}$, and $\{11, 12\}, \{9, 0\}$ (by (1)) do not belong to $[r]$. Hence, the edges that can be in $[r]$ are $\{3, 4\}, \{5, 6\}, \{10, 11\}$ and $\{13, 0\}$. Suppose $\{13, 0\} \subseteq [r]$. Then $\{7, 8\}, \{13, 0\} \subseteq [r]$ implies that $\{3, 4\} \not\subseteq [r]$ by item (ii) of Lemma 3.2, and $\{10, 11\}, \{13, 0\} \subseteq [r]$.

implies that \( \{1, 2\} \not\in [r] \) by item (i) of Lemma 3.2 which is a contradiction. Therefore, if \( \{13, 0\} \in [r] \), \( |[r]| = 4 \) which is a contradiction. Hence, \( \{13, 0\} \not\in [r] \). By symmetry \( \{10, 11\} \not\in [r] \), yielding that \( |[r]| \leq 4 \) which is a contradiction.

Since in every case we obtain a contradiction we conclude that for each chromatic class \( |[r]| \leq 4 \) which implies that \( |R| \geq 6 \). ■

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