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Numerical analysis of some Dual-Phase-Lag models

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Abstract

In this paper we analyze, from the numerical point of view, two dual-phase-lag models appearing in the heat conduction theory. Both models are written as linear partial differential equations of third order in time. The variational formulations, written in terms of the thermal acceleration, lead to linear variational equations, for which existence and uniqueness results, and energy decay properties, are recalled. Then, fully discrete approximations are introduced for both models using the finite element method to approximate the spatial variable and the implicit Euler scheme to discretize the time derivatives. Discrete stability properties are proved, and a priori error estimates are obtained, from which the linear convergence of the approximations are derived. Finally, some numerical simulations are described in one and two dimensions to demonstrate the accuracy of the approximations and the behaviour of the solutions.

Key words: Heat conduction, dual-phase-lag, finite elements, a priori estimates, numerical simulations.

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1 Introduction

For the Fourier heat equation theory the thermal disturbances at some point will be felt instantly anywhere for every instant. This is a paradox of the model and many authors have tried to overcome this drawback proposing different alternative constitutive equations. Most known theory is the proposed by the Maxwell-Cattaneo law that brings to an hyperbolic damped equation for the heat conduction [6]. It is also worth recalling the models obtained by Green and Naghdi [15–17] which assume alternative heat conduction theories.

In 1995, Tzou [29,30] suggested a modification of the Fourier constitutive equation. He proposed a theory of thermal flux with delay parameters. Since these pioneering works many papers have been published dealing with mathematical issues as existence and uniqueness, energy decay, spectral analysis, spatial behaviour and so on (see, for instance, [2,7,8,12,18,19,21–26,34,35]). In fact, this is a field under intensive study. The main idea of this model is based on the assumption that the temperature gradient at a certain time results in a heat flux vector at a different time. This proposition is usually understood in terms of the microstructure of the material. We believe that the mathematical and physical study of this theory is needed to clarify its applicability. Recently, it has been tested that these modified constitutive equations are able to simulate better the heat transport in some special cases as, for example, micro/nanoscales ([33]), ultra fast laser-pulsed processes ([32]) or living tissues ([37]).

Even if there are numerous papers dealing with mathematical issues, to our knowledge there are few works providing the numerical simulation of these models (see, e.g., [4,5,14,20,27,28]). In this work, we revisit the models considered in [21,24,26], where the existence and uniqueness of solution was proved, and the energy decay was analyzed, deriving the necessary conditions on the time-relaxation parameters. Here, our aim is to introduce the fully discrete approximation of these models by using the finite element method and the implicit Euler scheme, to obtain discrete stability properties, to prove some a priori error estimates and to present some numerical simulations in one- and two-dimensional examples in order to demonstrate the accuracy of the approximations and the behaviour of the solutions.

The paper is outlined as follows. The mathematical models are described in

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Section 2 following [21,24,26], deriving their variational formulation. Existence and uniqueness results, and energy decay properties, proved in [21,24,26], are also stated. Then, in Section 3 two numerical schemes are introduced, based on the finite element method to approximate the spatial domain and the backward Euler scheme to discretize the time derivatives. Discrete stability properties are proved, a priori error estimates are deduced for the approximative solutions and, under suitable regularity assumptions, the linear convergence of the algorithms is obtained. Finally, some one- and two-dimensional numerical simulations are presented in Section 4, and some conclusions are shown in Section 5.

2 Mathematical models

In this section, we present briefly the models, the required assumptions and the variational formulations of the mechanical problems, and we state existence and uniqueness results. We refer the reader to [21,24,26] for details.

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, be the thermal domain, assumed to be bounded, and denote by $[0, T]$, $T > 0$, the time interval of interest. The boundary of the body $\Gamma = \partial\Omega$ is assumed to be Lipschitz, with outward unit normal vector $\boldsymbol{\nu} = (\nu_i)_{i=1}^d$. For the sake of simplicity, we assume that, on the whole boundary Γ, the temperature is prescribed to a given temperature θ^b. Moreover, let $\boldsymbol{x} \in \Omega$ and $t \in [0, T]$ be the spatial and time variables, respectively. In order to simplify the writing, we do not indicate the dependence of the functions on $\boldsymbol{x}$ and t, and the subscript t indicates the derivative with respect to the time variable.

Denote by $\theta(\boldsymbol{x}, t)$ the temperature of the body at point $\boldsymbol{x} \in \overline{\Omega}$ and time $t \in [0, T]$.

Following the pioneeering work [29], the heat equation is written as follows,

$$\theta_t + \operatorname{div} \boldsymbol{q} = 0, \tag{1}$$

where $\boldsymbol{q}$ is the heat flux vector and the classical Fourier law is replaced by an approximation of the equation

$$\boldsymbol{q}(\boldsymbol{x}, t + \tau_q) = -\kappa \nabla\theta(\boldsymbol{x}, t + \tau_\theta),$$

where $\kappa > 0$ is the thermal diffusion coefficient, and τ_q and τ_θ are the phase-lag parameters of the heat flux and the phase-lag of the gradient of the temperature, respectively. If we adjoin these equations the problem becomes ill-posed in the sense of Hadamard [10]. At the same time as it is pointed out in [11], this model is not in agreement with the Second Law of Thermodynamics.

In fact, the solutions have a very explosive behaviour and we may conclude that the problem proposed cannot be a good candidate to describe the heat conduction nor from a mathematical point of view neither from the thermomechanical point of view. Nevertheless many people have been attracted for the theories obtained when we substitute the proposed constitutive equation by the Taylor approximations to the delay equations and currently many authors accept and study them. In [29] the following second-order approximation for $\boldsymbol{q}$ and the first-order approximation for $\nabla\theta$ were used,

$$\boldsymbol{q} + \tau_q \boldsymbol{q}_t + \frac{\tau_q^2}{2}\boldsymbol{q}_{tt} = -\kappa\nabla\theta - \kappa\tau_\theta\nabla\theta_t. \tag{2}$$

It is clear that the Fourier model is recovered when $\tau_q = \tau_\theta = 0$.

Combining (1) and (2) and assuming that the body is isotropic and homogeneous, we obtain a first thermal problem.

Problem $\mathbf{P}^1$. *Find the temperature $\theta : \overline{\Omega} \times [0, T] \to \mathbb{R}$ such that*

$$\frac{\tau_q^2}{2}\theta_{ttt} + \tau_q\theta_{tt} + \theta_t = \kappa\Delta\theta + \kappa\tau_\theta\Delta\theta_t \quad \textit{in} \quad \Omega \times (0, T), \tag{3}$$

$$\theta(\boldsymbol{x}, t) = \theta^b(\boldsymbol{x}, t) \quad \textit{for a.e.} \quad \boldsymbol{x} \in \Gamma,\ t \in [0, T], \tag{4}$$

$$\theta(\boldsymbol{x}, 0) = \theta_0(\boldsymbol{x}), \quad \theta_t(\boldsymbol{x}, 0) = e_0(\boldsymbol{x}), \quad \theta_{tt}(\boldsymbol{x}, 0) = \xi_0(\boldsymbol{x}) \quad \textit{for a.e.} \quad \boldsymbol{x} \in \Omega. \tag{5}$$

As it is pointed out in [30], second-order approximations can be used for both gradient of temperature and heat flux, and so equation (2) must be replaced accordingly leading to the second thermal problem, assuming again the body to be homogeneous and isotropic.

Problem $\mathbf{P}^2$. *Find the temperature $\theta : \overline{\Omega} \times [0, T] \to \mathbb{R}$ such that*

$$\frac{\tau_q^2}{2}\theta_{ttt} + \tau_q\theta_{tt} + \theta_t = \kappa\Delta\theta + \kappa\tau_\theta\Delta\theta_t + \kappa\frac{\tau_\theta^2}{2}\Delta\theta_{tt} \quad \textit{in} \quad \Omega \times (0, T), \tag{6}$$

$$\theta(\boldsymbol{x}, t) = \theta^b(\boldsymbol{x}, t) \quad \textit{for a.e.} \quad \boldsymbol{x} \in \Gamma,\ t \in [0, T], \tag{7}$$

$$\theta(\boldsymbol{x}, 0) = \theta_0(\boldsymbol{x}), \quad \theta_t(\boldsymbol{x}, 0) = e_0(\boldsymbol{x}), \quad \theta_{tt}(\boldsymbol{x}, 0) = \xi_0(\boldsymbol{x}) \quad \textit{for a.e.} \quad \boldsymbol{x} \in \Omega. \tag{8}$$

For the sake of simplicity in the presentation and the calculations developed in the next section, we assume that $\theta^b = 0$.

Remark 1 *We note that these proposed models, problems P^1 and P^2, are in agreement with the Second Law of Thermodynamics under some conditions on* *the phase-lag parameters (see [12,13]). Moreover, as it is pointed out in [36], the involvement of high-order terms in parameters τ_q and τ_θ are a natural consequence of the handling of systems in which multiple energy carriers are involved. An interesting discussion in this sense can be found in ([31] p. 376).*

In order to obtain the variational formulation of problems P^1 and P^2, let $Y = L^2(\Omega)$, $H = [L^2(\Omega)]^d$ and denote by $(\cdot,\cdot)_Y$ and $(\cdot,\cdot)_H$ the respective scalar products in these spaces, with corresponding norms $\|\cdot\|_Y$ and $\|\cdot\|_H$. Moreover, let us define the variational space E as follows,

$$E = \{z \in H^1(\Omega)\,;\, z = 0 \quad \text{on} \quad \Gamma\},$$

with respective scalar product $(\cdot,\cdot)_E$ and norm $\|\cdot\|_E$.

By using Green's formula, we write the variational formulation of problems P^1 and P^2 in terms of the thermal velocity $e = \theta_t$ and the thermal acceleration $\xi = \theta_{tt}$.

First, we have the corresponding weak formulation of Problem P^1.

Problem VP^1. *Find the thermal acceleration $\xi : [0,T] \to E$ such that $\xi(0) = \xi_0$, and, for a.e. $t \in (0,T)$,*

$$\begin{aligned} \frac{\tau_q^2}{2}(\xi_t(t),w)_Y + \tau_q(\xi(t),w)_Y + (e(t),w)_Y + \kappa(\nabla\theta(t),\nabla w)_H \\ +\kappa\tau_q(\nabla e(t),\nabla w)_H = 0 \quad \forall w \in E, \end{aligned} \tag{9}$$

where the temperature θ and the thermal velocity e are obtained from the respective equations

$$e(t) = \int_0^t \xi(s)\,ds + e_0, \quad \theta(t) = \int_0^t e(s)\,ds + \theta_0. \tag{10}$$

Similarly, we have the corresponding weak formulation of Problem P^2.

Problem VP^2. *Find the thermal acceleration $\xi : [0,T] \to E$ such that $\xi(0) = \xi_0$, and, for a.e. $t \in (0,T)$,*

$$\begin{aligned} \frac{\tau_q^2}{2}(\xi_t(t),w)_Y + \tau_q(\xi(t),w)_Y + (e,w)_Y + \kappa(\nabla\theta(t),\nabla w)_H + \kappa\tau_\theta(\nabla e(t),\nabla w)_H \\ +\kappa\frac{\tau_\theta^2}{2}(\nabla\xi(t),\nabla w)_H = 0 \quad \forall w \in E, \end{aligned} \tag{11}$$

where again the temperature θ and the thermal velocity e are obtained from *the respective equations*

$$e(t) = \int_0^t \xi(s)\,ds + e_0, \quad \theta(t) = \int_0^t e(s)\,ds + \theta_0. \tag{12}$$

In [21] Problem VP^1 was considered and the following result, which states the existence of a unique solution and an energy decay property, was proved.

Theorem 2 *Let the assumptions*

$$\tau_q > 0, \quad \tau_\theta > 0, \quad \kappa > 0 \tag{13}$$

hold. Therefore, Problem VP^1 has a unique solution with the following regularity

$$\theta \in C^1([0,T];E), \quad e \in C^1([0,T];E), \quad \xi \in C^1([0,T];Y).$$

Moreover, if we also assume that

$$\tau_\theta > \tau_q/2 \tag{14}$$

then the energy given by

$$E(t) = \frac{1}{2}\Big\{ \|\theta(t) + \tau_q e(t) + \frac{\tau_q^2}{2}\xi(t)\|_Y^2 + \kappa(\tau_q + \tau_\theta)\|\nabla\theta(t)\|_H^2 + \kappa\frac{\tau_\theta\tau_q^2}{2}\|\nabla e(t)\|_H^2 \\ + \kappa\tau_q^2(\nabla e(t), \nabla\theta(t))_H \Big\}$$

decays exponentially; i.e. there exist $\omega > 0$ and $M > 0$ such that

$$E(t) \le M E(0) e^{-\omega t}, \quad t \ge 0.$$

The analysis of Problem VP^2 was done in [26], where the following result, which states the existence of a unique solution and an energy decay property, was proved.

Theorem 3 *Let the assumptions (13) hold and suppose that the initial conditions have the regularity*

$$\theta_0,\, e_0 \in E, \quad \xi_0 \in Y.$$

Therefore, Problem VP^2 has a unique solution with the following regularity

$$\theta \in C^1([0,T];E), \quad e \in C^1([0,T];E), \quad \xi \in C^1([0,T];Y).$$

Moreover, if we also assume that

$$\tau_\theta > \tau_q, \tag{15}$$

then the energy given by

$$E(t) = \|\xi(t)\|_Y^2 + \|e(t)\|_Y^2 + \|\theta(t)\|_Y^2$$

decays exponentially; i.e. there exist $\omega > 0$ *and* $M > 0$ *such that*

$$E(t) \leq ME(0)e^{-\omega t}, \quad t \geq 0.$$

In fact the exponential stability of the solutions of this model is currently under research, because from the spectral study one thinks that the range where the solutions of the equation are exponentially stable should be enlarged.

3 Fully discrete approximations: an a priori error analysis

In this section, we introduce a finite element algorithm for approximating solutions to variational problems VP^1 and VP^2. This is done in two steps. First, we consider the finite element space $E^h \subset E$ given by

$$E^h = \{\psi^h \in C(\overline{\Omega})\ ;\ \psi^h_{|_{Tr}} \in P_1(Tr) \quad \forall Tr \in \mathcal{T}^h, \quad \psi^h = 0 \text{ on } \Gamma\}, \tag{16}$$

where Ω is assumed to be a polyhedral domain, $\mathcal{T}^h$ denotes a triangulation of $\overline{\Omega}$, and $P_1(Tr)$ represents the space of polynomials of global degree less or equal to 1 in Tr. Here, $h > 0$ denotes the spatial discretization parameter.

Secondly, the time derivatives are discretized by using a uniform partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \ldots < t_N = T$, and let k be the time step size, $k = T/N$. Moreover, for a continuous function $f(t)$ we denote $f_n = f(t_n)$ and, for the sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/k$ its corresponding divided differences.

Using the classical backward Euler scheme, the fully discrete approximation of Problem VP^1 is the following.

Problem VP^{1,hk}. *Find the discrete thermal acceleration* $\xi^{hk} = \{\xi_n^{hk}\}_{n=0}^N \subset E^h$ *such that* $\xi_0^{hk} = \xi_0^h$ *and, for* $n = 1, \ldots, N$,

$$\begin{aligned}&\frac{\tau_q^2}{2}(\delta\xi_n^{hk}, w^h)_Y + \tau_q(\xi_n^{hk}, w^h)_Y + (e_n^{hk}, w^h)_Y + \kappa(\nabla\theta_n^{hk}, \nabla w^h)_H \\ &\quad + \kappa\tau_\theta(\nabla e_n^{hk}, \nabla w^h)_H = 0 \quad \forall w^h \in E^h,\end{aligned} \tag{17}$$

where the discrete temperature and the discrete thermal velocity are then re-

covered from the relations

$$e_n^{hk} = k \sum_{j=1}^{n} \xi_j^{hk} + e_0^h, \quad \theta_n^{hk} = k \sum_{j=1}^{n} e_j^{hk} + \theta_0^h. \tag{18}$$

Similarly, the fully discrete approximation of Problem VP^2 is the following.

Problem VP^{2,hk}. *Find the discrete thermal acceleration $\xi^{hk} = \{\xi_n^{hk}\}_{n=0}^N \subset E^h$ such that $\xi_0^{hk} = \xi_0^h$ and, for $n = 1, \dots, N$,*

$$\frac{\tau_q^2}{2}(\delta\xi_n^{hk}, w^h)_Y + \tau_q(\xi_n^{hk}, w^h)_Y + (e_n^{hk}, w^h)_Y + \kappa(\nabla\theta_n^{hk}, \nabla w^h)_H$$
$$+\kappa\tau_\theta(\nabla e_n^{hk}, \nabla w^h)_H + \kappa\frac{\tau_\theta^2}{2}(\nabla\xi_n^{hk}, \nabla w^h)_H = 0 \quad \forall w^h \in E^h, \tag{19}$$

where again the discrete temperature and the discrete thermal velocity are then recovered from the relations

$$e_n^{hk} = k \sum_{j=1}^{n} \xi_j^{hk} + e_0^h, \quad \theta_n^{hk} = k \sum_{j=1}^{n} e_j^{hk} + \theta_0^h. \tag{20}$$

Using the classical Lax-Milgram lemma we can easily prove that both problems VP^{1,hk} and VP^{2,hk} have a respective unique discrete solution. So, the aim of this section is to provide the numerical analysis of both problems.

3.1 A priori error estimates for Problem VP^{1,hk}

We have the following stability result.

Lemma 4 *Under the assumptions of Theorem 2, it follows that the sequences $\{\theta^{hk}, e^{hk}, \xi^{hk}\}$ generated by Problem $VP^{1,hk}$ satisfy the stability estimate:*

$$\|\xi_n^{hk}\|_Y^2 + \|e_n^{hk}\|_Y^2 + \|\nabla e_n^{hk}\|_H^2 + \|\theta_n^{hk}\|_Y^2 + \|\nabla\theta_n^{hk}\|_H^2 \le C,$$

where C is a positive constant which is independent of the discretization parameters h and k.

PROOF. For the sake of clarity in the writing of this proof, we remove the superscripts h and k in all the variables.

Taking $w^h = \xi_n$ as a test function in discrete variational equation (17) we have

$$\frac{\tau_q^2}{2}(\delta\xi_n, \xi_n)_Y + \tau_q(\xi_n, \xi_n)_Y + (e_n, \xi_n)_Y + \kappa(\nabla\theta_n, \nabla\xi_n)_H + \kappa\tau_\theta(\nabla e_n, \nabla\xi_n)_H = 0.$$

Thus, keeping in mind that

$$\begin{aligned}
&(\delta\xi_n, \xi_n)_Y \geq \frac{1}{2k}\left\{\|\xi_n\|_Y^2 - \|\xi_{n-1}\|_Y^2\right\},\\
&(e_n, \xi_n)_Y \geq \frac{1}{2k}\left\{\|e_n\|_Y^2 - \|e_{n-1}\|_Y^2\right\},\\
&(\nabla e_n, \nabla\xi_n)_H \geq \frac{1}{2k}\left\{\|\nabla e_n\|_H^2 - \|\nabla e_{n-1}\|_H^2\right\},\\
&(\nabla\theta_n, \nabla\xi_n)_H = \frac{1}{k}\{(\nabla\theta_n, \nabla e_n)_H - (\nabla\theta_{n-1}, \nabla e_{n-1})_H\} - (\nabla e_n, \nabla e_{n-1})_H,
\end{aligned}$$

it follows that

$$\begin{aligned}
&\frac{1}{2k}\left\{\|\xi_n\|_Y^2 - \|\xi_{n-1}\|_Y^2\right\} + \frac{1}{2k}\left\{\|e_n\|_Y^2 - \|e_{n-1}\|_Y^2\right\} + \frac{1}{2k}\left\{\|\nabla e_n\|_H^2 - \|\nabla e_{n-1}\|_H^2\right\}\\
&\quad + \frac{1}{k}\{(\nabla\theta_n, \nabla e_n)_H - (\nabla\theta_{n-1}, \nabla e_{n-1})_H\} \leq C\left(\|\nabla e_n\|_H^2 + \|\nabla e_{n-1}\|_H^2\right).
\end{aligned}$$

By induction we find that

$$\begin{aligned}
\|\xi_n\|_Y^2 + \|e_n\|_Y^2 + \|\nabla e_n\|_H^2 + (\nabla\theta_n, \nabla e_n)_H &\leq Ck\sum_{j=1}^n \|\nabla e_j\|_H^2\\
&+C(\|\xi_0\|_Y^2 + \|e_0\|_Y^2 + \|\nabla e_0\|_H^2 + \|\nabla\theta_0\|_H^2),
\end{aligned}$$

and so, taking into account that

$$\begin{aligned}
&|(\nabla\theta_n, \nabla e_n)_H| \leq C\|\nabla\theta_n\|_H^2 + \epsilon\|\nabla e_n\|_H^2,\\
&\|\theta_n\|_Y^2 \leq Ck\sum_{j=1}^n \|e_j\|_Y^2 + C\|\theta_0\|_Y^2,\\
&\|\nabla\theta_n\|_H^2 \leq Ck\sum_{j=1}^n \|\nabla e_j\|_H^2 + C\|\nabla\theta_0\|_H^2,
\end{aligned}$$

where we used Cauchy's inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon}b^2,\ a, b, \epsilon \in \mathbb{R},\ \epsilon > 0, \tag{21}$$

and ϵ is assumed small enough, we find that

$$\begin{aligned}
\|\xi_n\|_Y^2 + \|e_n\|_Y^2 + \|\nabla e_n\|_H^2 + \|\theta_n\|_Y^2 + \|\nabla\theta_n\|_H^2 &\leq Ck\sum_{j=1}^n \left[\|\nabla e_j\|_H^2 + \|e_j\|_Y^2\right]\\
&+C(\|\xi_0\|_Y^2 + \|e_0\|_Y^2 + \|\nabla e_0\|_H^2 + \|\nabla\theta_0\|_H^2).
\end{aligned}$$

Finally, the desired stability estimates are a straightforward consequence of the application of a discrete version of Gronwall's inequality (see, e.g., [3]).

Now, we have a discrete version of the energy decay property.

Lemma 5 *Under the assumptions of Theorem 2, if we define the discrete energy* E_n^{hk} *in the following form:*

$$E_n^{hk} = \frac{1}{2}\Big\{\tau_q\|\frac{\tau_q}{2}\xi_n^{hk} + e_n^{hk}\|_Y^2 + \frac{\tau_q}{2}\|e_n^{hk}\|_Y^2 + \kappa\tau_q(\nabla\theta_n^{hk}, \nabla e_n^{hk})_H \\ +\kappa\|\nabla\theta_n^{hk}\|_H^2 + \frac{\kappa\tau_\theta\tau_q}{2}\|\nabla e_n^{hk}\|_H^2\Big\}, \tag{22}$$

then it decays, i.e. $\dfrac{E_n^{hk} - E_{n-1}^{hk}}{k} \leq 0.$

PROOF.

First, we note that

$$\frac{\tau_q}{2}\delta\xi_n^{hk} + \xi_n^{hk} = \frac{\tau_q}{2}\frac{\xi_n^{hk} - \xi_{n-1}^{hk}}{k} + \frac{e_n^{hk} - e_{n-1}^{hk}}{k}.$$

Then, taking $w^h = \frac{\tau_q}{2}\xi_n^{hk} + e_n^{hk}$ as a test function in (17), it follows that

$$\begin{aligned}&\frac{\tau_q}{2k}\Big(\|\frac{\tau_q}{2}\xi_n^{hk} + e_n^{hk}\|_Y^2 - \|\frac{\tau_q}{2}\xi_{n-1}^{hk} + e_{n-1}^{hk}\|_Y^2\Big)\\ &\quad+\frac{\tau_q}{4k}\Big(\|e_n^{hk}\|_Y^2 - \|e_n^{hk}\|_Y^2\Big) + \|e_n^{hk}\|_Y^2\\ &\quad+\frac{\kappa}{2k}\Big(\|\nabla\theta_n^{hk} - \nabla\theta_{n-1}^{hk}\|_H^2 + \|\nabla\theta_n^{hk}\|_H^2 - \|\nabla\theta_{n-1}^{hk}\|_H^2\Big)\\ &\quad+\frac{\kappa\tau_q}{2}(\nabla\theta_n^{hk}, \nabla\xi_n^{hk})_H + \kappa\tau_\theta\|\nabla e_n^{hk}\|_H^2\\ &\quad+\frac{\kappa\tau_\theta\tau_q}{4k}\Big(\|\nabla e_n^{hk} - \nabla e_{n-1}^{hk}\|_H^2 - \|\nabla e_n^{hk}\|_H^2 - \|\nabla e_{n-1}^{hk}\|_H^2\Big) \leq 0.\end{aligned}$$

Now observe that

$$\begin{aligned}\frac{\kappa\tau_q}{2}(\nabla\theta_n^{hk}, \nabla\xi_n^{hk})_H &= \frac{\kappa\tau_q}{2}(\nabla\theta_{n-1}^{hk}, \nabla\xi_n^{hk})_H + \frac{\kappa\tau_q}{2}(\nabla\theta_n^{hk} - \nabla\theta_{n-1}^{hk}, \nabla\xi_n^{hk})_H\\ &= \frac{\kappa\tau_q}{2k}(\nabla\theta_n^{hk}, \nabla e_n^{hk})_H - \frac{\kappa\tau_q}{2k}(\nabla\theta_{n-1}^{hk}, \nabla e_{n-1}^{hk})_H\\ &\quad-\frac{\kappa\tau_q}{2}(\nabla e_n^{hk}, \nabla e_n^{hk})_H + \frac{\kappa\tau_q}{2}(\nabla\theta_n^{hk} - \nabla\theta_{n-1}^{hk}, \nabla\xi_n^{hk})_H\end{aligned}$$

and that

$$\frac{\kappa\tau_q}{2}(\nabla\theta_n^{hk} - \nabla\theta_{n-1}^{hk}, \nabla\xi_n^{hk})_H \leq \frac{\kappa}{2k}\|\nabla\theta_n^{hk} - \nabla\theta_{n-1}^{hk}\|_H^2 + \frac{\kappa\tau_q^2}{8k}\|\nabla e_n^{hk} - \nabla e_{n-1}^{hk}\|_H^2,$$

then we have

$$
\begin{aligned}
&\frac{\tau_q}{2k}\Big(\|\frac{\tau_q}{2}\xi_n^{hk}+e_n^{hk}\|_Y^2-\|\frac{\tau_q}{2}\xi_{n-1}^{hk}+e_{n-1}^{hk}\|_Y^2\Big)+\frac{\tau_q}{4k}\Big(\|e_n^{hk}\|_Y^2-\|e_n^{hk}\|_Y^2\Big)\\
&\quad+\|e_n^{hk}\|_Y^2+\frac{\kappa}{2k}\Big(\|\nabla\theta_n^{hk}\|_H^2-\|\nabla\theta_{n-1}^{hk}\|_H^2\Big)\\
&\quad+\kappa(\tau_\theta-\frac{\tau_q}{2})\|\nabla e_n^{hk}\|_H^2+\frac{\kappa\tau_q}{2k}\Big((\nabla\theta_n^{hk},\nabla e_n^{hk})_H-(\nabla\theta_{n-1}^{hk},\nabla e_{n-1}^{hk})_H\Big)\\
&\quad+\frac{\kappa\tau_q}{4k}(\tau_\theta-\frac{\tau_q}{2})\|\nabla e_n^{hk}-\nabla e_{n-1}^{hk}\|_H^2+\frac{\kappa\tau_\theta\tau_q}{4k}\Big(\|\nabla e_n^{hk}\|_H^2-\|\nabla e_{n-1}^{hk}\|_H^2\Big)\leq 0.
\end{aligned}
$$

Therefore, using the fact that $\tau_\theta>\tau_q/2$ we obtain the decay of the discrete energy.

Now, we obtain the following a priori error estimates result.

Theorem 6 *Under the assumptions of Lemma 4, if we denote by (e,ξ,θ) the solution to problem VP^1 and by $(e^{hk},\xi^{hk},\theta^{hk})$ the solution to problem $VP^{1,hk}$, then we have the following a priori error estimates, for all $w^h=\{w_j^h\}_{j=0}^N\subset E^h$,*

$$
\begin{aligned}
&\max_{0\leq n\leq N}\Big\{\|\xi_n-\xi_n^{hk}\|_Y^2+\|e_n-e_n^{hk}\|_E^2+\|\theta_n-\theta_n^{hk}\|_E^2\Big\}\\
&\quad\leq Ck\sum_{j=1}^N\Big(\|\nabla((\theta_t)_j-\delta\theta_j)\|_H^2+\|\nabla((e_t)_j-\delta e_j)\|_H^2+\|(\xi_t)_j-\delta\xi_j\|_Y^2\\
&\qquad+\|\xi_j-w_j^h\|_Y^2+\|\nabla(\xi_j-w_j^h)\|_H^2+I_j^2\Big)+C\max_{0\leq n\leq N}\|\xi_n-w_n^h\|_Y^2\\
&\qquad+C(J_n^2+I_n^2)+\frac{C}{k}\sum_{j=1}^{N-1}\|\xi_j-w_j^h-(\xi_{j+1}-w_{j+1}^h)\|_Y^2\\
&\qquad+C\Big(\|\xi_0-\xi_0^h\|_Y^2+\|e_0-e_0^h\|_E^2+\|\theta_0-\theta_0^h\|_E^2\Big),
\end{aligned}\tag{23}
$$

where I_j and J_n denotes the integration errors defined by

$$
I_j=\left\|\int_0^{t_j}\nabla e(s)\,ds-k\sum_{l=1}^j\nabla e_l\right\|_H,\quad J_n=\left\|\int_0^{t_n}e(s)\,ds-k\sum_{j=1}^n e_j\right\|_Y
$$

and C is a positive constant which is independent of the discretization parameters h and k.

PROOF. First, we subtract variational equation (9) at time $t=t_n$ for a test function $w=w^h\in E^h\subset E$ and discrete variational equation (17) to obtain, for all $w^h \in E^h$,

$$\frac{\tau_q^2}{2}((\xi_t)_n - \delta\xi_n^{hk}, w^h)_Y + \tau_q(\xi_n - \xi_n^{hk}, w^h)_Y + \kappa(\nabla(\theta_n - \theta_n^{hk}), \nabla w^h)_H$$
$$+(e_n - e_n^{hk}, w^h)_Y + \kappa\tau_\theta(\nabla(e_n - e_n^{hk}), \nabla w^h)_H = 0 \quad \forall w^h \in E^h,$$

and therefore, we find that, for all $w^h \in E^h$,

$$\begin{aligned}
&\frac{\tau_q^2}{2}((\xi_t)_n - \delta\xi_n^{hk}, \xi_n - \xi_n^{hk})_Y + \tau_q(\xi_n - \xi_n^{hk}, \xi_n - \xi_n^{hk})_Y + \kappa(\nabla(\theta_n - \theta_n^{hk}), \nabla(\xi_n - \xi_n^{hk}))_H \\
&\quad +(e_n - e_n^{hk}, \xi_n - \xi_n^{hk})_Y + \kappa\tau_\theta(\nabla(e_n - e_n^{hk}), \nabla(\xi_n - \xi_n^{hk}))_H \\
&= \frac{\tau_q^2}{2}((\xi_t)_n - \delta\xi_n^{hk}, \xi_n - w^h)_Y + \tau_q(\xi_n - \xi_n^{hk}, \xi_n - w^h)_Y + \kappa(\nabla(\theta_n - \theta_n^{hk}), \nabla(\xi_n - w^h))_H \\
&\quad +(e_n - e_n^{hk}, \xi_n - w^h)_Y + \kappa\tau_\theta(\nabla(e_n - e_n^{hk}), \nabla(\xi_n - w^h))_H.
\end{aligned}$$

Keeping in mind that

$$\begin{aligned}
&((\xi_t)_n - \delta\xi_n^{hk}, \xi_n - \xi_n^{hk})_Y \geq ((\xi_t)_n - \delta\xi_n, \xi_n - \xi_n^{hk})_Y \\
&\quad +\frac{1}{2k}\left\{\|\xi_n - \xi_n^{hk}\|_Y^2 - \|\xi_{n-1} - \xi_{n-1}^{hk}\|_Y^2\right\}, \\
&(e_n - e_n^{hk}, \xi_n - \xi_n^{hk})_Y \geq (e_n - e_n^{hk}, (e_t)_n - \delta e_n))_Y \\
&\quad +\frac{1}{2k}\left\{\|e_n - e_n^{hk}\|_Y^2 - \|e_{n-1} - e_{n-1}^{hk}\|_Y^2\right\}, \\
&(\nabla(e_n - e_n^{hk}), \nabla(\xi_n - \xi_n^{hk}))_Y \geq (\nabla(e_n - e_n^{hk}), \nabla((e_t)_n - \delta e_n))_H \\
&\quad +\frac{1}{2k}\left\{\|\nabla(e_n - e_n^{hk})\|_H^2 - \|\nabla(e_{n-1} - e_{n-1}^{hk})\|_H^2\right\}, \\
&\frac{1}{k}\Big\{(\nabla(\theta_n - \theta_n^{hk}), \nabla(e_n - e_n^{hk}))_H - (\nabla(\theta_{n-1} - \theta_{n-1}^{hk}), \nabla(e_{n-1} - e_{n-1}^{hk}))_H\Big\} \\
&\quad = \frac{1}{k}\Big\{(\nabla(\theta_n - \theta_n^{hk}), \nabla(e_n - e_n^{hk}))_H - (\nabla(\theta_n - \theta_n^{hk}), \nabla(e_{n-1} - e_{n-1}^{hk}))_H \\
&\quad +(\nabla(\theta_n - \theta_n^{hk}), \nabla(e_{n-1} - e_{n-1}^{hk}))_H - (\nabla(\theta_{n-1} - \theta_{n-1}^{hk}), \nabla(e_{n-1} - e_{n-1}^{hk}))_H\Big\} \\
&\quad = (\nabla(\theta_n - \theta_n^{hk}), \nabla(\delta e_n - \delta e_n^{hk}))_H + (\nabla(\delta\theta_n - \delta\theta_n^{hk}), \nabla(e_{n-1} - e_{n-1}^{hk}))_H \\
&\quad = (\nabla(\theta_n - \theta_n^{hk}), \nabla(\delta e_n - (e_t)_n))_H + (\nabla(\theta_n - \theta_n^{hk}), \nabla((e_t)_n - \delta e_n^{hk}))_H \\
&\quad +(\nabla(\delta\theta_n - (\theta_t)_n), \nabla(e_{n-1} - e_{n-1}^{hk}))_H + (\nabla((\theta_t)_n - e_n^{hk}), \nabla(e_{n-1} - e_{n-1}^{hk}))_H \\
&\quad = (\nabla(\theta_n - \theta_n^{hk}), \nabla(\delta e_n - (e_t)_n))_H + (\nabla(\theta_n - \theta_n^{hk}), \nabla(\xi_n - \xi_n^{hk}))_H \\
&\quad +(\nabla(\delta\theta_n - (\theta_t)_n), \nabla(e_{n-1} - e_{n-1}^{hk}))_H + (\nabla(e_n - e_n^{hk}), \nabla(e_{n-1} - e_{n-1}^{hk}))_H
\end{aligned}$$

we find that, for all $w^h \in E^h$,

$$\frac{1}{2k}\left\{\|\xi_n-\xi_n^{hk}\|_Y^2-\|\xi_{n-1}-\xi_{n-1}^{hk}\|_Y^2\right\}+\frac{1}{2k}\left\{\|e_n-e_n^{hk}\|_Y^2-\|e_{n-1}-e_{n-1}^{hk}\|_Y^2\right\}$$
$$+\frac{1}{2k}\left\{\|\nabla(e_n-e_n^{hk})\|_H^2-\|\nabla(e_{n-1}-e_{n-1}^{hk})\|_H^2\right\}$$
$$+\frac{1}{k}\Big\{(\nabla(\theta_n-\theta_n^{hk}),\nabla(e_n-e_n^{hk}))_H-(\nabla(\theta_{n-1}-\theta_{n-1}^{hk}),\nabla(e_{n-1}-e_{n-1}^{hk}))_H\Big\}$$
$$\leq C\Big(\|\nabla((\theta_t)_n-\delta\theta_n)\|_H^2+\|\nabla((e_t)_n-\delta e_n)\|_H^2+\|\nabla(e_n-e_n^{hk})\|_H^2$$
$$+\|\nabla(e_{n-1}-e_{n-1}^{hk})\|_H^2+\|(\xi_t)_n-\delta\xi_n\|_Y^2+\|e_n-e_n^{hk}\|_Y^2+\|\nabla(\theta_n-\theta_n^{hk})\|_H^2$$
$$+\|\xi_n-w^h\|_Y^2+\|\nabla(\xi_n-w^h)\|_H^2+(\delta\xi_n-\delta\xi_n^{hk},\xi_n-w^h)_Y\Big),$$

where we used several times inequality (21) and the notations $\delta\theta_n=(\theta_n-\theta_{n-1})/k$, $\delta e_n=(e_n-e_{n-1})/k$ and $\delta\xi_n=(\xi_n-\xi_{n-1})/k$, and we recall that $\xi_n^{hk}=\delta e_n^{hk}$ and $e_n^{hk}=\delta\theta_n^{hk}$.

Therefore, by induction it follows that

$$\|\xi_n-\xi_n^{hk}\|_Y^2+\|e_n-e_n^{hk}\|_Y^2+\|\nabla(e_n-e_n^{hk})\|_H^2$$
$$\leq Ck\sum_{j=1}^{n}\Big(\|\nabla((\theta_t)_j-\delta\theta_j)\|_H^2+\|\nabla((e_t)_j-\delta e_j)\|_H^2+\|\nabla(e_j-e_j^{hk})\|_H^2$$
$$+\|(\xi_t)_j-\delta\xi_j\|_Y^2+\|e_j-e_j^{hk}\|_Y^2+\|\nabla(\theta_j-\theta_j^{hk})\|_H^2+\|\xi_j-w_j^h\|_Y^2$$
$$+\|\nabla(\xi_j-w_j^h)\|_H^2+(\delta\xi_j-\delta\xi_j^{hk},\xi_j-w_j^h)_Y\Big)+C\|\nabla(\theta_n-\theta_n^{hk})\|_H^2$$
$$+C\Big(\|\xi_0-\xi_0^h\|_Y^2+\|e_0-e_0^h\|_Y^2+\|\nabla(e_0-e_0^h)\|_H^2+\|\nabla(\theta_0-\theta_0^h)\|_H^2\Big).$$

Finally, taking into account that

$$k\sum_{j=1}^{n}(\delta\xi_j-\delta\xi_j^{hk},\xi_j-w_j^h)_Y=\sum_{j=1}^{n}(\xi_j-\xi_j^{hk}-(\xi_{j-1}-\xi_{j-1}^{hk}),\xi_j-w_j^h)_Y$$
$$=(\xi_n-\xi_n^{hk},\xi_n-w_n^h)_Y+(\xi_0^h-\xi_0,\xi_1-\xi_1^h)_Y$$
$$+\sum_{j=1}^{n-1}(\xi_j-\xi_j^{hk},\xi_j-w_j^h-(\xi_{j+1}-w_{j+1}^h))_Y, \tag{24}$$
$$\|\nabla(\theta_n-\theta_n^{hk})\|_H^2\leq C\Big(I_n^2+\sum_{j=1}^{n}k\|\nabla(e_j-e_j^{hk})\|_H^2+\|\nabla(\theta_0-\theta_0^h)\|_H^2\Big),$$
$$\|\theta_n-\theta_n^{hk}\|_Y^2\leq C\Big(J_n^2+\sum_{j=1}^{n}k\|e_j-e_j^{hk}\|_Y^2+\|\theta_0-\theta_0^h\|_Y^2\Big),$$

where we recall that I_n and J_n are the integration errors defined previously, using the above estimates and a discrete version of Gronwall's inequality we conclude the proof.

We note that (23) is the basis to get the convergence order of the approximations given by Problem $\mathrm{VP}^{1,hk}$. Therefore, as an example, if we assume the following additional regularity

$$\theta \in C^2([0,T]; H^2(\Omega)) \cap H^3(0,T;E). \tag{25}$$

using the classical results on the approximation by finite elements (see, e.g., [9]) we have the following.

Corollary 7 *Let the assumptions of Theorem 6 hold. Under the additional regularity (25) it follows that the approximations obtained by Problem $VP^{1,hk}$ are linearly convergent; that is, there exist a positive constant C, independent of the discretization parameters h and k such that*

$$\max_{0\leq n\leq N} \left\{ \|\xi_n - \xi_n^{hk}\|_Y + \|e_n - e_n^{hk}\|_E + \|\theta_n - \theta_n^{hk}\|_E \right\} \leq C(h+k).$$

3.2 A priori error estimates for Problem $VP^{2,hk}$

We proceed now in a similar form to analyze the approximations provided by Problem $\mathrm{VP}^{2,hk}$.

We have the following stability result.

Lemma 8 *Under the assumptions of Theorem 3, it follows that the sequences $\{\theta^{hk}, e^{hk}, \xi^{hk}\}$ generated by Problem $VP^{2,hk}$ satisfy the stability estimate:*

$$\|\xi_n^{hk}\|_Y^2 + \|e_n^{hk}\|_Y^2 + \|\nabla e_n^{hk}\|_H^2 + \|\theta_n^{hk}\|_Y^2 + \|\nabla \theta_n^{hk}\|_H^2 \leq C,$$

where C is a positive constant which is independent of the discretization parameters h and k.

PROOF. For the sake of clarity in the writing of this proof, we remove the superscripts h and k in all the variables.

Taking $w^h = \xi_n$ as a test function in discrete variational equation (19) we have

$$\frac{\tau_q^2}{2}(\delta\xi_n, \xi_n)_Y + \tau_q(\xi_n, \xi_n)_Y + (e_n, \xi_n)_Y + \kappa(\nabla\theta_n, \nabla\xi_n)_H + \kappa\tau_\theta(\nabla e_n, \nabla\xi_n)_H$$
$$+\kappa\frac{\tau_\theta^2}{2}(\nabla\xi_n, \nabla\xi_n)_H = 0.$$

Thus, keeping in mind that

$$
\begin{aligned}
&(\delta\xi_n,\xi_n)_Y \geq \frac{1}{2k}\left\{\|\xi_n\|_Y^2-\|\xi_{n-1}\|_Y^2\right\},\\
&(e_n,\xi_n)_Y \geq \frac{1}{2k}\left\{\|e_n\|_Y^2-\|e_{n-1}\|_Y^2\right\},\\
&(\nabla e_n,\nabla\xi_n)_H \geq \frac{1}{2k}\left\{\|\nabla e_n\|_H^2-\|\nabla e_{n-1}\|_H^2\right\},\\
&(\nabla\theta_n,\nabla\xi_n)_H = \frac{1}{k}\Big\{(\nabla\theta_n,\nabla e_n)_H-(\nabla\theta_{n-1},\nabla e_{n-1})_H\Big\}-(\nabla e_n,\nabla e_{n-1})_H,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&\frac{1}{2k}\left\{\|\xi_n\|_Y^2-\|\xi_{n-1}\|_Y^2\right\}+\frac{1}{2k}\left\{\|e_n\|_Y^2-\|e_{n-1}\|_Y^2\right\}+\frac{1}{2k}\left\{\|\nabla e_n\|_H^2-\|\nabla e_{n-1}\|_H^2\right\}\\
&\quad+\frac{1}{k}\Big\{(\nabla\theta_n,\nabla e_n)_H-(\nabla\theta_{n-1},\nabla e_{n-1})_H\Big\}\leq C\|\nabla e_n\|_H^2+C\|\nabla e_{n-1}\|_H^2.
\end{aligned}
$$

By induction we find that

$$
\begin{aligned}
\|\xi_n\|_Y^2+\|e_n\|_Y^2+\|\nabla e_n\|_H^2+(\nabla\theta_n,\nabla e_n)_H &\leq Ck\sum_{j=1}^n\|\nabla e_j\|_H^2\\
&+C\left(\|\xi_0\|_Y^2+\|e_0\|_Y^2+\|\nabla e_0\|_H^2+\|\nabla\theta_0\|_H^2\right),
\end{aligned}
$$

and so, taking again into account that

$$
\begin{aligned}
&|(\nabla\theta_n,\nabla e_n)_H|\leq C\|\nabla\theta_n\|_H^2+\epsilon\|\nabla e_n\|_H^2,\\
&\|\theta_n\|_Y^2\leq Ck\sum_{j=1}^n\|e_j\|_Y^2+C\|\theta_0\|_Y^2,\\
&\|\nabla\theta_n\|_H^2\leq Ck\sum_{j=1}^n\|\nabla e_j\|_H^2+C\|\nabla\theta_0\|_H^2,
\end{aligned}
$$

where $\epsilon>0$ is assumed small enough, proceeding as in the proof of Lemma 4 we obtain the desired stability estimates.

As in the analysis of Problem $VP^{1,hk}$, we have the decay of the discrete energy.

Lemma 9 *Under the assumptions of Theorem 3, if we define the discrete energy E_n^{hk} in the following form:*

$$
\begin{aligned}
E_n^{hk}=\frac{1}{2}\Big\{&A\|\frac{\tau_\theta^2}{2}\xi_n^{hk}+\tau_\theta e_n^{hk}+\theta_n^{hk}\|_Y^2+(B\tau_\theta+\frac{C\tau_\theta^2}{2})\|e_n^{hk}\|_Y^2\\
&+C\|\theta_n^{hk}\|_Y^2+2B(e_n^{hk},\theta_n^{hk})_Y\Big\},
\end{aligned}
\tag{26}
$$

where

$$
A=\frac{\tau_q^2}{\tau_\theta^2},\quad B=\tau_q-\frac{\tau_q^2}{\tau_\theta},\quad C=1-\frac{\tau_q^2}{\tau_\theta^2},
$$

then it decays, i.e. $\dfrac{E_n^{hk} - E_{n-1}^{hk}}{k} \leq 0.$

PROOF. First, we follow [26] to rewrite the discrete variational equation (19) as

$$A(\frac{\tau_\theta}{2}\delta\xi_n^{hk} + \tau_\theta\xi_n^{hk} + e_n^{hk}, w^h)_Y + B(\xi_n^{hk}, w^h)_Y + C(e_n^{hk}, w^h)_Y + \kappa(\nabla\theta_n^{hk}, \nabla w^h)_H$$
$$+\kappa\tau_\theta(\nabla e_n^{hk}, \nabla w^h)_H + \kappa\frac{\tau_\theta^2}{2}(\nabla\xi_n^{hk}, \nabla w^h)_H = 0 \quad \forall w^h \in E^h,$$

where coefficients A, B and C were defined above. We note that, from this definition, $C\tau_\theta - B > 0$.

Next, we take $w^h = \frac{\tau_\theta^2}{2}\xi_n^{hk} + \tau_\theta e_n^{hk} + \theta_n^{hk}$ as a test function in (19) to obtain

$$\begin{aligned}
&\frac{A}{2k}\Big\{\|\frac{\tau_\theta^2}{2}\xi_n^{hk} + \tau_\theta e_n^{hk} + \theta_n^{hk}\|_Y^2 - \|\frac{\tau_\theta^2}{2}\xi_{n-1}^{hk} + \tau_\theta e_{n-1}^{hk} + \theta_{n-1}^{hk}\|_Y^2\Big\} + \frac{B\tau_\theta^2}{2}\|\xi_n^{hk}\|_Y^2\\
&\quad+\Big(\frac{B\tau_\theta}{2} + \frac{C\tau_\theta^2}{4}\Big)\frac{1}{k}\Big\{\|e_n^{hk} - e_{n-1}^{hk}\|_Y^2 + \|e_n^{hk}\|_Y^2 - \|e_{n-1}^{hk}\|_Y^2\Big\} + B(\xi_n^{hk}, \theta_n^{hk})_Y\\
&\quad+C\tau_\theta\|e_n^{hk}\|_Y^2 + \frac{C}{2k}\Big\{\|\theta_n^{hk} - \theta_{n-1}^{hk}\|_Y^2 + \|\theta_n^{hk}\|_Y^2 - \|\theta_{n-1}^{hk}\|_Y^2\Big\}\\
&\quad+k\left\|\nabla\left(\frac{\tau_\theta^2}{2}\xi_n^{hk} + \tau_\theta e_n^{hk} + \theta_n^{hk}\right)\right\|_H^2 \leq 0.
\end{aligned}$$

Taking into account that

$$(\xi_n^{hk}, \theta_n^{hk})_Y = \frac{1}{k}(e_n^{hk}, \theta_n^{hk})_Y - \frac{1}{k}(e_{n-1}^{hk}, \theta_{n-1}^{hk})_Y - (e_n^{hk}, e_n^{hk})_Y + (\theta_n^{hk} - \theta_{n-1}^{hk}, \xi_n^{hk})_Y,$$
$$(\theta_n^{hk} - \theta_{n-1}^{hk}, \xi_n^{hk})_Y \leq \frac{1}{\tau_\theta 2k}\|\theta_n^{hk} - \theta_{n-1}^{hk}\|_Y^2 + \frac{\tau_\theta}{2k}\|e_n^{hk} - e_{n-1}^{hk}\|_Y^2,$$

it follows that

$$\begin{aligned}
&\frac{A}{2k}\Big\{\|\frac{\tau_\theta^2}{2}\xi_n^{hk} + \tau_\theta e_n^{hk} + \theta_n^{hk}\|_Y^2 - \|\frac{\tau_\theta^2}{2}\xi_{n-1}^{hk} + \tau_\theta e_{n-1}^{hk} + \theta_{n-1}^{hk}\|_Y^2\Big\}\\
&\quad+\Big(\frac{B\tau_\theta}{2} + \frac{C\tau_\theta^2}{4}\Big)\frac{1}{k}\Big\{\|e_n^{hk}\|_Y^2 - \|e_{n-1}^{hk}\|_Y^2\Big\} + \frac{B}{k}\Big\{(e_n^{hk}, \theta_n^{hk})_Y - (e_{n-1}^{hk}, \theta_{n-1}^{hk})_Y\Big\}\\
&\quad+\frac{C}{2k}\Big\{\|\theta_n^{hk}\|_Y^2 - \|\theta_{n-1}^{hk}\|_Y^2\Big\} \leq 0.
\end{aligned}$$

Finally, keeping in mind that $\tau_\theta > \tau_q$ and the previous condition on coefficients A, B and C, we conclude the desired discrete energy decay.

Now, we obtain the following a priori error estimates result.

Theorem 10 *Under the assumptions of Lemma 8, if we denote by (e,ξ,θ) the solution to problem VP^2 and by $(e^{hk},\xi^{hk},\theta^{hk})$ the solution to problem $VP^{2,hk}$, then we have the following a priori error estimates, for all $w^h=\{w_j^h\}_{j=0}^N\subset E^h$,*

$$
\begin{aligned}
&\max_{0\le n\le N}\Big\{\|\xi_n-\xi_n^{hk}\|_Y^2+\|e_n-e_n^{hk}\|_E^2+\|\theta_n-\theta_n^{hk}\|_E^2\Big\}\\
&\quad\le Ck\sum_{j=1}^N\Big(\|(\xi_t)_j-\delta\xi_j\|_Y^2+\|\xi_j-w_j^h\|_Y^2+\|\nabla(\xi_j-w_j^h)\|_H^2+I_j^2\Big)\\
&\quad +C(I_n^2+J_n^2)+C\max_{0\le n\le N}\|\xi_n-w_n^h\|_Y^2+\frac{C}{k}\sum_{j=1}^{N-1}\|\xi_j-w_j^h-(\xi_{j+1}-w_{j+1}^h)\|_Y^2\\
&\quad +C\Big(\|\xi_0-\xi_0^h\|_Y^2+\|e_0-e_0^h\|_E^2+\|\theta_0-\theta_0^h\|_E^2\Big),
\end{aligned}
\tag{27}
$$

where we recall that I_j and J_n denote the integration errors defined by

$$
I_j=\left\|\int_0^{t_j}\nabla e(s)\,ds-k\sum_{l=1}^j\nabla e_l\right\|_H,\quad J_n=\left\|\int_0^{t_n}e(s)\,ds-k\sum_{j=1}^n e_j\right\|_Y,
$$

and C is again a positive constant which is independent of the discretization parameters h and k.

PROOF. First, we subtract variational equation (11) at time $t=t_n$ for a test function $w=w^h\in E^h\subset E$ and discrete variational equation (19) to obtain, for all $w^h\in E^h$,

$$
\begin{aligned}
&\frac{\tau_q^2}{2}((\xi_t)_n-\delta\xi_n^{hk},w^h)_Y+\tau_q(\xi_n-\xi_n^{hk},w^h)_Y+\kappa(\nabla(\theta_n-\theta_n^{hk}),\nabla w^h)_H\\
&\quad+(e_n-e_n^{hk},w^h)_Y+\kappa\tau_\theta(\nabla(e_n-e_n^{hk}),\nabla w^h)_H+\kappa\frac{\tau_\theta^2}{2}(\nabla(\xi_n-\xi_n^{hk}),\nabla w^h)_H=0,
\end{aligned}
$$

and therefore, we find that, for all $w^h\in E^h$,

$$
\begin{aligned}
&\frac{\tau_q^2}{2}((\xi_t)_n-\delta\xi_n^{hk},\xi_n-\xi_n^{hk})_Y+\tau_q(\xi_n-\xi_n^{hk},\xi_n-\xi_n^{hk})_Y\\
&\quad+\kappa(\nabla(\theta_n-\theta_n^{hk}),\nabla(\xi_n-\xi_n^{hk}))_H+(e_n-e_n^{hk},\xi_n-\xi_n^{hk})_Y\\
&\quad+\kappa\tau_\theta(\nabla(e_n-e_n^{hk}),\nabla(\xi_n-\xi_n^{hk}))_H+\kappa\frac{\tau_\theta^2}{2}(\nabla(\xi_n-\xi_n^{hk}),\nabla(\xi_n-\xi_n^{hk}))_H\\
&=\frac{\tau_q^2}{2}((\xi_t)_n-\delta\xi_n^{hk},\xi_n-w^h)_Y+\tau_q(\xi_n-\xi_n^{hk},\xi_n-w^h)_Y\\
&\quad+\kappa(\nabla(\theta_n-\theta_n^{hk}),\nabla(\xi_n-w^h))_H+(e_n-e_n^{hk},\xi_n-w^h)_Y\\
&\quad+\kappa\tau_\theta(\nabla(e_n-e_n^{hk}),\nabla(\xi_n-w^h))_H+\kappa\frac{\tau_\theta^2}{2}(\nabla(\xi_n-\xi_n^{hk}),\nabla(\xi_n-w^h))_H.
\end{aligned}
$$

Keeping in mind that

$$
\begin{aligned}
&((\xi_t)_n-\delta\xi_n^{hk},\xi_n-\xi_n^{hk})_Y\geq((\xi_t)_n-\delta\xi_n,\xi_n-\xi_n^{hk})_Y\\
&\qquad+\frac{1}{2k}\left\{\|\xi_n-\xi_n^{hk}\|_Y^2-\|\xi_{n-1}-\xi_{n-1}^{hk}\|_Y^2\right\},\\
&(e_n-e_n^{hk},\xi_n-\xi_n^{hk})_Y\geq(e_n-e_n^{hk},(e_t)_n-\delta e_n)_Y\\
&\qquad+\frac{1}{2k}\left\{\|e_n-e_n^{hk}\|_Y^2-\|e_{n-1}-e_{n-1}^{hk}\|_Y^2\right\},\\
&(\nabla(e_n-e_n^{hk}),\nabla(\xi_n-\xi_n^{hk}))_H\geq(\nabla(e_n-e_n^{hk}),\nabla((e_t)_n-\delta e_n))_H\\
&\qquad+\frac{1}{2k}\left\{\|\nabla(e_n-e_n^{hk})\|_H^2-\|\nabla(e_{n-1}-e_{n-1}^{hk})\|_H^2\right\},\\
&\|\nabla(\theta_n-\theta_n^{hk})\|_H^2\leq C\Big(I_n^2+\sum_{j=1}^n k\|\nabla(e_j-e_j^{hk})\|_H^2+\|\nabla(\theta_0-\theta_0^h)\|_H^2\Big),
\end{aligned}
$$

where we use the notations $\delta e_n=(e_n-e_{n-1})/k$ and $\delta\xi_n=(\xi_n-\xi_{n-1})/k$, and we recall that $\xi_n^{hk}=\delta e_n^{hk}$ and I_n is the previously defined integration error, applying several times inequality (21) we have

$$
\begin{aligned}
&\frac{1}{2k}\left\{\|\xi_n-\xi_n^{hk}\|_Y^2-\|\xi_{n-1}-\xi_{n-1}^{hk}\|_Y^2\right\}+\frac{1}{2k}\left\{\|e_n-e_n^{hk}\|_Y^2-\|e_{n-1}-e_{n-1}^{hk}\|_Y^2\right\}\\
&\quad+\frac{1}{2k}\left\{\|\nabla(e_n-e_n^{hk})\|_H^2-\|\nabla(e_{n-1}-e_{n-1}^{hk})\|_H^2\right\}\\
&\leq C\Big(\|(\xi_t)_n-\delta\xi_n\|_Y^2+\|\xi_n-w^h\|_Y^2+\|\nabla(\xi_n-w^h)\|_H^2+\|e_n-e_n^{hk}\|_Y^2\\
&\quad+\|\nabla(e_n-e_n^{hk})\|_H^2+(\delta\xi_n-\delta\xi_n^{hk},\xi_n-w^h)_Y+I_n^2+\|\nabla(\theta_0-\theta_0^h)\|_H^2\\
&\quad+\sum_{j=1}^n k\|\nabla(e_j-e_j^{hk})\|_H^2\Big)\quad\forall w^h\in E^h.
\end{aligned}
$$

Therefore, by induction we obtain, for all $w^h=\{w_j^h\}_{j=0}^n\subset E^h$,

$$
\begin{aligned}
&\|\xi_n-\xi_n^{hk}\|_Y^2+\|e_n-e_n^{hk}\|_Y^2+\|\nabla(e_n-e_n^{hk})\|_H^2\\
&\quad\leq Ck\sum_{j=1}^n\Big(\|(\xi_t)_j-\delta\xi_j\|_Y^2+\|\xi_j-w_j^h\|_Y^2+\|\nabla(\xi_j-w_j^h)\|_H^2+\|e_j-e_j^{hk}\|_Y^2\\
&\quad+\|\nabla(e_j-e_j^{hk})\|_H^2+(\delta\xi_j-\delta\xi_j^{hk},\xi_j-w_j^h)_Y+I_j^2\Big)+C\Big(\|\xi_0-\xi_0^h\|_Y^2\\
&\quad+\|e_0-e_0^h\|_Y^2+\|\nabla(e_0-e_0^h)\|_H^2+\|\nabla(\theta_0-\theta_0^h)\|_H^2\Big).
\end{aligned}
$$

Finally, using (24) and proceeding as in the proof of Theorem 6 we find the desired a priori error estimates.

We note that (27) can be used to obtain a convergence analysis of the approximations given by Problem $\mathrm{VP}^{2,hk}$. Therefore, as an example, if we assume the following additional regularity

$$\theta \in C^2([0,T];H^2(\Omega)) \cap H^3(0,T;E). \tag{28}$$

using again classical results on the approximation by finite elements (see, e.g., [9]) we have the following.

Corollary 11 *Let the assumptions of Theorem 10 hold. Under the additional regularity (28) it follows that the approximations obtained by Problem VP^{2,hk} are linearly convergent; that is, there exist a positive constant C, independent of the discretization parameters h and k, such that*

$$\max_{0\leq n\leq N} \left\{ \|\xi_n - \xi_n^{hk}\|_Y + \|e_n - e_n^{hk}\|_E + \|\theta_n - \theta_n^{hk}\|_E \right\} \leq C(h+k).$$

4 Numerical simulations

In this final section, we describe the numerical scheme implemented in MATLAB for solving problems VP^{1,hk} and VP^{2,hk}, and we show some numerical examples to demonstrate the accuracy of the approximations and the behaviour of the solutions.

Let the finite element space E^h be defined in (16), for $n = 1, 2, \ldots, N$ and given θ_{n-1}^{hk}, e_{n-1}^{hk}, $\xi_{n-1}^{hk} \in E^h$, the discrete thermal acceleration ξ_n^{hk} for Problem VP^{1,hk} is obtained from equation (17), that is, we solve the following linear problem, for all $w^h \in E^h$,

$$\begin{aligned}
&\frac{\tau_q^2}{2}(\xi_n^{hk}, w^h)_Y + \tau_q k(\xi_n^{hk}, w^h)_Y + k^2(\xi_n^{hk}, w^h)_Y + \kappa k^3(\nabla\xi_n^{hk}, \nabla w^h)_H \\
&\qquad +\kappa k^2 \tau_\theta(\nabla\xi_n^{hk}, \nabla w^h)_H \\
&= \frac{\tau_q^2}{2}(\xi_{n-1}^{hk}, w^h)_Y - k(e_{n-1}^{hk}, w^h)_Y - \kappa k(\nabla\theta_{n-1}^{hk}, \nabla w^h)_H \\
&\qquad -\kappa k^2(\nabla e_{n-1}^{hk}, \nabla w^h)_H - \kappa k \tau_\theta(\nabla e_{n-1}^{hk}, \nabla w^h)_H.
\end{aligned}$$

Similarly, the discrete thermal acceleration ξ_n^{hk} for Problem VP^{2,hk} is obtained from equation (19), that is, we solve the following linear problem, for all $w^h \in E^h$,

$$\frac{\tau_q^2}{2}(\xi_n^{hk}, w^h)_Y + \tau_q k(\xi_n^{hk}, w^h)_Y + k^2(\xi_n^{hk}, w^h)_Y + \kappa k^3(\nabla\xi_n^{hk}, \nabla w^h)_H$$
$$+\kappa k^2\tau_\theta(\nabla\xi_n^{hk}, \nabla w^h)_H + \kappa k\frac{\tau_\theta^2}{2}(\nabla\xi_n^{hk}, \nabla w^h)_H$$
$$= \frac{\tau_q^2}{2}(\xi_{n-1}^{hk}, w^h)_Y - k(e_{n-1}^{hk}, w^h)_Y - \kappa k(\nabla\theta_{n-1}^{hk}, \nabla w^h)_H$$
$$-\kappa k^2(\nabla e_{n-1}^{hk}, \nabla w^h)_H - \kappa k\tau_\theta(\nabla e_{n-1}^{hk}, \nabla w^h)_H.$$

In both cases, the discrete temperature and the discrete thermal velocity are then recovered from the relations

$$e_n^{hk} = e_{n-1}^{hk} + k\xi_n^{hk}, \quad \theta_n^{hk} = \theta_{n-1}^{hk} + ke_n^{hk}.$$

The two numerical schemes were implemented on a 3.2 Ghz PC using MATLAB, and a typical run ($h = k = 0.01$) took about 3.7 seconds of CPU time, being almost the same time for both models.

4.1 Numerical convergence for the approximation of Problem P^1

We will consider the following academic problem:

Problem $\mathbf{P}^{ex,1}$. *Find the temperature $\theta : [0,1] \times [0,1] \to \mathbb{R}$ such that*

$$\frac{1}{2}\theta_{ttt} + \theta_{tt} + \theta_t = 2\theta_{xx} + 2\theta_{txx} + F_1 \quad in \quad (0,1)\times(0,1),$$
$$\theta(0,t) = 0, \quad \theta(1,t) = e^t \quad for\ all \quad t \in [0,1],$$
$$\theta(x,0) = x^2, \quad \theta_t(x,0) = x^2, \quad \theta_{tt}(x,0) = x^2 \quad for\ all \quad x \in (0,1),$$

where the artificial volume force F_1 is given by

$$F_1(x,t) = e^t\left(\frac{5x^2}{2} - 8\right).$$

We note that Problem $\mathrm{P}^{ex,1}$ corresponds to problem P^1 with the following data:

$$\Omega = (0,1), \quad T = 1, \quad \kappa = 2, \quad \tau_\theta = 1, \quad \tau_q = 1,$$

and the initial conditions:

$$\theta_0 = e_0 = \xi_0 = x^2.$$

The exact solution to Problem P^1 can be easily obtained and it has the following form

$$\theta(x,t) = e^t x^2.$$

Thus, the approximation errors estimated by

$$\max_{0\le n\le N} \left\{ \|\xi_n - \xi_n^{hk}\|_Y + \|e_n - e_n^{hk}\|_E + \|\theta_n - \theta_n^{hk}\|_E \right\}$$

are presented in Table 1 for several values of the discretization parameters h and k. Moreover, the evolution of the error depending on the parameter $h+k$ is plotted in Fig. 1. We notice that the convergence of the algorithm is clearly observed, and the linear convergence, stated in Corollary 7, is achieved.

$h\downarrow k\rightarrow$	0.01	0.005	0.002	0.001	0.0005	0.0002	0.0001
$1/2^3$	0.450935	0.433689	0.423098	0.419301	0.417395	0.416302	0.415958
$1/2^4$	0.242509	0.220683	0.210061	0.206386	0.204258	0.202859	0.202404
$1/2^5$	0.150809	0.121275	0.107430	0.103932	0.102127	0.100725	0.100140
$1/2^6$	0.116802	0.077552	0.058395	0.053562	0.051741	0.050672	0.050167
$1/2^7$	0.111817	0.063176	0.036609	0.029581	0.026837	0.025679	0.025331
$1/2^8$	0.121294	0.064305	0.029830	0.019415	0.015156	0.013269	0.012868
$1/2^9$	0.141453	0.073920	0.031413	0.017222	0.010636	0.007470	0.006742
$1/2^{10}$	0.174042	0.090775	0.037686	0.019460	0.010369	0.005307	0.003991
$1/2^{11}$	0.223281	0.116512	0.048021	0.024393	0.012428	0.005312	0.003142
$1/2^{12}$	0.295402	0.154280	0.063393	0.032010	0.016095	0.006500	0.003381
$1/2^{13}$	0.399290	0.208716	0.085637	0.043119	0.021562	0.008539	0.004228

Table 1
Example 1-model P^1: Numerical errors for some h and k.

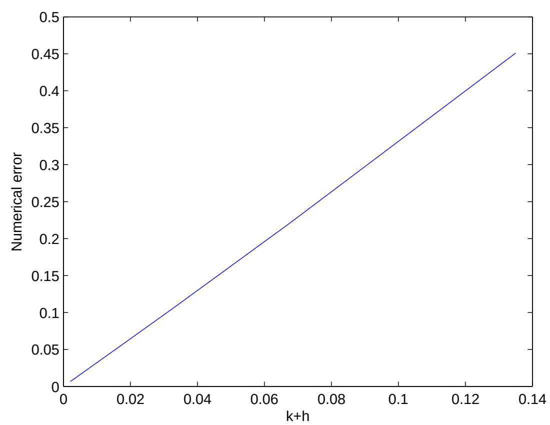

Fig. 1. Example 1-model P^1: Asymptotic constant error.

If we assume now that there are not volume forces, and we use the final time $T = 10$, the following data

$$\kappa = 2, \quad \tau_\theta = 0.003, \quad \tau_q = 0.005,$$

and the initial conditions

$$\theta_0 = x(x-1), \quad e_0 = \xi_0 = 0,$$

taking the discretization parameters $h = k = 10^{-3}$, the evolution in time of the discrete energy E_n^{hk}, defined in (22), is plotted in Fig. 2 in both natural (left) and semi-log (right) scales. As can be seen, it converges to zero and an exponential decay seems to be achieved.

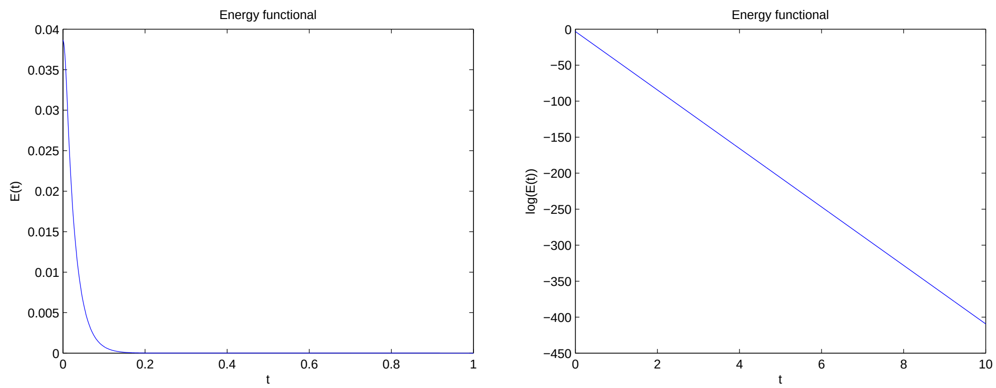

Fig. 2. Example 1-model P^1: Evolution in time of the discrete energy in natural (left) and semi-log (right) scales.

4.2 Numerical convergence for the approximation of Problem P^2

Now, we will consider the following academic problem:

Problem $\mathbf{P}^{ex,2}$. *Find the temperature $\theta : [0,1] \times [0,1] \to \mathbb{R}$ such that*

$$\frac{1}{2}\theta_{ttt} + \theta_{tt} + \theta_t = 2\theta_{xx} + 4\theta_{txx} + 2\theta_{ttxx} + F_2 \quad in \quad (0,1) \times (0,1), \tag{29}$$

$$\theta(0,t) = 0, \quad \theta(1,t) = e^t \quad for\ all \quad t \in [0,1], \tag{30}$$

$$\theta(x,0) = x^2, \quad \theta_t(x,0) = x^2, \quad \theta_{tt}(x,0) = x^2 \quad for\ all \quad x \in (0,1), \tag{31}$$

where the artificial volume force F_2 is given by

$$F_2 = e^t \left(\frac{5x^2}{2} - 16 \right).$$

We note that Problem $\mathrm{P}^{ex,2}$ corresponds to problem P^2 with the following data:

$$\Omega = (0,1), \quad T = 1, \quad \kappa = 2, \quad \tau_\theta = 2, \quad \tau_q = 1,$$

and the initial conditions:

$$\theta_0 = e_0 = \xi_0 = x^2.$$

The exact solution to Problem $\mathrm{P}^{ex,2}$ is again

$$\theta(x,t) = \mathrm{e}^t\, x^2.$$

Thus, the approximation errors estimated by

$$\max_{0\le n\le N} \left\{ \|\xi_n - \xi_n^{hk}\|_Y + \|e_n - e_n^{hk}\|_E + \|\theta_n - \theta_n^{hk}\|_E \right\}$$

are presented in Table 2 for several values of the discretization parameters h and k. Moreover, the evolution of the error depending on the parameter $h+k$ is plotted in Fig. 3. We notice that the convergence of the algorithm is clearly observed, and the linear convergence, stated in Corollary 11, is achieved.

$h \downarrow k \rightarrow$	0.01	0.005	0.002	0.001	0.0005	0.0002	0.0001
$1/2^3$	0.374698	0.354243	0.343695	0.340571	0.339109	0.338269	0.337996
$1/2^4$	0.208172	0.183186	0.170840	0.167261	0.165622	0.164713	0.164427
$1/2^5$	0.135771	0.103379	0.088449	0.084372	0.082527	0.081507	0.081193
$1/2^6$	0.111186	0.068698	0.048969	0.044044	0.041955	0.040832	0.040484
$1/2^7$	0.109795	0.058174	0.031411	0.024660	0.022037	0.020752	0.020374
$1/2^8$	0.119490	0.059719	0.026097	0.016404	0.012556	0.010848	0.010407
$1/2^9$	0.137349	0.067441	0.027224	0.014537	0.008833	0.006145	0.005520
$1/2^{10}$	0.164612	0.080241	0.031708	0.015997	0.008424	0.004334	0.003282
$1/2^{11}$	0.204480	0.099195	0.038938	0.019364	0.009734	0.004171	0.002524
$1/2^{12}$	0.261881	0.126527	0.049500	0.024518	0.012177	0.004887	0.002573
$1/2^{13}$	0.343974	0.165587	0.064619	0.031952	0.015810	0.006215	0.003084

Table 2
Example 1-model P^2: Numerical errors for some h and k.

If we assume now that there are not volume forces, and we use the final time $T = 10$, the following data

$$\kappa = 2, \quad \tau_\theta = 0.05, \quad \tau_q = 0.03,$$

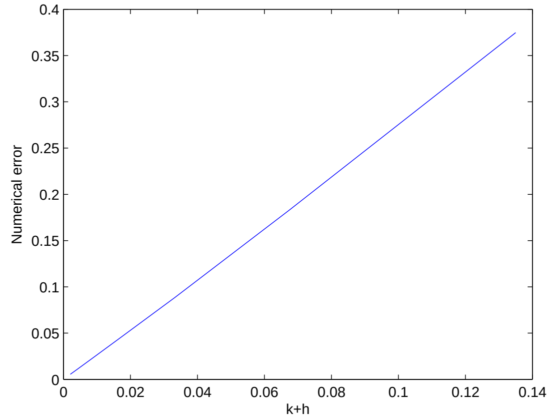

Fig. 3. Example 1-model P^2: Asymptotic constant error.

and the initial conditions

$$\theta_0 = x(x-1), \quad e_0 = \xi_0 = 0,$$

taking the discretization parameters $k = h = 10^{-3}$, the evolution in time of the discrete energy E_n^{hk}, defined in (26), is plotted in Fig. 4. As can be seen, it converges to zero and an exponential decay seems to be achieved.

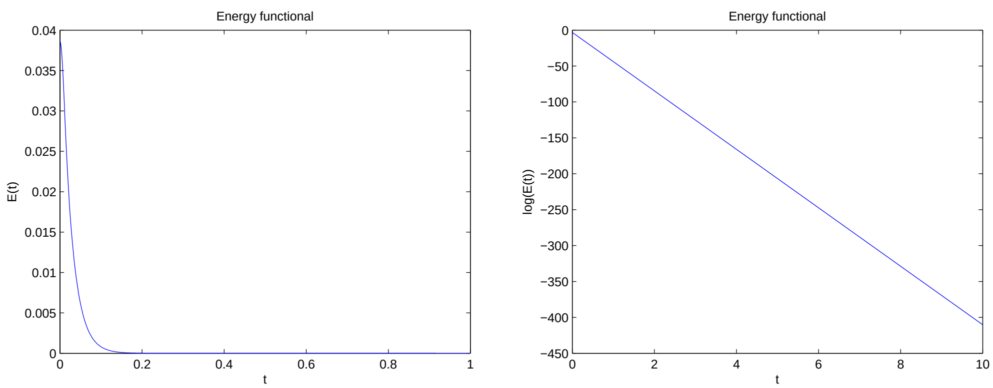

Fig. 4. Example 1-model P^2: Evolution in time of the discrete energy in natural (left) and semi-log (right) scales.

4.3 *Comparison with a first-order dual-phase-lag model*

In this last one-dimensional example, we will compare the solution obtained with these second-order dual-phase-lag models and a first-order dual-phase-lag model. In this case, the thermal problem is then written as follows,

$$\tau_q\theta_{tt} + \theta_t = \kappa\theta_{xx} + \kappa\tau_\theta\theta_{txx} \quad \text{in} \quad (0,1)\times(0,5),$$
$$\theta(x,t) = 0 \quad \text{for} \quad x = 0, 1,\ t \in [0,5],$$
$$\theta(x,0) = \theta_0(x) = x^3(x-1)^3, \quad \theta_t(x,0) = e_0(x) = x^3(x-1)^3 \quad \text{for} \quad x \in (0,1).$$

In order to compare the solution obtained with the second-order dual-phase-lag model, we note that the third initial condition for that model (i.e. the definition of ξ_0) is calculated from the above constitutive equation, that is,

$$\xi_0 = \frac{1}{\tau_q}\left(\kappa(\theta_0)_{xx} + \kappa\tau_\theta(e_0)_{xx} - e_0\right).$$

In this example, we have used the following data:

$$\Omega = (0,1), \quad T = 5, \quad \kappa = 2, \quad \tau_\theta = 3, \quad \tau_q = 1.$$

Using the discretization parameters $h = k = 0.001$, in Fig. 5 we plot the thermal velocity (left) and the temperature (right), obtained with the two models, at final time. As we can see, all the solutions have a quadratic behaviour, being greater the solutions with the second-order approximation. We also note that we do not show the results obtained with model P^2 because its solutions are similar to those provided by model P^1.

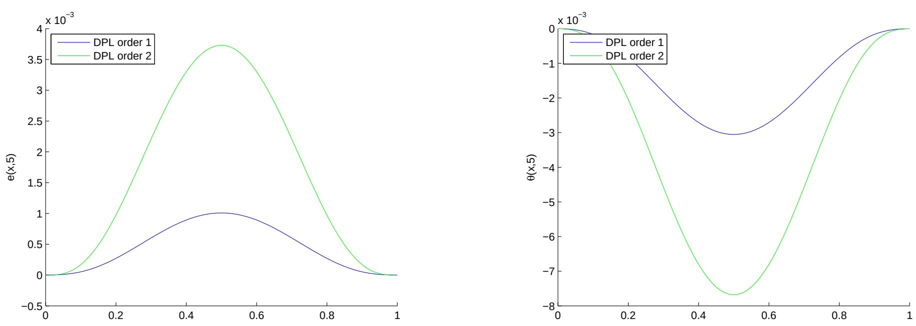

Fig. 5. Example 2-model P^1: Comparisons with a first-order dual-phase-lag model.

4.4 A first two-dimensional example: a surface heat force

As a first example, we consider the domain $(0,1)\times(0,1)$ where, on the boundary defined by $x = 1$, $y \in [0,1]$, a surface heat force is applied, quadratic in the vertical direction and linearly increasing with respect to the time. On the rest of the boundary, the temperature is assumed zero.

The following data are used:

$$\Omega = (0,1) \times (0,1), \quad T = 1, \quad \kappa = 2, \quad \tau_\theta = 2, \quad \tau_q = 1,$$

with the initial conditions

$$\theta_0 = e_0 = \xi_0 = 0,$$

and the boundary conditions:

$$\theta(x,0,t) = \theta(x,1,t) = \theta(0,y,t) = 0 \quad \text{for all} \quad x,\, y \in (0,1),\, t \in (0,1),$$
$$\nabla\theta(1,y,t) \cdot \boldsymbol{\nu} = 200y(y-1)t \quad \text{for all} \quad t \in (0,1),$$

where we recall that $\boldsymbol{\nu}$ represents the outward unit normal vector to Γ.

Using the time discretization parameter $k = 10^{-3}$, in Fig. 6 we plot the temperature, the thermal velocity and the thermal acceleration at final time using the model described in Problem P^1. The same results are plotted in Fig. 7 for Problem P^2. We can observe that the temperature and thermal velocity are rather similar and they have a quadratic behaviour as expected. However, concerning the thermal acceleration there are great differences between the two models, mainly due to the diffusion effect into the second one, when the results are more smooth.

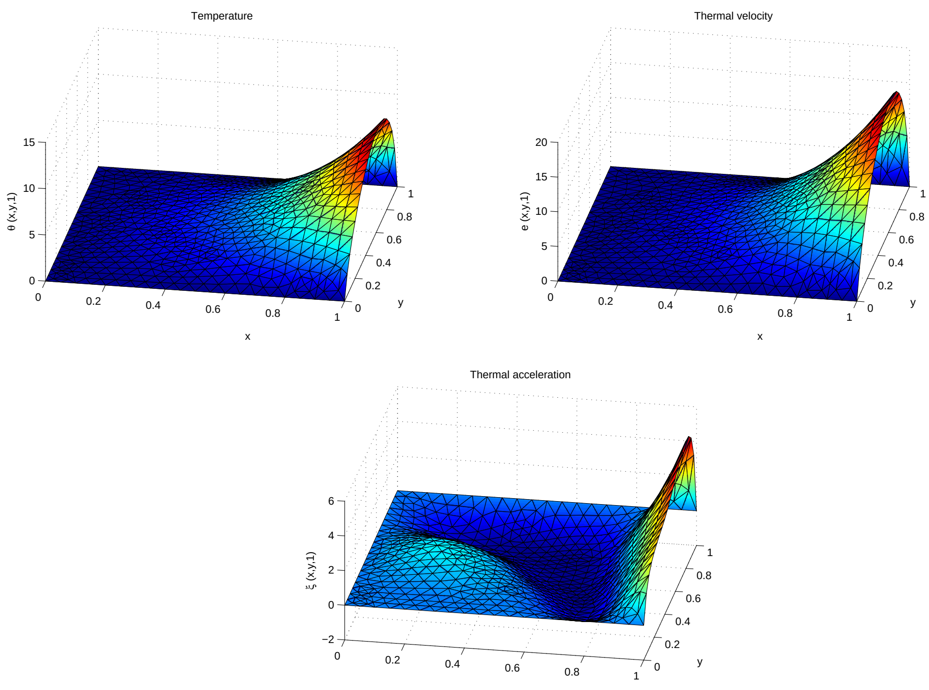

Fig. 6. Example 3-model P^1: Temperature, thermal velocity and thermal acceleration at final time.

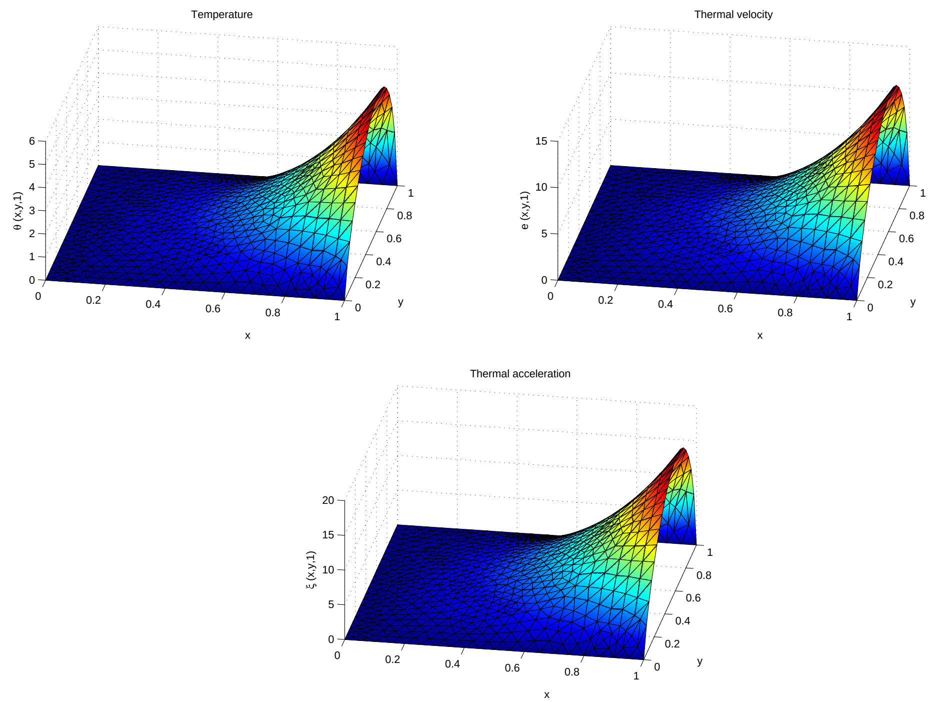

Fig. 7. Example 3-model P^2: Temperature, thermal velocity and thermal acceleration at final time.

4.5 A second two-dimensional example: a volume heat force

As a second two-dimensional example we consider a similar setting than in the previous one. The main difference is that we assume now a volume heat force S is applied instead of a surface one. The analysis of this slightly modified problem is rather similar to that presented in Sections 2 and 3. So, in this simulation the following data are used:

$$\kappa = 2, \quad \tau_\theta = 0.05, \quad \tau_q = 0.03,$$

with the initial conditions

$$\theta_0 = e_0 = \xi_0 = 0,$$

and the Dirichlet boundary conditions:

$$\theta(x, 0, t) = \theta(x, 1, t) = \theta(0, y, t) = \theta(1, y, t) = 0 \quad \text{for all} \quad x,\, y \in (0,\, 1),\, t \in (0, 1).$$

Moreover, the volume heat force is defined as $S(x, y, t) = 100t$.

Using the discretization parameter $k = 10^{-3}$, in Figs. 8 and 9 we plot the temperature, the thermal velocity and the thermal acceleration at final time using the models described in Problem P^1 and P^2, respectively. As in the previous example, we can observe that the temperature and the thermal velocity are similar for both models, and they have a quadratic behaviour, as expected because of the application of the volume heat force and the application of the Dirichlet conditions. Again, the differences are clearly noticed for the thermal acceleration. As can be seen, the diffusion effect in the second model produces a smoother surface and, in fact, in the first model some oscillations seem to be produced.

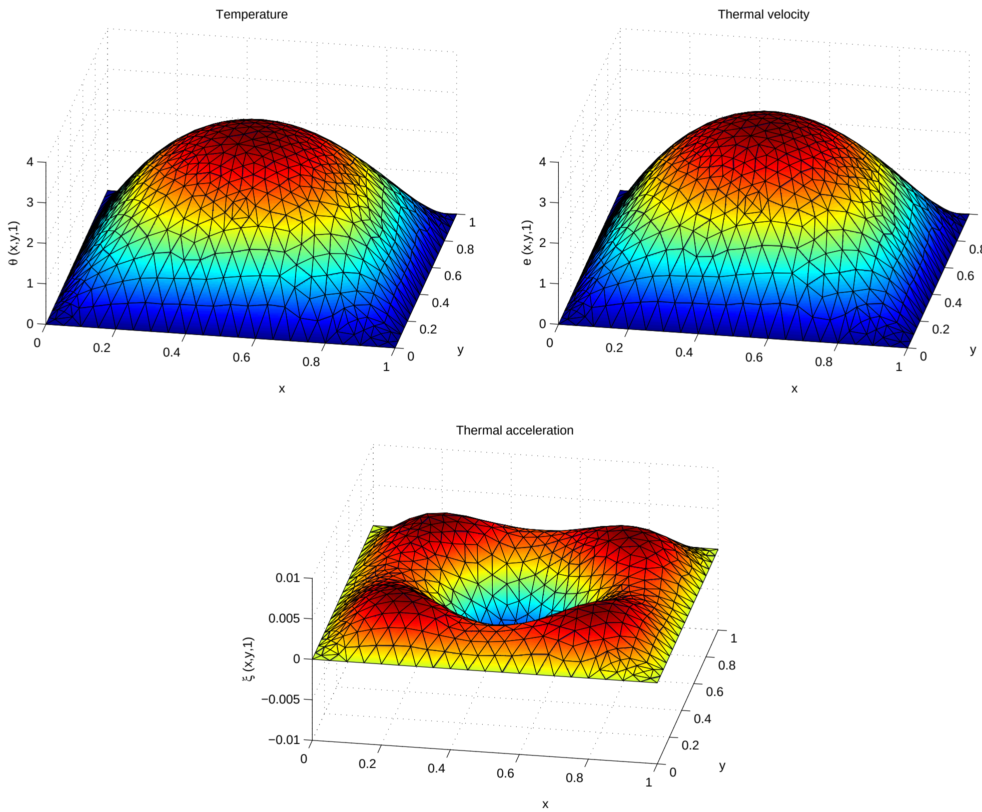

Fig. 8. Example 4-model P^1: Temperature, thermal velocity and thermal acceleration at final time.

5 Conclusions

In this paper we have analyzed, from the numerical point of view, two Dual-Phase-Lag models, which lead to linear partial differential equations of third order in time. We have introduced fully discrete schemes using the finite element method to approximate the spatial variable and the implicit Euler scheme to discretize the time derivatives. We have provided discrete stability results (Lemmata 4 and 8), which lead to the decay of the discrete energy (Lemmata 5 and 9), and a priori error estimates (Theorem 6 and 10). Finally, we have presented some one-dimensional numerical simulations to demonstrate the convergence of the numerical schemes, the decay of the discrete energies

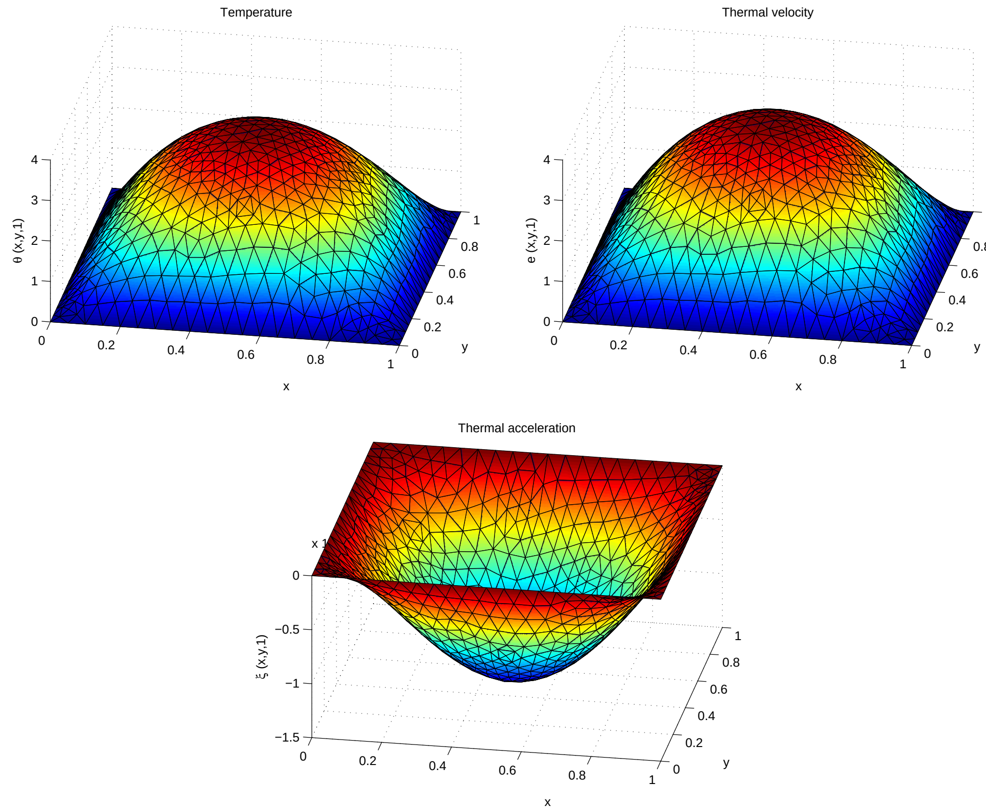

Fig. 9. Example 4-model P^2: Temperature, thermal velocity and thermal acceleration at final time.

and a comparison with a first-order dual-phase-lag model (Examples 4.1, 4.2 and 4.3), and some two-dimensional numerical simulations involving a surface heat force (Example 4.4) and a volume heat force (Example 4.5).

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