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# Convexity of the moment map image for torus actions on $b^m$-symplectic manifolds

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We prove a convexity theorem for the image of the moment map of a Hamiltonian torus action on a $b^m$-symplectic manifold.

This article is part of the theme issue 'Finite dimensional integrable systems: new trends and methods'.

## 1. Introduction

The purpose of this paper is to prove a convexity theorem for the image of the moment map of a Hamiltonian torus action on a $b^m$-symplectic manifold. $b^m$-Symplectic manifolds are Poisson manifolds where the Poisson structure is invertible on the complement of a hypersurface $Z$, and has a singularity of order $m$ on $Z$, where $m \geq 1$ is an integer. They are a generalization of $b$-symplectic [1–3] (or log-symplectic) manifolds and were introduced in the thesis of Scott [4].

In the case of Hamiltonian torus actions on compact symplectic manifolds, the convexity theorem of Atiyah [5] and Guillemin & Sternberg [6,7] is a classical fact about integrable systems on compact manifolds. A convexity theorem for the moment image of a Hamiltonian torus action on a $b$-symplectic manifold was proved in [8]. In this case, the moment image is governed by the singularity of the moment map in the neighbourhood of $Z$, encoded in the *modular weight* [9]. If this modular weight is non-zero, the image is not only convex, but, on each component of the complement of $Z$, has the form of a product of a convex polytope with a ray or the real line, possibly modified by symplectic cutting.

We show that the moment image has a similar form where $m > 1$. The argument requires some care, since in this case the moment map has an asymptotic series near $Z$ which involves $m$ modular weights. The leading term in this series gives the moment map a form as in [8]. We show that the sub-leading terms preserve this structure.

As a technical device to state our convexity theorem, we use the desingularization of $b^m$-symplectic forms in [10]. This simplifies the statements since we can deal with families of compact manifolds, with compact moment images.

In a companion paper [3], we use the convexity theorem to study formal geometric quantization of $b^m$-symplectic manifolds equipped with Hamiltonian torus actions. The form of the moment image will give rise to a very simple asymptotics of this quantization.

## 2. $b^m$-Manifolds

Let $M$ be a compact manifold, and let $f \in \mathcal{C}^\infty(M)$ have a transverse zero at a hypersurface $Z \subset M$. Let $m$ be a positive integer; the $m$-germ of $f$ at $Z$ gives rise to a sheaf, and therefore a vector bundle, whose sections are given by

$$\Gamma({}^{b^m}TM) = \{v \in \Gamma(TM) : v(f) \text{ vanishes to order } m \text{ at } Z\}.$$

By considering sections of the wedge powers $\Lambda^k({}^{b^m}T^*M)$, we obtain a complex $({}^{b^m}\Omega^k(M), d)$ of differential forms with singularities at $Z$.

The cohomology associated with this complex is given by the following theorem.

**Theorem 2.1 ($b^m$-Mazzeo-Melrose, [4]).**

$$ {}^{b^m}H^p(M) \cong H^p(M) \oplus (H^{p-1}(Z))^m. \tag{2.1}$$

This theorem comes from a Laurent expansion in a neighbourhood of $Z$ which for 2-forms has the form

$$\omega = \sum_{j=1}^{m} \frac{\mathrm{d}f}{f^j} \wedge \pi^*(\alpha_j) + \beta, \tag{2.2}$$

where $\alpha_j$ are closed 1-forms on $Z$, $\beta$ is a closed 2-form on $M$, and $\pi : U \longrightarrow Z$ is the projection.

A two $b$-form $\omega \in {}^{b^m}\Omega^2(M)$ is $b^m$-symplectic if it is closed and non-degenerate as an element of $\Lambda^2({}^{b^m}T^*M)$.

**Theorem 2.2.** *There exists a neighbourhood $U = (-\epsilon, \epsilon) \times Z$ of $Z$ and a diffeomorphism $\psi : U \longrightarrow U$ preserving $Z$ and the m-germ of $f$ such that*

$$\psi^*(\omega) = \sum_{j=1}^{m} \frac{\mathrm{d}f}{f^j} \wedge \pi^*(\alpha_j) + \pi^*(\beta), \tag{2.3}$$

*where $\pi : U \longrightarrow Z$ is the projection, and $\alpha_i$ and $\beta$ are closed.*

*Proof.* We know by [10, proposition 2.3] that

$$\omega = \sum_{j=1}^{m} \frac{\mathrm{d}f}{f^j} \wedge \pi^*(\alpha_j) + \beta.$$

Let $i : Z \longrightarrow U$ be the restriction. Then

$$\pi^* i^* \alpha_i - \alpha_i = \mathrm{d}a_i,$$

$$\pi^* i^* \beta - \beta = \mathrm{d}b,$$

and for $\epsilon$ sufficiently small, letting

$$\omega_0 = \sum_{j=1}^{m} \frac{\mathrm{d}f}{f^j} \wedge \pi^* i^*(\alpha_j) + \pi^* i^*(\beta),$$

we have that $\omega_t = t\omega_0 + (1-t)\omega$ is symplectic for all $t \in [0,1]$.

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_t = \mathrm{d}b_t,$$

where

$$b_t = \iota_{v_t}\omega_t,$$

for $v_t$ a section of ${}^{b^m}TM$.

By shrinking $\epsilon$, we may chose $v_t$ compactly supported on $U$ and $\|v_t\| < \delta$ for any fixed $\delta > 0$. Thus, the existence theorem for ordinary differential equations shows that $v_t$ integrates to a family of diffeomorphisms, $\psi_t$, vanishing to order $m$ on $Z$ such that $\psi_t^*(\omega_0) = \omega$. ∎

## 3. Torus actions on $b^m$-manifolds

Now let us assume a torus $T$ acts on $(M, Z, f)$ preserving $\omega$. We denote ${}^b\mathcal{C}^\infty(M) = \{a \log|f| + g, g \in \mathcal{C}^\infty(M)\}$ the space of smooth functions with logarithmic singularity at $Z$ and write ${}^{b^m}\mathcal{C}^\infty(M) = f^{-(m-1)}\mathcal{C}^\infty(M) + {}^b\mathcal{C}^\infty(M)$.

**Definition 3.1.** The action of $T$ is *Hamiltonian* if there exists a moment map $\mu \in {}^{b^m}\mathcal{C}^\infty(M) \otimes \mathfrak{t}^*$ with

$$\langle \mathrm{d}\mu, X\rangle = \iota_{X^M}\omega$$

for any $X \in \mathfrak{t}$ where $X^M$ is the fundamental vector field generated by $X$ on $M$.

We will prove the following result.

**Lemma 3.2.** *There exists a neighbourhood $U = Z \times (-\epsilon, \epsilon)$ where the moment map $\mu : M \longrightarrow \mathfrak{t}^*$ is given by*

$$\mu = a_1 \log|f| + \sum_{i=2}^{m} a_i \frac{f^{-(i-1)}}{i-1} + \mu_0,$$

*with $a_i \in \mathfrak{t}^{\circ}{}_L$, and $\mu_0$ is the moment map for the $T_L$-action on the symplectic leaves of the foliation.*

*Proof.* Note that theorem 2.2 holds equivariantly so we may assume that $\omega$ can be written as

$$\omega = \sum_{j=1}^{m} \frac{\mathrm{d}f}{f^j} \wedge \pi^*(\alpha_j) + \pi^*(\beta), \tag{3.1}$$

where $\pi : U \longrightarrow Z$ is the projection, and $\alpha_i$ and $\beta$ are closed and $T$-invariant.

The moment map $\mu$ therefore has the form

$$\mu = a_1 \log|f| + \sum_{i=2}^{m} a_i \frac{f^{-(i-1)}}{i-1} + \mu_0, \tag{3.2}$$

where $\langle a_i, X\rangle = \alpha_i(X^M)$ and $\mu_0$ is the moment map for the action of $T$ on the regular Poisson manifold $Z$. ∎

The form $\alpha_m$ is nowhere vanishing and determines the symplectic foliation of $Z$. We now make the following two assumptions.

**Assumption 3.3.** There exists $\xi \in \mathfrak{t}$ such that $a_m(\xi) \neq 0$. Remark: without loss of generality, we may assume also that $\xi$ generates a circle subgroup $S^1_\xi \in T$.

**Assumption 3.4.** The foliation given by $\alpha_m$ has a compact leaf $L$. Thus $(L, \beta)$ is a compact symplectic manifold acted on by the Hamiltonian torus action of $T_L = T/S^1_\xi$, with moment map $\mu_{0|\mathfrak{t}_L}$. This would follow from integrality of $\omega$; see [11].

## 4. Desingularization

Recall from [10] the following result.

**Theorem 4.1.** *Given a $b^m$-symplectic structure $\omega$ on a compact manifold $M^{2n}$, let $Z$ be its critical hypersurface.*

— *If $m$ is even, there exists a family of symplectic forms $\omega_\epsilon$ which coincide with the $b^m$-symplectic form $\omega$ outside an $\epsilon$-neighbourhood of $Z$ and for which the family of bivector fields $(\omega_\epsilon)^{-1}$ converges in the $C^{m-1}$-topology to the Poisson structure $\omega^{-1}$ as $\epsilon \to 0$.*
— *If $m$ is odd, there exists a family of folded symplectic forms $\omega_\epsilon$ which coincide with the $b^m$-symplectic form $\omega$ outside an $\epsilon$-neighbourhood of $Z$.*

If a torus acts, this family of forms $\omega_\epsilon$ may be chosen equivariantly.

**Remark 4.2.** Observe that even if the initial action is Hamiltonian in the $b^m$-sense, the desingularized action need not be Hamiltonian in the standard sense because its moment map might be circle valued.

## 5. The local convexity theorem

**Theorem 5.1.** *Let $Z_i$ be a connected component of $Z$. Then there exists a neighbourhood $(-\epsilon, \epsilon) \times Z_i$ where the image of the moment map for the $T$-action on the desingularized family $(M, \omega_\epsilon)$ is*

— *$\Delta_i \times (-a_\epsilon, a_\epsilon)$ for even $m$,*
— *$\Delta_i \times (-a_\epsilon, a_\epsilon)/\psi$ for odd $m$,*

*where $\Delta_i$ is the image of the moment map for the $T_{L_i}$-action on $L_i$, a symplectic leaf on $Z$ and*

— *$a_\epsilon \to \infty$ as $\epsilon \to 0$,*
— *$\psi : (-a_\epsilon, a_\epsilon) \mapsto (-a_\epsilon, a_\epsilon)$ is the involution $x \mapsto -x$.*

**Remark 5.2.** This implies that the image of the moment map is locally convex.

*Proof.* Recall that the moment map is given by the expression (3.2) with $a_i \in \mathfrak{t}^*$ constant. We claim that in (3.2) $\langle a_i, \xi \rangle = 0$ for all $\xi \in \mathfrak{t}_L$. To see this, suppose there exists $\xi \in \mathfrak{t}_L$ such that $\langle a_i, \xi \rangle \neq 0$. Since $T_L$ is a torus action on a compact Hamiltonian $T_L$-space, $L$, it must have a fixed point $p \in Z$. At this fixed point $\xi^M_{|p} = 0$ so $\langle a_i, \xi \rangle = \alpha_i(\xi^M_{|p}) = 0$. Thus $\langle a_i, \xi \rangle = 0$. ■

## 6. Global convexity theorem

In this section, we prove the following theorem.

**Theorem 6.1.** *Let $(M, Z, f)$ be a $b^m$-symplectic manifold. Let $M \setminus Z = \bigsqcup_{i=1}^r M_i$. Then the image of the moment map for the desingularized symplectic form $\omega_\epsilon$ on $\bar{M}_i$ is given by either*

(1) *a product $\Delta \times [-a_\epsilon, a_\epsilon]$, where $a_\epsilon \to \infty$ as $\epsilon \to 0$, or*
(2) *a convex polytope which has the form of a product $\Delta \times [0, a_\epsilon]$ in the neighbourhood of $Z$; in other words, a polytope of the form*

$$\Delta \times [-a_\epsilon, a_\epsilon] \cap H_1 \cap \cdots \cap H_n.$$

*Here $\Delta$ is the image of the moment map for the $T_L$-action on $L$ and $H_1, \ldots, H_n$ are half-spaces. In particular, the image polytopes $\Delta_i$ coincide.*

*Proof.* By the local convexity theorem, we know that if $\partial M_i = Z_i \sqcup Z_{i+1}$, then in a neighbourhood of $Z_i$, respectively, $Z_{i+1}$, the image is of the form $\Delta_i \times [-a_\epsilon, c]$, respectively, $\Delta_{i+1} \times [c', a_\epsilon]$, where $a_\epsilon \to \infty$ as $\epsilon \to 0$ and we may take $c$ and $c'$ positive.

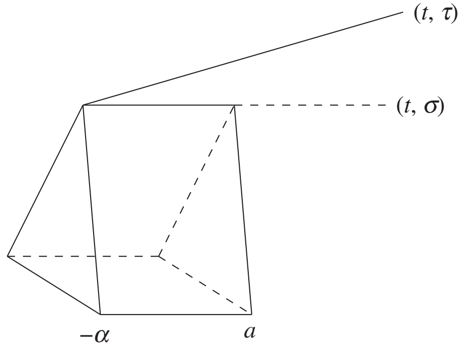
Let $\nu = \mu_{|M_i}$ denote the restriction of $\mu$ to $M_i$ and let $P = \nu(M_i)$ be the image of the moment map. By the convexity theorem for torus actions on symplectic manifolds, the image $P$ of the moment map restricted to $M_i$ is convex.

Therefore $P \cap \nu^{-1}([-a_\epsilon, -c]) = [-a_\epsilon, -c] \times \Delta_i$ and $P \cap \nu^{-1}([c', a_\epsilon]) = [c', a_\epsilon] \times \Delta_{i+1}$ for all $c, c'$ sufficiently close to $a_\epsilon$.

Let us see that this implies $P \cap \nu^{-1}([-a_\epsilon, a_\epsilon]) = [-a_\epsilon, a_\epsilon] \times \Delta_i$.

We first prove $P \cap [-a_\epsilon, a_\epsilon] \subset [-a_\epsilon, a_\epsilon] \times \Delta_i$. Suppose $p \in P \cap \nu^{-1}([-a_\epsilon, a_\epsilon])$ but $p \notin [-a_\epsilon, a_\epsilon] \times \Delta_i$; then $p = (t, \tau)$ where $t \in [-a_\epsilon, a_\epsilon]$ with $\tau \notin \Delta_i$.

Let $\sigma$ be a point of $\Delta_i$ at minimal distance from $\tau$. The line $\ell$ connecting $(t, \tau)$ and $(-\alpha, \sigma)$ must lie in $P$. Thus for $a$ sufficiently close to $-\alpha$, $\ell \cap [-\alpha, a] \times \mathbb{R}^{d-1}$ must lie in $[-\alpha, a] \times \Delta_i$. Hence $\ell$ must intersect the plane $(a, x)_{x \in \mathbb{R}^{d-1}}$ at a point $\sigma' \in \Delta_i$ closer to $\tau$ than $\sigma$ (see picture below). This is a contradiction.




Thus, we have proved that $P \subset [-a_\epsilon, a_\epsilon] \times \Delta_i$; also $P \subset [-a_\epsilon, a_\epsilon] \times \Delta_{i+1}$. Therefore, $P \subset [-a_\epsilon, a_\epsilon] \times (\Delta_i \cap \Delta_{i+1})$. In particular for $c$ sufficiently close to $a_\epsilon$, $[-a_\epsilon, c] \times \Delta_i \subset [-a_\epsilon, c] \times (\Delta_i \cap \Delta_{i+1})$ and $[c, a_\epsilon] \times \Delta_i \subset [c, a_\epsilon] \times (\Delta_i \cap \Delta_{i+1})$ so $\Delta_i \subset (\Delta_i \cap \Delta_{i+1})$ and $\Delta_{i+1} \subset (\Delta_i \cap \Delta_{i+1})$; thus $\Delta_i = \Delta_{i+1}$. As $P$ contains all lines $[-a_\epsilon, a_\epsilon] \times \Delta_i$, it must be equal to $[-a_\epsilon, a_\epsilon] \times \Delta_i$. On the other hand, if $\partial M_i$ is connected the image of the moment map must be of the form (1) or (2) of the theorem and can be written as $\Delta \times [-a_\epsilon, a_\epsilon] \cap H_1 \cap \cdots \cap H_n$ for some half-spaces $H_1, \ldots H_n$. ■

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# References

1. Guillemin V, Miranda E, Pires AR. 2011 Codimension one symplectic foliations and regular Poisson structures. *Bull. Brazilian Math. Soc. New Series* **42**, 1–17. (doi:10.1007/s00574-011-0001-z)
2. Guillemin V, Miranda E, Pires AR. 2014 Symplectic and Poisson geometry on *b*-manifolds. *Adv. Math.* **264**, 864–896. (doi:10.1016/j.aim.2014.07.032)
3. Guillemin V, Miranda E, Weitsman J. 2018 On geometric quantization of *b*-symplectic manifolds. *Adv. Math.* **331**, 941–951. (doi:10.1016/j.aim.2018.04.003)
4. Scott G. 2016 The geometry of $b^k$ manifolds. *J. Symplectic Geom.* **14**, 71–95. (doi:10.4310/JSG.2016.v14.n1.a3)

5. Atiyah MF. 1982 Convexity and commuting Hamiltonians. *Bull. London Math. Soc.* **14**, 1–15. (doi:10.1112/blms/14.1.1)
6. Guillemin V, Sternberg S. 1982 Convexity properties of the moment mapping. *Invent. Math.* **67**, 491–513. (doi:10.1007/BF01398933)
7. Guillemin V, Sternberg S. 1984 Convexity properties of the moment mapping II. *Invent. Math.* **77**, 533–546. (doi:10.1007/BF01388837)
8. Guillemin V, Miranda E, Pires AR, Scott G. 2017 Convexity for Hamiltonian torus actions on $b$-symplectic manifolds. *Math. Res. Lett.* **24**, 363–377. (doi:10.4310/mrl.2017.v24.n2.a5)
9. Guillemin V, Miranda E, Pires AR, Scott G. 2015 Toric actions on $b$-symplectic manifolds. *Int. Math. Res. Notices* **2015**, 5818–5848. (doi:10.1093/imrn/rnu108)
10. Guillemin V, Miranda E, Weitsman J. 2017 Desingularizing $b^m$-symplectic manifolds. *Int. Math. Res. Not.* rnx126. (doi:10.1093/imrn/rnx126)
11. Guillemin V, Miranda E, Weitsman J. 2017 On geometric quantization of $b^m$-symplectic manifolds. Preprint.