J. Eur. Math. Soc. 13, 709–731

DOI 10.4171/JEMS/265

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ACM bundles on cubic surfaces

Received April 15, 2008

Abstract. We prove that, for every $r \geq 2$, the moduli space $M_X^s(r; c_1, c_2)$ of rank r stable vector bundles with Chern classes $c_1 = rH$ and $c_2 = \frac{1}{2}(3r^2 - r)$ on a nonsingular cubic surface $X \subset \mathbb{P}^3$ contains a nonempty smooth open subset formed by ACM bundles, i.e. vector bundles with no intermediate cohomology. The bundles we consider for this study are extremal for the number of generators of the corresponding module (these are known as Ulrich bundles), so we also prove the existence of indecomposable Ulrich bundles of arbitrarily high rank on X.

1. Introduction

One has known since the work of Mumford that to have a reasonable parameter space for families of algebraic vector bundles on an algebraic variety X, it is necessary to impose a condition of stability. The existence of a coarse moduli space of stable vector bundles has been known since the work of Maruyama [25], Gieseker [14] and Simpson [28]. The moduli space of vector bundles of all ranks on algebraic curves has been studied extensively. Bundles of rank 2 on higher dimensional varieties have often played an important role—think of the work [27] of Mukai on the moduli of stable rank 2 bundles on a K3 surface. By means of the Serre correspondence, rank 2 bundles on any variety are related to the study of codimension 2 subvarieties (see for example Beauville [4]). However, little is known about bundles of higher rank.

A particular class of vector bundles that have been studied in recent years is the class of arithmetically Cohen–Macaulay (ACM) bundles. A vector bundle $\mathcal{E}$ on a projective variety X is ACM if all its intermediate cohomology groups $H^i(X, \mathcal{E}(m))$ are zero for $0 < i < \dim X$ and all $m \in \mathbb{Z}$. These bundles correspond to Maximal Cohen–Macaulay modules (MCM) over the associated graded ring. In the algebraic context, MCM modules have been extensively studied (see for example the book of Yoshino [31]), as they reflect relevant properties of the corresponding ring. There has also been recent work on ACM bundles of small rank on particular varieties such as Fano 3-folds, quartic threefolds and Grassmann varieties (see [1] and the references therein).

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From the point of view of representation theory, one can often divide varieties (or rings) into three classes depending on the behavior of the category of ACM bundles (or MCM modules). A variety is of *finite representation type* if there are only finitely many indecomposable ACM bundles (up to twist). The projective varieties of finite representation type have been completely classified into a small list (see [31]). For example, on a nonsingular quadric hypersurface X in $\mathbb{P}^n$, there are only one (or two) indecomposable ACM bundles (not counting the structure sheaf $\mathcal{O}_X$), depending on the parity of n. A variety is of *tame representation type* if, for each rank r, the indecomposable ACM bundles of rank r form a finite number of families of dimension at most one. For example, according to the work of Atiyah [2], on an elliptic curve, for each rank and degree there is a single family of indecomposable bundles, parameterized by the curve itself. A variety is of *wild representation type* if there exist n-dimensional families of nonisomorphic indecomposable ACM bundles for arbitrarily large n. Since all vector bundles on projective curves are ACM, we see that a nonsingular curve in projective space is of finite, tame, or wild representation type according as its genus is 0, 1, or ≥ 2. Drozd and Greuel [10] have shown that the category of MCM modules over the complete local ring of a reduced curve singularity is either finite, tame or wild. However, we cannot expect such a trichotomy in general, because for example a quadric cone in $\mathbb{P}^3$ has an infinite discrete set of indecomposable ACM bundles of rank 2 [9, 6.1].

In this paper we exhibit families of stable ACM bundles on a nonsingular cubic surface in $\mathbb{P}^3$, of arbitrarily high rank and dimension. Thus the cubic is of wild representation type. As far as we know, these are the first examples of indecomposable ACM bundles of arbitrarily high rank on any varieties except curves.

Our bundles have another particular characteristic: they are extremal for the number of generators of the corresponding module. In the case of MCM modules, this phenomenon was discovered by Ulrich [29]. He showed that for MCM modules over a local ring, there is a bound on the number of generators of the module in terms of the multiplicity and the rank. Since then, modules with the maximum number of generators have been called Ulrich modules, and correspondingly Ulrich bundles in the geometric case. Even the existence of Ulrich bundles on projective varieties is not known in general, though in our case of a cubic surface in $\mathbb{P}^3$ it is easy, because the ideal sheaf of a twisted cubic curve is one. Also rank two ACM bundles on the cubic surface have been classified by Faenzi [13], and in his classification there are stable Ulrich bundles with $c_1 = 2H$, $c_2 = 5$ that form a family of dimension 5.

Here is our main result, (5.7).

Theorem. *Let X be a nonsingular cubic surface in $\mathbb{P}^3$ over an algebraically closed field k. Then for every $r \geq 2$ there are stable Ulrich bundles of rank r with $c_1 = rH$ and $c_2 = \frac{1}{2}(3r^2 - r)$. They form a smooth open subset of an irreducible component of dimension $r^2 + 1$ of the moduli space $M^s_X(r; c_1, c_2)$ of rank r stable vector bundles with Chern classes c_1, c_2 on X.*

The paper is organized as follows. In Section 2 we recall properties of ACM sheaves including the theory of matrix factorizations due to Eisenbud. In Section 3 we recall the definition and some properties of Ulrich sheaves including a new proof of the bound on the number of generators of the graded module.

In Section 4 we give the construction of our Ulrich bundles over the cubic surface, based on an earlier work of the first author on minimal free resolutions of sets of points in general position, which in turn depends on earlier work of the second author on Gorenstein liaison of zero-schemes on the cubic surface. In Section 5 we show that our bundles are stable, and compute the dimension of the coarse moduli space.

In the last Section 6 we show that even though our surface is of wild representation type, our bundles are not so wild after all. We show that these bundles are stably equivalent to *layered* ACM bundles (6.10), where layered means a successive extension of rank 1 ACM bundles (corresponding to ACM curves in the surface). In particular our bundles are direct summands of successive extensions of ACM line bundles.

2. Generalities on ACM sheaves

Throughout this section, X denotes an integral hypersurface in $\mathbb{P}^n$ of dimension ≥ 2 defined by a homogeneous polynomial f of degree d, R is the polynomial ring $k[x_0, \dots, x_n]$ over an algebraically closed field k, and R_X is the coordinate ring of X. If $\mathcal{F}$ is an $\mathcal{O}_X$-module, $H^i_*(\mathcal{F})$ denotes $\bigoplus_{l \in \mathbb{Z}} H^i(X, \mathcal{F}(l))$ and $h^i(\mathcal{F})$ is the dimension of $H^i(\mathcal{F})$. The minimal number of generators of $\mathcal{F}$ will refer to the minimal number of generators of $H^0_*(\mathcal{F})$ and will be denoted by $\mu(\mathcal{F})$. The dual of a module M is $M^\vee = \mathrm{Hom}_{R_X}(M, R_X)$, and analogously for sheaves. For a torsion-free coherent sheaf $\mathcal{F}$, $\mathrm{rk}(\mathcal{F})$ will denote its rank. When $\mathcal{F} \cong \mathcal{O}_X(a_1) \oplus \cdots \oplus \mathcal{O}_X(a_r)$ we will say that $\mathcal{F}$ is a *dissocié* sheaf.

We recall that a subscheme $X \subset \mathbb{P}^n$ is *arithmetically Cohen–Macaulay*, *ACM* for short, (respectively *arithmetically Gorenstein*, *AG* for short) if its homogeneous coordinate ring R_X is a *local graded Cohen–Macaulay ring (resp. Gorenstein ring), in the sense of [**?**]. For example, when $X \subset \mathbb{P}^n$ is a hypersurface of degree d, then it is arithmetically Gorenstein.

We will deal both with maximal Cohen–Macaulay R_X-modules and arithmetically Cohen–Macaulay coherent sheaves over X. We recall the definition here.

Definition 2.1. A coherent sheaf $\mathcal{E}$ on X is an *ACM sheaf* if it is locally Cohen–Macaulay on X and $H^i_*(\mathcal{E}) = 0$ for $0 < i < \dim X$. A graded R_X-module E is a *maximal Cohen–Macaulay module* (MCM from now on) if depth $E = \dim E = \dim R_X$.

There is a one-to-one correspondence between ACM sheaves on arithmetically Cohen –Macaulay schemes X and graded MCM R_X-modules sending $\mathcal{E}$ to $H^0_*(\mathcal{E})$ (see [9, 2.1]). When X is nonsingular, ACM sheaves are locally free, so we will be speaking about vector bundles in this case.

Definition 2.2. For any coherent sheaf $\mathcal{E}$ on the hypersurface X we define the *syzygy sheaf* $\mathcal{E}^\sigma$ to be the sheafification of the kernel of a minimal free presentation of $E = H^0_*(\mathcal{E})$ over R_X. In other words, let $F_0 \to E \to 0$ be a minimal free cover, and sheafify to get

$$0 \to \mathcal{E}^\sigma \to \mathcal{F}_0 \to \mathcal{E} \to 0$$

with $\mathcal{F}_0$ free and $H^0_*(\mathcal{F}_0) \to H^0_*(\mathcal{E})$ surjective.

When X is a hypersurface defined by a polynomial f, Eisenbud proved in [11] that there is a one-to-one correspondence between MCM R_X-modules and matrix factorizations of f. We recall here how it works.

If E is a MCM R_X-module, then as an R-module it has projective dimension 1 and therefore it has a minimal free resolution over R:

$$0 \to F \xrightarrow{\varphi} G \to E \to 0$$

where F and G are free R-modules of the same rank and φ is a degree zero morphism (i.e. φ corresponds to a *homogeneous* matrix). As E is annihilated by f, $fG \subseteq \mathrm{im}(\varphi)$, so there exists a morphism $\psi : G \to F$ so that $\varphi\psi = f\,\mathrm{Id}_G$. Then $\varphi\psi\varphi = f\varphi$ and as φ is injective, this implies that $\psi\varphi = f\,\mathrm{Id}_F$.

Definition 2.3 ([11]). A *matrix factorization* of $f \in R$ is an ordered pair of morphisms of free R-modules ($\varphi : F \to G, \psi : G \to F$) such that $\varphi\psi = f\,\mathrm{Id}_G$ and $\psi\varphi = f\,\mathrm{Id}_F$.

If f is a non-zero-divisor, then one equality implies the other.

We just saw that to each MCM R_X-module E we can associate a matrix factorization (φ, ψ) with $E \cong \mathrm{Coker}(\varphi)$. Moreover, if f is irreducible, then $\det\varphi = f^r$ where $r = \mathrm{rk}(E)$ [11, 5.6]. A matrix factorization $(\varphi : F \to G, \psi : G \to F)$ is *reduced* if $\varphi(F) \subset \mathfrak{m}G$ and $\psi(G) \subset \mathfrak{m}F$ or, in other words, there is no scalar entry in φ or ψ different from 0 (here $\mathfrak{m}$ denotes the maximal irrelevant ideal of R). The correspondence between matrix factorizations and MCM modules over hypersurface rings can be summarized as follows:

Theorem 2.4 ([11, 6.3]). *Let $X \subset \mathbb{P}^n$ be an integral hypersurface defined by a form f. There are bijections between the sets of*

(i) *Equivalence classes of reduced matrix factorizations (φ, ψ) of f over R.*
(ii) *Isomorphism classes of nontrivial 2-periodic minimal free resolutions over R_X,*

$$\cdots \to F(-d) \xrightarrow{\varphi} G(-d) \xrightarrow{\psi} F \xrightarrow{\varphi} G \to M = \mathrm{Coker}\,\varphi \to 0.$$

(iii) *MCM R_X-modules $M = \mathrm{Coker}\,\varphi$ without free summands.*

The correspondence between (i) *and* (iii) *sends (φ, ψ) to $M := \mathrm{Coker}\,\varphi$. Moreover* $\det(\varphi) = f^r$ *where* $r = \mathrm{rk}(M)$.

Lemma 2.5. *If $X \subset \mathbb{P}^n$ is an integral hypersurface of degree d and $\mathcal{E}$ is an ACM sheaf over X corresponding to a matrix factorization (φ, ψ) then*

(i) *$\mathcal{E}^\sigma$ is also ACM and has no direct free summands.*
(ii) *If $\mathcal{E}$ is indecomposable, then so is $\mathcal{E}^\sigma$.*
(iii) *If $\mathcal{E}$ has no direct free summands, then (ψ, φ) is a matrix factorization for $\mathcal{E}^\sigma$.*
(iv) *$\mathcal{E}^\vee$ is also ACM and $\mu(\mathcal{E}) = \mu(\mathcal{E}^\vee)$. Furthermore $\mathcal{E}$ is reflexive.*
(v) *If $\mathcal{E}$ has no direct free summands, then $(\varphi^\vee, \psi^\vee)$ is a matrix factorization for $\mathcal{E}^\vee$.*
(vi) *If $\mathcal{E}$ has no direct free summands, then $\mathcal{E}^{\sigma\sigma} \cong \mathcal{E}(-d)$, $\mathcal{E}^{\sigma\vee} \cong \mathcal{E}^{\vee\sigma}(d)$, $\mathcal{E}^{\sigma\vee\sigma\vee} \cong \mathcal{E}$.*

(vii) *If* $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ *is an exact sequence of ACM sheaves, then there is a dissocié sheaf* $\mathcal{M}$ *and an exact sequence*

$$0 \to \mathcal{E}'^{\sigma} \to \mathcal{E}^{\sigma} \oplus \mathcal{M} \to \mathcal{E}''^{\sigma} \to 0.$$

Proof. (i) and (ii) follow from [8, 4.2]. Statement (iii) follows from 2.4. From [9, 2.3] we know that $\mathcal{E}^{\vee}$ is also ACM and reflexive. The equality $\mu(\mathcal{E}) = \mu(\mathcal{E}^{\vee})$ can be found in [20, 1.5] and the proof goes as follows. Let $E = H^0_*(\mathcal{E})$, $m = \mu(E)$ and let $0 \to F \to G \to E \to 0$ be a minimal free R-resolution of E with $\mathrm{rk}(G) = \mathrm{rk}(F) = m$. Then dualizing we obtain $m = \mu(\mathrm{Ext}^1_R(E, R))$. On the other hand, applying $\mathrm{Hom}(E, \cdot)$ to the exact sequence $0 \to R(-d) \xrightarrow{f} R \to R_X \to 0$ we obtain the exact sequence

$$0 \to \mathrm{Hom}_R(E, R_X) \to \mathrm{Ext}^1_R(E, R(-d)) \xrightarrow{f} \mathrm{Ext}^1_R(E, R).$$

As f annihilates E, the first module is isomorphic to $\mathrm{Hom}_{R_X}(E, R_X) = E^{\vee}$ and the last morphism is 0. Therefore $m = \mu(E^{\vee})$ and statement (iv) follows.

Theorem 2.4 and (iv) imply item (v).

Statement (vi) is a consequence of (iii) and (v) and of the 2-periodic resolution in Theorem 2.4.

The last claim was proved in [8, 4.1]. □

3. Generalities on Ulrich sheaves

The minimal number of generators $\mu(\mathcal{E})$ of an ACM $\mathcal{O}_X$-module $\mathcal{E}$ of positive rank over an integral scheme X of degree d is bounded above.

Theorem 3.1. *Let* $X \subset \mathbb{P}^n$ *be an integral subscheme and* $\mathcal{E}$, *an ACM sheaf on* X. *Then* $\mu(\mathcal{E})$ (*the minimal number of generators of* $H^0_*(\mathcal{E})$) *is bounded by*

$$\mu(\mathcal{E}) \le \deg(X)\,\mathrm{rk}(\mathcal{E}). \tag{1}$$

Remark 3.2. For Cohen–Macaulay local rings this theorem was proved in [29] and [5].

Proof. Let $m = \dim X$, and choose a finite projection $\pi : X \to \mathbb{P}^m$. Then $\pi_*(\mathcal{E})$ is a coherent sheaf on $\mathbb{P}^m$ which is locally Cohen–Macaulay because depth $\mathcal{E}$ can be calculated on X or on $\mathbb{P}^m$ and is the same. Hence $\pi_*(\mathcal{E})$ is locally free on $\mathbb{P}^m$.

Moreover, $\pi_*(\mathcal{E})$ is ACM on $\mathbb{P}^m$. Indeed, since π is a finite morphism, $H^i(X, \mathcal{E}(l)) = H^i(\mathbb{P}^m, \pi_*(\mathcal{E})(l))$ for all i, l. Then by the Theorem of Horrocks [22], $\pi_*(\mathcal{E})$ is dissocié. Now $\mu(\mathcal{E})$ is the minimal number of generators of $E = H^0_*(\mathcal{E})$ as an $R_X = k[x_0, \dots, x_n]/I_X$-module. Inside R_X we have the polynomial ring $S = k[x_0, \dots, x_m]$ coming from $\mathbb{P}^m$. Since $\pi_*(\mathcal{E})$ is dissocié of rank $\deg(X)\,\mathrm{rk}(\mathcal{E})$, it is minimally generated by exactly $\deg(X)\,\mathrm{rk}(\mathcal{E})$ elements. These elements also generate $H^0_*(\mathcal{E})$ over R_X but may no longer be minimal, so $\mu(\mathcal{E}) \le \deg(X)\,\mathrm{rk}(\mathcal{E})$. □

Modules attaining the upper bound were studied by Ulrich in [29] and named after him in [20] (in [5] they were called MGMCM).

Definition 3.3. Let $X \subset \mathbb{P}^n$ be an integral scheme. An ACM sheaf $\mathcal{E}$ over X (resp. an MCM R_X-module E) is an *Ulrich sheaf* (resp. *Ulrich module*) if the minimal number of generators of $\mathcal{E}$ (resp. E) is $\mu(\mathcal{E}) = \deg(X)\,\mathrm{rk}(\mathcal{E})$ (resp. $\mu(E) = \deg(X)\,\mathrm{rk}(E)$).

When X is nonsingular, $\mathcal{E}$ is locally free, so we call it an *Ulrich bundle*.

Definition 3.4. Let $\mathcal{E}$ (resp. E) be a coherent sheaf on X (resp. a finitely generated graded R_X-module E). Then we say that $\mathcal{E}$ (resp. E) is *normalized* if $H^0(X, \mathcal{E}) \neq 0$ and $H^0(X, \mathcal{E}(-1)) = 0$ (resp. $E_0 \neq 0$, $E_{-1} = 0$).

Corollary 3.5. *Let $X \subset \mathbb{P}^n$ be an integral subscheme of degree d, and $\mathcal{E}$ an ACM sheaf on X. Assume furthermore that $\mathcal{E}$ is normalized. Then $h^0(\mathcal{E}) \leq d\,\mathrm{rk}(\mathcal{E})$ and equality implies that $\mathcal{E}$ is an Ulrich sheaf.*

Proof. Since $\mathcal{E}$ is normalized, the elements of $H^0(\mathcal{E})$ are part of a minimal system of generators for $\mathcal{E}$. Then $h^0(\mathcal{E}) \leq \mu(\mathcal{E}) \leq d\,\mathrm{rk}(\mathcal{E})$. Equalities imply that $\mathcal{E}$ is an Ulrich sheaf. □

Here we state some properties of Ulrich ACM sheaves.

Lemma 3.6. *Let $\mathcal{E}$ be a rank r ACM sheaf over an integral hypersurface $X \subset \mathbb{P}^n$ of degree $d \geq 2$. Then the following holds:*

(a) *If $\mathcal{E}$ is Ulrich then it has no free summands.*
(b) *$\mathcal{E}$ is Ulrich if and only if $\mathcal{E}^\vee$ is Ulrich.*

Proof. (a) If $\mathcal{E}$ is Ulrich and can be decomposed as $\mathcal{E} = \mathcal{F} \oplus \mathcal{M}$, where $\mathcal{M}$ is dissocié, then $\mu(\mathcal{E}) = \mu(\mathcal{F}) + \mathrm{rk}(\mathcal{M})$ and $\mathcal{F}$ is an ACM sheaf of rank $r - \mathrm{rk}(\mathcal{M})$. Then inequality (1) applied to $\mathcal{F}$ yields $\mu(\mathcal{E}) \leq d(r - \mathrm{rk}(\mathcal{M})) + \mathrm{rk}(\mathcal{M})$. But $\mu(\mathcal{E}) = dr$ by assumption, so this is a contradiction if $d > 1$.

(b) is a consequence of (a) and 2.5(iv)&(v). □

Proposition 3.7. *Let $\mathcal{E}$ be an ACM sheaf of rank r over an integral hypersurface $X \subset \mathbb{P}^n$ of degree $d \geq 2$.*

(a) *If $\mathcal{E}$ is Ulrich and is not an extension of lower rank Ulrich bundles then it has a linear minimal free resolution over $\mathcal{O}_{\mathbb{P}^n}$:*

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-a-1)^{dr} \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}^n}^{dr}(-a) \to \mathcal{E} \to 0.$$

(b) *If $\mathcal{E}$ is normalized then the following are equivalent:*
  (i) *$\mathcal{E}$ has a linear minimal free resolution over $\mathcal{O}_{\mathbb{P}^n}$.*
  (ii) *$\mathcal{E}$ is Ulrich and generated by its global sections.*
  (iii) *$h^0(\mathcal{E}) = dr$.*

Proof. We denoted by E the module $H^0_*(\mathcal{E})$.

(a) Let $0 \to \bigoplus_{j=1}^{dr} R(b_j) \xrightarrow{\varphi} \bigoplus_{i=1}^{dr} R(a_i) \to E \to 0$ be the minimal free resolution of E as an R-module, with $a_1 \geq \cdots \geq a_{dr}$ and $b_1 \geq \cdots \geq b_{dr}$, (φ, ψ) a matrix factorization of f. After choosing bases, φ is a matrix with entries of degree $u_{i,j} = a_i - b_j$, which decrease from top to bottom and from right to left. As $\mathcal{E}$ has no free summands,

φ is reduced by Theorem 2.4, i.e. any entry of degree 0 is actually 0. In particular, if $u_{i,i} \leq 0$ then $\det(\varphi) = 0$, which is impossible because (φ, ψ) is a matrix factorization for f. Arguing similarly for E^σ we find that $u_{i,i}$ must be strictly smaller than d, so $1 \leq u_{i,i} \leq d-1$. Moreover, if $u_{i,i-1} \leq 0$, then φ has the form

$$\varphi = \begin{pmatrix} A_1 & A \\ 0 & A_2 \end{pmatrix}.$$

If we write

$$\psi = \begin{pmatrix} B_1 & C \\ D & B_2 \end{pmatrix}$$

where B_i has the same size as A_i, then from $\varphi\psi = f\,\mathrm{Id}$ and $\psi\varphi = f\,\mathrm{Id}$ we see that (A_1, B_1) and (A_2, B_2) are matrix factorizations of f because $B_1A_1 = f\,\mathrm{Id}$, $A_2B_2 = f\,\mathrm{Id}$. Therefore there is an extension sequence

$$0 \to \operatorname{Coker} A_1 \to E \to \operatorname{Coker} A_2 \to 0.$$

Moreover $\operatorname{Coker} A_1$ and $\operatorname{Coker} A_2$ are Ulrich because if A_i is an $l_i \times l_i$ matrix, then $\mu(E) \leq \mu(\operatorname{Coker} A_1) + \mu(\operatorname{Coker} A_2) \leq dl_1 + dl_2 = dr$ and all inequalities become equalities.

Therefore $u_{i,i-1} \geq 1$ and $dr = \sum_{i=2}^{dr} u_{i,i-1} + u_{1,dr} \geq dr - 1 + u_{1,dr}$, which implies that $u_{1,dr} = 1$. As φ decreases from top to bottom and right to left, we have $u_{i,i-1} = 1$. Since φ is homogeneous we can conclude that all the entries in φ have degree 1.

We prove (b).

(i)⇒(ii). If $\mathcal{E}$ has a linear minimal resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(t-1)^s \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}^n}(t)^s \to \mathcal{E} \to 0,$$

then φ is a reduced matrix factorization with $\det(\varphi) = f^r$, $r = \mathrm{rk}(\mathcal{E})$ and the degree of $\det(\varphi)$ is s. Hence $s = \deg(X)r$ and $\mathcal{E}$ is Ulrich. Moreover as $\mathcal{E}$ is normalized, this implies that $t = 0$ and $\mathcal{E}$ is generated by its global sections.

(ii)⇒(i). Let

$$0 \to \mathcal{F} = \bigoplus_{j=1}^{dr} \mathcal{O}_{\mathbb{P}^n}(b_j) \xrightarrow{\varphi} \mathcal{G} = \bigoplus_{i=1}^{dr} \mathcal{O}_{\mathbb{P}^n}(a_i) \to \mathcal{E} \to 0$$

be the minimal free resolution of $\mathcal{E}$ as $\mathcal{O}_{\mathbb{P}^n}$-module, with $a_1 \geq \cdots \geq a_{dr}$ and $b_1 \geq \ldots \geq b_{dr}$, (φ, ψ) a reduced matrix factorization of f (note that by 3.6, $\mathcal{E}$ has no free direct summand). As $H^0(\mathcal{E}(-1)) = 0$, we have $H^0(\mathcal{F}(-1)) \cong H^0(\mathcal{G}(-1))$. Therefore we have an equality $\{b_j \mid b_j \geq 1\} = \{a_i \mid a_i \geq 1\}$, and in particular $a_1 = b_1$ if these sets are nonempty. But then, as φ is reduced, its last column would be zero, which is impossible because φ is a matrix factorization. Hence, $a_i \leq 0$ and $b_i < 0$ for all $i = 1, \ldots, m$, and as $H^0(\mathcal{E}) \neq 0$, $a_1 = 0$.

Now, $\mathcal{E}$ is generated by global sections if and only if $a_{dr} = 0$, which is now equivalent to saying that all a_i are zero. But the degree of the determinant of φ must be dr, so this is equivalent to saying that $\mathcal{E}$ has a linear resolution.

(iii)⇒(i). We assume now that $\mathcal{E}$ is normalized with $h^0(\mathcal{E}) = dr$. Then by Corollary 3.5, $\mathcal{E}$ is Ulrich. To see that it has a linear minimal free resolution we proceed by induction on $r = \mathrm{rk}(\mathcal{E})$. For $r = 1$, $\mathcal{E}$ is not an extension of lower rank Ulrich sheaves, so we conclude by (a).

We assume now that $r > 1$. If $\mathcal{E}$ is not an extension of lower rank Ulrich sheaves, then by (a) we are done. If $\mathcal{E}$ is an extension of Ulrich sheaves $\mathcal{E}_i$ of lower rank r_i, $i = 1, 2$,

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0,$$

then $h^0(\mathcal{E}_i) = dr_i$ because of the bound given in Corollary 3.5. In particular, $\mathcal{E}_i$ is Ulrich and normalized, and by induction hypothesis it has a linear minimal resolution. We then apply the horseshoe lemma of [30, 2.2.8] to obtain a linear resolution for $\mathcal{E}$. The remaining implication is clear. □

The existence of Ulrich sheaves on a projective scheme X is not known in general. Below we present some known examples of Ulrich sheaves.

Example 3.8. (a) The existence of an Ulrich sheaf of rank 1 on a hypersurface defined by a form f is related to the possibility of writing f as the determinant of a $d \times d$-matrix of linear forms. For example, on a smooth cubic surface the existence of twisted cubic curves in X allows writing the equation of f as a linear determinant (see [3, Corollary 6.4]).

(b) The existence of orientable rank 2 Ulrich bundles on a hypersurface defined by a form f is related to the possibility of writing f as the Pfaffian of a linear skew-symmetric matrix, and by Serre's correspondence, to the existence of certain arithmetically Gorenstein subschemes of codimension 2 of X. On a smooth cubic surface, the Serre correspondence applied to a set of five general points in X proves the existence of a rank 2 Ulrich bundle ([3, Proposition 7.6]).

(c) The existence of Ulrich sheaves on special varieties has been studied by many authors. It is known for example that there exists at least one Ulrich MCM module on complete intersections (see [21] and the references therein). In [12], Eisenbud and Schreyer proved the existence of Ulrich bundles on any algebraic curve, on Veronese varieties and of rank two Ulrich sheaves on Del Pezzo surfaces. Although rank one Ulrich sheaves are rare in general, it is not difficult to prove that there exists a rank one Ulrich sheaf on any ACM rational surface S in $\mathbb{P}^4$ (in this case S is either a cubic scroll, a Del Pezzo surface, a Castelnuovo surface or a Bordiga surface, and the existence of a rational quartic curve on them leads to the existence of a rank one Ulrich sheaf).

4. Ulrich vector bundles on the cubic surface

From now on, X will be a nonsingular cubic surface in $\mathbb{P}^3$ defined by a degree 3 homogeneous polynomial $f \in R = k[x_0, x_1, x_2, x_3]$ and R_X will denote the ring $R/(f)$. Then $p_a(X) = 0$ and $\omega_X = \mathcal{O}_X(-1)$. Moreover, as X is regular, any ACM sheaf on X is locally free. Let $\mathcal{E}$ be a vector bundle of rank r on X. Let $c_1(\mathcal{E})$ and $c_2(\mathcal{E})$ denote its Chern classes, and $\deg(\mathcal{E}) = \deg c_1(\mathcal{E})$ its degree. Then the Riemann–Roch theorem says that

$$\chi(\mathcal{E}) = r + \frac{c_1(\mathcal{E}).H}{2} + \frac{c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})}{2}.$$

$\mathcal{E}$ is called *orientable* if $\det(\mathcal{E})$ is isomorphic to $\mathcal{O}_X(l)$ for some $l \in \mathbb{Z}$, or in other words, $c_1(\mathcal{E}) = lH$.

Remark 4.1. Let $\mathcal{E}$ be an ACM bundle of rank r on the cubic surface X, and let H be a general hyperplane section. Then from the exact sequence

$$0 \to \mathcal{E}(-1) \to \mathcal{E} \to \mathcal{E}_H \to 0$$

we find $h^0(\mathcal{E}) \geq h^0(\mathcal{E}_H)$ and $\deg(\mathcal{E}) = \deg(\mathcal{E}_H)$. By Riemann–Roch on the elliptic cubic curve we have $h^0(\mathcal{E}_H) \geq \deg(\mathcal{E}_H)$ and so $\deg(\mathcal{E}) \leq h^0(\mathcal{E})$.

On the other hand, we know by Corollary 3.5 that if $\mathcal{E}$ is ACM and normalized, $h^0(\mathcal{E}) \leq 3r$. Therefore, if $\mathcal{E}$ is a normalized ACM bundle on X, then $\deg(\mathcal{E}) \leq 3r$ and equality implies that $\mathcal{E}$ is an Ulrich bundle with $h^0(\mathcal{E}) = 3r$.

Lemma 4.2. *Let $\mathcal{E}$ be a normalized ACM bundle of rank r on X. Then the following are equivalent:*

(i) *$\mathcal{E}$ has a linear minimal free resolution.*
(ii) *$\mathcal{E}$ is Ulrich and generated by its global sections.*
(iii) $h^0(\mathcal{E}) = 3r$.
(iv) $\deg(\mathcal{E}) = 3r$.

Proof. (i), (ii) and (iii) are equivalent by Lemma 3.7.

(i)⇒(iv). From the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1)^{3r} \to \mathcal{O}_{\mathbb{P}^3}^{3r} \to \mathcal{E} \to 0$$

we obtain $\chi(\mathcal{E}) = 3r$ and $\chi(\mathcal{E}(1)) = 9r$. For a general hyperplane H, the hyperplane section of X is an integral elliptic curve and we have the exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}(1) \to \mathcal{E}_H(1) \to 0.$$

Therefore $\chi(\mathcal{E}(1)) - \chi(\mathcal{E}) = \chi(\mathcal{E}_H(1))$. By the Riemann–Roch theorem on an elliptic curve, this last term is equal to $\deg(\mathcal{E}_H(1))$. As $c_1(\mathcal{E}_H(1)) = c_1(\mathcal{E}_H) + rH$, we have $\deg(\mathcal{E}_H(1)) = \deg(\mathcal{E}) + 3r$ and this implies that $\chi(\mathcal{E}(1)) - \chi(\mathcal{E}) = \deg(\mathcal{E}) + 3r$. As the term in the left is equal to $6r$, we obtain $\deg(\mathcal{E}) = 3r$ as desired.

(iv)⇒(ii) by Remark 4.1. Therefore all conditions are equivalent. □

We recall the following result that was proved by the first author in [7].

Proposition 4.3 (see [7]). *Let Z be a set of $n = \frac{1}{2}(3r^2 - r)$ general points on X, $r \geq 2$. Then the minimal free resolution over R of the saturated ideal of Z in X is*

$$0 \to R(-r-3)^{r-1} \to R(-r-1)^{3r} \to R(-r)^{2r+1} \to I_{Z,X} \to 0.$$

In the next theorem we prove the existence of Ulrich bundles of any rank.

Theorem 4.4. (a) *If $\mathcal{E}$ is a normalized orientable Ulrich bundle of rank $r \geq 2$ generated by global sections, then there is an exact sequence*

$$0 \to \mathcal{O}_X^{r-1} \to \mathcal{E} \to \mathcal{J}_{Z,X}(r) \to 0 \tag{2}$$

where Z is a zero-scheme of degree $n = \frac{1}{2}(3r^2 - r)$, and $h^0(\mathcal{J}_{Z,X}(r-1)) = 0$.

(b) *Conversely, if $r \geq 2$ and Z is a sufficiently general set of $n = \frac{1}{2}(3r^2 - r)$ points on X, then there exists an extension of $\mathcal{J}_{Z,X}(r)$ by $\mathcal{O}_X^{r-1}$ as above, where $\mathcal{E}$ is a normalized orientable Ulrich bundle of rank r generated by global sections.*

Proof. (a) As $\mathcal{E}$ is generated by global sections, we can take $r-1$ general sections of $\mathcal{E}$ so that the quotient of $\mathcal{E}$ by $\mathcal{O}_X^{r-1}$ is torsion free. Therefore we have an exact sequence

$$0 \to \mathcal{O}_X^{r-1} \to \mathcal{E} \to \mathcal{J}_Z(D) \to 0$$

where $D = c_1(\mathcal{E})$ is a certain divisor on X and Z is a zero-scheme of degree equal to $c_2(\mathcal{E})$. By Lemma 4.2(iv), D has degree $3r$. As $\mathcal{E}$ is orientable we have $c_1(\mathcal{E}) = rH$, and the Riemann–Roch theorem applied to $\mathcal{E}$ gives $c_2(\mathcal{E}) = \frac{1}{2}(3r^2 - r)$. Note that $h^0(\mathcal{J}_{Z,X}(r-1)) = 0$ since $\mathcal{E}$ is normalized.

(b) Let Z be a set of n points, sufficiently general so that the minimal free resolution of $\mathcal{J}_{Z,X}$ is given by Proposition 4.3. By a generalization of the well-known Serre correspondence we obtain an extension of $\mathcal{J}_{Z,X}(r)$ by $\mathcal{O}_X^{r-1}$ in the following way.

If R_X, R_Z denote the homogeneous coordinate rings of X and Z, and $I_{Z,X}$ the saturated ideal of Z in X, then from the exact sequence

$$0 \to I_{Z,X} \to R_X \to R_Z \to 0,$$

we obtain $\operatorname{Ext}^1(I_{Z,X}, R_X(-1)) \cong \operatorname{Ext}^2(R_Z, R_X(-1))$. As the canonical module of X is isomorphic to $R_X(-1)$, we see that $\operatorname{Ext}^2(R_Z, R_X(-1))$ is isomorphic to the canonical module K_Z of Z, because 2 is precisely the codimension of Z in X.

On the other hand, a minimal free resolution for the canonical module K_Z can be obtained by applying $\operatorname{Hom}_R(\cdot, K_{\mathbb{P}^3})$ to the minimal free resolution of I_Z given in Proposition 4.3 (see [26, 1.2.4]). Therefore there is a minimal resolution of K_Z as follows:

$$\cdots \to R(r-3)^{3r} \to R(r-1)^{r-1} \to K_Z \to 0,$$

and K_Z is generated in degree $1-r$ by elements $\nu_1, \ldots, \nu_{r-1}$. These generators provide an extension

$$0 \to R_X^{r-1} \to E \to I_{Z,X}(r) \to 0 \tag{3}$$

via the isomorphism $K_Z \cong \operatorname{Ext}^1(I_{Z,X}, R_X(-1))$. To prove that this module E is a maximal Cohen–Macaulay R_X-module we just need to prove that $\operatorname{Ext}^1(E, K_X) = 0$. This follows by applying $\operatorname{Hom}_{R_X}(\cdot, K_X)$ to the exact sequence (3). Indeed, this leads to an exact sequence

$$\operatorname{Hom}(R_X^{r-1}, K_X) \cong R_X(-1)^{r-1} \to \operatorname{Ext}^1(I_{Z,X}(r), K_X) \cong K_Z(-r) \\ \to \operatorname{Ext}^1(E, K_X) \to 0$$

where the first morphism is an epimorphism because it is defined by the generators $\nu_1, \ldots, \nu_{r-1}$.

Now observe that $\mathcal{E} := \widetilde{E}$ is normalized. Indeed, it can be deduced from the exact sequence (3) that $h^0\mathcal{E} = 3r$ and that $h^0(\mathcal{E}(-1)) = 0$ ($h^0(\mathcal{J}_{Z,X}(r))$ and $h^0(\mathcal{J}_{Z,X}(r-1))$ can be computed from the resolution given in Proposition 4.3). Therefore, by Lemma 4.2, $\mathcal{E}$ is Ulrich and generated by global sections. □

Corollary 4.5. *On a nonsingular cubic surface $X \subseteq \mathbb{P}^3$, for every $r \geq 2$, there exist normalized orientable Ulrich bundles $\mathcal{E}$ of rank r generated by global sections with $c_1(\mathcal{E}) = rH$ and $c_2(\mathcal{E}) = \frac{1}{2}(3r^2 - r)$*

5. Stability of general Ulrich bundles

Our goal in this section is to prove that the Ulrich bundles constructed in the previous section are stable. Following the terminology of [23], we recall that a vector bundle $\mathcal{E}$ on a nonsingular projective variety X is *semistable* if for every nonzero coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ we have the inequality

$$P(\mathcal{F})/\mathrm{rk}(\mathcal{F}) \leq P(\mathcal{E})/\mathrm{rk}(\mathcal{E}),$$

where $P(\mathcal{F})$ and $P(\mathcal{E})$ are the Hilbert polynomials of the sheaves. It is *stable* if one always has strict inequality above. With these definitions, one knows (cf. [23, 4.3.4]) that there is a projective coarse moduli scheme $M^{ss}(P)$ whose closed points are in one-to-one correspondence with certain equivalence classes of semistable sheaves, and there is an open subscheme $M^{s}(P)$ whose points correspond to the isomorphism classes of stable vector bundles.

There is another definition, more adapted to calculations, using the *slope* of $\mathcal{E}$, which is defined as $\deg(c_1(\mathcal{E}))/\mathrm{rk}(\mathcal{E})$, where $c_1(\mathcal{E})$ is the first Chern class. We say that $\mathcal{E}$ is *μ-semistable* if for every subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $0 < \mathrm{rk}\,\mathcal{F} < \mathrm{rk}\,\mathcal{E}$, $\mathrm{slope}(\mathcal{F}) \leq \mathrm{slope}(\mathcal{E})$. We say $\mathcal{E}$ is *μ-stable* if strict inequality always holds. The two definitions are related as follows:

$$\mu\text{-stable} \Rightarrow \text{stable} \Rightarrow \text{semistable} \Rightarrow \mu\text{-semistable}.$$

To begin, we need a summary of Atiyah's classification of vector bundles on an elliptic curve.

Remark 5.1 (Vector bundles on an elliptic curve). Atiyah proved in [2] that the following statements hold for vector bundles on a nonsingular elliptic curve Y.

- For every $r \geq 1$ and $d \in \mathbb{Z}$, there is a 1-dimensional family of indecomposable bundles $\mathcal{E}_{r,d}$ of rank r and degree d parameterized by the points of Y.
- $h^0(\mathcal{E}_{r,d}) = \begin{cases} d & \text{if } d > 0, \\ 0 \text{ or } 1 & \text{if } d = 0, \\ 0 & \text{if } d < 0. \end{cases}$
- $\mathcal{E}_{r,d}$ is semistable for all r, d and it is stable if $(r, d) = 1$.
- Every vector bundle on Y is a direct sum of indecomposable bundles $\mathcal{E}_{r,d}$.

Now let $Y \subset \mathbb{P}^2$ be a nonsingular cubic curve. From the above it is easy to see that the only normalized Ulrich bundles on Y are $\mathcal{E}_{r,d}$ with $d = 3r$ in the case for which $h^0(\mathcal{E}_{r,d}(-1)) = 0$. These correspond to an open subset of the curve Y. They are semistable but not stable and satisfy $h^0(\mathcal{E}_{r,d}) = 3r$. (Note that on a curve the two definitions of stability and semistability coincide).

We come back to the nonsingular cubic surface X.

Proposition 5.2. *Any bundle $\mathcal{E}$ satisfying the conclusion of* 4.4(a) *is μ-semistable.*

Proof. Since $\mathcal{E}$ is normalized and ACM with $h^0(\mathcal{E}) = 3r$ it follows that the general hyperplane section $\mathcal{E}_H$ is also normalized of degree $3r$ and $h^0(\mathcal{E}_H) = 3r$. By Atiyah's classification, $\mathcal{E}_H = \bigoplus_i \mathcal{E}_{r_i,d_i}$. Since $h^0(\mathcal{E}_{r_i,d_i}(-1)) = 0$ it follows that $d_i \le 3r_i$. Summing up, we must have equality for each i. Since the bundles $\mathcal{E}_{r,d}$ on the elliptic curve are all semistable, we find that $\mathcal{E}_H$ is semistable of slope 3.

Now if $\mathcal{F}$ is a coherent subsheaf of $\mathcal{E}$, then $\mathcal{F}_H$ is a coherent subsheaf of $\mathcal{E}_H$, so $\mathcal{F}_H$ has slope ≤ 3. Hence $\mathcal{F}$ has slope ≤ 3 and $\mathcal{E}$ is μ-semistable. □

Theorem 5.3. *If $\mathcal{E}$ is a vector bundle on X satisfying the conclusion of* 4.4(a) *and Z is sufficiently general, then $\mathcal{E}$ is μ-stable.*

Proof. Let $\mathcal{F} \subset \mathcal{E}$ be a coherent subsheaf. Since $\mathcal{E}$ is μ-semistable, slope$(\mathcal{F}) \le$ slope$(\mathcal{E})$ $= 3$. We only need to eliminate the case slope$(\mathcal{F}) = 3$.

By pulling-back torsion if necessary, we may assume that $\mathcal{E}/\mathcal{F}$ is torsion free, in which case $\mathcal{F}$ is locally free. So we only need to show the nonexistence of a semistable locally free proper subsheaf $\mathcal{F}$ of slope 3.

From the inclusion $\mathcal{F} \subset \mathcal{E}$ we find Hom$(\mathcal{F}, \mathcal{E}) \neq 0$, so $H^0(\mathcal{F}^\vee \otimes \mathcal{E}) \neq 0$. Tensoring the sequence (2) of 4.4(a) with $\mathcal{F}^\vee$ we find

$$0 \to (\mathcal{F}^\vee)^{r-1} \to \mathcal{E} \otimes \mathcal{F}^\vee \to \mathcal{J}_Z(r) \otimes \mathcal{F}^\vee \to 0.$$

Since $\mathcal{F}^\vee$ is μ-semistable of negative degree, it has no sections. So we will achieve a contradiction by showing that $h^0(\mathcal{J}_Z(r) \otimes \mathcal{F}^\vee) = 0$ for Z sufficiently general. At this point we need a lemma.

Lemma 5.4. *Let $\mathcal{G}$ be a μ-semistable vector bundle on X of rank s and degree $3sp$ for some $p > 0$. Suppose also that $\mathcal{G}_H$ is semistable for a general plane section. Then $h^0(\mathcal{G}) \le \frac{3}{2}sp(p+1) + s$.*

Proof. From the exact sequence

$$0 \to \mathcal{G}(-1) \to \mathcal{G} \to \mathcal{G}_H \to 0$$

we have $h^0(\mathcal{G}) \le h^0(\mathcal{G}(-1)) + h^0(\mathcal{G}_H)$. Applying the same to $\mathcal{G}(-1), \mathcal{G}(-2), \dots$ and summing we find

$$h^0(\mathcal{G}) \le \sum_{i=0}^{\infty} h^0(\mathcal{G}_H(-i)).$$

Note that $\deg(\mathcal{G}(-i)) = 3s(p-i)$. Since $\mathcal{G}_H(-i)$ is semistable, all its indecomposable summands have the same slope. Then using Atiyah's results we obtain

$$\begin{aligned} h^0(\mathcal{G}_H(-i)) &= 3s(p-i) && \text{for } 3s(p-i) > 0,\\ h^0(\mathcal{G}_H(-i)) &\leq \mathrm{rk} = s && \text{for } 3s(p-i) = 0,\\ h^0(\mathcal{G}_H(-i)) &= 0 && \text{for } 3s(p-i) < 0. \end{aligned}$$

Hence $h^0(\mathcal{G}) \leq 3s(1+2+\cdots+p)+s = \frac{3}{2}sp(p+1)+s$. □

Proof of Theorem 5.3, continued.. We apply the lemma to $\mathcal{F}^\vee(r)$, which has rank $s < r$ and degree $-3s+3rs = 3s(r-1)$. Hence $h^0(\mathcal{F}^\vee(r)) \leq \frac{3}{2}sr(r-1)+s$.

If $H^0(\mathcal{F}^\vee(r)) = 0$, we are done.

If $H^0(\mathcal{F}^\vee(r)) \neq 0$, suppose first that $H^0(\mathcal{F}^\vee(r))$ generates a subsheaf of rank s of $\mathcal{F}^\vee(r)$. Then each general point P will impose s conditions on a section of $\mathcal{F}^\vee(r)$ to lie in $\mathcal{J}_P \otimes \mathcal{F}^\vee(r)$. Since Z consists of $n = \frac{1}{2}(3r^2-r)$ general points, we have $\frac{1}{2}s(3r^2-r)$ potential conditions. This number is greater than $h^0(\mathcal{F}^\vee(r))$, so we conclude that $h^0(\mathcal{J}_Z \otimes \mathcal{F}^\vee(r)) = 0$ as required.

Suppose on the other hand that the sections of $\mathcal{F}^\vee(r)$ generate a subsheaf $\mathcal{G}$ of rank $t < s$. Then each general point P will impose only t conditions on a section to lie in $\mathcal{J}_P \otimes \mathcal{F}^\vee(r)$. But $\mathcal{G}$ and $\mathcal{G}_H$ will also be μ-semistable of slope ≤ 3, so the arguments of the lemma will give a similar bound with t in place of s, and we conclude as before. □

Notice that usually the existence of ACM bundles of high rank is only proved using extensions of lower rank ACM bundles. In the following result we show that the bundles constructed in Theorem 4.4 (or more generally, μ-stable Ulrich bundles on X) are not extensions of ACM bundles of lower rank.

Proposition 5.5. *Let $\mathcal{E}$ be a μ-stable Ulrich bundle of rank r with $c_1 = rH$. Then $\mathcal{E}$ is normalized and it is not an extension of lower rank ACM bundles. In particular, the bundles constructed in* 4.4(b) *are indecomposable and are not extensions of lower rank ACM bundles.*

Proof. We first prove that $\mathcal{E}$ is normalized.

As $\mathcal{E}$ is Ulrich and μ-stable, it is easy to see that it cannot be an extension of Ulrich bundles of lower rank. Therefore by Proposition 3.7(a), $\mathcal{E}$ has a linear resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(a-1)^{3r} \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}^n}^{3r}(a) \to \mathcal{E} \to 0.$$

Then $\chi(\mathcal{E}) = 3r\left(\binom{a+3}{3} - \binom{a+2}{3}\right)$ and $\chi(\mathcal{E}(-1)) = 3r\left(\binom{a+2}{3} - \binom{a+1}{3}\right)$. The difference between these two values is $\chi(\mathcal{E}) - \chi(\mathcal{E}(-1)) = 3r(a+1)$. On the other hand, the exact sequence

$$0 \to \mathcal{E}(-1) \to \mathcal{E} \to \mathcal{E}_H \to 0$$

tells us that $\chi(\mathcal{E}) - \chi(\mathcal{E}(-1)) = \chi(\mathcal{E}_H)$ and this last term is equal to $\deg(\mathcal{E})$ by Riemann–Roch on the elliptic curve. We conclude that $a = 0$ and $\mathcal{E}$ is normalized.

Now suppose there exists an extension

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

with $\mathcal{E}_i$ an ACM bundle of rank r_i, $r = r_1 + r_2$. As $\mathcal{E}$ is μ-stable, the slope of $\mathcal{E}_1$ is < 3 so $\deg(\mathcal{E}_1) < 3r_1$. On the other hand, by Remark 4.1 we have $\deg(\mathcal{E}_2) \leq h^0(\mathcal{E}_2)$ and by Corollary 3.5, $h^0(\mathcal{E}_2) \leq 3r_2$ (this last bound still holds in case $h^0(\mathcal{E}_2) = 0$). Therefore $3r = 3r_1 + 3r_2 > \deg(\mathcal{E}_1) + \deg(\mathcal{E}_2) = \deg(\mathcal{E})$, which is precisely $3r$, and we obtain a contradiction. □

Lemma 5.6. *With $\mathcal{E}$ as in* 4.4(a) *we have* $\chi(\mathcal{E} \otimes \mathcal{E}^\vee) = -r^2$.

Proof. Use additivity of χ on the sequences

$$0 \to (\mathcal{E}^\vee)^{r-1} \to \mathcal{E} \otimes \mathcal{E}^\vee \to \mathcal{J}_Z(r) \otimes \mathcal{E}^\vee \to 0,$$
$$0 \to \mathcal{O}_X(-r) \to \mathcal{E}^\vee \to \mathcal{O}_X^{r-1} \to \mathrm{Ext}^1(\mathcal{J}_Z(r), \mathcal{O}_X) \to 0,$$
$$0 \to \mathcal{J}_Z(r) \otimes \mathcal{E}^\vee \to \mathcal{E}^\vee(r) \to \mathcal{E}^\vee(r) \otimes \mathcal{O}_Z \to 0,$$
$$0 \to \mathcal{O}_X \to \mathcal{E}^\vee(r) \to \mathcal{O}_X^{r-1}(r) \to \mathrm{Ext}^1(\mathcal{J}_Z(r), \mathcal{O}_X(r)) \to 0$$

and the fact that the last sheaf in the second and fourth sequences has length n and the last sheaf in the third sequence has length rn. The calculations are left to the reader. □

Summing up we obtain the main result of this paper.

Theorem 5.7. *Let X be a nonsingular cubic surface in $\mathbb{P}^3$ over an algebraically closed field k. Let $r \geq 2$. The normalized orientable Ulrich bundles of rank r generated by global sections are all μ-semistable. Among these, the μ-stable Ulrich bundles correspond to a nonempty open subset of dimension $r^2 + 1$ of the moduli space $M^s(r; c_1, c_2)$ of stable vector bundles on X with Chern classes $c_1 = rH$ and $c_2 = \frac{1}{2}(3r^2 - r)$.*

Proof. The first statement follows from 5.2. If $\mathcal{E}$ is μ-stable then it is stable and we have $h^0(\mathcal{E} \otimes \mathcal{E}^\vee) = 1$. By duality $h^2(\mathcal{E} \otimes \mathcal{E}^\vee) = h^0(\mathcal{E} \otimes \mathcal{E}^\vee(-1)) = 0$. Hence from the previous lemma we find $h^1(\mathcal{E} \otimes \mathcal{E}^\vee) = r^2 + 1$.

Since $h^2(\mathcal{E} \otimes \mathcal{E}^\vee) = 0$, there are no obstructions, so at the point corresponding to $\mathcal{E}$, the moduli space is smooth of dimension $h^1(\mathcal{E} \otimes \mathcal{E}^\vee)$ (cf. [23, 4.5.2]).

It remains to show that the μ-stable Ulrich bundles with the given Chern classes form an open subset of the moduli space. This open set will be nonempty by 4.4 and 5.3.

In any flat family of vector bundles, the condition of being ACM is an open condition by the semicontinuity theorem [16, III, 12.8]. The condition of being Ulrich is not necessarily open, but the condition of being μ-stable is open and here we prove that μ-stable ACM bundles with the given Chern classes are Ulrich.

Indeed, let $\mathcal{E}$ be an ACM μ-stable bundle on X with $c_1 = rH$, $c_2 = \frac{1}{2}(3r^2 - r)$. As $\mathcal{E}$ is μ-stable of slope 3, $\mathcal{E}(-1)$ cannot have sections. $\mathcal{E}^\vee$ is also μ-stable of slope -3, so we have $h^0(\mathcal{E}^\vee(-1)) = 0$. By Serre duality we obtain $h^2(\mathcal{E}) = 0$ and then by the Riemann–Roch theorem on X it follows that $h^0(\mathcal{E}) = 3r$. Thus $\mathcal{E}$ is normalized with $h^0(\mathcal{E}) = 3r$ and therefore it is Ulrich by Lemma 4.2. □

6. Filtrations of general ACM bundles

Although we have seen in Proposition 5.5 that the bundles $\mathcal{E}$ constructed in Theorem 4.4 are not extensions of lower rank ACM bundles, we shall prove that for a suitable dissocié sheaf $\mathcal{L}$, the bundle $\mathcal{E} \oplus \mathcal{L}$ is indeed an extension of ACM bundles of rank 1.

Our main technique in this section is Gorenstein liaison theory or, more precisely, strict G-links (see Definition 6.1 below). In this case the links are performed by a divisor of type $mH_Y - K_Y$ on an ACM scheme Y that is a divisor on a scheme $X \subseteq \mathbb{P}^n$. To put this in a more general context, recall that if Y is an ACM scheme satisfying G_0, we can define the *anticanonical divisor* $M = M_Y$, given by an embedding of ω_Y as a fractional ideal in the sheaf of total quotient rings $\mathcal{K}_Y$, even if Y does not have a well-defined canonical divisor [19, 2.7]. Recall also that if Y is an ACM scheme in $\mathbb{P}^n$ satisfying G_0, and if G is an effective divisor on Y, linearly equivalent to $mH + M_Y$ for some $m \in \mathbb{Z}$, then G is AG in $\mathbb{P}^n$ [19, 3.4], [24, 5.2, 5.4].

Definition 6.1. Let $Y \subset \mathbb{P}^n$ be an arithmetically Cohen–Macaulay scheme that is Gorenstein in codimension 0. We say that two equidimensional subschemes Z_1, Z_2 of codimension 1 of Y without embedded components are *strictly G-linked* by a subscheme $G \subset Y$ if G contains Z_1 and Z_2, $\mathcal{J}_{Z_i,G} \cong \mathrm{Hom}(\mathcal{O}_{Z_j}, \mathcal{O}_G)$ for $i, j = 1, 2, i \neq j$, and there is an $m \in \mathbb{Z}$ such that G is linearly equivalent to $mH_Y + M_Y$ where M_Y is the anticanonical divisor [19].

In our case, Y will be an ACM curve contained in the nonsingular cubic surface X and therefore it will have a well-defined canonical divisor K_Y so that $M_Y = -K_Y$. Divisors linearly equivalent to $mH - K_Y$ are AG schemes [19, 3.3].

To associate ACM bundles to sets of points we work with $\mathcal{N}$-type resolutions (or so-called Bourbaki sequences, cf. [5]). We recall the definition here.

Definition 6.2. Let $X \subset \mathbb{P}^n$ be an equidimensional scheme and $Z \subset X$ be a codimension 2 subscheme without embedded components. An *$\mathcal{N}$-type resolution* of Z is an exact sequence

$$0 \to \mathcal{L} \to \mathcal{N} \to \mathcal{J}_{Z,X} \to 0$$

with $\mathcal{L}$ dissocié and $\mathcal{N}$ a coherent sheaf satisfying $H^1_*(\mathcal{N}^\vee) = 0$ and $\mathrm{Ext}^1(\mathcal{N}, \mathcal{O}_X) = 0$.

When X satisfies Serre's condition S_2 and $H^1_*(\mathcal{O}_X) = 0$, then an $\mathcal{N}$-type resolution of Z exists ([15, 2.12]). An $\mathcal{N}$-type resolution is not unique but it is well known that any two $\mathcal{N}$-type resolutions of the same subscheme are *stably equivalent* (see [15, 1.10]). In other words, if $\mathcal{N}$ and $\mathcal{N}'$ are two sheaves appearing in the middle of an $\mathcal{N}$-type resolution of a subscheme Z, then there exist dissocié sheaves $\mathcal{L}_1$, $\mathcal{L}_2$ and an integer a such that

$$\mathcal{N} \oplus \mathcal{L}_1 \cong \mathcal{N}'(a) \oplus \mathcal{L}_2.$$

See 6.7 and 6.8 below for examples of $\mathcal{N}$-type resolutions.

When X is an AG scheme and we have an $\mathcal{N}$-type resolution as above, then $\mathcal{N}$ is an ACM sheaf if and only if Z is an ACM scheme. In particular, if Z is a 0-dimensional scheme, then any sheaf appearing in an $\mathcal{N}$-type resolution of Z will be ACM. We already saw examples of $\mathcal{N}$-type resolutions in Theorem 4.4.

We study when an AG scheme G of codimension 2 in an AG scheme $X \subset \mathbb{P}^n$ occurs as a divisor $mH_Y - K_Y$ on some ACM divisor $Y \subset X$.

Proposition 6.3. *Let X be an AG scheme with $\omega_X \cong \mathcal{O}_X(\ell)$, and let G be an AG subscheme of codimension 2 in X. Then the following conditions are equivalent:*

(i) *There is an ACM divisor $Y \subseteq X$ satisfying G_0 and containing G and an integer m so that $G \sim mH + M_Y$ on Y, where M_Y is the anticanonical divisor.*
(ii) *G has an $\mathcal{N}$-type resolution with $\mathcal{N}$ an ACM sheaf of rank 2 that is an extension of two rank 1 ACM sheaves on X. In this case we have two exact sequences:*

$$0 \to \mathcal{O}_X(\ell - m) \to \mathcal{N} \to \mathcal{J}_{G,X} \to 0,$$

$$0 \to \mathcal{J}_Y \to \mathcal{N} \to \mathcal{O}_X(Y + \ell - m) \to 0.$$

Proof. (i)⇒(ii). Since $G \sim mH + M_Y$ on Y, we have $\mathcal{J}_{G,Y} \cong \omega_Y(-m)$. On the other hand, comparing with the ideal sheaf $\mathcal{J}_G$ of G on X, we have an exact sequence

$$0 \to \mathcal{J}_Y \to \mathcal{J}_G \to \mathcal{J}_{G,Y} \to 0. \tag{4}$$

We combine this with the natural exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}(Y) \to \omega_Y \otimes \omega_X^\vee \to 0 \tag{5}$$

of [17, 2.10]. Since $\omega_X \cong \mathcal{O}_X(\ell)$, twisting by $a = \ell - m$ we get

$$0 \to \mathcal{O}_X(a) \to \mathcal{O}(Y + a) \to \omega_Y(-m) \to 0.$$

Since $\omega_Y(-m) \cong \mathcal{J}_{G,Y}$, we can do the fibered sum construction with the sequence (4) above and obtain two short exact sequences

$$0 \to \mathcal{O}_X(a) \to \mathcal{N} \to \mathcal{J}_G \to 0, \tag{6}$$

$$0 \to \mathcal{J}_Y \to \mathcal{N} \to \mathcal{O}(Y + a) \to 0.$$

The first is an $\mathcal{N}$-type resolution of $\mathcal{J}_G$, and the second shows that $\mathcal{N}$ is an extension of two rank 1 ACM sheaves on X.

(ii)⇒(i). Conversely, suppose given an $\mathcal{N}$-type resolution of the form (6) above, and suppose that $\mathcal{N}$ is an extension

$$0 \to \mathcal{L} \to \mathcal{N} \to \mathcal{M} \to 0$$

where $\mathcal{L}$, $\mathcal{M}$ are rank 1 ACM sheaves on X. The composed map $\mathcal{L} \to \mathcal{I}_G \to \mathcal{O}_X$ shows that $\mathcal{L}$ is isomorphic to the ideal sheaf $\mathcal{I}_Y$ of an ACM divisor Y containing G. Then by comparing Chern classes we find $\mathcal{M} \cong \mathcal{O}(Y + a)$. Dividing the sequence (6) by $\mathcal{I}_Y$ in the second and third place we obtain

$$0 \to \mathcal{O}_X(a) \to \mathcal{O}(Y + a) \to \mathcal{J}_{G,Y} \to 0.$$

Comparing with the sequence (5) above, we conclude that $\mathcal{I}_{G,Y} \cong \omega_Y(a - \ell) = \omega_Y(-m)$. Therefore $G \sim mH + M_Y$ on Y. Note from the isomorphism $\mathcal{I}_{G,Y} \cong \omega_Y(a - \ell)$ that ω_Y is locally free at the generic points of Y, so Y satisfies G_0. □

From now on X will be a nonsingular cubic surface, C will denote any nonsingular conic on X, Γ any twisted cubic on X, and L any of the 27 lines in X. The curves C, Γ and L are arithmetically Cohen–Macaulay curves and, according to the proof of [18, 2.4], any arithmetically Cohen–Macaulay curve on X is linearly equivalent to $C + aH_X$, $\Gamma + aH$, $L + aH$ or aH, for some $a \in \mathbb{N}$. The sheaves $\mathcal{J}_{C,X}$, $\mathcal{J}_{\Gamma,X}$ and $\mathcal{J}_{L,X}$ are ACM bundles and their minimal free $\mathcal{O}_{\mathbb{P}^3}$-resolutions are:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3)^2 \xrightarrow{\varphi_1} \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{J}_{C,X} \to 0,$$
$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3)^3 \xrightarrow{\varphi_2} \mathcal{O}_{\mathbb{P}^3}(-2)^3 \to \mathcal{J}_{\Gamma,X} \to 0,$$
$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-3) \xrightarrow{\varphi_3} \mathcal{O}_{\mathbb{P}^3}(-1)^2 \to \mathcal{J}_{L,X} \to 0.$$

Note that $\mathcal{O}(C) \cong \mathcal{J}_L(1)$, $\mathcal{O}(L) \cong \mathcal{J}_C(1)$ if C and L are contained in the same plane, and $\mathcal{O}(\Gamma) \cong \mathcal{J}_{\Gamma'}(2)$ if $\Gamma + \Gamma' = 2H$.

Recall the definition 2.2 of the syzygy sheaf $\mathcal{F}^\sigma$ of a sheaf $\mathcal{F}$.

Proposition 6.4. *Let $X \subset \mathbb{P}^3$ be a nonsingular cubic surface.*

(a) *If L is a line on X, then $\mathcal{J}^\sigma_{L,X} \cong \mathcal{J}_C(-1)$ for a conic C in a plane with L.*
(b) *Conversely, $\mathcal{J}^\sigma_C \sim \mathcal{J}_L(-2)$.*
(c) *If Γ is a twisted cubic on X, then $\mathcal{J}^\sigma_{\Gamma,X}$ is a rank 2 ACM sheaf that is an extension of two ACM line bundles. Indeed, there is an exact sequence*

$$0 \to \mathcal{O}_X(L) \to \mathcal{J}^\sigma_{\Gamma,X}(3) \to \mathcal{O}_X(C) \to 0$$

where L and C are a line and a conic such that $\Gamma \sim C + L$.

Proof. (a) and (b) are elementary. For (c) we proceed as follows. Let L, C be a line and a conic such that $L + C \cong \Gamma$. Then one finds $C.L = 1$, and one can compute $\mathrm{Ext}^1(\mathcal{O}_X(C), \mathcal{O}_X(L)) \cong H^1(\mathcal{O}_X(L - C)) \cong H^1(\mathcal{O}_X(L + L' - H))$ where $L' = H - C$. By duality this is dual to $H^1(\mathcal{J}_{L+L',X})$. But L and L' are two skew lines, so $H^1 \neq 0$ and there exists a nonsplit extension

$$0 \to \mathcal{O}_X(L) \to \mathcal{E} \to \mathcal{O}_X(C) \to 0.$$

From this we see that $\mathcal{E}$ has three global sections, and we can write a diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \to & \mathcal{F} & \to & \mathcal{J}_{C,X} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_X & \to & \mathcal{O}_X^3 & \to & \mathcal{O}_X^2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_X(L) & \to & \mathcal{E} & \to & \mathcal{O}_X(C) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathcal{O}_L(-1) & \to & \mathcal{G} & \to & 0 & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

where $\mathcal{F}$ and $\mathcal{G}$ are the kernel and the cokernel of the map $\mathcal{O}_X^3 \to \mathcal{E}$. By the snake lemma there is a map $\delta : \mathcal{J}_{C,X} \to \mathcal{O}_L(-1)$ joining the top and bottom rows into a long exact sequence. If $\delta = 0$ then the extension defining $\mathcal{E}$ splits, contrary to hypothesis. Therefore $\delta \neq 0$ and its image must be $\mathcal{J}_{C,X} \otimes \mathcal{O}_L \cong \mathcal{O}_L(-1)$ since $L.C = 1$. Thus $\mathcal{G} = 0$ and we find that $\mathcal{E}$ is generated by global sections, $\mathcal{F} \cong \mathcal{J}_{\Gamma,X}$, and so $\mathcal{E}^\sigma \cong \mathcal{J}_{\Gamma,X}$. This is not quite what we want. But $\mathcal{E}$ is ACM and has no free summands because $\mathrm{Ext}^1(\mathcal{O}_X(a), \mathcal{J}_{\Gamma,X}) = 0$ for all $a \in \mathbb{Z}$, so Lemma 2.5(vi) applies and we obtain $\mathcal{J}_{\Gamma,X}^\sigma \cong \mathcal{E}(-3)$. □

We recall the following definition of [9].

Definition 6.5. An ACM sheaf $\mathcal{E}$ on a normal ACM scheme X is *layered* if there exists a filtration

$$0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_r = \mathcal{E}$$

whose quotients $\mathcal{E}_i/\mathcal{E}_{i-1}$ are rank 1 ACM sheaves on X for $i = 1, \ldots, r$.

Proposition 6.6. *Let $X \subset \mathbb{P}^n$ be a normal arithmetically Gorenstein scheme. Then the following holds:*

(a) *The dual of a layered sheaf is layered.*
(b) *Any extension of layered sheaves is layered.*
(c) *On a nonsingular cubic surface $X \subset \mathbb{P}^3$, a syzygy of a layered ACM sheaf is stably equivalent to a layered sheaf.*

Proof. (a) and (b) are obvious. For (c) we note that every rank 1 ACM sheaf on X is a twist of $\mathcal{J}_{L,X}$, $\mathcal{J}_{C,X}$, $\mathcal{J}_{\Gamma,X}$, and the syzygy of any of these is layered. The condition then follows from [8, 4.1(d)]. □

Example 6.7. Take two points Q, R on the cubic surface X. They have an $\mathcal{N}$-type resolution

$$0 \to \mathcal{O}_X(-1) \to \mathcal{N}_{Q+R} \to \mathcal{J}_{Q+R} \to 0,$$

where $\mathcal{N}_{Q+R}$ is an ACM bundle by Serre's correspondence. It is easy to prove that $\mathcal{N}_{Q+R}$ is indecomposable. Indeed, if $\mathcal{N}_{Q+R}$ decomposes as $\mathcal{O}(C_1) \oplus \mathcal{O}(C_2)$, then C_1 is an ACM divisor and $C_2 \sim -C_1 - H$. Composing with the map $\mathcal{N}_{Q+R} \to \mathcal{J}_{Q+R} \hookrightarrow \mathcal{O}_X$ we see that $-C_1$ and $C_1 + H$ must be effective and both divisors contain $Q + R$. Numerically, this can only happen if $-C_1$ is linearly equivalent to a line or a conic in X. But as Q and R are general points in X, $Q \cup R$ cannot be contained in a line or a conic.

However, the following argument proves that $\mathcal{N}_{Q+R}$ is layered. Two general points Q, R are contained in a twisted cubic curve Γ, and then $Q + R \sim -K_\Gamma$. Therefore by 6.3, $\mathcal{N}_{Q+R}$ is an extension

$$0 \to \mathcal{O}_X(-\Gamma) \to \mathcal{N}_{Q+R} \to \mathcal{O}_X(\Gamma - 1) \to 0. \tag{7}$$

As $H - \Gamma$ is not effective, this extension does not split.

Note that $\mathcal{N}_{Q+R}$ has the following minimal free resolution over $\mathcal{O}_X$ of period 2 (see Theorem 2.4):

$$\cdots \to \mathcal{O}_X(-2) \oplus \mathcal{O}_X(-3)^3 \to \mathcal{O}_X(-1)^3 \oplus \mathcal{O}_X(-2) \to \mathcal{N}_{Q+R} \to 0. \tag{8}$$

Example 6.8. Let P be a point on a nonsingular cubic surface X in $\mathbb{P}^3$. Then P has an $\mathcal{N}$-type resolution on X,

$$0 \to \mathcal{O}_X \to \mathcal{N}_P \to \mathcal{J}_P \to 0.$$

One can show that for a general point $P \in X$, the sheaf $\mathcal{N}_P$ is not an extension of ACM line bundles. Indeed, as $c_1(\mathcal{N}_P) = 0$, such an extension would be of the form $0 \to \mathcal{O}(-C) \to \mathcal{N}_P \to \mathcal{O}(C) \to 0$. Composing with the map $\mathcal{N}_P \to \mathcal{J}_P \hookrightarrow \mathcal{O}_X$ we see that C must be an effective ACM divisor containing P. As $h^0(\mathcal{N}_P) = 1$, we have $h^0(\mathcal{O}_X(C)) = 1$ and $h^0(\mathcal{O}_X(C-H)) = 0$, so C must be linearly equivalent to a line (lines are the only ACM curves C on X for which $h^0(\mathcal{O}_X(C)) = 1$). But a general point P is not contained in a line in X, so such an extension cannot exist. This argument also proves that $\mathcal{N}_P$ is indecomposable. Therefore $\mathcal{N}_P$ is an indecomposable rank 2 ACM sheaf that is not layered.

Note that $\mathcal{N}_P$ has the following 2-periodic minimal free resolution as an $\mathcal{O}_X$-module:

$$\ldots \to \mathcal{O}_X(-2)^3 \oplus \mathcal{O}_X(-3) \to \mathcal{O}_X \oplus \mathcal{O}_X(-1)^3 \to \mathcal{N}_P \to 0. \tag{9}$$

Let us look at the syzygy of $\mathcal{N}_P$. We consider a complete intersection of two planes containing P meeting X properly. Then P is linked to two points $Q \cup R$ by a complete intersection Y which has the following resolution in X

$$0 \to \mathcal{O}_X(-2) \to \mathcal{O}_X(-1) \oplus \mathcal{O}_X(-1) \to \mathcal{J}_Y \to 0.$$

By [8, 3.2], $Q \cup R$ has an $\mathcal{N}$-type resolution as follows:

$$0 \to \mathcal{M} \to \mathcal{G} \to \mathcal{J}_{Q+R,X} \to 0$$

where $\mathcal{G}$ is an extension $0 \to \mathcal{O}_X(1)^2 \oplus \mathcal{O}_X \to \mathcal{G} \to \mathcal{N}_P^{\sigma\vee} \to 0$ and $\mathcal{M}$ is a dissocié sheaf. As ${\mathcal{N}_P}^{\sigma\vee}$ is ACM, this extension splits so $\mathcal{G} \cong \mathcal{O}_X(1)^2 \oplus \mathcal{O}_X \oplus \mathcal{N}_P^{\sigma\vee}$. On the other hand, $Q \cup R$ has an $\mathcal{N}$-type resolution as in Example 6.7, and as any two $\mathcal{N}$-type resolutions are stably equivalent, we have

$$\mathcal{O}_X(1)^2 \oplus \mathcal{O}_X \oplus \mathcal{N}_P^{\sigma\vee} \oplus \mathcal{L}_1 \cong \mathcal{N}_{Q+R}(a) \oplus \mathcal{L}_2$$

for some twist $a \in \mathbb{Z}$ and some dissocié sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$. But $\mathcal{N}_P^{\sigma\vee}$ and $\mathcal{N}_{Q+R}(a)$ are indecomposable, so $\mathcal{N}_P^{\sigma\vee} \cong \mathcal{N}_{Q+R}(a)$. Looking at the resolutions (8) and (9) we find $\mathcal{N}_P^{\sigma\vee} \cong \mathcal{N}_{Q+R}(2)$. As the first Chern class of $\mathcal{N}_{Q+R}$ is $-H$ we obtain $\mathcal{N}_P^{\sigma} \cong {\mathcal{N}_{Q+R}}^{\vee}(-2) \cong \mathcal{N}_{Q+R}(-1)$.

Now by 2.5(vi), $\mathcal{N}_P \cong \mathcal{N}_P^{\sigma\sigma}(3)$, which in turn is isomorphic to $\mathcal{N}_{Q+R}^{\sigma}(2)$. As $\mathcal{N}_{Q+R}$ is layered, it follows from 6.6 that $\mathcal{N}_P$ is stably equivalent to a layered sheaf even though it is not layered itself.

Theorem 6.9. *Let $\mathcal{N}$ be the vector bundle in an $\mathcal{N}$-type resolution of a set of general points $Z \subset X$. Then $\mathcal{N}$ is stably equivalent to a layered sheaf.*

Proof. In [18] the second author proved that any set of general points Z can be strictly G-linked to a general point P. We do induction on the number t of links needed.

If $t = 0$, we have one general point P and an $\mathcal{N}$-type resolution

$$0 \to \mathcal{O}_X \to \mathcal{N}_P \to \mathcal{J}_{P,X} \to 0.$$

Any bundle $\mathcal{N}$ corresponding to another $\mathcal{N}$-type resolution of P is stably equivalent to $\mathcal{N}_P$. We have seen in Example 6.8 that $\mathcal{N}_P$ is the syzygy of a rank 2 layered sheaf, so this case is finished.

If $t \geq 1$, there is a strict G-link from Z to Z' such that the induction hypothesis applies to the sheaf $\mathcal{N}'$ belonging to an $\mathcal{N}$-type resolution of Z',

$$0 \to \mathcal{L}' \to \mathcal{N}' \to \mathcal{J}_{Z',X} \to 0.$$

The strict G-link is performed by a Gorenstein scheme W having an $\mathcal{N}$-type resolution

$$0 \to \mathcal{O}_X(-a) \to \mathcal{E} \to \mathcal{J}_{W,X} \to 0$$

where $\mathcal{E}$ is an extension

$$0 \to \mathcal{O}_X(-Y) \to \mathcal{E} \to \mathcal{O}_X(Y - aH) \to 0 \tag{10}$$

for a certain ACM curve $Y \subset X$ and $a \in \mathbb{Z}$ (see 6.3). Then by [8, Proposition 3.2] we know that there is an $\mathcal{N}$-type resolution of Z

$$0 \to \mathcal{L} \to \mathcal{G} \to \mathcal{J}_{Z,X} \to 0,$$

such that $\mathcal{G}$ is an extension

$$0 \to \mathcal{L}'^{\vee} \oplus \mathcal{E}^{\vee} \to \mathcal{G} \to \mathcal{N}'^{\sigma\vee} \to 0.$$

Any other sheaf $\mathcal{N}$ appearing in an $\mathcal{N}$-type resolution

$$0 \to \mathcal{P} \to \mathcal{N} \to \mathcal{J}_{Z,X} \to 0$$

is stably equivalent to $\mathcal{G}$, so it is enough to prove that $\mathcal{G}$ is stably equivalent to a layered sheaf. By induction hypothesis there is a dissocié sheaf $\mathcal{M}$ such that the bundle $\mathcal{N}' \oplus \mathcal{M}$ is layered. By Proposition 6.6(c), we find that $(\mathcal{N}' \oplus \mathcal{M})^{\sigma}$, which is equal to $\mathcal{N}'^{\sigma}$, is also stably equivalent to a layered sheaf. Therefore $\mathcal{N}'^{\sigma\vee}$ is also stably equivalent to a layered sheaf and so is $\mathcal{G}$. □

Corollary 6.10. *Any Ulrich bundle corresponding to a set of general points is stably equivalent to a layered sheaf.*

Let $\mathcal{C}$ be the category of ACM bundles on X and let $G(\mathcal{C})$ be the corresponding Grothendieck group (i.e. we say that in $G(\mathcal{C})$, $\mathcal{E} = \mathcal{E}' + \mathcal{E}''$ whenever there is an exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$). We regard $G(\mathcal{C})$ as a $\mathbb{Z}[h]$-module with the operation $h \cdot \mathcal{E} = \mathcal{E}(1)$ and define the quotient group $G' = G(\mathcal{C})/(1-h)G(\mathcal{C})$. In other words, in G' we identify a sheaf with all of its twists.

Corollary 6.11. *Let $\mathcal{N}$ be the vector bundle in an $\mathcal{N}$-type resolution of a set of general points $Z \subset X$. Then, in G', $\mathcal{N}$ belongs to the subgroup generated by rank one ACM bundles and it is equivalent to $r\mathcal{O}_X$, $r = \mathrm{rk}(\mathcal{N})$.*

Proof. It can be seen in the proof of Theorem 6.9 that, in G', $\mathcal{N}$ is equivalent to

$$\sum_i (\mathcal{O}(L_i) + \mathcal{O}(-L_i)) + \sum_j (\mathcal{O}(C_j) + \mathcal{O}(-C_j)) + \sum_k (\mathcal{O}(\Gamma_k) + \mathcal{O}(-\Gamma_k))$$

where L_i are lines, C_j are conics and Γ_k are twisted cubics in X.

As the syzygy of $\mathcal{O}_X(-L_i)$ (resp. $\mathcal{O}_X(-C_j)$) is a twist of $\mathcal{O}_X(L_i)$ (resp. $\mathcal{O}_X(C_j)$), the first two summands are equivalent to sums of $\mathcal{O}_X$ in G'.

Let Γ be a twisted cubic in X. We need to prove that $\mathcal{O}_X(-\Gamma) + \mathcal{O}_X(\Gamma) = 2\mathcal{O}_X$ in G'. Take a line L meeting Γ in one point. Then there are two conics D_1, D_2 such that $L + \Gamma \sim D_1 + D_2$ and it is easy to prove that $(\Gamma - D_1).(\Gamma - D_2) = 0$. Therefore there is an exact sequence

$$0 \to \mathcal{O}_X(\Gamma - L) \to \mathcal{O}_X(\Gamma - D_1) \oplus \mathcal{O}_X(\Gamma - D_2) \to \mathcal{O}_X \to 0$$

and tensoring by $\mathcal{O}_X(L)$ we obtain the exact sequence

$$0 \to \mathcal{O}_X(\Gamma) \to \mathcal{O}_X(D_2) \oplus \mathcal{O}_X(D_1) \to \mathcal{O}_X(L) \to 0.$$

Therefore, in G' we have $\mathcal{O}_X(-\Gamma) + \mathcal{O}_X(\Gamma) = \mathcal{O}_X(-D_1) + \mathcal{O}_X(-D_2) - \mathcal{O}_X(-L) + \mathcal{O}_X(D_1) + \mathcal{O}_X(D_2) - \mathcal{O}_X(L)$. Moreover as the syzygy of $\mathcal{O}_X(-D_i)$ (resp. $\mathcal{O}_X(L)$) is a twist of $\mathcal{O}_X(D_i)$ (resp. of $\mathcal{O}_X(L)$), we obtain $\mathcal{O}_X(-\Gamma) + \mathcal{O}_X(\Gamma) = 2\mathcal{O}_X$. □

Added in proof. Since the proof of Theorem 5.3 depends on general position arguments that are hard to justify, we give another proof of this result in our forthcoming paper, ”Stable Ulrich bundles” (arXiv:1102.0878), in which we also construct nonorientable stable Ulrich bundles on the cubic surface, and stable Ulrich bundles of all even ranks on a general cubic threefold in $\mathbb{P}^4$.

Acknowledgments. Research of the first author was partially supported by 2009 SGR 1284 and MTM2009-14163-C02-02.

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