

SINGULAR FIBERS OF THE GELFAND–CETLIN SYSTEM ON $\mathfrak{u}(n)^*$

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ABSTRACT. In this paper, we show that every singular fiber of the Gelfand–Cetlin system on coadjoint orbits of unitary groups is a smooth isotropic submanifold which is diffeomorphic to a 2-stage quotient of a compact Lie group by free actions of two other compact Lie groups. In many cases, these singular fibers can be shown to be homogeneous spaces or even diffeomorphic to compact Lie groups. We also give a combinatorial formula for computing the dimensions of all singular fibers, and give a detailed description of these singular fibers in many cases, including the so-called (multi-)diamond singularities. These (multi-)diamond singular fibers are degenerate for the Gelfand–Cetlin system, but they are Lagrangian submanifolds diffeomorphic to direct products of special unitary groups and tori. Our methods of study are based on different ideas involving complex ellipsoids, Lie groupoids, and also general ideas coming from the theory of singularities of integrable Hamiltonian systems.

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1. INTRODUCTION

The Gelfand–Cetlin system is a famous integrable Hamiltonian system on the coadjoint orbits of unitary groups, which was found and studied by Guillemin and Sternberg in early 1980s [GS83a, GS83b], using the so-called Thimm’s method of collective motions [Thi81], and related to the classical work of Gelfand and Cetlin in representation theory [GC50]. A result of Alekseev and Meinrenken [AM07] says that this

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system is also equivalent to an integrable system found by Flaschka and Ratiu [FR96], via the so-called Ginzburg-Weinstein transformation. Compared to many other integrable systems, especially those arising in classical mechanics and physics (see, e.g., [BF04, Zun96]), the Gelfand-Cetlin has some very special topological and geometric properties:

- Its base space (i.e. the space of connected fibers of the momentum map) is (affinely equivalent to) a convex polytope, similar to the case of toric systems (whose base spaces are the so-called Delzant polytopes [Del88]), even though the system is *not* toric.
- In fact, the Gelfand-Cetlin momentum map of the system (which consists of eigenvalue functions of a chain of matrices) generates a toric action, but only on a dense open set of the symplectic manifold in question. This momentum map is *not* globally smooth, though it can be changed into a smooth momentum map with the same fibers (i.e. preimages) by taking the symmetric functions of the eigenvalue functions. However, this smooth momentum map does not generate a torus action.
- Unlike the toric case, where the singularities are all elliptic nondegenerate (in the sense of Vey–Eliasson; see, e.g., [Eli90, Zun96, MZ04, Mir14, Mir03] for nondegenerate singularities), the Gelfand-Cetlin system admits many degenerate singularities.
- The degenerate singular fibers of the Gelfand-Cetlin system are very peculiar in the sense that they are all smooth isotropic submanifolds, as will be shown in this paper (see also Cho–Kim–Oh [CKO18] where the same result is obtained by different methods), while many degenerate singular fibers of other integrable Hamiltonian systems are singular varieties. (See, e.g., [BGK18, BF04, Zun00, Zun03] for various results about degenerate singularities).
- It turns out that the Gelfand-Cetlin systems can be obtained by the method of toric degenerations, see Nishinou–Nohara–Ueda [NNU10]. This method, which comes from algebraic geometry, is now known to generate a lot of “artificial” integrable Hamiltonian systems (see, e.g. Harada–Kaveh [HK15] and references therein). This toric degeneration nature of the Gelfand-Cetlin system may be strongly related to its topological and geometrical particularities.

This paper is the result of a project dating back to 2006 to study the singularities of the Gelfand-Cetlin system, from the point of view of the general topological theory of integrable Hamiltonian systems and their singularities. Unlike some other papers on the subject like [NNU10, CKO18], which are mainly motivated by considerations from algebraic geometry, our work is mainly motivated by considerations from dynamical systems. The first results of this project appeared in the form of a PhD thesis in 2009 of Iman Alamiddine [Ala09], who did it under the supervision of N.T. Zung and with the help of E. Miranda. The main result of this thesis is a complete description of a degenerate singular fiber in $\mathfrak{u}(3)^*$, which is a Lagrangian submanifold diffeomorphic to S^3 , together with a neighborhood of it (i.e. a symplectic normal form for the system around this degenerate fiber). At that time, we conjectured that all the fibers of the Gelfand-Cetlin system (in any dimension) are smooth isotropic. In particular, N.T.

Zung gave a talk on this subject at IMPA in 2009, where a method for studying the topology of singular fibers of the Gelfand-Cetlin system using the complex ellipsoids was presented and some results were announced¹.

Inspired by ideas coming from the Gelfand-Cetlin system, D. Bouloc proved in 2015 [Bou15] a similar conjecture for all singular fibers of the Kapovich-Millson system of bending flows of 3D polygons [KM96], and also of another similar integrable system studied by Nohara and Ueda [NU14] on the 2-Grassmannian manifold. Namely, he showed that all these fibers are isotropic submanifolds (if the ambient symplectic variety itself is a manifold) or orbifolds (in special situations when the ambient symplectic spaces are orbifolds but not manifolds). Remark that these systems of Kapovich-Millson and Nohara-Ueda can also be obtained by toric degenerations (see Foth-Hu [FH05]).

Encouraged by the results of [Bou15], we have a more general conjecture about the singular fibers of integrable systems which can be obtained via toric degenerations, and have worked out the case of Gelfand-Cetlin system for the present paper. In particular, we will show in this paper that every singular fiber of the Gelfand-Cetlin system on coadjoint orbits of unitary groups is a smooth isotropic submanifold which is diffeomorphic to a 2-stage quotient of a compact Lie group by free actions of two other compact Lie groups (Theorem 4.16 and Corollary ??). In many cases, these singular fibers can be shown to be homogeneous spaces or even diffeomorphic to compact Lie groups. We also give a combinatorial formula for computing the dimensions of all singular fibers (Proposition 5.3), and give a detailed description of these singular fibers in many cases, including the so-called (multi-)diamond singularities. These (multi-)diamond singular fibers are degenerate for the Gelfand-Cetlin system, but they are Lagrangian submanifolds diffeomorphic to direct products of special unitary groups and tori.

We remark that Cho, Kim and Oh in a recent preprint [CKO18] already proved the smooth isotropic character of the singular fibers of the Gelfand-Cetlin system and gave a combinatorial formula for the dimensions of these fibers. We found out [CKO18] by chance during the preparation of our paper. Their paper and ours are independent, and complement each other, because the motivations are completely different (Cho, Kim and Oh came to the problem from the point of view of pure symplectic geometry and mirror symmetry, while our project was motivated by problems coming from dynamical systems), and the methods used are also very different. In particular, in our work we use the variety of complete flags of complex ellipsoids which is not present in [CKO18]. It is precisely a kind of duality between such a variety of complete flags of complex ellipsoids and a coadjoint orbit of the unitary group (see Figure 2) that gives us a geometric description of the fibers of the Gelfand-Cetlin system.

We notice that some related partial results on the topology of collective integrable systems have been obtained by Lane in [Lan17]. We remark also that, even though the singularities of the Gelfand-Cetlin system are rather special from the point of view of general integrable Hamiltonian systems, there are still many similarities with

¹See http://strato.impa.br/videos/workshop_geometry/geometry_070809_03.avi; this talk contains some good ideas and some errors.

other singularities that we encountered before. In particular, there is still a topological decomposition into direct products of simpler singularities, as will be seen in Section 5. One can also talk about the (real) *toric degree* of these singularities (see [Zun16] and references therein for the notion of toric degree), a topic that we will not discuss in this paper.

Organization of this paper. The rest of this paper is organized as follows:

In section 2 we give the basic notions of Gelfand–Cetlin systems.

In Section 3 we construct a variety of complete flags of complex ellipsoids, which is dual to a symplectic phase space of the Gelfand-Cetlin system (i.e., a coadjoint orbit of the unitary group), and “move” the Gelfand-Cetlin from the coadjoint orbit to a “dual system” on the variety of complete flags of complex ellipsoids. The “dual Gelfand-Cetlin momentum map” on this “dual Grassmannian” also consists of eigenvalue functions of appropriate matrices, and the two momentum maps (the Gelfand-Cetlin momentum map and its “dual map” have the same image).

In Section 4 we study the symmetry group of an ellipsoid flag with fixed eigenvalues, which gives a new geometrical interpretation of the Gelfand–Cetlin system, and use this machinery to define the symmetry groupoid of ellipsoid flags. These objects yield a good understanding of the geometry of the Gelfand–Cetlin system: The main result of this section is Theorem 4.16 where we prove that the fibers of Gelfand–Cetlin system are smooth submanifolds and identify these manifolds with a quotient manifold constructed using this symmetry groupoid of ellipsoid flags.

Based on the results of Section 3 and Section 4, in Section 5 we give a combinatorial formula for the dimensions of the fibers of the Gelfand-Cetlin system, and show a necessary and sufficient condition for a fiber to be of maximal possible dimension, or equivalently, to be a Lagrangian submanifold (Proposition 5.3). We also give many concrete examples of the fibers, together with their topological description. In particular, there is an interesting family of degenerate singular fibers, which we call (*multiple-*)*diamond singularities* because the equalities in the corresponding Gelfand-Cetlin triangles form “diamonds”, and which are Lagrangian submanifolds diffeomorphic to compact Lie groups of the type

$$(1.1) \quad SU(l_1) \times \dots \times SU(l_s) \times \mathbb{T}^{N - \sum (l_i^2 + 1)},$$

where s is the number of diamonds and l_1, \dots, l_s are their sizes. A particular case of diamond singularities worked out in detail in this section is the case of a degenerate fiber diffeomorphic to the 3-dimensional sphere $S^3 \approx SU(2)$ in $\mathfrak{u}(3)^*$, which has been studied before in Alamiddine’s thesis [Ala09].

Finally, in Section 6 we prove the isotropic character of the fibers of the Gelfand-Cetlin theorem (Proposition 6.1) by direct computations. We believe that this is a general phenomenon, which should be true not only for the Gelfand-Cetlin system and the systems studied by Bouloc [Bou15], but also for many other systems obtained via the toric degeneration method as well. In this relation, we present in Proposition 6.3 a general sufficient condition for the isotropicness of the singular fibers of an integrable Hamiltonian system. Unfortunately, this proposition is still not general enough: there are some degenerate singular fibers of the Gelfand-Cetlin system (especially those which

are Lagrangian) which do not satisfy the conditions of this last proposition. So one will need a more general proposition in order to avoid direct case-by-case computations.

2. THE GELFAND–CETLIN SYSTEM ON COADJOINT ORBITS OF $\mathfrak{u}(n)^*$

2.1. **Coadjoint orbits of $\mathfrak{u}(n)^*$.** Consider the *unitary group*

$$(2.1) \quad U(n) = \{M \in \mathcal{M}_n(\mathbb{C}) \mid MM^* = I_n = M^*M\},$$

and its Lie algebra

$$(2.2) \quad \mathfrak{u}(n) = \{B \in \mathcal{M}_n(\mathbb{C}) \mid B + B^* = 0\}.$$

The dual space $\mathfrak{u}(n)^*$ is identified with the space $\mathcal{H}(n) = \sqrt{-1}\mathfrak{u}(n)$ of *Hermitian matrices* of size n via the map $\varphi : \mathcal{H}(n) \rightarrow \mathfrak{u}(n)^*$ defined for all $A \in \mathcal{H}(n)$ and $B \in \mathfrak{u}(n)$ by

$$(2.3) \quad \varphi(A)(B) = -\sqrt{-1} \operatorname{tr}(AB).$$

Under this identification, the coadjoint representation of $U(n)$ is simply the matrix conjugation: $\operatorname{Ad}_C^*(A) = CAC^*$ for all $C \in U(n)$ and $A \in \mathcal{H}(n)$. The spectrum of a Hermitian matrix is real. Therefore in this paper, by a coadjoint orbit of $U(n)$ we always mean its identification with the set $\mathcal{O}(\lambda) \subset \mathcal{H}(n)$ of all Hermitian matrices with a fixed real spectrum $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$.

Each coadjoint orbit $\mathcal{O}(\lambda)$ is equipped with a natural symplectic form, called the **Kirillov–Kostant–Souriau form**, of which we recall here the expression. Note that if $A \in \mathcal{O}(\lambda)$, then the tangent space of the coadjoint orbit at A can be written as

$$(2.4) \quad T_A\mathcal{O}(\lambda) = \{[B, A] \mid B \in \mathfrak{u}(n)\},$$

and the Kirillov–Kostant–Souriau form ω on $\mathcal{O}(\lambda)$ take the form

$$(2.5) \quad \omega_A([H_1, A], [H_2, A]) = \sqrt{-1} \operatorname{tr}(A[H_1, H_2])$$

for $H_1, H_2 \in \mathfrak{u}(n)$. Moreover, each coadjoint orbit $\mathcal{O}(\lambda)$ is diffeomorphic to a model given by the following proposition.

Proposition 2.1 ([Aud12, Prop. II.1.15]). *Suppose that the spectrum λ consists of k distinct eigenvalues of respective multiplicities n_1, \dots, n_k , that is*

$$(2.6) \quad \underbrace{\lambda_1 = \dots = \lambda_{d_1}}_{n_1} > \underbrace{\lambda_{d_1+1} = \dots = \lambda_{d_2}}_{n_2} > \dots > \underbrace{\lambda_{d_{k-1}+1} = \dots = \lambda_n}_{n_k}.$$

Then the coadjoint orbit $\mathcal{O}(\lambda)$ is diffeomorphic to the homogeneous space $U(n)/U(\lambda)$, where

$$(2.7) \quad U(\lambda) \approx U(n_1) \times \dots \times U(n_k)$$

is the subgroup of block-diagonal unitary matrices of size n , with diagonal blocks of respective sizes n_1, \dots, n_k .

From the above proposition we obtain that the orbit $\mathcal{O}(\lambda)$ has dimension $n^2 - (n_1^2 + \dots + n_k^2)$. In particular, we say that λ (and the corresponding orbit $\mathcal{O}(\lambda)$) is *generic* if all the eigenvalues are distinct, that is

$$(2.8) \quad \lambda_1 > \dots > \lambda_n.$$

In this case, $U(\lambda)$ is a n -dimensional torus $\mathbb{T}^n = U(1)^n$ and the generic orbit $\mathcal{O}(\lambda)$ has maximal dimension $n^2 - n = n(n-1)$.

2.2. The Gelfand–Cetlin system. Fix a spectrum $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$. For any matrix $A \in \mathcal{O}(\lambda)$ and any integer $1 \leq k \leq n$, denote by A_k the upper-left submatrix of A of size $k \times k$. Since A_k is a Hermitian matrix of size k , it admits k real eigenvalues

$$(2.9) \quad F_{1,k}(A) \geq \dots \geq F_{k,k}(A).$$

The family of functions

$$(2.10) \quad F = \{F_{i,j} : \mathcal{O}(\lambda) \rightarrow \mathbb{R} \mid 1 \leq i \leq j < n\}$$

satisfies the **Gelfand–Cetlin triangle (diagram) of inequalities** [GS83a, Prop. 5.3] shown in Figure 1.

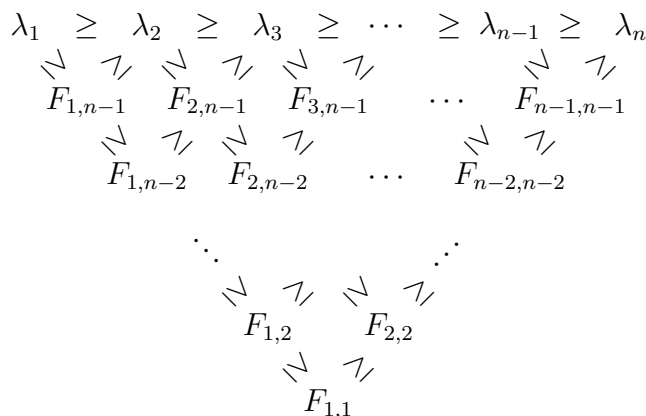


FIGURE 1. The Gelfand–Cetlin triangle (diagram).

Moreover, these functions commute pairwise under the Poisson bracket induced by the Kirillov–Kostant–Souriau form on $\mathcal{O}(\lambda)$, and we have the following theorem due to Guillemin and Sternberg [GS83b]:

Theorem 2.2 ([GS83b]). *On any coadjoint orbit $\mathcal{O}(\lambda)$, the non-constant functions in F define a completely integrable Hamiltonian system. The singular values of this system are the values for which there is a non-trivial equality in the Gelfand–Cetlin diagram on Figure 1.*

Indeed, the family F contains $n(n-1)/2$ functions, which is precisely half the dimension of a generic coadjoint orbit. If the coadjoint orbit $\mathcal{O}(\lambda)$ is not generic because some of the eigenvalues $\lambda_1, \dots, \lambda_n$ are equal, then the Gelfand–Cetlin inequalities imply that some of the functions in F are constant. But one can check that the number of remaining non-constant functions is, again, half the dimension of $\mathcal{O}(\lambda)$.

Remark that we can also define the functions $F_{1,n}, \dots, F_{n,n}$ on $\mathfrak{u}(n)^*$ in the same way; they will be constant on $\mathcal{O}(\lambda)$: $F_{1,n} = \lambda_1, \dots, F_{n,n} = \lambda_n$.

By abusing the notation, for any given spectrum λ we will call **momentum map** of the Gelfand–Cetlin system, or **Gelfand–Cetlin map**, the following so-called *collective function*, which will be denoted by the same letter F : is given:

$$(2.11) \quad F = (F_{i,j})_{1 \leq i \leq j < n} : \mathcal{O}(\lambda) \rightarrow \mathbb{R}^{n(n-1)/2}.$$

Even though this map has redundant components when λ is non-generic, it has the same level sets as the usual momentum map of the Gelfand–Cetlin integrable system, which is enough for the matter of the present paper.

3. GEOMETRIC INTERPRETATION OF THE FIBERS

Fix an ordered real spectrum $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$, and denote by

$$(3.1) \quad D_\lambda = \text{diag}(\lambda_1, \dots, \lambda_n) \in \mathcal{O}(\lambda)$$

the diagonal matrix with entries $\lambda_1, \dots, \lambda_n$. We have a natural projection $\pi : U(n) \rightarrow \mathcal{O}(\lambda)$ given by

$$(3.2) \quad \pi(C) = CD_\lambda C^*.$$

3.1. Partial and complete flag manifolds. Recall that a *flag* in \mathbb{C}^n is a sequence of vector subspaces

$$(3.3) \quad V^\bullet = (\{0\} = V^0 \subset V^1 \subset \dots \subset V^k = \mathbb{C}^n)$$

with increasing dimensions $0 = d_0 < d_1 < \dots < d_k = n$ (we will often omit V^0). The k -tuple (d_1, \dots, d_k) is called the *signature* of the flag V^\bullet . A flag is *complete* if $k = n$ and $(d_1, d_2, \dots, d_k) = (1, 2, \dots, n)$, otherwise it is *partial*.

Fix a signature $d = (d_1, \dots, d_k)$. One can associate to any flag V^\bullet with signature d a basis (u_1, \dots, u_n) of \mathbb{C}^n such that for any $1 \leq i \leq k$, the space V^i is generated by the first d_i vectors of this basis. Up to a Gram–Schmidt process, the basis (u_1, \dots, u_n) can be supposed unitary, we then identify it with the matrix $C \in U(n)$ with columns u_1, \dots, u_n . Conversely, any matrix $C \in U(n)$ represents a unique flag V^\bullet with signature d such that for any $1 \leq i \leq k$, V^i is generated by the first d_i columns of C .

Note that two matrices C_1, C_2 represent the same flag if and only if there exists a block diagonal matrix $P \in U_d = U(n_1) \times \dots \times U(n_k)$ such that $C_2 = C_1 P$, where $n_i = d_i - d_{i-1}$ (with convention $d_0 = 0$). It is then standard to identify the set of all flags with signature d in \mathbb{C}^n with the homogeneous space

$$(3.4) \quad \mathcal{F}_d = U(n)/U_d = U(n)/(U(n_1) \times \dots \times U(n_k)).$$

In particular, the set of complete flags in \mathbb{C}^n is identified with $U(n)/\mathbb{T}^n$, where $\mathbb{T}^n = U(1) \times \dots \times U(1)$ is the usual n -dimensional torus.

Remark 3.1. In particular, Proposition 2.1 states that every coadjoint orbit $\mathcal{O}(\lambda)$ is diffeomorphic to a flag manifold \mathcal{F}_d , with signature d determined by the redundancies among the values $\lambda_1, \dots, \lambda_n$.

3.2. Complex ellipsoids. Let V be a finite-dimensional (complex) vector space with a Hermitian product $\langle \cdot | \cdot \rangle$ (the convention chosen in this paper for a Hermitian product is to be linear in the first variable and anti-linear the second variable). Recall that a linear transformation $\alpha : V \rightarrow V$ is called *Hermitian* if $\langle \alpha(v_1) | v_2 \rangle = \langle v_1 | \alpha(v_2) \rangle$ for any $v_1, v_2 \in V$, i.e., $\alpha^* = \alpha$. A Hermitian transformation is diagonalizable in a unitary basis: there exists a basis (v_1, \dots, v_k) of V , with $\langle v_i | v_j \rangle = \delta_{i,j}$, such that $\alpha(v_i) = \gamma_i v_i$. Moreover, its eigenvalues $\gamma_1, \dots, \gamma_k$ are real numbers. In particular, for any $v = x_1 v_1 + \dots + x_k v_k$ in V we have

$$(3.5) \quad \langle \alpha(v) | v \rangle = 1 \iff \gamma_1 |x_1|^2 + \dots + \gamma_k |x_k|^2 = 1.$$

By analogy with the real Euclidean case, we will set the following definition:

Definition 3.2. A *complex ellipsoid* in V is a subset of the form

$$(3.6) \quad E_\alpha = \{v \in V \mid \langle \alpha(v) | v \rangle = 1\}$$

where $\alpha : V \rightarrow V$ is a positive definite Hermitian transformation (recall that α is positive definite if $\langle \alpha(v) | v \rangle > 0$ for any $v \neq 0$, or equivalently if all its eigenvalues are positive). If (v_1, \dots, v_k) is a basis of eigenvectors for α and $\gamma_1, \dots, \gamma_k$ are the associated eigenvalues, then we say that E_α has axes $\mathbb{C}v_1, \dots, \mathbb{C}v_k$ and radii $1/\sqrt{\gamma_1}, \dots, 1/\sqrt{\gamma_k}$.

When $V = \mathbb{C}^k$ and $A \in \mathcal{H}(k)$, we simply write

$$(3.7) \quad E_A = \{x \in \mathbb{C}^k \mid \langle Ax | x \rangle = 1\}.$$

The complex ellipsoids satisfy the following immediate properties:

Lemma 3.3. Let α, β be positive definite Hermitian transformations of V , and $\phi : V \rightarrow W$ a unitary map. Then:

- (1) for any $v \neq 0$ in V , there exists $t > 0$ such that $tv \in E_\alpha$,
- (2) $E_\alpha = E_\beta$ if and only if $\alpha = \beta$,
- (3) $\phi(E_\alpha) = E_{\phi \circ \alpha \circ \phi^{-1}}$,
- (4) if $V = W$, then ϕ preserves E_α if and only if α and ϕ commute.

Proof. Fix $v \neq 0$. Since α is positive definite, $\langle \alpha(v) | v \rangle > 0$. Then $t = 1/\sqrt{\langle \alpha(v) | v \rangle}$ is well-defined and we have:

$$\langle \alpha(tv) | tv \rangle = t^2 \langle \alpha(v) | v \rangle = 1$$

which proves (1). But if $E_\alpha = E_\beta$, then we also have $\langle \beta(tv) | tv \rangle = 1$, hence $\langle \beta(v) | v \rangle = 1/t^2 = \langle \alpha(v) | v \rangle$. That is

$$\langle (\beta - \alpha)v | v \rangle = 0$$

for all $v \in \mathbb{C}^n$. Since $\beta - \alpha$ is again Hermitian, it is diagonalizable. But the above condition implies that all its eigenvalues are zero, so we conclude that $\alpha = \beta$, which proves (2). The equality (3) comes from:

$$\langle \alpha(v) | v \rangle = \langle \phi \circ \alpha(v) | \phi(v) \rangle = \langle \phi \circ \alpha \circ \phi^{-1}(\phi(v)) | \phi(v) \rangle.$$

Then (4) follows from (2) and (3). \square

The following proposition deals with the intersection of a complex ellipsoid with a lower dimensional vector subspace.

Proposition 3.4. *Let E_α be a complex ellipsoid in V , and W, W' two linear subspaces of V . Then:*

- (1) $E_\alpha \cap W$ is a complex ellipsoid in W : there exists a positive definite Hermitian map $\beta : W \rightarrow W$ such that $E_\alpha \cap W = E_\beta$.
- (2) In particular, if $V = \mathbb{C}^n$, $W = \mathbb{C}^k \times \{0\}^{n-k}$ then $E_\alpha \cap W = E_{A_k}$.
- (3) If $E_\alpha \cap W = E_\alpha \cap W'$, then $W = W'$.

Proof. For (2), it suffices to remark that $\langle Ai(x) | i(x) \rangle_n = \langle A_k x | x \rangle_k$ for every $x \in \mathbb{C}^k$, where $i : \mathbb{C}^k \rightarrow \mathbb{C}^k \times \{0\}^{n-k}$ is the canonical identification. Now for (1), choose $\phi : \mathbb{C}^n \rightarrow V$ unitary such that $\phi(\mathbb{C}^k \times \{0\}^{n-k}) = W$. Define $A \in \mathcal{H}(n)$ by $Ax = \phi^{-1} \circ \alpha \circ \phi(x)$, hence $w \in E_\alpha$ if and only if $\phi^{-1}(w) \in E_A$. Now define $\beta : W \rightarrow W$ by $\beta(w) = \phi(A_k \phi^{-1}(w))$. We obtain $w \in E_\alpha \cap W$ if and only if $\phi^{-1}(w) \in E_A \cap (\mathbb{C}^k \times \{0\}^{n-k})$, which is equivalent to $w \in E_\beta$.

For (3), consider $w \in W$. By Lemma 3.3, there exists $t > 0$ such that $tw \in E_\alpha$. If $E_\alpha \cap W = E_\alpha \cap W'$ then, tw , and hence w , lie in W' and so $W \subset W'$. With a symmetric argument we conclude that $W = W'$. \square

The above properties motivate the following definitions:

Definition 3.5. *An **ellipsoid flag** in \mathbb{C}^n is a triple $(E^\bullet, V^\bullet, A)$ where:*

- i) V^\bullet is a (vector space) flag in \mathbb{C}^n ,
- ii) A is a positive definite Hermitian matrix of size n ,
- iii) $E^\bullet = (E^1 \subset E^2 \subset \dots \subset E^k = E_A)$ is the increasing sequence of ellipsoids defined by $E^k = E_A \cap V^k$.

The **signature** of $(E^\bullet, V^\bullet, A)$ is the signature of V^\bullet , and we say that $(E^\bullet, V^\bullet, A)$ is **complete** if V^\bullet is complete.

We will sometimes denote the ellipsoid flag $(E^\bullet, V^\bullet, A)$ by $E^\bullet = E_A \cap V^\bullet$, or simply by E^\bullet .

Recall that, for each $1 \leq k \leq n$, the complex ellipsoid $E^k = E_{\alpha_k}$ is defined by a unique positive definite Hermitian map $\alpha_k : V^k \rightarrow V^k$. For brevity, we will say that the flag E^\bullet is defined by the family $\alpha_\bullet = (\alpha_1, \dots, \alpha_n)$.

Definition 3.6. *Let $(E^\bullet, V^\bullet, A)$ be a complete ellipsoid flag. We call **eigenvalues** of E^\bullet the $(n(n+1)/2)$ -tuple*

$$(3.8) \quad \Gamma(E^\bullet) = (\gamma_{i,j}(E^\bullet) \mid 1 \leq j \leq n, 1 \leq i \leq j),$$

where for any $1 \leq j \leq n$,

$$(3.9) \quad \gamma_{1,j}(E^\bullet) \geq \gamma_{2,j}(E^\bullet) \geq \dots \geq \gamma_{j,j}(E^\bullet)$$

are the eigenvalues of the defining maps $\alpha_j : V^j \rightarrow V^j$ of E^\bullet .

Fix a positive spectrum $\lambda = (\lambda_1 \geq \dots \geq \lambda_n > 0)$ and consider a matrix $A \in \mathcal{O}(\lambda)$. Let $C \in U(n)$ such that $A = \pi(C) = CD_\lambda C^*$. As above, for $1 \leq k \leq n$ denote by A_k the upper-left submatrix of A and by $i_k : \mathbb{C}^k \hookrightarrow \mathbb{C}^n$ the canonical inclusion. Consider

$$(3.10) \quad V_{\text{std}}^\bullet = (i_1(\mathbb{C}) \subset i_2(\mathbb{C}^2) \subset \dots \subset i_n(\mathbb{C}^n))$$

the standard complete flag of \mathbb{C}^n , and V_C^\bullet its image by the linear map $x \mapsto C^*.x$. By Proposition 3.4, the complete ellipsoid flag $(\tilde{E}_A^\bullet, V_{\text{std}}^\bullet, A)$, and hence its image under Φ denoted by flag $(E_C^\bullet, V_C^\bullet, D_\lambda)$, have eigenvalues

$$(3.11) \quad F(A) = \{F_{i,j}(A) \mid 1 \leq i \leq j \leq n\}.$$

Note that $V_C^\bullet = Q(C^*)$ where $Q : U(n) \rightarrow \mathcal{F}$ is the map which sends each nondegenerate matrix to the flag defined by its columns, so the ellipsoid flag E_C^\bullet depends on the diagonalization C chosen for A .

Introduce the eigenvalue map

$$(3.12) \quad \begin{array}{ccc} \Gamma_\lambda : & \mathcal{F} & \longrightarrow \mathbb{R}^N \\ & V^\bullet & \longmapsto \Gamma(E_{D_\lambda} \cap V^\bullet) \end{array}$$

and denote by $t : U(n) \rightarrow U(n)$ the involution $t(C) = C^* = C^{-1}$. Then the above remarks are summed up in the commutative diagram given in Figure 2.

$$\begin{array}{ccc} U(n) & \xrightarrow{t: C \mapsto C^*} & U(n) \\ \downarrow \pi: C \mapsto CD_\lambda C^* & & \downarrow Q: \text{flag generated by the columns} \\ \mathcal{O}(\lambda) & & \mathcal{F} \\ \swarrow F: \text{Gelfand-Cetlin map} & & \nwarrow \Gamma_\lambda: V^\bullet \mapsto \Gamma(E_{D_\lambda} \cap V^\bullet) \\ & \mathbb{R}^N & \end{array}$$

FIGURE 2. From the Gelfand–Cetlin system to the map Γ_λ on the flag manifold \mathcal{F} .

Definition 3.7. *This this paper, by a **fiber**, we will mean either a preimage of the map F on $\mathcal{O}(\lambda)$ (i.e. a fiber of the Gelfand–Cetlin system on a given coadjoint orbit), or a preimage of the map Γ_λ in the diagram in Figure 2.*

4. THE GEOMETRY OF ELLIPSOID FLAGS WITH FIXED EIGENVALUES

Fix $\lambda = (\lambda_1 \geq \dots \geq \lambda_n > 0)$ a positive spectrum. As before, denote by D_λ the diagonal matrix with entries $\lambda_1, \dots, \lambda_n$, and by E_{D_λ} the corresponding ellipsoid in \mathbb{C}^n .

4.1. Symmetry group of an ellipsoid flag. Let V^\bullet in \mathcal{F} be a given complete flag and $E^\bullet = E_{D_\lambda} \cap V^\bullet$ the ellipsoid flag obtained by intersecting V^\bullet with the “standard” ellipsoid E_{D_λ} . For a fixed $1 \leq k \leq n$, let us describe the subgroup G_k of $U(V^k)$ consisting of those elements which the k -dimensional complex ellipsoid $E^k = E_{\alpha_k} \subset V^k$. We recall the following result from linear algebra.

Lemma 4.1. *Let V be a finite dimensional inner product space, $\alpha : V \rightarrow V$ a positive definite Hermitian transform, and $V = W_1 \oplus \dots \oplus W_r$ the decomposition of V into eigenspaces of α . Then the subgroup of unitary transformations $\varphi \in U(V)$ which commute with α is exactly*

$$(4.1) \quad G = U(W_1) \oplus \dots \oplus U(W_r).$$

Remark 4.2. In the above expression, by $\varphi = \varphi_1 \oplus \cdots \oplus \varphi_r$ with $\varphi_i \in U(W_i)$ we mean the map $\varphi : V \rightarrow V$ defined by

$$(4.2) \quad \varphi(v) = \varphi_1(v_1) \oplus \cdots \oplus \varphi_r(v_r)$$

for any $v = v_1 \oplus \cdots \oplus v_r \in W_1 \oplus \cdots \oplus W_r$.

This implies the following:

Proposition 4.3. *The subgroup of unitary transformations of V^k that preserves E^k is exactly*

$$(4.3) \quad G_k = U(W_1) \oplus \cdots \oplus U(W_r) \subset U(V^k),$$

where $V^k = W_1 \oplus \cdots \oplus W_r$ is the decomposition of V^k into eigenspaces of α_k .

Each W_i has dimension n_i determined by the eigenvalues

$$(4.4) \quad \underbrace{\gamma_{1,k} = \cdots = \gamma_{d_1,k}}_{n_1} > \underbrace{\gamma_{d_1+1,k} = \cdots = \gamma_{d_2,k}}_{n_2} > \cdots > \underbrace{\gamma_{d_{r-1}+1,k} = \cdots = \gamma_{k,k}}_{n_r}$$

of α_k . (“Horizontal” equalities in the Gelfand–Cetlin diagram).

Proof. The first part of the proposition is immediate: by Lemma 3.3, $\phi \in U(V^k)$ preserves E_{α_k} if and only if ϕ commutes with α_k . We conclude using Lemma 4.1. Since V^\bullet is in $\mathcal{F}(c)$, α_k has eigenvalues $\gamma_{1,k} \geq \cdots \geq \gamma_{k,k}$, so the numbers n_i correspond indeed to the dimensions of the different eigenspaces. \square

For later use, we will need the subgroup $H_{k+1} \subset G_{k+1}$ of unitary transformations of V^{k+1} which preserve not only E^{k+1} but also V^k (or equivalently, E^k), and H'_{k+1} the subgroup of transformations in H_{k+1} whose restriction to L_k is the identity, where L_k denotes the orthogonal complement of V^k in V^{k+1} . Note that $\dim L_k = 1$.

Lemma 4.4. *Fix a vector $\ell \in L_k$ of length 1 and write $\alpha_{k+1}(\ell) = w \oplus a\ell$ with $w \in V^k$, $a \in \mathbb{C}$. Then for any $v \in V^k$ we have*

$$(4.5) \quad \alpha_{k+1}(v) = \alpha_k(v) \oplus \langle v | w \rangle \ell$$

Proof. Using the decomposition $V^{k+1} = V^k \oplus L_k$, write $\alpha_{k+1}(v) = \beta(v) \oplus \lambda(v)\ell$ with $\beta : V^k \rightarrow V^k$ and $\lambda : V^k \rightarrow \mathbb{C}$. Since V^k and L_k are orthogonal, we have

$$\lambda(v) = \langle \alpha_{k+1}(v) | \ell \rangle = \langle v | \alpha_{k+1}(\ell) \rangle = \langle v | w \rangle.$$

It remains to show that $\beta = \alpha_k$. To do so, remark that for any $v \in V^k \subset V^{k+1}$,

$$\langle \alpha_{k+1}(v) | v \rangle = \langle \beta(v) | v \rangle$$

hence $E_{\alpha_k} = E_{\alpha_{k+1}} \cap V^k = E_\beta$. By Lemma 3.3, it follows that $\alpha_k = \beta$. \square

We are now able to describe the groups H'_{k+1} and H_{k+1} .

Proposition 4.5. *Let $V^k = W_1 \oplus \cdots \oplus W_r$ be the decomposition of V^k into the eigenspaces of α_k , and*

$$(4.6) \quad V^{k+1} = W_1 \oplus \cdots \oplus W_r \oplus L_k$$

the induced decomposition of V^{k+1} , where L_k is the orthogonal complement of V^k in V^{k+1} . Denote by $\text{proj}_{W_i} : V^{k+1} \rightarrow W_i$ the orthogonal projection on W_i . Then the group H_{k+1} of unitary maps on V^{k+1} which preserve E^k and E^{k+1} is the set of elements

$$(4.7) \quad \phi_1 \oplus \cdots \oplus \phi_r \oplus \xi \cdot \text{id}_{L_k} \in U(W_1) \oplus \cdots \oplus U(W_r) \oplus U(L_k)$$

such that for any $1 \leq i \leq n$ and $w_i \in \text{proj}_{W_i}(\alpha_{k+1}(L_k))$ we have $\phi_i(w_i) = \xi w_i$.

The subgroup H'_{k+1} is the set of all above elements with ξ equal to 1.

Proof. Fix a unit vector $\ell \in L_k$ (the orthonormal complement of V^k in V^{k+1}), and write

$$\alpha_{k+1}(\ell) = w_1 \oplus \cdots \oplus w_r \oplus a\ell$$

with $w_i \in W_i$ and $a \in \mathbb{C}$. Then, by Lemma 4.4, for any $v_i \in W_i$,

$$\alpha_{k+1}(v_i) = \alpha_k(v_i) \oplus \langle v_i | w_1 \oplus \cdots \oplus w_r \rangle \ell = \gamma_{ik} v_i \oplus \langle v_i | w_i \rangle \ell.$$

Note that w_i depends of the choice of $\ell \in L_k$, but $\mathbb{C}w_i$ can be determined intrinsically as the subspace $\text{proj}_{W_i}(\alpha_{k+1}(L_k))$.

Let $\phi \in U(V^{k+1})$. Suppose ϕ preserves V^k , and more precisely E^k . Then

$$\phi = \phi_1 \oplus \cdots \oplus \phi_r \oplus \xi \cdot \text{id}_{L_k} \in U(W_1) \oplus \cdots \oplus U(W_r) \oplus U(L_k).$$

On the other hand ϕ preserves E^{k+1} if and only if ϕ commutes with α_{k+1} . For $v_i \in W_i$ we have

$$\begin{cases} \alpha_{k+1}(\phi(v_i)) = \gamma_{ik} \phi(v_i) \oplus \langle \phi(v_i) | w_i \rangle \ell = \gamma_{ik} \phi(v_i) \oplus \langle v_i | \phi_i^*(w_i) \rangle \ell, \\ \phi(\alpha_{k+1}(v_i)) = \gamma_{ik} \phi(v_i) \oplus \langle v_i | w_i \rangle \xi \ell = \gamma_{ik} \phi(v_i) \oplus \langle v_i | \xi w_i \rangle \ell. \end{cases}$$

Similarly,

$$\begin{cases} \alpha_{k+1}(\phi(\ell)) = \alpha_{k+1}(\xi \ell) = \xi w_1 \oplus \cdots \oplus \xi w_r \oplus \xi a \ell, \\ \phi(\alpha_{k+1}(\ell)) = \phi(w_1 \oplus \cdots \oplus w_r \oplus a \ell) = \phi_1(w_1) \oplus \cdots \oplus \phi_r(w_r) \oplus \xi a \ell. \end{cases}$$

It follows that ϕ preserves E^{k+1} if and only if for any $1 \leq i \leq r$, $\phi_i(w_i) = \xi w_i$. \square

Remark 4.6. The space $\text{proj}_{W_i}(\alpha_{k+1}(L_k))$ is trivial if $\alpha_{k+1}(L_k)$ is orthogonal to W_i . In this case, for any $w_i \in W_i$ we have $\alpha_{k+1}(w_i) = \alpha_k(w_i)$, so W_i is also an eigenspace of α_{k+1} associated to the same eigenvalue, and there must be a vertical equality in the Gelfand–Cetlin diagram for E^\bullet .

By contraposition, if the eigenvalue $\gamma_{i,k}$ associated to W_i appears only once in the Gelfand–Cetlin diagram, then the space $\text{proj}_{W_i}(\alpha_{k+1}(L_k))$ has necessarily dimension 1.

However, note that the converse is not true. A vertical inequality in the Gelfand–Cetlin diagram does not imply that the corresponding space $\text{proj}_{W_i}(\alpha_{k+1}(L_k))$ is trivial.

Remark 4.7. If W'_i denotes the orthogonal complement of $W''_i = \text{proj}_{W_i}(\alpha_{k+1}(L_k))$ in W_i , then H_{k+1} can be written as the group of all elements of the form

$$\begin{aligned} & (\phi'_1 \oplus \xi \cdot \text{id}_{W''_1}) \oplus \cdots \oplus (\phi'_r \oplus \xi \cdot \text{id}_{W''_r}) \oplus \xi \text{id}_{L_k} \\ & \in (U(W'_1) \oplus U(W''_1)) \oplus \cdots \oplus (U(W'_r) \oplus U(W''_r)) \oplus U(L_k). \end{aligned}$$

This group is clearly isomorphic to

$$U(W'_1) \times \cdots \times U(W'_r) \times U(1)$$

and each W'_i has codimension at most 1 in W_i . In particular, the subgroup H'_{k+1} is isomorphic to

$$U(W'_1) \times \cdots \times U(W'_r)$$

Consider the group

$$(4.8) \quad G(E^\bullet) = G_1 \times \cdots \times G_n,$$

where for each $1 \leq k \leq n$, G_k is the group of unitary transformations of V^k preserving $E^k = E_{D_\lambda} \cap V^k$. Consider also the subgroups $H'(E^\bullet) < H(E^\bullet) < G(E^\bullet)$ defined by

$$(4.9) \quad H'(E^\bullet) = H'_1 \times \cdots \times H'_n \quad \text{and} \quad H(E^\bullet) = H_1 \times \cdots \times H_n$$

where for each $1 \leq k \leq n$, H_k is the group of unitary transformations of V^k preserving both E^k and E^{k-1} (or equivalently, both E^k and V^{k-1}) and H'_k is the group of transformations in H_k whose restriction to $(V^{k-1})^\perp$ is the identity.

Definition 4.8. *The group $G(E^\bullet)$ is called the **coarse symmetry group** of the ellipsoid flag E^\bullet . The quotient manifold*

$$(4.10) \quad S(E^\bullet) = G(E^\bullet)/H'(E^\bullet)$$

with respect to the free right $H'(E^\bullet)$ -action on $G(E^\bullet)$ defined by

$$(4.11) \quad \phi \cdot f = ((f_2^{-1})|_{V^1} \circ \phi_1 \circ f_1, \dots, (f_n^{-1})|_{V^{n-1}} \circ \phi_{n-1} \circ f_{n-1}, \phi_n \circ f_n)$$

for any $\phi \in G(E^\bullet)$, $f \in H'(E^\bullet)$, is called the **reduced symmetry space** of E^\bullet .

4.2. Push-forward of flags. In this subsection, V^\bullet is a complete flag in \mathbb{C}^n and $E^\bullet = E_{D_\lambda} \cap V^\bullet$ is its intersection with the standard ellipsoid. Fix an element $C^* \in U(n)$ such that $Q(C^*) = V^\bullet$, i.e. columns (u_1, \dots, u_n) of C^* form a unitary basis of V^\bullet .

Definition 4.9. *Let $\phi = (\phi_1, \dots, \phi_n)$ in $G(E^\bullet)$.*

(1) *The **push-forward of V^\bullet by ϕ** is the complete flag*

$$(4.12) \quad \phi_* V^\bullet = (V_\phi^1 \subset \cdots \subset V_\phi^n = \mathbb{C}^n)$$

defined by

$$(4.13) \quad V_\phi^k = \phi_n(\phi_{n-1}(\dots \phi_k(V^k) \dots)) \quad \forall 1 \leq k \leq n.$$

(2) *The **push-forward of $C^* = (u_1, \dots, u_n)$ by ϕ** is the family*

$$(4.14) \quad \phi_* C^* = (u_1^\phi, \dots, u_n^\phi)$$

defined by

$$(4.15) \quad u_k^\phi = \phi_n(\phi_{n-1}(\dots \phi_k(u_k) \dots)) \quad \forall 1 \leq k \leq n..$$

The following proposition shows that the above definition makes sense.

Proposition 4.10. *For any $\phi \in G(E^\bullet)$ and $C^* \in U(n)$ such that $Q(C^*) = V^\bullet$:*

- (1) $\phi_* C^*$ is a unitary matrix,
- (2) $Q(\phi_* C^*) = \phi_* V^\bullet$.

In other words, the map $Q : U(n) \rightarrow \mathcal{F}$ intertwines the two push-forward operations defined above.

Proof. Recall that (u_1, \dots, u_n) is a unitary basis. Since the maps ϕ_1, \dots, ϕ_n are unitary transformations, $u_1^\phi, \dots, u_n^\phi$ are also unit vectors. Moreover, for any $1 \leq i < j \leq n$,

$$\begin{aligned} \langle u_i^\phi | u_j^\phi \rangle &= \langle \phi_n(\phi_{n-1}(\dots \phi_i(u_i) \dots)) | \phi_n(\phi_{n-1}(\dots \phi_j(u_j) \dots)) \rangle \\ &= \langle \phi_{j-1}(\dots \phi_i(u_i) \dots) | u_j \rangle \\ &= 0 \end{aligned}$$

since $\phi_{j-1}(\dots \phi_i(u_i) \dots)$ lies in $V^{j-1} = \text{Span}(u_1, \dots, u_{j-1})$. It follows that $\phi_* C^*$ is a unitary matrix. Moreover, it is clear that each u_i lies in V_ϕ^i so $\phi_* C^*$ is a basis of the flag $\phi_* V^\bullet$. \square

The push-forwards of flags and unitary bases can be rewritten more concisely in terms of applications $\bar{\phi}_1, \dots, \bar{\phi}_n$ defined in the following lemma.

Lemma 4.11. *Let $\phi \in G(E^\bullet)$. For $1 \leq k \leq n$, define $\bar{\phi}_k : V^k \rightarrow V_\phi^k$ recursively by:*

$$(4.16) \quad \begin{cases} \bar{\phi}_n = \phi_n, \\ \bar{\phi}_k = (\bar{\phi}_{k+1})|_{V_\phi^k} \circ \phi_k \quad \text{for } 1 \leq k < n, \end{cases}$$

Then for any $1 \leq k \leq n$:

- (1) $u_k^\phi = \bar{\phi}_k(u_k)$,
- (2) $V_\phi^k = \bar{\phi}_k(V^k) = \bar{\phi}_{k+1}(V^k)$,
- (3) $E_{D_\lambda} \cap V_\phi^k = \bar{\phi}_k(E^k) = \bar{\phi}_{k+1}(E^k)$

(with the convention that $\bar{\phi}_{n+1}$ is the identity map on \mathbb{C}^n).

Proof. By definition of $\bar{\phi}_k$, for any subset $S \subset V^k$ we have

$$\bar{\phi}_k(S) = \phi_n(\phi_{n-1}(\dots \phi_k(S) \dots)).$$

Taking $S = \{u_k\}$ proves (1) while $S = V^k = \phi_k^{-1}(V^k)$ proves (2). Now, recall that by definition $\phi_k : V^k \rightarrow V^k$ preserves $E^k = E_{D_\lambda} \cap V^k$. Then for any $\ell \geq k$, by writing

$$E^k = E_{D_\lambda} \cap V^k = E_{D_\lambda} \cap V^\ell \cap V^k$$

we get

$$\phi_\ell(E^k) = \phi_\ell(E_{D_\lambda} \cap V^\ell) \cap \phi_\ell(V^k) = E_{D_\lambda} \cap V^\ell \cap \phi_\ell(V^k) = E_{D_\lambda} \cap \phi_\ell(V^k).$$

Iterating this process, we can show that $\bar{\phi}_\ell(E^k) = E_{D_\lambda} \cap \bar{\phi}_\ell(V^k)$. Take $\ell = k$ or $\ell = k+1$ to prove (3). \square

The push-forwards defined in this section do not provide a one-to-one correspondence between the group of symmetries $G(E^\bullet)$ and the set of flags with given eigenvalues. Actually, two symmetries give the same push-forwards if they are equal up to an element in the groups $H'(E^\bullet)$ or $H(E^\bullet)$ defined above. More precisely, we have:

Proposition 4.12. *For any $\phi, \psi \in G(E^\bullet)$,*

- (1) $\phi_* V^\bullet = \psi_* V^\bullet$ if and only if $\psi = \phi \cdot f$ for some $f \in H(E^\bullet)$,
- (2) $\phi_* C^* = \psi_* C^*$ if and only if $\psi = \phi \cdot f$ for some $f \in H'(E^\bullet)$,

where the actions of $H(E^\bullet)$ and $H'(E^\bullet)$ on $G(E^\bullet)$ are given by Formula (4.11) in Definition 4.8.

Proof. (1) We have $V_\phi^n = \mathbb{C}^n = V_\psi^n$. Suppose that $V_\phi^k = V_\psi^k$ for all $1 \leq k \leq n$. Then $f_k = \bar{\phi}_k^{-1} \circ \bar{\psi}_k : V^k \rightarrow V^k$ is well-defined and preserves V^{k-1} . By Lemma 4.11, f_k preserves also E^k , hence $f_k \in H_k$. We then have, for any $1 \leq k < n$,

$$f_k^{-1} = \bar{\psi}_k^{-1} \circ \bar{\phi}_k = ((\bar{\psi}_{k+1})|_{V^k}^{V_\psi^k} \circ \psi_k)^{-1} \circ ((\bar{\phi}_{k+1})|_{V^k}^{V_\phi^k} \circ \phi_k) = \psi_k^{-1} \circ (f_{k+1}^{-1})|_{V^k}^{V^k} \circ \phi_k,$$

that is $\psi_k = (f_{k+1}^{-1})|_{V^k}^{V^k} \circ \phi_k \circ f_k$.

Conversely, if $\psi = \phi \cdot f$ for some $f = (f_1, \dots, f_n) \in H(E^\bullet)$, then for any $1 \leq k \leq n$, $\bar{\psi}_k = \bar{\phi}_k \circ f_k$ and hence

$$V_\psi^k = \bar{\psi}_{k+1}(V^k) = \bar{\phi}_{k+1} \circ f_{k+1}(V^k) = \bar{\phi}_{k+1}(V^k) = V_\phi^k.$$

(2) If $\phi_* C^* = \psi_* C^*$, then $\phi_* V = \psi_* V$ by Proposition 4.10, and hence $\psi = \phi \cdot f$ for some $f \in H(E^\bullet)$. By definition of the action of $H(E^\bullet)$ on $G(E^\bullet)$ we have

$$u_k^\psi = \bar{\psi}_k(u_k) = \bar{\phi}_k \circ f_k(u_k).$$

Finally, for any $1 \leq k \leq n$,

$$u_k^\psi = u_k^\phi \iff \bar{\phi}_k \circ f_k(u_k) = \bar{\phi}_k(u_k) \iff f_k(u_k) = u_k,$$

hence $\psi_* C^* = \phi_* C^*$ if and only if f_k is the identity on $\text{Span}(u_k) = (V^{k-1})^{\perp V^k}$ for any k , that is $f \in H'(E^\bullet)$. \square

We want to prove now that the push-forward operation allows, starting from a flag V^\bullet , to reach all the flags lying in the same fiber (in the sense of Definition 3.7) as V^\bullet , and similarly for a unitary basis formed by the columns of C^* .

To do so, we first need the following result, stating that if a complex ellipsoid E contains two codimension 1 ellipsoids E_1 and E_2 with same radii, then there exists a symmetry of E mapping E_1 onto E_2 .

Lemma 4.13. *Let V_1, V_2 be two codimension 1 subspaces of some Hermitian space V . Consider $\alpha : V \rightarrow V$ a positive definite Hermitian form and E_α the corresponding ellipsoid in V . For $i = 1, 2$, let $\beta_i : V_i \rightarrow V_i$ be the positive Hermitian form defining the ellipsoid $E_{\beta_i} = E_\alpha \cap V_i$. Then there exists a unitary map $\varphi : V \rightarrow V$ such that*

$$(4.17) \quad \begin{cases} \varphi(E_\alpha) = E_\alpha, \\ \varphi(V_1) = V_2. \end{cases}$$

if and only if β_1 and β_2 have same eigenvalues.

Proof. The “only if” part is obvious. Let us now assume that β_1 and β_2 have the same eigenvalues $\lambda_1 \geq \dots \geq \lambda_n$, where $n = \dim V - 1$. For $i = 1, 2$, denote by (v_i^1, \dots, v_i^n) a unitary basis of eigenvectors of β_i . Fix a vector ℓ_i of length 1 in $L_i = (V_i)^\perp$ and set $\mathcal{B}_i = (v_i^1, \dots, v_i^n, \ell_i)$. The matrix of α in the unitary basis \mathcal{B}_i has the form

$$A_i = \left(\begin{array}{c|c} D_\lambda & x_i \\ \hline x_i^* & a_i \end{array} \right)$$

where $D_\lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$, $x_i = (x_i^1, \dots, x_i^n) \in \mathbb{C}^n$ and $a_i \in \mathbb{R}$.

For every $X \in \mathbb{R} \setminus \{\lambda_1, \dots, \lambda_n\}$, $D_\lambda - XI_n$ is invertible. Set $y_i = (D_\lambda - XI_n)^{-1}x_i$. Then $y_i^* = x_i^*(D_\lambda - XI_n)^{-1}$. It follows that

$$\left(\begin{array}{c|c} I_n & 0 \\ \hline y_i^* & (a_i - X) - y_i^*x_i \end{array} \right) \left(\begin{array}{c|c} D_\lambda - XI_n & x_i \\ \hline 0 & 1 \end{array} \right) = A_i - XI_{n+1}.$$

Thus we obtain the following relation between the characteristic polynomial $P_\alpha(X)$ of α and the characteristic polynomial $P_\lambda(X) = (\lambda_1 - X) \cdots (\lambda_n - X)$ of β_i :

$$P_\alpha(X) = ((a_i - X) - y_i^*x_i)P_\lambda(X).$$

But observe that

$$a_i = \text{tr } A_i - \text{tr } D_\lambda = \text{tr } \alpha - (\lambda_1 + \cdots + \lambda_n)$$

does not depend on $i \in \{1, 2\}$. It follows that $y_1^*x_1 = y_2^*x_2$, that is

$$\sum_{j=1}^n \frac{|x_1^j|^2}{\lambda_j - X} = \sum_{j=1}^n \frac{|x_2^j|^2}{\lambda_j - X}.$$

Write $\{1, \dots, n\} = I_1 \sqcup \cdots \sqcup I_k$ such that j, j' are in a same I_p if and only if $\lambda_j = \lambda_{j'} =: \lambda_{I_p}$. Define $\psi : V^1 \rightarrow V^2$ as follows. Identifying the coefficients in the above equality between rational functions, we obtain that for any $1 \leq p \leq k$,

$$r = \|x_i^{I_p}\|^2 = \sum_{j \in I_p} |x_i^j|^2$$

does not depend on i . In other words $x_1^{I_p}$ and $x_2^{I_p}$ lie in sphere of same radius in $\mathbb{C}^{|I_p|}$, hence there exists a matrix $g^p \in U(|I_p|)$ such that $x_2^{I_p} = g^p \cdot x_1^{I_p}$. More precisely, for any $j \in I_p$,

$$x_2^j = \sum_{j' \in I_p} g_{j',j}^p x_1^{j'}.$$

Then set for any $j \in I_p$,

$$\psi(v_1^j) = \sum_{j' \in I_p} g_{j',j}^p v_2^{j'}$$

(note that $\psi(v_1^j)$ is an eigenvector of β_2 associated to the eigenvalue λ_{I_p}). Then ψ is unitary and satisfies $\psi(x_1^1 v_1^1 + \cdots + x_1^n v_1^n) = x_2^1 v_2^1 + \cdots + x_2^n v_2^n$.

Now extend ψ to a map $\varphi : V \rightarrow V$ by setting for all $v_1 \in V_1$ and $a \in \mathbb{C}$,

$$\varphi(v_1 \oplus al_1) = \psi(v_1) \oplus al_2.$$

It is clear that φ maps V_1 onto V_2 . Let us check now that it preserves E_α , or equivalently, that $\varphi \circ \alpha = \alpha \circ \varphi$.

For any eigenvector $v_1^j \in V_1$, $j \in I_p$,

$$\alpha \circ \varphi(v_1^j) = \alpha \left(\sum_{j' \in I_p} g_{j',j}^p v_2^{j'} \right) = \sum_{j' \in I_p} g_{j',j}^p (\lambda_{I_p} v_2^{j'} \oplus \bar{x}_2^{j'} l_2) = \lambda_j \psi(v_1^j) \oplus c_j l_2$$

with $c_j = \sum_{j' \in I_p} g_{j',j}^p \bar{x}_2^{j'}$. But since the matrix $(g_{j,j'})$ is in $U(|I_p|)$, its inverse is $(\bar{g}_{j',j})$ and we have $c_j = \bar{x}_1^j$. Hence, $\alpha \circ \varphi(v_1^j)$ is equal to

$$\varphi \circ \alpha(v_1^j) = \varphi(\lambda_j v_1^j \oplus \bar{x}_1^j \ell_1) = \psi(\lambda_j v_1^j) \oplus \bar{x}_1^j \ell_2.$$

Moreover,

$$\alpha \circ \varphi(\ell_1) = \alpha(\ell_2) = \sum_{j=1}^n x_2^j v_2^j \oplus a_2 \ell_2 = \varphi \left(\sum_{j=1}^n x_1^j v_1^j \oplus a_2 \ell_1 \right) = \varphi \circ \alpha(\ell_1)$$

so $\phi : V \rightarrow V$ is the unitary transform given by the lemma. \square

Proposition 4.14. *Let $V_1^\bullet, V_2^\bullet \in \mathcal{F}$ be two complete flags and $C_1^*, C_2^* \in U(n)$ bases for these flags, i.e., $Q(C_i^*) = V_i^\bullet$. Then the following three conditions are equivalent:*

- (1) $\Gamma_\lambda(V_1^\bullet) = \Gamma_\lambda(V_2^\bullet)$,
- (2) $V_2^\bullet = \phi_* V_1^\bullet$ for some $\phi \in G(E_1^\bullet)$,
- (3) $C_2^* = \phi_* C_1^*$ for some $\phi \in G(E_1^\bullet)$.

Proof. Set $E_i^\bullet = E_{D_\lambda} \cap V_i^\bullet$, so $\Gamma_\lambda(V_i^\bullet) = \Gamma(E_i^\bullet)$, and let $\alpha_\bullet = (\alpha_1, \dots, \alpha_n)$ (res. $\beta_\bullet = (\beta_1, \dots, \beta_n)$) be the family of unitary maps defining E_1^\bullet (res. E_2^\bullet).

(3) \Rightarrow (2) is just a consequence of Proposition 4.10.

(2) \Rightarrow (1): if $V_2^\bullet = \phi_* V_1^\bullet$, then by Lemma 4.11,

$$E_{\beta_k} = E_2^k = E_{D_\lambda} \cap V_2^k = \bar{\phi}_k(E_1^k) = \bar{\phi}_k(E_{\alpha_k})$$

hence $\beta_k = \bar{\phi}_k \circ \alpha_k \circ \bar{\phi}_k^*$ and so α_k and β_k have same eigenvalues.

(1) \Rightarrow (2): suppose $\Gamma_\lambda(V_1^\bullet) = \Gamma_\lambda(V_2^\bullet)$ and let us construct $\phi \in G(E^\bullet)$ recursively. Applying Lemma 4.13 to $V_1^{n-1}, V_2^{n-1} \subset \mathbb{C}^n$, we obtain a map $\phi_n : \mathbb{C}^n \rightarrow \mathbb{C}^n$ preserving E_1^n and such that $\phi_n(V_1^{n-1}) = V_2^{n-1}$.

Suppose constructed $\phi_n, \phi_{n-1}, \dots, \phi_{k+1}$ such that $V_2^{i-1} = \bar{\phi}_i(V_1^{i-1})$ for all $k < i \leq n$. Then

$$W_2^{k-1} = \phi_{k+1}^{-1}(\phi_{k+2}^{-1}(\dots \phi_n^{-1}(V_2^{k-1}) \dots)) \subset V_1^k$$

gives a ellipsoid $E_1^k \cap W_2^{k-1}$ in V_1^k with same eigenvalues as E_2^{k-1} , and then same eigenvalues as E_1^{k-1} . Applying Lemma 4.13, we obtain a map $\phi_k : V_1^k \rightarrow V_1^k$ preserving E_1^k and such that $\phi_k(V_1^{k-1}) = W_2^{k-1}$, or equivalently,

$$\phi_n(\phi_{n-1}(\dots \phi_k(V_1^{k-1}) \dots)) = V_2^{k-1}.$$

(1) \Rightarrow (3): suppose $\Gamma_\lambda(V_1^\bullet) = \Gamma_\lambda(V_2^\bullet)$, then we just showed that there exists $\phi \in G(E_1^\bullet)$ such that $V_2^\bullet = \phi_* V_1^\bullet$. Set $C_2'^* = \phi_* C_1^*$. We have

$$Q(C_2'^*) = \phi_* Q(C_1^*) = \phi_* V_1^\bullet = V_2^\bullet = Q(C_2^*)$$

hence there exists $T = \text{diag}(\xi_1, \dots, \xi_n)$ such that $C_2^* = C_2'^* T$. For any $1 \leq i \leq n$, let $f_i : V_1^i \rightarrow V_1^i$ be the multiplication by the scalar $\xi_i \in U(1)$. Then $f = (f_1, \dots, f_n)$ is an element of $H(E_1^\bullet)$. Consider $\psi = \phi \cdot f \in G(E_1^\bullet)$. We have clearly $\bar{\psi}_i = \xi_i \bar{\phi}_i$, hence $\psi_* C_1^* = (\phi_* C_1^*) T = C_2'^* T = C_2^*$. \square

4.3. The symmetry groupoid of ellipsoid flags. Let us fix a value $\Lambda = q(V_0^\bullet) \in \mathbb{R}^N$ given by a complete flag V_0^\bullet , and denote by $\mathcal{F}_\Lambda \ni V^\bullet$ the preimage of Λ under the map Q . For each complete flag $V^\bullet \in \mathcal{F}$, let us denote by $G_\lambda(V^\bullet) = G(E^\bullet)$ the symmetry group of the ellipsoid flag $E^\bullet = E_{D_\lambda} \cap V^\bullet$, and let

$$(4.18) \quad \mathcal{G}_\Lambda = \sqcup_{V^\bullet \in \mathcal{F}_\Lambda} G_\lambda(V^\bullet) = \{(V^\bullet, \phi) \mid V^\bullet \in \mathcal{F}_\Lambda, \phi \in G_\lambda(V^\bullet)\}$$

be the disjoint union of those groups as V^\bullet varies in \mathcal{F}_Λ . Define the source and the target maps $s, t : \mathcal{G}_\Lambda \rightarrow \mathcal{F}_\Lambda$ by the following formula for (V^\bullet, ϕ) in \mathcal{G}_Λ :

$$(4.19) \quad \begin{cases} s(V^\bullet, \phi) = V^\bullet, \\ t(V^\bullet, \phi) = \phi_* V^\bullet, \end{cases}$$

where $\phi_* V^\bullet$ denotes the push-forward of flags defined by Formula (4.13). The composition $\phi \circ \psi$ of two elements (V^\bullet, ϕ) and (W^\bullet, ψ) such that $V = \psi_* W^\bullet$ is defined in an obvious way. For example, $(\phi \cdot \psi)_n = (\phi_n \circ \psi)_n$, $(\phi \cdot \psi)_{n-1} = (\psi_n|_{W_{n-1}})^{-1} \circ \phi_{n-1} \circ (\psi_n|_{W_{n-1}}) \circ \psi_{n-1}$. Similarly, there is a natural way to inverse the elements in \mathcal{G}_Λ . We have the following proposition, whose proof follows immediately from the previous propositions:

Proposition 4.15. *\mathcal{G}_Λ with the above natural maps and a natural smooth structure is a transitive Lie groupoid.*

We refer to [Mac87] for the general theory of transitive Lie groupoids. We will call $\mathcal{G}_\Lambda \rightrightarrows \mathcal{F}_\Lambda$ the **symmetry groupoid of ellipsoid flags** for a given Λ . According to Proposition 4.14, the base manifold \mathcal{F}_Λ of this groupoid is diffeomorphic to the quotient of the group $G_\lambda(V_0^\bullet)$ by the right action of the group $H_\lambda(V_0^\bullet)$ given by Formula (4.11).

Denote by $U(n)_\Lambda^*$ the set of all $C^* \in U(n)$ such that the flag $Q(C^*)$ generated by its columns belongs to \mathcal{F}_Λ . According to Proposition 4.14, we have that $U(n)_\Lambda^*$ is diffeomorphic to the quotient $S(E_0^\bullet) = G_\lambda(V_0^\bullet)/H'_\lambda(V_0^\bullet)$ of the group $G_\lambda(V_0^\bullet)$ by the right action of the group $H'_\lambda(V_0^\bullet)$ given by Formula (4.11). Recall that this quotient space is called the reduced symmetry space of the ellipsoid flag E_0^\bullet , by Definition 4.8. Moreover, the map q restricted to $U(n)_\Lambda^*$ is a submersion from $U(n)_\Lambda^*$ to \mathcal{F}_Λ .

There is a natural smooth action of the Lie groupoid $\mathcal{G}_\Lambda \rightrightarrows \mathcal{F}_\Lambda$ on the submersion $Q : U(n)_\Lambda^* \rightarrow \mathcal{F}_\Lambda$ defined as follows: if $\phi \in G_\lambda(V^\bullet)$ and $C^* \in U(n)_\Lambda^*$ such that $Q(C^*) = V^\bullet$ then

$$(4.20) \quad \phi \cdot C^* = \phi_* C^*.$$

Moreover, this action is transitive on $U(n)_\Lambda^*$.

The above discussions lead to the following result on the smoothness of singular fibers of the Gelfand-Cetlin system:

Theorem 4.16. *For any value Λ of the Gelfand-Cetlin map F_λ , the fiber $F_\lambda^{-1}(\Lambda)$ of the Gelfand-Cetlin system on $\mathcal{O}(\lambda)$ is an embedded smooth manifold diffeomorphic to the quotient manifold*

$$(4.21) \quad U(n)_\Lambda^*/U_\lambda$$

where U_λ is the group of unitary matrices commuting with $D_\lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ and its free (right) action on $U(n)_\Lambda^*$ is defined by

$$(4.22) \quad C^* \cdot P = P^* C^*$$

for all $P \in U_\lambda$ and $C^* \in U(n)_\Lambda^*$.

Remark that, instead of looking at $U(n)_\Lambda^*$, we can also look at its inversion $U(n)_\Lambda$ in $U(n)$, and write the above quotient as $U(n)_\Lambda/U_\lambda$ with the right action $C \cdot P = CP$.

Proof. The proof is straightforward: two matrices $C_1, C_2 \in U(n)_\Lambda$ give rise to the same element in the fiber $F^{-1}(\Lambda) \in \mathcal{O}(\lambda)$ of the Gelfand–Cetlin system if and only if $C_1 D_\lambda C_1^* = C_2 D_\lambda C_2^*$, which means that $P D_\lambda = D_\lambda P$, where $P = C_2^* C_1$, that is, $P \in U_\lambda$. (Recall that $C^* = C^{-1}$ for unitary matrices C). \square

5. DIMENSIONS, DECOMPOSITION AND EXAMPLES OF THE FIBERS

In this section, based on the results of the previous section, we compute the dimensions of the fibers of Gelfand–Cetlin systems. We also show a topological decomposition of these fibers into direct products, and describe explicitly some examples.

As before, we denote by $\lambda = (\lambda_1 \geq \dots \geq \lambda_n > 0)$ a fixed positive spectrum, by $2N$ the dimension of the coadjoint orbit $\mathcal{O}(\lambda)$ and by $F_\lambda : \mathcal{O}(\lambda) \rightarrow \mathbb{R}^N$ the momentum map of the Gelfand–Cetlin system on $\mathcal{O}(\lambda)$. We also consider \mathcal{F} the set of complete flags in \mathbb{C}^n and $\Gamma_\lambda : \mathcal{F} \rightarrow \mathbb{R}^N$ the map that associates to a flag V^\bullet the eigenvalues $\Gamma_\lambda(V^\bullet)$ of the ellipsoid flag $E_{D_\lambda} \cap V^\bullet$. We denote by $p : U(n) \rightarrow \mathcal{O}(\lambda)$ and $q : U(n) \rightarrow \mathcal{F}$ the natural projections. Together with the involution $t : U(n) \rightarrow U(n)$, $t(C) = C^*$, these maps define a commutative diagram given in Figure 2.

In each case we fix a value $\Lambda = (\lambda_{i,j})_{1 \leq i \leq j < n}$ in \mathbb{R}^N , and we fix a flag V^\bullet in $\Gamma_\lambda^{-1}(\Lambda)$. We denote by $\alpha_\bullet = (\alpha_1, \dots, \alpha_n)$ the family of Hermitian operators $\alpha_k : V^k \rightarrow V^k$ defining the ellipsoid flag $E^\bullet = E_{D_\lambda} \cap V^\bullet$. By our assumptions, for each $1 \leq k \leq n$, α_k has eigenvalues $\lambda_{1,k} \geq \lambda_{2,k} \geq \dots \geq \lambda_{k,k}$.

5.1. Dimensions of the fibers. Using the notations and the results of the previous section, we have the following formula:

$$(5.1) \quad \dim F^{-1}(\Lambda) = \dim S(E^\bullet) - \dim U_\lambda = \dim G_\lambda(V^\bullet) - \dim H'_\lambda(V^\bullet) - \dim U_\lambda,$$

which can also be written as

$$(5.2) \quad \dim F^{-1}(\Lambda) = \sum_{k=1}^n (\dim G_k - \dim H'_k) - \dim U_\lambda,$$

where G_k is the group of unitary transformations of V^k preserving $E^k = E_{D_\lambda} \cap V^k$; H_k is the group of unitary transformations of V^k preserving both E^k and E^{k-1} and such that their restriction to $(V^{k-1})^\perp$ is the identity; U_λ is the subgroup of unitary matrices which commute with $\text{diag}(\lambda_1, \dots, \lambda_n)$.

5.2. Regular fibers on generic coadjoint orbits. In the regular case on a generic coadjoint orbit, i.e. where all the inequalities in the Gelfand-Cetlin triangle are strict, all the eigenvalues of each ellipsoid in the ellipsoid flags are distinct and are different from those of the subsequent ellipsoid (i.e. the one of one dimension smaller), for each $k = 1, \dots, n$ we have that H'_k is trivial and $G_k = U(W_{1,k}) \oplus \dots \oplus U(W_{k,k})$ where each $W_{i,k}$ is a 1-dimensional eigenspace of α_k , so we have $G_k \cong U(1)^k \cong \mathbb{T}^k$. Similarly, U_λ consists of only diagonal unitary matrices, so $U_\lambda \cong \mathbb{T}^n$. Thus, in the regular case on a generic coadjoint orbit, we have

$$(5.3) \quad \dim F^{-1}(\Lambda) = \left(\sum_{k=1}^n k \right) - n = \frac{n(n+1)}{2} - n = \frac{n(n-1)}{2} = N,$$

which is exactly half the dimension $n(n-1)$ of a generic coadjoint orbit of $U(n)$. This fact is not surprising, since we know that connected regular fibers of an integrable Hamiltonian systems are Lagrangian tori whose dimension is equal to half the dimension of the symplectic manifold. In our regular generic case, $S(E^\bullet) \cong U(n)_\Lambda$ is a torus of dimension $n(n+1)/2$ on which a torus $U(n)_\lambda \cong \mathbb{T}^n$ of dimension n acts freely, and the quotient space $F^{-1}(\Lambda)$ is a torus of dimension $n(n+1)/2 - n = n(n-1)/2 = N$.

5.3. Elliptic nondegenerate singular fibers. In the case when all the horizontal inequalities in the Gelfand-Cetlin triangle are strict, but there are some diagonal inequalities (of the types $\lambda_{i,k} = \lambda_{i-1,k-1}$ or $\lambda_{i,k} = \lambda_{i,k-1}$), we still have $G_k \cong \mathbb{T}^k$ for each k and $U_\lambda \cong \mathbb{T}^n$, but now some of the groups H'_k are non-trivial tori. For example, if $\lambda_{i,k} = \lambda_{i-1,k-1}$ then it means that the corresponding eigenspaces of α_{k-1} and α_k coincide: $W_{i,k} = W_{i-1,k-1}$ which implies that H'_k contains $U(W_{i,k})$. In general, if there are $s(k) \geq 0$ equalities between the numbers $\lambda_{1,k}, \dots, \lambda_{k,k}$ and the numbers $\lambda_{1,k-1}, \dots, \lambda_{k-1,k-1}$ then the group H'_k is isomorphic to $\mathbb{T}^{s(k)}$. Thus, in this case we have

$$(5.4) \quad S(E^\bullet) \cong \mathbb{T}^{\frac{n(n+1)}{2} - \sum_k s(k)}$$

and the fiber

$$(5.5) \quad F^{-1}(\Lambda) \cong \mathbb{T}^{N - \sum_k s(k)}$$

is a torus of dimension $N - \sum_k s(k)$, where $N = \frac{n(n-1)}{2}$ is the dimension of regular fibers and $\sum_k s(k)$ is the total number of diagonal equalities in the Gelfand-Cetlin triangle.

Remark that, in this case, the momentum map F is smooth at $F^{-1}(\Lambda)$, and $F^{-1}(\Lambda)$ is an elliptic singularity of corank $\sum_k s(k)$, so the fact that it is a torus of dimension $N - \sum_k s(k)$ fits well with the general theory of nondegenerate singularities of integrable Hamiltonian systems (see [Zun96]).

5.4. Spherical singularity of a generic orbit of $\mathfrak{u}(3)^*$. Take $n = 3$ and consider a generic coadjoint orbit of dimension $2N = 6$. Choose $\Lambda = (\lambda_{1,1}, \lambda_{2,1}, \lambda_{2,2})$ such that $\lambda_{2,1} = \lambda_{2,2}$. The inequalities in the Gelfand-Cetlin diagram implies that we have $\lambda_{1,1} = \lambda_{2,1} = \lambda_{2,2} = \lambda_2$.

Computation of $G(E^\bullet)$. We simply apply Proposition 4.3 to obtain:

- $G_1 = U(V^1)$,
- $G_2 = U(V^2)$,
- $G_3 = U(W_1) \oplus U(W_2) \oplus U(W_3)$ where each W_i has dimension 1.

Computation of $H'(E^\bullet)$. For $0 \leq k < 3$, let L_k be the orthogonal complement of V^k in V^{k+1} . Set $A_k = \text{proj}_{V^k}(\alpha_{k+1}(L_k))$ and denote by A'_k its orthogonal complement in V^k . By Proposition 4.5, H'_{k+1} is the set of all transforms

$$\phi_k \oplus \text{id}_{L_k}$$

with $\phi_k \in U(V^k)$ satisfying $\phi_k(v) = v$ for any $v \in A_k$. We then compute the dimension of A_k to determine whether the latter condition on ϕ_k is trivial or not.

- By definition, $H_1 = G_1 = \{\xi \cdot \text{id}_{V^1}\}$ and $H'_1 = \{\text{id}_{V^1}\}$.
- Fix ℓ_1 a unit vector in L_1 and denote $\alpha_2(\ell_1) = w_1 \oplus a_2 \ell_1$. Then by Lemma 4.4, for all $v_1 \in V^1$,

$$\alpha_2(v_1) = \alpha_1(v_1) \oplus \langle v_1 | w_1 \rangle \ell_1.$$

But remark that both α_1 and α_2 are simply the scalar multiplication by λ_2 , so finally we have $\langle v_1 | w_1 \rangle = 0$ for all $v_1 \in V^1$, that is $w_1 = 0$. It follows that $A_1 = \text{Span}(w_1) = \{0\}$, and then

$$H'_2 = \{\phi_1 \oplus \text{id}_{L_1} \mid \phi_1 \in U(V^1)\}.$$

- Since A_2 has dimension at most 1, its complement A'_2 has dimension 1 or 2. But recall that H_3 is diffeomorphic to $U(A'_2) \times U(1)$, so

$$(\dim_{\mathbb{C}} A'_2)^2 + 1 = \dim_{\mathbb{R}} H_3 \leq \dim_{\mathbb{R}} G_3 = 3.$$

It follows that $\dim_{\mathbb{C}} A'_2 = 1 = \dim_{\mathbb{C}} A$. Hence H'_3 rewrites as the subgroup

$$H'_3 = \{\xi_3 \cdot \text{id}_{A'_2} \oplus \text{id}_{A_2} \oplus \text{id}_{L_3} \mid \xi_3 \in U(1)\}.$$

Quotient space $S(E^\bullet) = G(E^\bullet)/H'(E^\bullet)$. We are going to write all the transformations in $G_1, G_2, G_3, H'_2, H'_3$ in the same basis $C = (u_1, u_2, u_3)$, which by definition satisfies $V^k = \text{Span}(u_1, \dots, u_k)$ for all $1 \leq k \leq 3$.

First let us remark that $W_2 \subset V^2$ and is actually equal to A'_2 . Indeed suppose $v_3 \in W_2$ and write

$$v_3 = v_2 \oplus c \ell_2, \quad v_2 \in V^2, \quad c \in \mathbb{C}$$

where ℓ_2 is some fixed unit vector in L_2 . Recall that by Lemma 4.4, if we write $\alpha_3(\ell_2) = w_2 \oplus a_3 \ell_2$ with $w_2 \in V^2$ and $a_3 \in \mathbb{R}$, then $\alpha_3(v_2) = \lambda_2 v_2 \oplus \langle v_2 | w_2 \rangle \ell_2$. Since $v_3 \in W_2$, it satisfies $\alpha_3(v_3) = \lambda_2 v_3$, which rewrites as

$$\begin{cases} \lambda_2 v_2 + c w_2 = \lambda_2 v_2, \\ \langle v_2 | w_2 \rangle + c a_3 = \lambda_2 c. \end{cases}$$

We saw previously that $A_2 = \text{proj}_{V^2}(\alpha_3(L_2)) = \text{Span}(w_2)$ has dimension 1, that is $w_2 \neq 0$. It follows that $c = 0$, hence $v_3 = v_2 \in V^2$. Moreover the above system gives also $\langle v_3 | w_2 \rangle = \langle v_2 | w_2 \rangle = 0$, that is v_3 is orthogonal to A_2 . Since the dimensions agree, we obtain that $W_2 = (A_2)^{\perp V^2} = A'_2$.

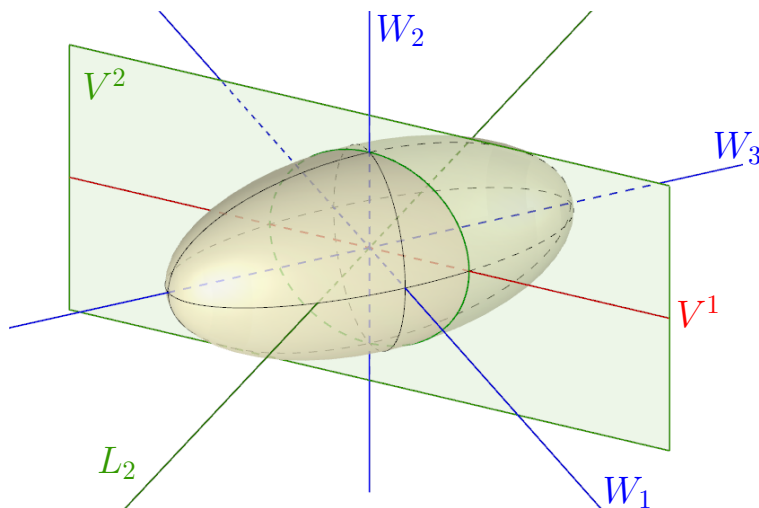


FIGURE 3. A complete flag in \mathbb{C}^3 corresponding to a matrix in the spherical singular fiber on $\mathfrak{u}(3)^*$

Then, note that without loss of generality we can assume that $V^1 = A_2$. Indeed, take $\phi_2 \in U(V^2)$ mapping V^1 onto A_2 , and set $\phi = (\text{id}_{V^1}, \phi_2, \text{id}_{V^3}) \in G(E^\bullet)$. The push-forward $\phi_* V^\bullet$ satisfies $V_\phi^2 = V^2$ and $V_\phi^3 = V^3$, hence only the 1-dimensional ellipsoid E^1 has been modified. In particular the spaces L_k, A_k, A'_k remain unchanged for $k \geq 2$. But according to the Gelfand–Cetlin diagram, the eigenvalue of E_ϕ^1 is necessarily λ , hence $C' = \phi_* C$ lies in the same fiber of Γ_λ as C .

Figure 3 provides a geometric interpretation of the above two facts. Consider E^3 a generic ellipsoid in \mathbb{C}^3 with semi-principal axes v_1, v_2, v_3 of lengths $a_1 < a_2 < a_3$ (with $a_i = 1/\sqrt{\lambda_i}$). If $E^2 = E^3 \cap V^2$ is a circle of radius a_2 contained in E^3 , then v_2 must be an axis of E^2 , giving the condition $W_2 = \text{Span}(v_2) \subset V^2$. Moreover, any ellipsoid $E^1 = E^2 \cap V^1$ has radius a_2 too, so the value $c = \Gamma_\lambda(E^\bullet)$ does not depend on the choice of V^1 .

Finally, if we denote by J_1 the identity on $A_2 = V_1 = \text{Span}(u_1)$, by J_2 the identity on $W_2 = A'_2 = L_1 = \text{Span}(u_2)$ and by J_3 the identity on $L_2 = \text{Span}(u_3)$, we have:

- $G_1 = \{\zeta_1 J_1\}$,
- $G_2 = U(V^2) = U(\mathbb{C}u_1 \oplus \mathbb{C}u_2)$,
- $G_3 = \{\zeta_3 J_2 \oplus \psi_3 \mid \zeta_3 \in U(1), \psi_3 \in T\}$,
- $H'_2 = \{\xi_2 J_1 \oplus J_2 \mid \xi_2 \in U(1)\}$,
- $H'_3 = \{J_1 \oplus \xi_3 J_2 \oplus J_3 \mid \xi_3 \in U(1)\}$.

where T is a subgroup of $U(\mathbb{C}u_1 \oplus \mathbb{C}u_3)$ diffeomorphic to \mathbb{T}^2 . Define a map $\tilde{\varphi} : G(E^\bullet) \rightarrow G_2 \times T$ by setting for all $\phi = (\zeta_1 J_1, \phi_2, \zeta_3 J_2 \oplus \psi_3) \in G(E^\bullet)$,

$$\tilde{\varphi}(\phi) = ((J_1 \oplus \zeta_3 J_2) \circ \phi_2 \circ (\zeta_1 J_1 \oplus J_2), \psi_3).$$

For any $f = (J_1, \xi_2 J_1 \oplus J_2, J_1 \oplus \xi_3 \oplus J_3) \in H'(E^\bullet)$,

$$\phi \cdot f = (\xi_2^* \zeta_1 J_1, (J_1 \oplus \xi_3^* J_2) \circ \phi_2 \circ (\xi_2 J_1 \oplus J_2), \xi_3 \zeta_3 J_2 \oplus \psi_3).$$

The map $\tilde{\varphi}$ is $H'(E^\bullet)$ -invariant and induces a diffeomorphism

$$\varphi : S(E^\bullet) \xrightarrow{\sim} G_2 \times T \approx U(2) \times \mathbb{T}^2$$

Quotient space $S(E^\bullet)/U_\lambda$. Note that modulo $H'(E^\bullet)$, any $\phi \in G(E^\bullet)$ can be written

$$\phi = (J_1, \phi_2, J_2 \oplus \psi_3), \quad \phi_2 \in G_2 = U(V^2), \quad \psi_3 \in T$$

in a unique way, providing an explicit inverse $\varphi^{-1} : G_2 \times T \rightarrow S(E^\bullet)$. For such a ϕ , the push-forward $\phi_*C = (u_1^\phi, u_2^\phi, u_3^\phi)$ is given by

$$\begin{cases} u_3^\phi = \psi_3(u_3), \\ u_2^\phi = (J_2 \oplus \psi_3)(\phi_2(u_2)), \\ u_1^\phi = (J_2 \oplus \psi_3)(\phi_2(u_1)). \end{cases}$$

This can be written in the more convenient form:

$$\forall 1 \leq k \leq 3, \quad u_k^\phi = (J_2 \oplus \psi_3) \circ (\phi_2 \oplus J_3)(u_k).$$

Let us now describe the action of U_λ on $S(E^\bullet)$ explicitly. For any $P = \text{diag}(\xi_1^*, \xi_2^*, \xi_3^*) \in U_\lambda$, the matrix $C' = P^*(\phi_*C)$ has columns (u'_1, u'_2, u'_3) given by the relation

$$\forall 1 \leq k \leq 3, \quad u'_k = (\xi_1 I_1 \oplus \xi_2 I_2 \oplus \xi_3 I_3)(u_k^\phi),$$

where I_j denote the identity map on $W_j = \mathbb{C}e_j$. Note that $J_2 = I_2$ and recall that $\psi_3 = \zeta_1 I_1 \oplus \zeta_3 I_3$ for some $\zeta_1, \zeta_2 \in U(1)$. It follows that

$$(\xi_1 I_1 \oplus \xi_2 I_2 \oplus \xi_3 I_3) \circ (J_2 \oplus \psi_3) \circ (\phi_2 \oplus J_3) = (J_2 \oplus \psi'_3) \circ (\phi'_2 \oplus J_3)$$

where $\psi'_3 = (\xi_1 I_1 \oplus \xi_3 I_3) \circ \psi_3 = \xi_1 \zeta_1 I_1 \oplus \xi_3 \zeta_3 I_3 \in T$ and

$$\phi'_2 = (J_1 \oplus \xi_2 J_2) \circ \phi_2 \in G_2.$$

Hence $P^*(\phi_*C) = \phi'_*C$ with $\phi' = (J_1, \phi'_2, J_2 \oplus \psi'_3)$.

Consider the smooth map

$$\tilde{\Psi} \left| \begin{array}{l} S(E^\bullet) \longrightarrow \mathbb{S}^3 \\ (J_1, \phi_2, J_2 \oplus \psi_3) \longmapsto \phi_2^{-1}(u_1) \end{array} \right.$$

with values in the 3-sphere \mathbb{S}^3 . It is clearly onto. With the above notation, note that

$$(\phi'_2)^{-1}(u_1) = \phi_2^{-1} \circ (J_1 \oplus \xi_2^*)(u_1) = \phi_2^{-1}(u_1),$$

so $\tilde{\Psi}$ is constant along the orbits of the action of U_λ on $S(E^\bullet)$. Moreover, suppose $(\phi'_2)^{-1}(u_1) = \phi_2^{-1}(u_1) = v_1$ for two maps $\phi_2, \phi'_2 \in G_2 = U(V^2)$. Let $v_2 = \phi_2^{-1}(u_2)$ and $v'_2 = (\phi'_2)^{-1}(u_2)$. Since $\phi_2 \in U(V^2)$, (v_1, v_2) and (v_1, v'_2) are two unitary bases of V^2 . It follows that there exists $\xi_2^* \in U(1)$ such that $v'_2 = \xi_2^* v_2$, hence $(\phi'_2)^{-1} = \phi_2^{-1} \circ (J_1 \oplus \xi_2^* J_2)$. Finally, the level-sets of $\tilde{\Psi}$ are exactly the orbits of U_λ since for any $\psi_3 = \zeta_1 I_1 \oplus \zeta_3 I_3$ and $\psi'_3 = \zeta'_1 I_1 \oplus \zeta'_3 I_3$ we have $\psi'_3 = (\xi_1^* I_1 \oplus \xi_3^* I_3) \circ \psi_3$. Hence $\tilde{\Psi}$ induces a diffeomorphism $\Psi : S(E^\bullet)/U_\lambda \rightarrow \mathbb{S}^3$ and by Theorem 4.16 we recover the following result of the thesis of Alamiddine [Ala09]:

Proposition 5.1. *Let $F_\lambda : \mathcal{O}(\lambda) \rightarrow \mathbb{R}^3$ be the Gelfand–Cetlin system on the generic coadjoint orbit $\mathfrak{u}(3)^*$ corresponding to eigenvalues $\lambda_1 > \lambda_2 > \lambda_3$. Then the singular fiber $F_\lambda^{-1}(\lambda_2, \lambda_2, \lambda_2)$ is an embedded submanifold of $\mathcal{O}(\lambda)$ diffeomorphic to the 3-dimensional sphere S^3 .*

Remark 5.2. An important contribution of [Ala09] is the symplectic normal form proved for a neighborhood of these fibers: The symplectic model is that of the geodesic flow of the canonical (symmetric) metric on S^3 .

5.5. Diamond singularities. Consider now the case when there are some horizontal inequalities in the Gelfand-Cetlin triangle of eigenvalues. Say $\lambda_{i,k} = \lambda_{i+1,k}$. In this case, the corresponding fiber $F^{-1}(\Lambda)$ will be called a **degenerate singularity** of the Gelfand-Cetlin system. Notice that the components $F_{i,k}$ and $F_{i+1,k}$ of the Gelfand-Cetlin momentum map (whose values at $F^{-1}(\Lambda)$ are equal, $\lambda_{i,k} = \lambda_{i+1,k}$) are non-smooth at $F^{-1}(\Lambda)$, though their sum and their product are still smooth functions there, but if we change $F_{i,k}$ and $F_{i+1,k}$ by $F_{i,k} + F_{i+1,k}$ and $F_{i,k}F_{i+1,k}$ in the momentum map to make it smooth then the new momentum map will have a degenerate singularity at $F^{-1}(\Lambda)$ (see [Eli90, Zun96, MZ04] for the notion of nondegenerate singularities of smooth integrable systems). That’s why we call $F^{-1}(\Lambda)$ a degenerate singularity.

Assume now, for example, that we have

$$(5.6) \quad \dots > \lambda_{i-1,k} > \lambda_{i,k} = \lambda_{i+1,k} = \dots = \lambda_{i+l-1,k} > \lambda_{i+l,k} > \dots,$$

i.e. we have exactly l equal eigenvalues $\lambda_{i,k} = \lambda_{i+1,k} = \dots = \lambda_{i+l-1,k} = \gamma$ on line k of the Gelfand-Cetlin triangle, $l \geq 2$. Then automatically we must have a diamond of equalities as shown on Figure 4 (assuming that our coadjoint orbit is generic, i.e. on line n of the Gelfand-Cetlin triangle there is no equality).

$$\begin{array}{c}
 \lambda_{i+l-1,k+l-1} \\
 \parallel \quad \parallel \\
 \vdots \quad \parallel \quad \parallel \quad \vdots \\
 \lambda_{i+1,k+1} = \dots = \lambda_{n-1,n-1} \\
 \parallel \quad \parallel \quad \parallel \quad \parallel \\
 \lambda_{i,k} = \lambda_{i+1,k} = \dots = \lambda_{i+l-1,k} \\
 \vdots \quad \parallel \quad \parallel \quad \parallel \quad \parallel \quad \vdots \\
 \lambda_{i,k-l+2} = \lambda_{i+1,k-l+2} \\
 \parallel \quad \parallel \\
 \lambda_{i,k-l+1}
 \end{array}$$

FIGURE 4. A diamond of equalities

Assume that there are no other equalities except the ones in the diamond on Figure 4. Then we will say that Λ is a diamond singular value and $F^{-1}(\Lambda)$ a **diamond**

singularity of the Gelfand-Cetlin system. In this case, the groups G_j and H'_j ($1 \leq j \leq n$) are as follows (with the above indices i, k, l such that $i + l - 1 \leq k$ and $k + l - 1 \leq n$):

- $G_j \cong \mathbb{T}^j$ for $j \leq k - l + 1$ and for $j \geq k + l - 1$; $G_j \cong U(l - |k - j|) \times \mathbb{T}^{k-l+|k-j|}$ for $k - l + 1 < j < k + l - 1$. (The number $l - |k - j|$, when it's positive, is the number of eigenvalues on line j in the Gelfand-Cetlin triangle which are equal to the given value γ).
- H'_j is trivial for $j \leq k - l + 1$ and for $j \geq k + l$; $H'_j \cong U(l - 1/2 - |k - j + 1/2|)$ for $k - l + 1 < j < k + l$. For example, $H'_{k+l-1} \cong H'_{k-l+2} \cong U(1) \cong \mathbb{T}^1$, and $H'_k \cong H'_{k+1} \cong U(l - 1)$.

It follows from the above computations that in this case we have

$$(5.7) \quad S(E^\bullet) \approx U(l) \times \mathbb{T}^{N+n-l^2} \approx SU(l) \times \mathbb{T}^{N+n-l^2+1}$$

and the diamond singular fiber

$$(5.8) \quad F^{-1}(\Lambda) \approx SU(l) \times \mathbb{T}^{N-l^2+1}$$

is of dimension N , equal to the dimension of a regular fiber.

The spherical singularity studied in detail in Subsection 5.4 is a special case of diamond singularities, where we have $n = 3, i = 1, k = 2, l = 2, N = 3, N - l^2 + 1 = 0$, and the fiber is diffeomorphic to $SU(2) \approx S^3$.

The above computations can be generalized to the case of multiple diamonds, when the set of equalities in the Gelfand-Cetlin triangle form exactly $s \geq 1$ diamonds of sizes l_1, \dots, l_s respectively which do not intersect each other. We may call it a **multiple-diamond singularity**. A multiple-diamond fiber will be diffeomorphic to

$$(5.9) \quad SU(l_1) \times \dots \times SU(l_s) \times \mathbb{T}^{N - \sum(l_i^2 + 1)}$$

which is still of the same dimension N as the regular fibers.

5.6. General degenerate singularities. For a general degenerate singularity, instead of diamonds of equalities in the Gelfand-Cetlin triangle, we have parallelograms of equalities which may overlap each other in a vertical fashion to create connected chains of equalities. (The parallelograms do not overlap each other horizontally, in the sense that if we take a connected chain of overlapping parallelograms of equalities, then the intersection of that chain with each line of the Gelfand-Cetlin triangle is connected if not empty). The topology of the singular fibers can be described by these chains of parallelograms: each point in the Gelfand-Cetlin triangle which does not belong to any chain and is not in the uppermost line corresponds to a factor \mathbb{T}^1 , each chain corresponds to some complicated factor, and $F^{-1}(\Lambda)$ is diffeomorphic to the direct product of all these factors.

In the case when there is just one parallelogram of equal eigenvalues of size $a \times b$ (whose vertices in the Gelfand-Cetlin triangle have indices, say, $(p, q), (p, q + a - 1), (p + b - 1, q + b - 1)$ and $(p + b - 1, q + a + b - 2)$ for some p, q), and $\mathcal{O}(\lambda)$ is a generic coadjoint orbit (i.e., the inequalities in the top line of the Gelfand-Cetlin triangle are strict), then

by computations of the groups G_j and H'_j similar to the previous subsection, we get the formula

$$(5.10) \quad \dim F^{-1}(\Lambda) = N - \min(a, b) \cdot |a - b|$$

In particular, if $a \neq b$ then $\dim F^{-1}(\Lambda) < N$, and if $a = b$ then we have a diamond singularities and $\dim F^{-1}(\Lambda) = N$.

In general, we always have

$$(5.11) \quad \dim F^{-1}(\Lambda) \leq N$$

for any fiber $\dim F^{-1}(\Lambda)$. Instead of proving this directly via combinatorial formulas in the general case, one can deduce it from Corollary ?? proved in the next section, which says that all the fibers of the Gelfand-Cetlin system are isotropic.

We observe that *symmetrically overlapping diamonds* also give rise to singular fibers of the same dimension as regular fibers. For example, consider the case with $\lambda_{1,1} = \lambda_{1,2} = \lambda_{2,2} = \lambda_{1,3} = \lambda_{2,3} = \lambda_{3,3} = \lambda_{2,4} = \lambda_{3,4} = \lambda_{2,5} = \lambda_{3,5} = \lambda_{4,5} = \lambda_{3,6} = \lambda_{4,6} = \lambda_{4,7}$ and no other equalities. This is a case of 2 overlapping diamonds of size 3×3 . In this case, we have $\dim G_1 = 1$, $\dim H'_1 = 0$ (as always), $\dim G_2 = 4$, $\dim H'_2 = 1$, $\dim G_3 = 9$, $\dim H'_3 = 4$, $\dim G_4 = 4 + 2$, $\dim H'_4 = 4$, $\dim G_5 = 9 + 2$, $\dim H'_5 = 4$, $\dim G_6 = 4 + 4$, $\dim H'_6 = 4$, $\dim G_7 = 1 + 6$, $\dim H'_7 = 1$. (Most perfect squares in these formulas are dimensions of corresponding unitary groups, and the other numbers are dimensions of tori). It implies that $\sum_{i=1}^7 (\dim G_i - \dim H'_i) = 28 = 1 + 2 + 3 + 4 + 5 + 6 + 7$, which is exactly what one gets in the case of regular fibers, so the dimension of this overlapping diamonds fiber is equal to the dimension of regular fibers.

In general, the dimension of $\dim F^{-1}(\Lambda)$ by the following combinatorial formula. For simplicity, we will show how to deduce this explicit formula only for the case of generic coadjoint orbits; the more general case can be done in a similar way.

To each connected chain of equalities in the Gelfand-Cetlin triangle we associate a sequence of positive numbers

$$(5.12) \quad l_1, l_2, \dots, l_s$$

where $l_1 = 1$ is the number of eigenvalues in the intersection of the chain with the first line (from bottom up) for which this intersection is non-empty (and hence automatically contains exactly one eigenvalue), l_2 is the number of eigenvalues in the intersection of the chain with the next line, and so on, and $l_s = 1$ is for the last line with non-empty intersection with the chain. For example, for the above example of overlapping diamonds, this sequence is 1, 2, 3, 2, 3, 2, 1. In general, for any two consecutive numbers, l_{i-1} and l_i in this chain, there are only 3 possibilities: $l_i - l_{i-1} = 1$ (increase by 1), $l_i - l_{i-1} = 0$ (remains constant), and $l_i - l_{i-1} = -1$ (decrease). Of course, for each increase there must a decrease.

Notice now that if $l_i - l_{i-1} = 1$ then

$$(5.13) \quad \dim G_{(i)} - \dim G_{(i)}^{xchain} - (\dim H'_{(i)} - \dim H'_{(i)}^{xchain}) = l_i - 1 = l_{i-1}$$

where $G_{(i)}$ is the group G_{i+s} , where the line number $i + s$ has the relative index i in the above sequence (i.e., the line number $s + 1$ is the first line which has non-trivial intersection with our chain), and similarly for $H'_{(i)}$, $\dim G_{(i)}^{xchain}$ is the dimension of G_{i+s}

in the case when this chain of equalities disappears (but the other chains of equalities in the Gelfand–Cetlin triangles remain the same), and similarly for $H'_{(i)}{}^{xchain}$.

On the other hand, if $l_i - l_{i-1} = -1$ or $l_i - l_{i-1} = 0$ then

$$(5.14) \quad \dim G_{(i)} - \dim G_{(i)}^{xchain} - (\dim H'_{(i)} - \dim H'_{(i)}{}^{xchain}) = -l_i,$$

Now, if there is no equality of the type $l_i - l_{i-1}$, then since for each increase there is exactly one decrease, all of the above numbers add up to 0, i.e. we have

$$(5.15) \quad \sum_{i=1}^s (\dim G_{(i)} - \dim G_{(i)}^{xchain} - (\dim H'_{(i)} - \dim H'_{(i)}{}^{xchain})) = 0,$$

For example, in the above example of two overlapping diamonds, the above sum is $1 + 2 + (-2) + s + (-2) + (-1) = 0$. But in the more general case, when there are equalities $l_i = l_{i-1}$, they will contribute negative terms to the sum, and we get:

$$(5.16) \quad \sum_{i=1}^s (\dim G_{(i)} - \dim G_{(i)}^{xchain} - (\dim H'_{(i)} - \dim H'_{(i)}{}^{xchain})) = - \sum_{l_i=l_{i-1}} l_i,$$

The above discussions imply the following formula:

Proposition 5.3. *i) For any value Λ of the momentum map of the Gelfand–Cetlin system on any coadjoint orbit $\mathcal{O}(\lambda)$ we have*

$$(5.17) \quad \dim F^{-1}(\Lambda) = \dim F^{-1}(\Lambda_0) - \sum_{l_\alpha=l_{\alpha-1}} l_\alpha \leq \dim F^{-1}(\Lambda_0),$$

where $\dim F^{-1}(\Lambda_0)$ means a regular Lagrangian fiber of the system, and l_α in the sum runs through all the intersections of the connected chains of equalities of eigenvalues with the horizontal lines in the Gelfand–Cetlin triangles which satisfy the equation $l_i = l_{i-1}$ (with the above notations).

ii) The equality $\dim F^{-1}(\Lambda) = \dim F^{-1}(\Lambda_0)$ holds if and only if each connected chain of equalities of eigenvalues in the Gelfand–Cetlin triangle is axially symmetric (so that we have $l_i \neq l_{i-1}$ everywhere in every chain). In the case of a generic coadjoint orbit, it means that each chain is a union of symmetrically overlapping diamonds.

Remark 5.4. Proposition 5.3 was also obtained by Cho–Kim–Oh [CKO18] by a different method and using a different language.

6. ISOTROPIC CHARACTER OF THE FIBERS

The dimension formula (5.17) shows that the dimension of every fiber is less or equal to the dimension of a regular fiber, i.e. half the dimension of the coadjoint orbit, so it supports the conjecture that every fiber is isotropic. Indeed, we have:

Proposition 6.1. *All the fibers of the Gelfand–Cetlin system on any coadjoint orbit of $U(n)$ are isotropic submanifolds.*

Proof. We will show a proof of the above proposition by direct computations, which are a bit technical but straightforward.

Let $A \in \dim F^{-1}(\Lambda_0) \subset \mathcal{O}(\lambda) \subset \mathcal{H}(n)$ be a generic point in a given singular level set, and let $[Y, A]$ and $[Z, A]$ ($Y, Z \in \mathfrak{u}(n)$) be two arbitrary tangent vectors at A of $\mathcal{O}(\lambda)$ (by the notations of Section 2) which are tangent to $F^{-1}(\Lambda_0)$ at A . By Formula (2.3) of the symplectic form, the isotropicness of $\dim F^{-1}(\Lambda_0)$ means that

$$(6.1) \quad \text{tr}(A[Y, Z]) = 0.$$

This is the equality that we will prove here. Notice that we can write $\text{tr}(A[Y, Z]) = \text{tr}(AYZ - AZY) = \text{tr}(AYZ - YAZ) = \text{tr}([A, Y]Z)$, where $[A, Y]$ is a tangent vector to the fiber $\dim F^{-1}(\Lambda_0)$ at A .

Recall that transformation $A \mapsto CAC^* \in \dim F^{-1}(\Lambda_0)$ by a unitary matrix C can be decomposed into a composition of transformations by matrices $C_k \oplus I_{n-k}$ (where $C_k \in U(k)$ and I_{n-k} is the identity matrix of size $(n-k) \times (n-k)$) for $k = 1, \dots, n$ such that the conjugation of A_k by C_k leaves A_{k-1} intact, where A_k is the upper left submatrix of size k of A , i.e. the upper left submatrix of size $(k-1) \times (k-1)$ of $C_k A_k C_k^*$ is equal to A_{k-1} . At the level of tangent vector, it means that we have a decomposition

$$(6.2) \quad Y = \sum_{k=1}^n Y_k \oplus 0_{n-k}$$

where 0_{n-k} means the zero square matrix of size $(n-k) \times (n-k)$, $Y_k \in \mathfrak{u}(k)$ such that the upper left submatrix of size $(k-1) \times (k-1)$ of $[A_k, Y_k]$ is zero. The same decomposition holds for Z : $Z = \sum_{k=1}^n Z_k \oplus 0_{n-k}$.

Now, it is easy to see immediately that if $k \neq h$ then $\text{tr}([A, Y_k \oplus 0_{n-k}](Z_h \oplus 0_{n-h})) = 0$. So in order to show that $\text{tr}([A, Y]Z) = 0$, we just need to show that $\text{tr}([A_k, Y_k]Z_k) = 0$ for each k . It is sufficient to show it for $k = n$, because for $k < n$ the proof will be exactly the same. So now we can assume that $Y = Y_n$ and $Z = Z_n$ such that the upper left submatrices of size $(n-1) \times (n-1)$ of $[A, Y]$ and $[A, Z]$ are zero.

By conjugating simultaneously A, Y, Z by a unitary matrix of the type $C_{n-1} \oplus I_1$ which does not change $\text{tr}([A, Y]Z)$ and the above condition on $[A, Y]$ and $[A, Z]$, we can reduce the problem to the case when $A_{n-1} = \text{diag}(\gamma_1, \dots, \gamma_{n-1})$ is diagonal. In this case, we can write A as:

$$(6.3) \quad A = \begin{pmatrix} \gamma_1 & & 0 & a_1 \\ & \ddots & & \vdots \\ 0 & & \gamma_{n-1} & a_{n-1} \\ \overline{a_1} & \dots & \overline{a_{n-1}} & a_n \end{pmatrix}$$

where $a_i = a_{in}$ and $a_{ni} = \overline{a_{in}} = \overline{a_i}$. The condition that the upper left submatrix of size $(n-1) \times (n-1)$ of $[A, Y]$ is equal to zero means that the entries y_{ij} of Y satisfy the following equations for all $i, j \leq n-1$:

$$(6.4) \quad (\gamma_i - \gamma_j)y_{ij} = y_{in}\overline{a_j} - a_i y_{nj}.$$

In particular, when $i = j \leq n-1$ we have:

$$(6.5) \quad y_{in}\overline{a_i} = a_i y_{ni} = -a_i \overline{y_{in}},$$

which means that $y_{in}\overline{a_i} \in \sqrt{-1}\mathbb{R}$ is pure imaginary. The same equalities hold for the entries of $Z = (z_{ij})$, and in particular we have $z_{in}\overline{a_i} \in \sqrt{-1}\mathbb{R}$ for every $i \leq n-1$.

By direct computations, we have

$$(6.6) \quad \operatorname{tr}([A, Y]Z) = 2\sqrt{-1}\Im\left(\sum_{i=1}^{n-1}(a_i(y_{ii} - y_{nn}) - \gamma_i y_{in}) + \sum_{j < n, j \neq i} y_{ij} a_j\right) \overline{z_{in}},$$

so the point is to show that

$$(6.7) \quad \left(\sum_{i=1}^{n-1}(a_i(y_{ii} - y_{nn}) - \gamma_i y_{in}) + \sum_{j < n, j \neq i} y_{ij} a_j\right) \overline{z_{in}} \in \mathbb{R}.$$

Recall that A is supposed to be a generic element in $\mathcal{O}(\gamma)$. If A is generic but $a_i = 0$ then it means that $a_i = 0$ for all the other A in $\mathcal{O}(\gamma)$ after a conjugation to the form (6.3), which implies that the entry of index (i, n) in $[A, Y]$ is zero, i.e. $\sum_{i=1}^{n-1}(a_i(y_{ii} - y_{nn}) - \gamma_i y_{in}) + \sum_{j < n, j \neq i} y_{ij} a_j = 0$, and so we can delete such terms (corresponding to the values of i such that $a_i = 0$) from (6.7). In other words, we just need to prove that

$$(6.8) \quad \left(\sum_{i < n, a_i \neq 0} (a_i(y_{ii} - y_{nn}) - \gamma_i y_{in}) + \sum_{j < n, j \neq i} y_{ij} a_j\right) \overline{z_{in}} \in \mathbb{R}.$$

Notice that when $a_i \neq 0$ then it follows from $z_{in} \overline{a_i} \in \sqrt{-1}\mathbb{R}$ and $y_{in} \overline{a_i} \in \sqrt{-1}\mathbb{R}$ that we have $y_{in} \overline{z_{in}} \in \mathbb{R}$, hence $\gamma_i y_{in} \overline{z_{in}} \in \mathbb{R}$ because $\gamma_i \in \mathbb{R}$. Moreover we have $a_i(y_{ii} - y_{nn}) \overline{z_{in}} \in \mathbb{R}$ because $a_i \overline{z_{in}}, y_{ii}, y_{nn} \in \sqrt{-1}\mathbb{R}$. Hence (6.7) follows from the following claim:

$$(6.9) \quad \left(\sum_{i, j < n, a_i \neq 0, j \neq i} y_{ij} a_j \overline{z_{in}}\right) \in \mathbb{R}.$$

Consider the terms in the sum in (6.9). If $\gamma_i \neq \gamma_j$ then by (6.4) we have $y_{ij} = \frac{y_{in} \overline{a_j} - a_i y_{nj}}{\gamma_i - \gamma_j}$ and

$$(6.10) \quad y_{ij} a_j \overline{z_{in}} = \frac{1}{\gamma_i - \gamma_j} (y_{in} \overline{a_j} - a_i y_{nj}) a_j \overline{z_{in}} = \frac{1}{\gamma_i - \gamma_j} (\overline{a_j} a_j y_{in} \overline{z_{in}} + (a_i \overline{z_{in}}) (a_j \overline{y_{jn}})) \in \mathbb{R}$$

because $\gamma_i - \gamma_j \in \mathbb{R}$, $y_{in} \overline{z_{in}} \in \mathbb{R}$, $a_i \overline{z_{in}} \in \sqrt{-1}\mathbb{R}$ and $a_j \overline{y_{jn}} \in \sqrt{-1}\mathbb{R}$.

If $\gamma_i = \gamma_j$ then by symmetry between the indices i and j , and the genericity of A , if $a_i \neq 0$ then we also have $a_j \neq 0$. It follows from (6.4) that in this case we have $y_{in} \overline{a_j} = a_i y_{nj} = -a_i \overline{y_{jn}}$, and similarly $z_{in} \overline{a_j} = -a_i \overline{z_{jn}}$. Regrouping the terms of indices (i, j) and (j, i) in the sum (6.10) together, we get:

$$(6.11) \quad y_{ij} a_j \overline{z_{in}} + y_{ji} a_i \overline{z_{jn}} = y_{ij} a_j \overline{z_{in}} - y_{ji} \overline{a_j} z_{in} = y_{ij} a_j \overline{z_{in}} + \overline{y_{ij} a_j} z_{in} \in \mathbb{R}.$$

Thus, the sum in (6.9) is a real number, and we have finished the proof of Proposition 6.1. \square

Remark 6.2. In [CKO18] Cho, Kim and Oh also gave a proof of Proposition 6.1. Their proof is based on their description of each fiber of the Gelfand–Cetlin system as a tower of consecutive fibrations, and is also computational.

In relation to Proposition 6.1, let us mention the following Proposition 6.3 about a sufficient condition for the isotropicness of the singular fibers of analytic integrable systems. The Gelfand-Cetlin system is also analytic. (The original momentum map of the Gelfand-Cetlin system is not globally smooth, but it can be made into a globally analytic system by taking the symmetric functions of the eigenvalues instead of the eigenvalues themselves). Unfortunately, not all fibers of its satisfy this sufficient condition, and since we don't have a more general proposition yet, we had to do a direct computational proof for the Gelfand-Cetlin case.

Proposition 6.3. *Let $c \in \mathbb{R}^\times$ be a singular value of the momentum map $G : (M^{2n}, \omega) \rightarrow \mathbb{R}^n$ of a real analytic momentum map of an integrable Hamiltonian system, such that the fiber $G^{-1}(c)$ of a complexification $G_{\mathbb{C}} : M_{\mathbb{C}}^{2n} \rightarrow \mathbb{C}^n$ of G is an analytic variety of dimension n on which the set of singular points of the complexified momentum map $G_{\mathbb{C}}$ is of positive codimension. ($M_{\mathbb{C}}^{2n} \supset M^{2n}$ is a local complexification of $M_{\mathbb{C}}^{2n}$). Then we have:*

i) If $G^{-1}(c)$ is a submanifold of M^{2n} then it is isotropic.

ii) If $G^{-1}(c)$ is a singular analytic variety then the strata of its Whitney stratification are isotropic.

Proof. Consider a complex singular level set $G_{\mathbb{C}}^{-1}(c) \supset G^{-1}(c)$ where c is a singular value of G . According to Whitney's stratification theorem (see, e.g., [Sch66]) and the above conditions, $G_{\mathbb{C}}^{-1}(c)$ admits a stratification, whose highest dimensional strata are of dimension n and are regular with respect to the map $G_{\mathbb{C}}$, hence they are Lagrangian. By Whitney's regularity condition (b), it implies by induction that all the smaller-dimensional strata are isotropic. In the case when $G^{-1}(c)$ is regular, the set of points of $G^{-1}(c)$ which admits an open neighborhood in $G^{-1}(c)$ lying entirely in a stratum is a dense subset of $G^{-1}(c)$, and at those points we have that $G^{-1}(c)$ is isotropic. By continuity, $G^{-1}(c)$ is isotropic everywhere. The case of when $G^{-1}(c)$ is a singular variety is similar, and in that case every stratum of $G^{-1}(c)$ is isotropic. \square

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