On the parametrization of the controllability subspaces of a controllable pair

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Abstract

Given a controllable linear control system defined by a pair of constant matrices (A, B), the set of controllability subspaces is a stratified submanifold of the set of (A, B)-invariant subspaces. We parametrize each strata by means of coordinate charts. This parametrization has significant differences to that of (A, B) invariant subspaces, showing a more complex geometric structure.

1 Introduction

Consider a time-invariant, linear multivariable system

$$\dot{x} = Ax + Bu$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $m \leq n$. If F is a state feedback and G is a nonsingular matrix, the controllable subspace of $(A + BF, BG)$ is called a *controllability subspace* of the original pair (A, B). Controllability subspaces play an important role in geometric control theory (significant references are [5], [10] and [11]). In [6] the geometry of the set of controllability subspaces of a given dimension has been studied. More precisely it is shown that the set of controllability subspaces $\mathcal{S}$ of a given dimension d, $\mathrm{Ctr}_d(A, B)$, can be stratified according to the controllability indices h of the restriction of (A, B) to $\mathcal{S}$. As shown in [6], the controllability subspaces are precisely those subspaces for which the restriction is controllable (see section 1). So, we have a finite partition

$$\mathrm{Ctr}_d(A, B) = \bigcup_h \mathrm{Ctr}_h(A, B)$$

where each $\mathrm{Ctr}_h(A, B)$ is an orbit space with a structure similar to that of $\mathrm{Inv}_h(B^t, A^t)$ (see [3] and [6]). However, since the restriction defining $\mathrm{Ctr}_h(A, B)$ is not the dual to that defined in a natural way by (B^t, A^t) (see [3]), the geometry of $\mathrm{Ctr}_h(A, B)$ and that of $\mathrm{Inv}_h(B^t, A^t)$ have

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significant differences. In particular, the coordinate atlas obtained in [7] can not be "translated" to the set of controllability subspaces. Our aim in this paper is to obtain a coordinate atlas parameterizing each one of the strata $\mathrm{Ctr}_h(A,B)$. We point out that, in contrast with [4] where the structure of linked and non linked parameters shows that $\mathrm{Inv}_h(B^t,A^t)$ is a vector bundle on a flag manifold (see also [8]), in $\mathrm{Ctr}_h(A,B)$ the situation is much more involved.

In this paper we make use of the following notation. $\mathbb{K}$ is the field of either the complex or real numbers. $\mathcal{M}_{p,q}$ denotes the set of $p\times q$ matrices with entries in $\mathbb{K}$ and $\mathcal{M}^*_{p,q}$ the set of full rank ones. If $p=q$ we write simply $\mathcal{M}_p$ and $\mathcal{M}^*_p$, respectively. The latter set is also denoted $\mathrm{Gl}(p)$. If $X\in\mathcal{M}_{p,q}$ we identify X with the linear map $\mathbb{K}^q \longrightarrow \mathbb{K}^p$ defined in a natural way.

A general reference on differentiable manifolds is included in [9].

2 Preliminaries

We fix a controllable pair (A,B) with $A\in\mathcal{M}_n$ and $B\in\mathcal{M}_{n,m}$ and controllability indices $\underline{k}=(k_1\geq\cdots\geq k_r)$. We will assume without lost of generality that B has full column rank m.

We recall that a subspace $\mathcal{S}$ of $\mathbb{K}^n$ is an (A,B)-invariant subspace if $A(\mathcal{S})\subset\mathcal{S}+\operatorname{Im}B$. The subspace $\mathcal{S}$ is said to be a *controllability subspace* of (A,B) if there exist matrices $F\in\mathcal{M}_{n,m}$ and $G\in\mathcal{M}_{m,\ell}$ such that

$$\mathcal{S}=\operatorname{Im}BG+\operatorname{Im}(A+BF)G+\cdots+\operatorname{Im}(A+BF)^{n-1}BG\,.$$

It is clear that a controllability subspace of (A,B) is an (A,B)-invariant subspace.

A characterization of controllability subspaces in terms of a restriction on (A,B)-invariant subspaces is given in [6]. We recall now the definition of this restriction in an equivalent manner.

Let $\mathcal{S}$ be an (A,B)-invariant subspace and let $F\in\mathcal{M}_{m,n}$ such that $(A+BF)\mathcal{S}\subset\mathcal{S}$. Let $s=\dim(\mathcal{S}\cap\operatorname{Im}B)$ and $\mathcal{S}\cap\operatorname{Im}B=\operatorname{Im}(BG)$ with G an $m\times s$ full rank matrix. If $\mathcal{S}=\operatorname{Im}X$ where X is an $n\times d$ full rank matrix we have from the above relations that $(A+BF)X=X\overline{A}$ and $BG=X\overline{B}$ where $\overline{A}\in\mathcal{M}_d$ and $\overline{B}\in\mathcal{M}_d$ are uniquely determined by these equalities.

Lemma 2.1 *The pair $(\overline{A},\overline{B})$ is well defined modulo feedback equivalence.*

Proof. Let $F'\in\mathcal{M}_{m,n}$, $P\in\mathcal{M}^*_d$, $Q\in\mathcal{M}^*_s$ and $\overline{A}'$, $\overline{B}'$ be such that $(A+BF')XP=XP\overline{A}'$, $BGQ=XP\overline{B}'$. We have to show that $(\overline{A}',\overline{B}')$ is feedback equivalent to $(\overline{A},\overline{B})$. If we keep the matrix F and change X and G by XP and GQ, respectively, our statement follows easily. So, we can suppose that $P=I_d$, $Q=I_s$. Then we can write $(A+BF')X=X\overline{A}'$ as

$$(A+BF)X+BHX=X\overline{A}'$$

with $H=F'-F$. But, $(A+BF)X=X\overline{A}$. Hence

$$X(\overline{A}'-\overline{A})=BHX\,.$$

So, $\operatorname{Im}(BHX)\subset\mathcal{S}\cap\operatorname{Im}B=\operatorname{Im}BG$, and we can define a linear map $\overline{F}:\mathbb{R}^d\longrightarrow\mathbb{R}^s$ such that $BHX=BG\overline{F}$ (recall that BG has full rank). Then

$$X(\overline{A}'-\overline{A})=BG\overline{F}=X\overline{BF}$$

and the lemma follows. ■

Definition 2.2 *With the above notation we define* $(\overline{A},\overline{B})$ *to be a* restriction *of* (A,B) *to* $\mathcal{S}$. *It is well defined modulo feedback equivalence.*

Remark 2.3 *One can check that the relations defining* $(\overline{A},\overline{B})$ *are equivalent to the existence of matrices* $Y \in \mathcal{M}_{m,d}$ *and* $G \in \mathcal{M}_{m,s}$ *making commutative the following diagram*

$$\begin{array}{ccc} \mathbb{K}^d \times \mathbb{K}^s & \xrightarrow{(\overline{A},\overline{B})} & \mathbb{K}^d \\ {\scriptstyle \begin{pmatrix} X & 0 \\ Y & G \end{pmatrix}} \Big\downarrow & & \Big\downarrow {\scriptstyle X} \\ \mathbb{K}^n \times \mathbb{K}^m & \xrightarrow[(A,B)]{} & \mathbb{K}^n \end{array}$$

where $s = \dim(\mathcal{S} \cap \operatorname{Im} B)$ *and the vertical arrows are full rank matrices (we can always put* $Y = FX$ *for a suitable* $F : \mathbb{K}^m \longrightarrow \mathbb{K}^n$*). Then,*

$$\operatorname{Im} \begin{pmatrix} X & 0 \\ Y & G \end{pmatrix} = \{(x,y) \in \mathcal{S} \times \mathbb{K}^m;\ Ax + By \in \mathcal{S}\} \ .$$

In fact, the inclusion $\subset$ *follows from the commutativity of the diagram. Conversely, let* $(x,y) \in \mathcal{S} \times \mathbb{K}^m$ *such that* $Ax+By \in \mathcal{S}$. *Since* $x \in \mathcal{S}$ *we have that* $x = Xu$ *for some* $u \in \mathbb{K}^d$. *As* $\begin{pmatrix} X & 0 \\ Y & G \end{pmatrix}$ *has full column rank,* $y = Yz + Gv$ *for some vectors* $z \in \mathbb{K}^d$ *and* $v \in \mathbb{K}^s$. *The commutativity of the diagram, which is equivalent to the equalities* $AX + BY = X\overline{A}$ *and* $BG = X\overline{B}$*, implies that* $BY(z-u) \in \mathcal{S}$. *But* $\mathcal{S} \cap \operatorname{Im} B = \operatorname{Im} BG$ *and* B *is injective. Therefore,* $Y(z-u) = Gw$ *for some* $w \in \mathbb{K}^d$*, and so* $y = Yu + G(v+w)$ *following our assertion.*

Remark 2.4 *Let* f, π *be the maps from* $\mathbb{K}^n \times \mathbb{K}^m$ *to* $\mathbb{K}^n$ *defined by* $f(x,y) = Ax + By$ *and* $\pi(x,y) = x$*, respectively. In [6] a more intrinsic definition of the above restriction is given in terms of the pair* (f,π). *In fact, the equality proved in the previous remark says that*

$$\operatorname{Im} \begin{pmatrix} X & 0 \\ Y & G \end{pmatrix} = \pi^{-1}(\mathcal{S}) \cap f^{-1}(\mathcal{S})$$

so that $(\overline{A},\overline{B})$ *is the matrix of the restriction of* (A,B) *to* $\pi^{-1}(\mathcal{S}) \cap f^{-1}(\mathcal{S}) \longrightarrow \mathcal{S}$ *in a suitable basis. This links definition 2.2 with the definition of restriction given in [6], which generalizes the one given in [1].*

In [2] all the possible controllability indices of $(\overline{A},\overline{B})$ with regard to those of (A,B) are described (see (1) and (2) below) . On the other hand, it is proved in [6] that an (A,B)-invariant subspace $\mathcal{S}$ is a controllability subspace if and only if $(\overline{A},\overline{B})$ is controllable. Moreover, if we denote by $\operatorname{Ctr}_h(A,B)$ the set of controllability subspaces $\mathcal{S}$ of (A,B) such that $\underline{h} = (h_1 \geq \cdots \geq h_s)$ are the controllability indices of any restriction $(\overline{A},\overline{B})$ of (A,B) to $\mathcal{S}$, $\operatorname{Ctr}_h(A,B)$ is described as an orbit space. Let us recall the main result. Let $s = \dim(\mathcal{S} \cap \operatorname{Im} B)$ and denote by $M(k,h)$ the set of matrices X such that

(a) $X \in \mathcal{M}^*_{n,d}$, $d = \dim S$.

(b) $X = [X_{ij}]$, $1 \leq i \leq r$, $1 \leq j \leq s$ with

$$X_{ij} = \begin{pmatrix} x^1_{i,j} & \dots & x^{h_j-k_i+1}_{i,j} & 0 & 0 & \dots & 0 \\ 0 & x^1_{i,j} & \dots & x^{h_j-k_i+1}_{i,j} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & x^1_{i,j} & \dots & x^{h_j-k_i+1}_{i,j} \end{pmatrix}$$

if $k_i \leq h_j$ or 0 otherwise.

Remark 2.5 *Notice that* $s = \dim(\mathcal{S} \cap \operatorname{Im} B)$ *is equivalent to* $\operatorname{rank} \begin{pmatrix} X & B \end{pmatrix} = d + m - s$. *Notice also that* $s = \operatorname{rank} \overline{B}$.

If $k = h$, we write $M(h,h) = G(h)$. Then, the following result is proved in [6]

Theorem 2.6 *With the above notation,*

(i) $G(h)$ *is a Lie subgroup of* $\mathrm{Gl}(d)$ *which acts freely on* $M(k,h)$ *on the right by matrix multiplication.*

(ii) The orbit space $M(k,h)/G(h)$ *has a differentiable structure such that the natural projection* $\pi : M(k,h) \longrightarrow M(k,h)/G(h)$ *is a submersion.*

(iii) The map $X \longmapsto \operatorname{Im} X$, *with* $X \in M(k,h)$ *induces a bijection between* $M(k,h)/G(h)$ *and* $\mathrm{Ctr}_h(A,B)$. *Through this bijection* $\mathrm{Ctr}_h(A,B)$ *is a differentiable manifold.*

(iv) $\dim \mathrm{Ctr}_h(A,B) = \dim M(k,h) - \dim G(h) =$

$$= \sum_{1 \leq i \leq r, 1 \leq j \leq s} \sup\{k_j - k_i + 1, 0\} - \sum_{1 \leq i,j \leq s} \sup\{h_j - h_i + 1, 0\} =$$

$$= \sum_{i=1}^{h} s_i((r_1 - s_1) - (r_{i+1} - s_{i+1}))$$

where $\underline{r} = (r_1 \geq \cdots \geq r_k)$, $\underline{s} = (s_1 \geq \cdots \geq s_h)$ *are the conjugate partitions of* $\underline{k}$ *and* $\underline{h}$, *respectively.*

Notice that $s_1 = \operatorname{rank} \overline{B} = \operatorname{rank}\,(BG) = \dim(S \cap \operatorname{Im} B) = s$.

If we reorder the Brunovsky bases we obtain a matrix representation of the subspaces in $\mathrm{Ctr}_h(A,B)$ more convenient for our purposes. We illustrate it with an example.

Consider $k = (4,3,3,1,1)$ and $h = (3,3,1)$. Then, $S = \operatorname{Im} X$ where $X \in M(k,h)$ has the form

$$X = \left(\begin{array}{ccc|ccc|c}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
x_1 & 0 & 0 & x_9 & 0 & 0 & 0 \\
0 & x_1 & 0 & 0 & x_9 & 0 & 0 \\
0 & 0 & x_1 & 0 & 0 & x_9 & 0 \\
\hline
x_2 & 0 & 0 & x_{10} & 0 & 0 & 0 \\
0 & x_2 & 0 & 0 & x_{10} & 0 & 0 \\
0 & 0 & x_2 & 0 & 0 & x_{10} & 0 \\
\hline
x_3 & x_4 & x_5 & x_{11} & x_{12} & x_{13} & x_{17} \\
\hline
x_6 & x_7 & x_8 & x_{14} & x_{15} & x_{16} & x_{18}
\end{array}\right)$$

Denote by

$$(v_{11}, v_{12}, v_{13}, v_{14};\ v_{21}, v_{22}, v_{23};\ v_{31}, v_{32}, v_{33};\ v_{41}, v_{51})$$

and

$$(u_{11}, u_{12}, u_{13};\ u_{21}, u_{22}, u_{23};\ u_{31})$$

the corresponding bases of $\mathbb{K}^n$ and S, respectively.

If we arrange the above bases in the following way

$$\begin{array}{lcl} v_{11}, v_{12}, v_{13}, v_{14} & & v_{51}, v_{41}, v_{33}, v_{23}, v_{14} \\ v_{21}, v_{22}, v_{23} & & v_{32}, v_{22}, v_{13} \\ v_{31}, v_{32}, v_{33} & \longrightarrow & v_{31}, v_{21}, v_{12} \\ v_{41} & & v_{11} \\ v_{51} & & \end{array}$$

$$\begin{array}{lcl} u_{11}, u_{12}, u_{13} & & u_{31}, u_{23}, u_{13} \\ u_{21}, u_{22}, u_{23} & \longrightarrow & u_{22}, u_{12} \\ u_{31} & & u_{21}, u_{11} \end{array}$$

the matrix representation of S in these bases is

$$Z = \left(\begin{array}{ccc|cc|cc} x_{18} & x_{16} & x_8 & x_{15} & x_7 & x_{14} & x_6 \\ x_{17} & x_{13} & x_5 & x_{12} & x_4 & x_{11} & x_3 \\ 0 & x_{10} & x_2 & 0 & 0 & 0 & 0 \\ 0 & x_9 & x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & x_{10} & x_2 & 0 & 0 \\ 0 & 0 & 0 & x_9 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & x_{10} & x_2 \\ 0 & 0 & 0 & 0 & 0 & x_9 & x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The row-block sizes are now $(5, 3, 3, 1)$ and the column-block sizes $(3, 2, 2)$. These are the conjugate partitions of k and h respectively. If $k = (k_1, \ldots, k_r)$ are the controllability indices of (A, B), its conjugate partition $r = (r_1, \ldots, r_k)$ are the *Brunovsky indices* of (A, B).

Remark 2.7 *Let P be the permutation matrix representing the above change of bases. Then,*

$$P^{-1}B = \begin{pmatrix} I_m \\ 0 \end{pmatrix},$$

so that $\operatorname{rank} \begin{pmatrix} X & B \end{pmatrix} = m + d - s_1$ *if and only if* $\operatorname{rank} Z_0 = d - s_1$, *$Z_0$ being the submatrix of Z obtained by removing the first r_1 rows and s_1 columns.*

Definition 2.8 *We denote by $\mathcal{M}(r, s)$ the set of matrices Z such that*

(α) $Z \in \mathcal{M}^*_{n,d}$, $d = \dim S$

(β) $Z = [Z_{ij}]$, $1 \le i \le k$, $1 \le j \le h$, *where $Z_{i,j}$ is a $r_i \times s_j$-matrix with*

(β_1) $Z_{ij} = 0$ *if* $1 \le j \le h$, $j \le i \le k$

(β_2) $Z_{ij} = [Z_{pq}^{j-i+1}]$, $i \leq p \leq k$, $j \leq q \leq h$ *with* Z_{pq}^{j-i+1} *of size* $(r_p - r_{p+1}) \times (s_q - s_{q+1})$ *and* $Z_{pq}^{j-i+1} = 0$ *if* $1 + i \leq p \leq k, \quad p < q \leq k$.

(γ) rank $Z_0 = d - s_1$, *where* $Z_0 = [Z_{ij}]$, $2 \leq i \leq k$ *and* $2 \leq j \leq h$ *(see remark 2.7).*

If $r = s$ *we write* $\mathcal{M}(r, s) = \mathcal{G}(s)$

Remark 2.9 *Notice that the matrix* Z *of the previous definition can be derived easily from the following two rules*

(i) Each block $Z_{i+1,j+1}$ *is obtained from* Z_{ij} *by removing the first* $r_i - r_{i+1}$ *rows and the first* $s_j - s_{j+1}$ *columns. Hence only different parameters can appear in the upper blocks* $Z_{11}, Z_{12}, \ldots, Z_{1h}$.

(ii) Z*, as well as each one of its* Z_{ij} *blocks, is an upper block triangular matrix.*

As already said, in [2] the compatibility conditions between the Brunovsky indices of a pair and its restriction to an (A, B)-invariant subspaces so as the set $\mathcal{M}(r, s)$ to be a nonempty set were described. These conditions are as follows (see [2, Corollary 3.3]):

$$(i) \qquad r_i \leq s_i + (r_1 - s_1 - 1), \quad i = 1, \ldots, n \tag{1}$$

and

$$(ii) \qquad \sum_{j=1}^{h_p} (r_j - s_j - p) \geq 0, \quad 1 \leq p \leq r_1 - s_1, \tag{2}$$

where $h_p := \max\{i : r_i - s_i \geq p\}$, $p = 1, \ldots, r_1 - s_1$.

Notice that the inequality in (1) extends up to n. It must be understood that $s_i := 0$ for $i > h_1$ and $r_i := 0$ for $i > k_1$. It should be noticed also that, unlike the case of $\binom{A}{B}$-invariant subspaces, it may happen that $h_1 \geq k_1$.

We will assume from now on that conditions (1) and (2) hold true.

In the above example the block decomposition of matrix Z would be

$$\begin{array}{c} \left(\begin{array}{ccc|cc|c}
Z_{11}^1 & Z_{12}^1 & Z_{13}^1 & Z_{12}^2 & Z_{13}^2 & Z_{13}^3 \\
0 & Z_{22}^1 & Z_{23}^1 & 0 & Z_{23}^2 & 0 \\
0 & 0 & Z_{33}^1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & Z_{22}^1 & Z_{23}^1 & Z_{23}^2 \\
0 & 0 & 0 & 0 & Z_{33}^1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & Z_{33}^1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\begin{array}{l} r_1 - r_2 \\ r_2 - r_3 \\ r_3 - r_4 \\ r_4 \\ r_2 - r_3 \\ r_3 - r_4 \\ r_4 \\ r_3 - r_4 \\ r_4 \\ r_4 \end{array} \\
\begin{array}{cccccc} s_1 - s_2 & s_2 - s_3 & s_3 & s_2 - s_3 & s_3 & s_3 \end{array} \end{array}$$

with $r = (5, 3, 3, 1)$, $s = (3, 2, 2)$ and

$$
\begin{array}{l}
Z^1_{11} = \begin{pmatrix} x_{18} \\ x_{17} \end{pmatrix}, \; Z^1_{12} = \emptyset, \; Z^1_{13} = \begin{pmatrix} x_{16} & x_8 \\ x_{13} & x_5 \end{pmatrix}, \\
Z^1_{22} = \emptyset, \; Z^1_{23} = \emptyset \\
Z^1_{33} = \begin{pmatrix} x_{10} & x_2 \\ x_9 & x_1 \end{pmatrix} \\
Z^2_{12} = \emptyset, \; Z^2_{13} = \begin{pmatrix} x_{15} & x_7 \\ x_{12} & x_4 \end{pmatrix} \\
Z^2_{23} = \emptyset \\
Z^3_{13} = \begin{pmatrix} x_{14} & x_6 \\ x_{11} & x_3 \end{pmatrix}
\end{array}
$$

Let $\underline{r} = (r_1 \geq \cdots \geq r_k)$ and $\underline{s} = (s_1 \geq \cdots \geq s_h)$ be the conjugate partitions of k and h, respectively. Then, the natural map

$$M(k, h) \longrightarrow \mathcal{M}(r, s)$$

consisting on a change of bases by fixed permutation matrices is a diffeomorphism inducing a bijection

$$M(k, h)/G(h) \cong \mathcal{M}(r, s)/\mathcal{G}(s).$$

Then, one can replace, in theorem 2.6, $M(k, h)$ and $G(h)$ by $\mathcal{M}(r, s)$ and $\mathcal{G}(s)$, respectively.

3 A reduced form

The manifold $\mathrm{Ctr}_h(A, B)$ can be parametrized through a set of coordinate charts obtained as a system of canonical representatives of the orbits of its matrix description $\mathcal{M}(r, s)/\mathcal{G}(s)$. The algorithm for reducing an element of $\mathcal{M}(r, s)$ to a canonical form is based on a sequence of elementary transformations defined by some subsets of $\mathcal{G}(s)$. Let us write explicitly an element $P \in \mathcal{G}(s)$. This matrix can be partitioned as $P = (P_{ij})$ with

$$
P_{ij} = \begin{pmatrix}
P^{\alpha}_{ij} & P^{\alpha}_{i,j+1} & \dots & \dots & P^{\alpha}_{i,h} \\
0 & P^{\alpha}_{i+1,j+1} & \dots & \dots & P^{\alpha}_{i+1,h} \\
0 & 0 & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & 0 & P^{\alpha}_{i+h-j,h} \\
0 & 0 & \dots & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots \\
0 & \dots & \dots & \dots & 0
\end{pmatrix}
$$

$1 \leq i, j \leq h$, $i \leq j$ and 0 otherwise ($\alpha = j - i + 1$).

From the action of P on $Z \in \mathcal{M}$ a canonical representative of the orbit $Z\mathcal{G}(s)$ can be derived. For convenience we introduce the following notation.

(i) If $Z = [Z_{ij}]$ and $Z_{ij} = [Z_{pq}^{j-i+1}]$ we write for $\ell = 1, \ldots, h$ and $q \geq \ell$

$$Z_q^\ell = \begin{pmatrix} Z_{1q}^\ell \\ Z_{2q}^\ell \\ \vdots \\ Z_{\delta_{q\ell},q}^\ell \end{pmatrix}$$

where $\delta_{q\ell} = \min(q - \ell + 1, k)$.

So,

$$Z_{1j} = \left(\begin{array}{c|c|c|c} Z_j^j & Z_{j+1}^j & \cdots & Z_h^j \\ \hline 0 & \hline 0 & & \hline 0 \end{array}\right) \qquad 1 \leq j \leq h$$

(ii) We denote by:

((ii)$_1$) $\prod_i$ a block diagonal matrix $P \in \mathcal{G}(s)$, such that

$$P_{11} = \mathrm{diag}(I_{s_1 - s_2}, I_{s_2 - s_3}, \ldots, P_{ii}^1, \ldots, I_{s_h}), \quad 1 \leq i \leq h$$

(Notice that from remark 2.9, $\prod_i$ is completely determined from its first diagonal block).

((ii)$_2$) $\overline{\prod}_{ij}^\alpha$ a matrix $P \in \mathcal{G}(s)$ such that the only possible non zero block is P_{ij}^α, $\alpha \geq 2, 1 \leq i \leq j - \alpha + 1$.

((ii)$_3$) $\prod_{ij}^\alpha = I_d + \overline{\prod}_{ij}^\alpha$.

We call the matrices $\prod_i$ and $\prod_{ij}^\alpha$ *elementary matrices* and the corresponding actions, *elementary actions*.

The following proposition, whose proof is an easy consequence of the previous definitions, describes the effect on a matrix $Z \in \mathcal{M}(r, s)$ of these elementary actions. In fact, taking into account remark 2.9, we can limit ourselves to consider the action on the upper blocks $Z_{11}, \ldots, Z_{1h}$.

Proposition 3.1 *With the above notation the following holds*

1. *The upper blocks of* $Z \prod_i$ *are the same as those of* Z *except the blocks* $Z_i^1, \ldots, Z_i^i$ *which become* $Z_i^1 P_{ii}^1, \ldots, Z_i^i P_{ii}^1$*, respectively.*
2. *The upper blocks of* $Z \prod_{ij}^\alpha$ *are the same as those of* Z *except the blocks* $Z_j^\alpha, \ldots, Z_j^{\alpha+i-1}$ *which become* $Z_j^\alpha + Z_i^1 P_{ij}^\alpha, \ldots, Z_j^{\alpha+i-1} + Z_i^i P_{ij}^\alpha$.

Notice that the second action consists of adding to a block Z_j^ℓ linear combinations of the columns of the blocks $Z_1^1 \ldots, Z_{j-l+1}^1$.

We proceed now to describe the reduction process for a matrix $Z \in \mathcal{M}(r, s)$.

Step 1. We begin with the block $Z_1^1 = Z_{11}^1$ of size $(r_1 - r_2) \times (s_1 - s_2)$. Since $s_1 - s_2 \leq r_1 - r_2$ because of the full rank condition of Z, we can choose $s_1 - s_2$ linearly independent rows, $n_{11} <$

$n_{12} < \cdots < n_{1s_1-s_2}$. Then we take P_{11}^1 so that the submatrix of $Z_{11}^1 P_{11}^1$ formed by these rows is the identity matrix. Now, we can find matrices $\prod_{1j}^1$ making zeros the rows $n_{11}, \dots, n_{1s_1-s_2}$ of the blocks Z_{1j}^1, $j = 2, \dots, h$. Similarly, with matrices $\prod_{1j}^\alpha$ we make zero the same rows of all blocks Z_{1j}^α.

Step 2. We look at the submatrix of Z_2^1 obtained by removing the first $r_1 - r_2$ rows (see remark 2.7) and the rows $n_{11}, \dots, n_{1s_1-s_2}$. This is actually the submatrix of the $(1,1)$-block of Z_0 obtained by removing the rows $n_{11}, \dots, n_{1s_1-s_2}$. Since Z_0 has full column rank, this submatrix has also full column rank $s_2 - s_3$. Thus we can choose $s_2 - s_3$ linearly independent rows $n_{21} < n_{22} < \cdots < n_{2s_2-s_3}$ with $n_{21} \geq r_1 - r_2$.

Then we take a matrix P_{22}^1 so that the submatrix of $Z_2^1 P_{22}^1$ formed by this second set of rows is the identity matrix. Then with matrices $\prod_{2j}^1$ we make zero the rows $n_{21}, \dots, n_{2s_2-s_3}$ of Z_j^1, j=3,…,h, and with matrices $\prod_{2j}^\alpha$ we make zero the same rows of the blocks Z_j^α. Notice that the unit vector of the rows $n_{21}, \dots, n_{2s_2-s_3}$ *we are not allowed* to make zero elements of the blocks $Z_{12}^2, Z_{13}^3, \dots$

Step 3. We look at the submatrix of Z_3^1 obtained by removing the first $r_1 - r_2$ rows and the rows $n_{11}, \dots, n_{1s_1-s_2}, n_{21}, \dots, n_{2s_2-s_3}$ and we proceed in an analogous way as in the previous step. ■

The process ends after a finite number of steps and proves the following result.

Theorem 3.2 *For every $Z \in \mathcal{M}(r,s)$ there exist both a set of positive integers pairwise different*

$$\begin{aligned} I = \{ n_{ij};\ & 1 \leq n_{11} \leq \cdots \leq n_{1s_1-s_2} \leq r_1 - r_2\,, \\ & r_1 - r_2 \leq n_{i1} \leq \cdots \leq n_{is_i-s_{i+1}} \leq r_2 - r_{i+1}\,,\ i = 2, \dots, h\} \end{aligned}$$

and a matrix $P \in \mathcal{G}(s)$ such that the matrix $Y = ZP$ satisfies the following conditions:

If $Y = (Y_{ij})$, with $Y_{ij} \in \mathcal{M}_{r_i,s_j}$, where $Y_{ij} = 0$ for $1 \leq j \leq h$, $j < i \leq k$ and $Y_{ij} = (Y_{pq}^{j-i+1})$, $i \leq p \leq k$, $j \leq q \leq h$, with

$$Y_{pq}^{j-i+1} = 0\,, \quad i+1 \leq p \leq k, q < p \leq k\,,$$

Y_{pq}^{j-i+1} of size $(r_p - r_{p+1}) \times (s_q - s_{q+1})$, then

(i) For $q \geq 1$, if

$$Y_q^1 = \begin{pmatrix} Y_{1q}^1 \\ \vdots \\ \vdots \\ Y_{qq}^1 \end{pmatrix}$$

the rows n_{ij} with $1 \leq i \leq q-1$, $1 \leq j \leq s_1 - s_q$ are zero and the rows $n_{q1}, \dots, n_{qs_q-s_{q+1}}$ are unit vectors.

(ii) For $\alpha = 2, \dots, h$ and $q \geq \alpha$, if

$$Y_q^\alpha = \begin{pmatrix} Y_{1q}^\alpha \\ \vdots \\ \vdots \\ Y_{q-\alpha+1,q}^\ell \end{pmatrix}$$

the rows n_{ij} with $1 \le i \le q-1$, $1 \le j \le s_1 - s_q$ are zero.

(iii) The matrix $Y_0 = (Y_{ij})$, $2 \le i \le k$, $2 \le j \le h$ must have full column rank.

Definition 3.3 *We call the matrix Y a* reduced form *of Z and the set of indices I verifying the conditions given in theorem 3.2 an* admissible set of indices *for Z.*

We illustrate the above theorem with two examples.

Example 3.4 *Let $Z \in \mathcal{M}((6,3,1),(4,2,1))$. Taking the set of admissible indices $n_{1,1} = 1$, $n_{1,2} = 3$, $n_{2,1} = 4$, $n_{3,1} = 5$, the corresponding reduced form is*

$$Y = \left(\begin{array}{cccc|cc|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1 & x_2 & x_3 & x_5 & x_7 & x_8 & x_{10} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & x_4 & 1 & 0 & x_9 & 0 \\
0 & 0 & 0 & x_6 & 0 & 0 & 0 \\
\hline
 & & & & 1 & 0 & 0 \\
 & & & & x_4 & 1 & x_9 \\
 & & & & 0 & x_6 & 0 \\
\hline
 & & & & & & x_6
\end{array}\right)$$

The number of parameters in Y is $N = 10$, which coincides with $\dim \mathrm{Ctr}_{(3,2,1,1)}(A,B)$ according to the formula given in theorem 2.6. The controllability indices of (A,B) in this example are $k = (3,2,1,1,1,1)$. Also, the matrix

$$Y_0 = \begin{pmatrix} 1 & 0 & 0 \\ x_4 & 1 & x_9 \\ 0 & x_6 & 0 \\ 0 & 0 & x_6 \end{pmatrix}$$

must have full rank. This condition is equivalent to $\det(Y_0^* Y_0) \neq 0$. In this case

$$\det(Y_0^* Y_0) \neq 0 \Leftrightarrow x_6 \neq 0,$$

condition that can be seen equivalent to $\operatorname{rank} Y_0 = 3$ *by direct inspection. We will say that x_6 is a* linked *parameter of Y. Thus we find out that there may be parameters of different nature in Y:* free *and* linked *parameters.*

Example 3.5 *If $Z \in \mathcal{M}((6,5,4),(4,3,3,2))$, taking the set of integers admissible for Z $n_{11} = 1$,*

$n_{31} = 2$, $n_{41} = 3$ y $n_{42} = 4$, the corresponding reduced form is as follows

$$
Y = \left(\begin{array}{cccc|ccc|ccc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & x_1 & 0 & 0 & 0 & x_2 & x_3 & 0 & 0 \\
0 & x_4 & 1 & 0 & 0 & x_5 & x_6 & 0 & 0 & 0 & 0 & 0 \\
0 & x_7 & 0 & 1 & 0 & x_8 & x_9 & 0 & 0 & 0 & 0 & 0 \\
0 & x_{10} & x_{11} & x_{12} & 0 & x_{13} & x_{14} & 0 & 0 & 0 & 0 & 0 \\
0 & x_{15} & x_{16} & x_{17} & 0 & x_{18} & x_{19} & 0 & 0 & 0 & 0 & 0 \\
\hline
 & & & & 1 & 0 & 0 & x_1 & 0 & 0 & x_2 & x_3 \\
 & & & & x_4 & 1 & 0 & 0 & x_5 & x_6 & 0 & 0 \\
 & & & & x_7 & 0 & 1 & 0 & x_8 & x_9 & 0 & 0 \\
 & & & & x_{10} & x_{11} & x_{12} & 0 & x_{13} & x_{14} & 0 & 0 \\
 & & & & x_{15} & x_{16} & x_{17} & 0 & x_{18} & x_{19} & 0 & 0 \\
\hline
 & & & & & & & x_4 & 1 & 0 & x_5 & x_6 \\
 & & & & & & & x_7 & 0 & 1 & x_8 & x_9 \\
 & & & & & & & x_{10} & x_{11} & x_{12} & x_{13} & x_{14} \\
 & & & & & & & x_{15} & x_{16} & x_{17} & x_{18} & x_{19}
\end{array}\right)
$$

The number of parameters in Y is 19 which coincides, as in the previous example, with the dimension of $\mathrm{Ctr}_{(4,4,3,1)}(A,B)$. The controllability indices of (A,B) in this example are $k = (3,3,3,3,2,1,1)$. Also

$$
Y_0 = \left(\begin{array}{ccc|ccc|cc}
1 & 0 & 0 & x_1 & 0 & 0 & x_2 & x_3 \\
x_4 & 1 & 0 & 0 & x_5 & x_6 & 0 & 0 \\
x_7 & 0 & 1 & 0 & x_8 & x_9 & 0 & 0 \\
x_{10} & x_{11} & x_{12} & 0 & x_{13} & x_{14} & 0 & 0 \\
x_{15} & x_{16} & x_{17} & 0 & x_{18} & x_{19} & 0 & 0 \\
\hline
 & & & x_4 & 1 & 0 & x_5 & x_6 \\
 & & & x_7 & 0 & 1 & x_8 & x_9 \\
 & & & x_{10} & x_{11} & x_{12} & x_{13} & x_{14} \\
 & & & x_{15} & x_{16} & x_{17} & x_{18} & x_{19}
\end{array}\right)
$$

must have full rank or, equivalently, $\det(Y_0^* Y_0) \neq 0$. In this example all parameters of Y are present in Y_0, but, as in the previous example, condition $\det(Y_0^* Y_0) \neq 0$ may be equivalent to only a proper subset of parameters be linked. In any case, it follows from condition $\det(Y_0^* Y_0) \neq 0$ that some parameters of Y form a Zariski open set of $\mathbb{R}^{N_2}$ with $N_2 \leq N$. We will say that N_2 is the number of linked parameters of Y and $N_1 = N - N_2$ is the number of free parameters of Y.

We summarize the information about the free and linked parameters in the following remark.

Remark 3.6 *If N is the number of parameters in Y, we can decompose N as $N = N_1 + N_2$, where N_1 is the number of free parameters and N_2 is the number of linked parameters of Y. These linked parameters are determined by the condition $\det(Y_0^* Y_0) \neq 0$, and form an open and dense set of $\mathbb{R}^{N_2}$ defined as the complementary of the solutions of a polynomial equation.*

Remark 3.7 *It is important to observe that taking into account how an admissible set of indices for Z has been obtained, it is clear that the condition of "being an admissible set" is generic. That is to say, if (n_{ij}) is an admissible set of indices for Z then it is also an admissible set of indices for small enough perturbations of Z. This implies, among other things, that the number* *of free and linked parameters and the polynomial equation linking them are the same for all reduced forms of matrices obtained from Z by sufficiently small perturbations.*

Similarly, if Y is a reduced form of Z we can write $Y = ZP_Z$, where $P_Z \in \mathcal{G}(s)$ depends differentiably on Z.

In the previous examples we see that the number of parameters of the reduced forms coincide with the dimension of $\mathrm{Ctr}_h(A, B)$. In fact, this is a general result as we next show.

Proposition 3.8 *With the notation theorem 3.2, if N is the number of parameters of Y, we have that*

$$N = \dim \mathrm{Ctr}_h(A, B)\,.$$

Proof. According to the description of Y in theorem 3.2,

$$\begin{aligned}
N &= \sum_{i=1}^{h}(s_i - s_{i+1})(r_1 - s_1 - r_{i+1} + s_{i+1}) + \\
&\quad + \sum_{i=2}^{h}(s_i - s_{i+1})(r_1 - s_1 - r_i + s_i) + \cdots + s_h(r_1 - r_2 - s_1 + s_2) = \\
&= \sum_{i=1}^{h}(s_i - s_{i+1})(r_1 - s_1 - r_{i+1} + s_{i+1}) + \\
&\quad + \sum_{i=1}^{h}(s_{i+1} - s_{i+2})(r_1 - s_1 - r_{i+1} + s_{i+1}) + \cdots \\
&\quad + \sum_{i=1}^{h}(s_{i+h-1} - s_{i+h})(r_1 - s_1 - r_{i+1} + s_{i+1}) = \\
&= \sum_{1=1}^{h} s_i((r_1 - s_1) - (r_{i+1} - s_{i+1})).
\end{aligned}$$

And this is, by theorem 2.6 (iv), the dimension of $\mathrm{Ctr}_h(A, B)$. ■

4 An atlas of coordinate charts of $\mathrm{Ctr}_h(A, B)$

In order to prove that the set of reduced forms constructed in the previous section defines an atlas of coordinate charts for $\mathcal{M}(r, s)/\mathcal{G}(s)$, we need the following lemmas

Lemma 4.1 *Let $Z \in \mathcal{M}(r, s)$ and $Q \in \mathcal{G}(s)$. If I is an admissible set of indices for Z, it is also an admissible set of indices for ZQ.*

Proof. Let $Y = ZP$ be a reduced form for Z corresponding to an admissible set of indices $I = (n_{ij})$. Then $ZQ = Y(P^{-1}Q)$. So, we can assume without loss of generality that Z is in reduced form, and it is sufficient to look at the block Z_{11}. Then, if

$$Z_{11} = \begin{pmatrix} Z^1_{11} & Z^1_{12} & Z^1_{13} & \dots \\ 0 & Z^1_{22} & Z^1_{23} & \dots \\ 0 & 0 & Z^1_{33} & \dots \\ \dots & \dots & \dots & \end{pmatrix}$$

$$Z_{11}Q = \begin{pmatrix} Z^1_{11}Q^1_{11} & Z^1_{11}Q^1_{12} + Z^1_{12}Q^1_{22} & Z^1_{11}Q^1_{13} + Z^1_{12}Q^1_{23} + Z^1_{13}Q^1_{33} & \dots \\ 0 & Z^1_{22}Q^1_{22} & Z^1_{22}Q^1_{23} + Z^1_{23}Q^1_{33} & \dots \\ 0 & 0 & Z^1_{33}Q^1_{33} & \dots \\ \dots & \dots & \dots & \end{pmatrix}$$

Since Q^1_{11} and Q^1_{22} are invertible, it is clear that n_{1j}, $1 \leq j \leq s_1 - s_2$ and n_{2j}, $1 \leq j \leq s_2 - s_3$ are admissible and we can assume that the corresponding rows in $Z^1_{11}Q^1_{11}$ and $Z^1_{22}Q^1_{22}$ are unit vectors, or what is equivalent, that $Q^1_{11} = I_{s_1-s_2}$ and $Q^1_{22} = I_{s_2-s_3}$.

Then with block Z^1_{22} we can make zero the block $Z^1_{22}Q^1_{23}$ so that taking into account that Q^1_{33} is invertible we see that n_{3j}, $1 \leq j \leq s_3 - s_4$ is also admissible for $Z_{11}Q$. That n_{4j}, n_{5j}, ... are admissible indices is proved by a similar procedure. ■

Lemma 4.2 *Let Y and $\overline{Y}$ be two matrices of $\mathcal{M}(r,s)$ in reduced form with the same set of indices I. If $\overline{Y} = YP$ with $P \in \mathcal{G}(s)$, then $P = I_d$.*

Proof. The equality $\overline{Y} = YP$ implies that

$$\begin{aligned} &\overline{Y}^1_{11} = Y^1_{11}P^1_{11} \\ &\overline{Y}^1_{12} = Y^1_{11}P^1_{12} + Y^1_{12}P^1_{22} \\ &\text{etc.} \end{aligned}$$

From these equalities and taking into account where the rows that are unit vectors or zero are placed, we conclude that $P = I_d$. ■

We are now ready to parametrize the manifold $\mathrm{Ctr}_h(A,B)$. More precisely we are going to describe a coordinate atlas of $\mathrm{Ctr}_h(A,B)$. As we have seen, every point of $\mathrm{Ctr}_h(A,B)$ can be identified with an orbit $Z\mathcal{G}(s)$ of $\mathcal{M}(r,s)$, so that taking into account the above lemmas we can associate to every point of $S \in \mathrm{Ctr}_h(A,B)$ a matrix in reduced form Y depending only on a set of admissible set of indices $I = (n_{ij})$ (definition 3.3).

Furthermore, from the process that we used to obtain a reduced form, we see that if $Z \in \mathcal{M}(r,s)$, there is an open neighborhood of Z in $\mathcal{M}(r,s)$ such that every matrix in this neighborhood has the same admissible set of indices I, the same number of free and linked parameters and the same polynomial equation linking them (see remark 3.7) .

So, if we denote $\bigwedge$ the set of indices $I = (n_{ij})$ verifying the conditions in theorem 3.2 and $\mathcal{U}_I$ is the set of matrices $Z \in \mathcal{M}(r,s)$ such that I is admissible for Z, one has that $\{\mathcal{U}_I;\, I \in \bigwedge\}$ is an open covering of $\mathcal{M}(r,s)$. Hence, if $\pi : \mathcal{M}(r,s) \longrightarrow \mathcal{M}(r,s)/\mathcal{G}(s)$ is the natural projection onto the orbit space, then $\{\pi(\mathcal{U}_I) = \widetilde{\mathcal{U}}_I;\, I \in \bigwedge\}$ is an open covering of $\mathcal{M}(r,s)/\mathcal{G}(s)$ and hence of $\mathrm{Ctr}_h(A,B)$.

Finally, let $\mathcal{V}$ be the subset of $\mathbb{K}^{N_2}$ formed by the linked parameters of Y. It is clear that, according to remark 3.6, $\mathcal{V}$ is an open and dense subset of $\mathbb{K}^{N_2}$. Our aim is to define a diffeomorphism θ_I between $\widetilde{\mathcal{U}}_I$ and $\mathbb{K}^{N_1} \times \mathcal{V}$. For this, let $\mathcal{U}'_I$ the subset of $\mathcal{U}_I$ formed by the reduced forms of the matrices in $\mathcal{U}_I$. Since to every orbit $Z\mathcal{G}(s)$ of $\mathcal{M}(r,s)$ we can associate a matrix in reduced form Y depending only on I, we can define a natural bijection $\alpha_I : \widetilde{\mathcal{U}}_I \longrightarrow \mathcal{U}'_I$ by $\alpha_I(Z\mathcal{G}(s)) = Y$; and if we fix an order in the set of parameters of the reduced forms, we can identify through a natural bijection, $\beta_I : \mathcal{U}'_I \longrightarrow \mathbb{K}^{N_1} \times \mathcal{V}$, $\mathcal{U}'_I$ and $\mathbb{K}^{N_1} \times \mathcal{V}$. Then, we define $\theta_I : \widetilde{\mathcal{U}}_I \longrightarrow \mathbb{K}^{N_1} \times \mathcal{V}$ by $\theta_I = \beta_I \circ \alpha_I$. We can now state the following result.

Theorem 4.3 *With the above notation θ_I is a diffeomorphism and $\{\widetilde{\mathcal{U}}_I,\, I \in \bigwedge\}$ is a coordinate atlas of $\mathcal{M}(r,s)/\mathcal{G}(s)$ and hence of* $\mathrm{Ctr}_h(A,B)$.

Proof. It is clear that θ_I is bijective. In order to see that it is a diffeomorphism let $\varphi_I : \mathcal{U}_I \longrightarrow \mathbb{K}^{N_1} \times \mathcal{V}$ be the mapping defined by $\varphi_I = \theta_I \circ \pi$. With the notation in remark 3.7, we have that $\varphi_I(Z) = ZP_Z$, where P_Z depends differentiably on Z, so that φ_I is differentiable. Then, if σ is a local section of π defined in an open neighborhood $\widetilde{V} \subset \widetilde{\mathcal{U}}_I$, $\sigma : \widetilde{V} \longrightarrow \mathcal{U}_I$, one has that $\varphi_I \circ \sigma = \theta_I$ restricted to $\widetilde{V}$. Hence θ_I is differentiable. The differentiability of θ^{-1} follows from the fact that θ_I^{-1} is the restriction of $\pi \circ \beta_I^{-1}$ to $\mathcal{U}'_I$.

Finally, let θ_I and θ_J be local coordinates charts corresponding to the open sets $\widetilde{\mathcal{U}}_I$ and $\widetilde{\mathcal{U}}_J$, respectively. Since θ_I and θ_J are diffeomorphisms, the map $\theta_J^{-1} \circ \theta_I$ defined in $\widetilde{\mathcal{U}}_I \cap \widetilde{\mathcal{U}}_J$ (which is non empty) is also a diffeomorphism. Hence the compatibility conditions hold. ∎

5 Conclusions

Each one of the reduced forms described in theorem 3.2 (depending on the set of admissible indices) parametrizes an open and dense set of controllability subspaces of $\mathrm{Ctr}_h(A,B)$, that is to say, "almost all" of them. The set $\mathrm{Ctr}_h(A,B)$ is a subset of all (A,B)-invariant subspaces (of dimension d) and one can obtain a parametrization of this set via the parametrization of (C,A)-invariant subspaces of dimension $n-d$ given, for example, in [7]. It is interesting to remark that, in contrast with this parametrization, the one of $\mathrm{Ctr}_h(A,B)$ obtained here has, in general, linked parameters, that is to say, we do not parametrize with $\mathbb{K}^N$, as in [7], but with the complementary of an algebraic variety, namely, $\mathbb{K}^{N_1} \times \mathcal{V}$.

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