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Spatial decay in transient heat conduction for general elongated regions

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Abstract

Zanaboni's procedure for establishing Saint-Venant's principle is extended to anisotropic homogeneous transient heat conduction on regions that are successively embedded in each other to become indefinitely elongated. No further geometrical restrictions are imposed. The boundary of each region is maintained at zero temperature apart from the common surface of intersection which is heated to the same temperature assumed to be of bounded time variation. Heat sources are absent. Subject to these conditions, the thermal energy, supposed bounded in each region, becomes vanishingly small in those parts of the regions sufficiently remote from the heated common surface. As with the original treatment, the proof involves certain monotone bounded sequences, and does not depend upon differential inequalities or the maximum principle. A definition is presented of an elongated region.

1 Introduction

Previous discussions of decay in transient heat conduction include that by Knowles [15], who is concerned with a semi-infinite cylinder and an energy

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function over space-time related to that introduced here. It is shown that the function in that part of the cylinder greater than a certain distance from the heated base possesses an upper bound that exponentially decays with respect to that distance provided the temperature vanishes at asymptotically large axial distance. Horgan, Payne and Wheeler [10], who briefly review other main contributions, consider the same problem but treat the cross-sectional spatial mean square norm of the temperature, and construct a similiar exponentially spatially decaying upper bound. They conclude that decay is at least as rapid as in the steady problem. Both studies involve a differential inequality for the respective measures, whereas the present approach relies upon the Cauchy convergence of a monotone sequence. An explicit distance function is not employed except in the definition of an elongated region. Consequently, decay estimates of the kind derived in [10, 15] are not to be expected.

Saint-Venant's principle as postulated by Zanaboni [19] for linear elasticity relates to a sequence of successively embedded regions that create an elongated body of arbitrary shape. The surfaces of the component regions have common intersection Γ which is the only part of the respective surfaces subjected to the same prescribed self-equilibrated load. Zero boundary data are prescribed over the remainder of the surface of each region which are in equilbrium under zero body force. Zanaboni's version of Saint-Venant's principle asserts that the strain energy becomes vanishingly small in those parts of the indefinitely elongated body sufficiently remote from Γ,

The objective of this paper is to prove the corresponding assertion for classical anisotropic homogeneous transient heat conduction in the absence of heat sources.. The common surface Γ of the sequence of enlarging regions is subject to the same prescribed temperature for all regions, in each of which the thermal energy is bounded. Furthermore, the time-derivative of the temperature over Γ is assumed bounded. The general procedure is that originally devised by Zanaboni [19] but modified to incorporate certain simplications introduced in [14]. Linear thermoelasticity is similarly treated in [12].

The argument is mainly algebraic and employs a fundamental inequality, derived using integration by parts and standard inequalities, to demonstrate that a space-time measure defined in terms of the thermal energy of the enlarging regions forms a monotonically decreasing sequence that is bounded below. Saint-Venant's principle as formulated by Zanaboni then follows from the Cauchy and other convergence theorems.

Section 2 details the geometric context of the problem. A bounded elongated region, defined in Section 3, is used to generate an unbounded elongated region. Part of the surface of this first region forms the common intersection Γ of the surfaces of all subsequent regions and is heated to a prescribed temperature which is the same for all regions. Complementary parts of the respective surfaces are at zero temperature. The initial boundary value problems are stated in Section 4, while the positive-definite thermal energy measures, assumed uniformly bounded and in terms of which the analysis is conducted, are introduced in Section 5. The main part of Section 5, however, is devoted to the construction of the fundamental inequality crucial for the proof in Section 7 of Zanaboni's version of Saint-Venant's principle. In Section 6, the fundamental inequality is employed to derive a monotonically decreasing bounded below sequence of the thermal energies which by the Cauchy and Bolzano-Weierstrass convergence theorems leads to bounds for the thermal energy in a region sufficiently remote from the common surface Γ. The appropriate form of Saint-Venant's principle is expressed as a theorem stated and proved in Section 7. Section 8 consists of some brief concluding remarks.

The usual conventions are adopted of summation over repeated subscripts, and a subscript comma to denote partial differentiation. Vector and tensor quantities are not typographically distinguished, while subscripts have the range $1, 2, 3$ apart from η which is used as an additional time variable. A solution of sufficient smoothnees is assumed always to exist.

2 Geometry

A bounded region of three dimensional Euclidean space is indefinitely enlarged by successive accretion, such that n accretions generate $(n+1)$ regions, each embedded in its successor. Accordingly, in terms of a notation convenient for later purposes, the sequence of open simply connected regions $\{\Omega_j\}, j = 1, \dots n+1$, satisfies the inclusions

$$\emptyset \neq \Omega_{n+1-r} \subset \Omega_{n+1-s}, \qquad 0 \leq s < r \leq n. \tag{2.1}$$

The final enlarged region in the sequence is Ω_{n+1}, while each accretion used in its construction is of size and shape that may be chosen appropriately to the problem under consideration. The surface $\partial\Omega_n$ of each region Ω_n is Lipschitz continuous.

Note that s, r are integers and unless equal thus differ by at least 1.

Let $\Omega_0 = \emptyset$. The accretions $D_i^{(n+1)}$ are defined by

$$D_i^{(n+1)} = \Omega_{n+1-i} \backslash \Omega_{n-i}, \qquad 0 \leq i \leq n, \tag{2.2}$$

$$D_n^{(n+1)} = \Omega_1 \backslash \Omega_0 = \Omega_1. \tag{2.3}$$

At the $(m+1)$ stage, where $m > n$, a further $(m-n)$ accretions have been added to Ω_{n+1} to form a region Ω_{m+1}. In consequence, the new region consists of new accretions $D_j^{m+1}, j = 0, 1, \dots (m-n)$ plus those used to form Ω_{n+1}, so that

$$D_{(m-n)+i}^{(m+1)} = D_i^{(n+1)}, \qquad i = 0, 1, 2 \dots n. \tag{2.4}$$

Correspondingly, regions in the sequence may be identified according to the relations

$$\Omega_{m+1-j} = \Omega_{n+1-i}, \qquad j = (m-n) + i,\ i = 0, 1, 2 \dots n. \tag{2.5}$$

The non-empty part of the boundary common to all $\partial\Omega_n$ is denoted by Γ where

$$\emptyset \neq \Gamma \subset \partial\Omega_n \cap \partial\Omega_{n+1}, \qquad n = 1, 2, 3, \dots.., \tag{2.6}$$

while that part of the surface $\partial\Omega_n$ contained in Ω_{n+1} is represented by Σ_n:

$$\Sigma_n = \partial\Omega_n \cap \Omega_{n+1}, \qquad n = 1, 2, 3, ... \tag{2.7}$$

We require the sequence of regions Ω_n, $n = 1, 2, 3, \ldots$ to be *elongated*. This term is defined in the next section.

3 Elongated regions

An intuitive understanding of what is meant by an elongated region is that it has at least one dimension much larger than the others. This description, however, fails to restrict the shape or connectivitly of the region, which becomes important when discussing Saint-Venant's principle. Cavities and cracks can create stress ooncentrations which invalidate the usual principle.

Our definition of an elongated region depends upon a basic region that has the following properties.

Definition 3.1 (The basic region) *The basic region is a bounded region Ω with smooth boundary $\partial\Omega$ for which the distance function is $d(x, y)$ for x, $y \in \Omega \cup \partial\Omega$. Let $\partial\Omega$ be decomposed into mutually disjoint but individually connected parts such that $\partial\Omega = \Gamma \cup \partial\Omega_1 \cup \partial\Omega_2$. When $x \in \Gamma$, $z \in \partial\Omega_1$, and $y \in \partial\Omega_2$ we require $d(x, z) \leq d(x, y)$. Moreover, let $\bar{x} \in \Gamma$ and $\bar{y} \in \partial\Omega_2$ be chosen to satisfy*

$$d(\bar{x}, \bar{y}) = \sup_{x,y} d(x, y).$$

Take planes perpendicular to the straight line joining $\bar{x}$, $\bar{y}$ and let the intersection with Ω of the plane through the point $\bar{x} + \lambda\bar{y}$ be the cross section $P(\lambda)$. Define

$$\underline{\lambda} = \min\,(\lambda : P(\lambda) \cap \partial\Omega_1 \neq \emptyset), \tag{3.1}$$
$$\bar{\lambda} = \max\,(\lambda : P(\lambda) \cap \partial\Omega_1 \neq \emptyset). \tag{3.2}$$

Now vary Γ, $\partial\Omega_1$, $\partial\Omega_2$ by the addition or subtraction of respective parts of the boundary $\partial\Omega$ such that the endpoints of Γ lie on $P(\underline{\lambda})$ and the endpoints of $\partial\Omega_2$ lie on $P(\bar{\lambda})$.

Let each $P(\lambda)$ be singly connected and satisfy the following conditions:

$$c \leq |P(\lambda)| < M < \infty, \qquad \underline{\lambda} \leq \lambda \leq \bar{\lambda}, \tag{3.3}$$
$$0 < |P(\lambda)| < M < \infty, \qquad 0 < \lambda \leq \underline{\lambda}, \tag{3.4}$$
$$0 = |P(0)|, \tag{3.5}$$
$$0 < |P(\lambda)| < M < \infty, \qquad \bar{\lambda} \leq \lambda < 1, \tag{3.6}$$
$$0 = |P(1)|, \tag{3.7}$$

where c, M are speciified positive constants, and $|P(\lambda)|$ denotes the diameter of $P(\lambda)$.

These conditions are designed to prevent cross-sections from either collapsing to zero or becoming unbounded. The assumption of single connectedness precludes Ω from containing cavities in $\mathbb{R}^3$ or (wide) slits in $\mathbb{R}^2$. Note that the ratio $d(\bar{x}, \bar{y})/M$ may be arbitrarily large but finite. For simplicity, wedges, thick infinite plates, quarter-spaces, cones, half-spaces, the whole space, and exterior regions are excluded, but such regions may be easily incorporated into our analysis. On the other hand, cavities and cracks are likely to require a modified treatment, and in consequence their consideration is postponed.

Definition 3.2 (Elongated region) *Let Ω_n, $n = 1, 2, 3, \ldots$ form the embedded seqeunce defined in Section 2. Then Ω_n, $n = 1, 2, 3, \ldots$ comprise a sequence of elongated regions provided each member of the sequence is the union of basic regions $\Omega_{n+1}\backslash\Omega_n$, $n = 0, 1, 2, \ldots$, and in addition satisfy*

$$\lim_{n\to\infty} |\Omega_n| \to \infty.. \tag{3.8}$$

Cylindrical regions, regions that spiral within a wedge or cone, or are helical in shape, or are non-contiguous (i.e., no self-contact) entangled knots are examples of a sequence of elongated regions in the sense of Definition 3.2.

4 The initial boundary value problems

Each region Ω_n, defined in the previous sections, is occupied by a homogeneous heat conducting material with the same heat conduction symmetric tensor κ. The treatment can be extended to spatially inhomogeneous heat conduction materials in an obvious manner. A related, but different, study of functionally graded heat conducting materials with similar nonhomogeneous properties is presented in [11] for a cylinder, while [17] examines, again by a different method, a certain nonlinear parabolic system for both a cylinder and cone. With respect to an orthogonal Cartesian $x_1x_2x_3$-coordinate system common to all regions, the components of κ are $\kappa_{ij} = \kappa_{ji}$, $i, j = 1, 2, 3$. It is supposed that κ is positive-definite in the sense that the following inequality holds for an assigned positive constant κ_0 and for each vector $\xi \in \mathbb{R}^3$:

$$\kappa_0\xi_i\xi_i \leq \kappa_{ij}\xi_i\xi_j. \tag{4.1}$$

The (positive) temperature in Ω_n is denoted by $u^{(n)}(x, t) \in \mathbb{R}$, where $(x, t) \in \Omega_n \times [0, T_n)$ and $[0, T_n)$ is the maximal time interval of existence for Ω_n. Assume that $T_n > 0$ and that $T = \min_n T_n > 0$. Assume further that $u^{(n)}$ is twice spatially and once temporally differentiable and define the second order linear partial differential operator L to be

$$L(u^{(n)}) = \left(\kappa_{ij}u^{(n)}_{,i}\right)_{,j}, \qquad (x, t) \in \Omega_n \times [0, T). \tag{4.2}$$

The generalised normal derivative on $\partial\Omega_n$ is denoted by

$$\frac{\partial u^{(n)}}{\partial n} = n_i\kappa_{ij}u^{(n)}_{,j}, \qquad (x, t) \in \partial\Omega_n \times [0, T), \tag{4.3}$$

where n_i are the Cartesian coordinates of the generic unit outward normal vector on $\partial\Omega_n$, $n = 1, 2, 3, \ldots$.

Let $w(x,t)$, $(x,t) \in \Gamma \times [0,T)$, be a prescribed function that for an assigned positive constant M_1 satisfies the assumption:

$$M_1^2 = \int_0^T \int_{\Gamma(\eta)} w_{,\eta}^2 \, dSd\eta. \tag{4.4}$$

In particular, we have $M_1 = 0$ when w is independent of time. This condition, however, is incompatible with the homogeneous initial data assumed below.

The sequence of initial boundary value problems to be studied is specified by

$$\begin{aligned} L\left(u^{(n)}\right) &= \dot{u}^{(n)}(x,t), \qquad (x,t) \in \Omega_n \times [0,T), & (4.5)\\ u^{(n)}(x,t) &= w(x,t), \qquad (x,t) \in \Gamma \times [0,T), & (4.6)\\ &= 0, \qquad (x,t) \in (\partial\Omega_n \backslash \Gamma) \times [0,T), & (4.7)\\ u^{(n)}(x,0) &= 0, \qquad x \in \Omega_n, & (4.8) \end{aligned}$$

where $n = 1, 2, 3, \ldots$, and a superposed dot indicates differentiation with respect to time. Homogeneous initial data are adopted in (4.8) for convenience.

5 Fundamental inequality

A fundamental inequality is derived which leads to a monotone sequence studied in Section 6. For this purpose, we introduce the bilinear function defined on a region Ω by

$$V_\Omega(u,v) = \int_0^t \int_{\Omega(\eta)} u_{,i}\kappa_{ij}v_{,j} \, dxd\eta + \frac{1}{2}\int_{\Omega(t)} uv \, dx, \tag{5.1}$$

where $u, v \in C^2(\Omega \times [0,T))$, and the notation $\Omega(t)$ indicates that relevant quantities are evaluated at time t. In particular, we consider the thermal energy function obtained when (5.1) is specialised to the form

$$V_{\Omega_n}\left(u^{(n)}, u^{(n+1)}\right) = \int_0^t \int_{\Omega_n(\eta)} u_{,i}^{(n)}\kappa_{ij}u_{,j}^{(n+1)} \, dxd\eta + \frac{1}{2}\int_{\Omega_n(t)} u^{(n)}u^{(n+1)} \, dx, \tag{5.2}$$

where $u^{(n)} \in C^2(\Omega_n \times [0,T))$. It is also supposed that

$$V_{\Omega_n}(u^{(n)}, u^{(n)}) \leq M_2, \qquad n = 1, 2, \ldots, \tag{5.3}$$

for prescribed positive constant M_2 independent of n.

The function $V_{\Omega_\infty \backslash \Omega_n}\left(u^{(\infty)}, u^{(\infty)}\right)$ is employed in [15] to establish exponential decay in a semi-infinite cylinder using an argument based upon differential inequalities

Repeated integration by parts and use of relations (4.5)-(4.8) together with definitions (4.2) of the linear operator L and (4.3) of the generalised normal derivative, yields an equivalent representation for (5.2). We have

$$\begin{aligned} V_{\Omega_n}(u^{(n)},u^{(n+1)}) &= \int_0^t\int_\Gamma w\frac{\partial u^{(n+1)}}{\partial n}\,dSd\eta - \int_0^t\int_{\Omega_n(\eta)} u^{(n)}L(u^{(n+1)})\,dxd\eta \\ &\quad +\frac{1}{2}\int_{\Omega_n(t)} u^{(n)}u^{(n+1)}\,dx \\ &= V_{\Omega_{n+1}}(u^{(n+1)},u^{(n+1)}) - \frac{1}{2}\int_{\Omega_n(t)} u^{(n)}u^{(n+1)}\,dx \\ &\quad +\int_0^t\int_{\Omega_n(\eta)} L\left(u^{(n)}\right)u^{(n+1)}\,dxd\eta. \end{aligned} \tag{5.4}$$

We examine the last term on the right of (5.4). Let $Q_n(t)=\Omega_n\times[0,t)$ and for differentiable functions ϕ, ψ set

$$[\phi,\psi]_{Q_n(t)} = \int_0^t\int_{\Omega_n(\eta)} \kappa_{ij}\phi_{,i}\psi_{,j}\,dxd\eta, \qquad (x,t)\in Q_n(t). \tag{5.5}$$

Integration by parts yields

$$\begin{aligned} \int_0^t\int_{\Omega_n(\eta)}\left(\kappa_{ij}u^{(n)}_{,i}\right)_{,j}u^{(n+1)}\,dxd\eta &= \int_0^t\int_{\Gamma(\eta)} u^{(n)}\frac{\partial u^{(n)}}{\partial n}\,dSd\eta \\ &\quad +\int_0^t\int_{\Sigma_n(\eta)} u^{(n+1)}\frac{\partial u^{(n)}}{\partial n}\,dSd\eta - \left[u^{(n)},u^{(n+1)}\right]_{Q_n(t)} \\ &= V_{\Omega_n}(u^{(n)},u^{(n)}) + \int_0^t\int_{\Sigma_n(\eta)} u^{(n+1)}\frac{\partial u^{(n)}}{\partial n}\,dSd\eta \\ &\quad -\left[u^{(n)},u^{(n+1)}\right]_{Q_n(t)}. \end{aligned} \tag{5.6}$$

Insertion of (5.6) into (5.4), after rearrangement gives

$$\begin{aligned} 2V_{\Omega_n}(u^{(n)},u^{(n+1)}) &= V_{\Omega_n}(u^{(n)},u^{(n)}) + V_{\Omega_{n+1}}(u^{(n+1)},u^{(n+1)}) \\ &\quad +\int_0^t\int_{\Sigma_n(\eta)} u^{(n+1)}\frac{\partial u^{(n)}}{\partial n}\,dSd\eta. \end{aligned} \tag{5.7}$$

The next step constructs an upper bound for $V_{\Omega_n}(u^{(n)},u^{(n+1)})$ by means of Young's, or the arithmetic-geometric mean, inequality. The result substituted in (5.7) generates the intermediate fundamental inequality

$$\begin{aligned} V_{\Omega_n}(u^{(n)},u^{(n)}) &+ V_{\Omega_{n+1}}(u^{(n+1)},u^{(n+1)}) \\ &\le \alpha_1 V_{\Omega_n}(u^{(n)},u^{(n)}) + \alpha_1^{-1}V_{\Omega_n}(u^{(n+1)},u^{(n+1)}) \\ &\quad -\int_0^t\int_{\Sigma_n(\eta)} u^{(n+1)}\frac{\partial u^{(n)}}{\partial n}\,dSd\eta, \end{aligned} \tag{5.8}$$

where α_1 is an arbitrary positive constant to be chosen. A bound must now be obtained for the surface integral appearing in the last term on the right of (5.8).

5.1 Subsidiary inequalities

Let n be fixed, and consider two subregions A_n, B_n that satisfy $A_n \subset \Omega_n$, $B_n \subset \Omega_{n+1} \backslash \Omega_n$, and let

$$\Sigma_n \subset \partial A_n, \qquad \Sigma_n \subset \partial B_n, \tag{5.9}$$

$$\begin{aligned} \partial A_n \backslash \left(\Sigma_n \cup \Sigma_{A_n}\right) &\neq 0, & (5.10)\\ \partial B_n \backslash \left(\Sigma_n \cup \Sigma_{B_n}\right) &\neq 0, & (5.11)\end{aligned}$$

where $\Sigma_n = \partial\Omega_n \cap \Omega_{n+1}$ (see (2.7), $\Sigma_{A_n} = \partial A_n \cap \Omega_n$, and $\Sigma_{B_n} = \partial B_n \cap (\Omega_{n+1} \backslash \Omega_n)$.

Schwarz's inequality applied to the last term on the right of (5.8) yields

$$\begin{aligned} \left|\int_0^t \int_{\Sigma_n(\eta)} u^{(n+1)} \frac{\partial u^{(n)}}{\partial n}\, dS d\eta\right| \leq & \left[\int_0^t \int_{\Sigma_n(\eta)} u^{(n+1)} u^{(n+1)}\, dS d\eta\right]^{1/2} \times \\ & \left[\int_0^t \int_{\Sigma_n(\eta)} \left(\frac{\partial u^{(n)}}{\partial n}\right)^2 dS d\eta\right]^{1/2}. \end{aligned} \tag{5.12}$$

To bound the first integral on the right we employ the Sobolev trace inequality (see, for example, [1, 9, 8, 16, 18]):

$$\int_\Sigma v^2\, dS \leq \int_{\partial D} v^2\, dS \leq C(D) \int_D \kappa_{ij} v_{,i} v_{,j}\, dx, \tag{5.13}$$

where D, a bounded region of three-dimensional Euclidean space $\mathbb{R}^3$, has Lipschitz continuous boundary ∂D such that $\Sigma \subset \partial D$, $S \subset \partial D$ are non-intersecting proper subsets of ∂D that satisfy $\partial D \backslash (\Sigma \cup S) \neq \emptyset$. The function $v \in W^{1,2}(D)$ vanishes on part of the boundary:

$$v(x) = 0, \qquad x \in \partial D \backslash (\Sigma \cup S), \tag{5.14}$$

and $C(D)$ is a computable positive constant.

Inequality (5.13) is applied to the subregion B_n with $\Sigma = \Sigma_n$.

Now let $v \in W^{2,2}(D)$ satisfy the boundary condition

$$v(x) = 0, \qquad x \in \partial D \backslash S. \tag{5.15}$$

The second integral on the right of (5.12) is treated by means of the inequality

$$\int_\Sigma \left(\frac{\partial v}{\partial n}\right)^2 \leq a(D) \int_D \kappa_{ij} v_{,i} v_{,j}\, dx + b(D) \int_D (L(v))^2\, dx, \tag{5.16}$$

where the operator L is defined in (4.2). and $a(D)$, $b(D)$ are positive constants. The proof relies upon a Rellich identiy (see, for example, [2, 3, 13, 16]). We now apply (5.16) to the subregion A_n with $S = \Sigma_{A_n}$ and $\Sigma = \Sigma_n$.

The respective embedding constants $C(D)$, $a(D)$, $b(D)$ depend upon the region D and therefore in the present particular case on B_n and A_n. Consequently, for each n, we postulate that the choice of A_n and B_n can be always adjusted such that, for example, $a(A_n) = a(A_1) = a$, $b(A_n) = b(A_1) = b$, and $C(B_n) = C(B_1) = C$, where a, b, and C as defined are positive constants. We obtain

$$\begin{aligned}
\left|\int_0^t \int_{\Sigma_n(\eta)} u^{(n+1)} \frac{\partial u^{(n)}}{\partial n}\, dS d\eta\right| &\leq \frac{C^{1/2}(\alpha_2+\alpha_3)}{2} \int_0^t \int_{B_n(\eta)} u_{,i}^{(n+1)} \kappa_{ij} u_{,j}^{(n+1)}\, dx d\eta \\
&\quad + \frac{aC^{1/2}}{2\alpha_2} \int_0^t \int_{A_n(\eta)} u_{,i}^{(n)} \kappa_{ij} u_{,j}^{(n)}\, dx d\eta \\
&\quad + \frac{b}{2\alpha_3} C^{1/2} \int_0^t \int_{A_n(\eta)} \left(u_{,\eta}^{(n)}\right)^2 dx d\eta \\
&\leq \frac{C^{1/2}(\alpha_2+\alpha_3)}{2} \int_0^t \int_{\Omega_{n+1}(\eta)\setminus\Omega_n(\eta)} u_{,i}^{(n+1)} \kappa_{ij} u_{,j}^{(n+1)}\, dx d\eta \\
&\quad + \frac{aC^{1/2}}{2\alpha_2} \int_0^t \int_{\Omega_n(\eta)} u_{,i}^{(n)} \kappa_{ij} u_{,j}^{(n)}\, dx d\eta \\
&\quad + \frac{b}{2\alpha_3} C^{1/2} \int_0^t \int_{\Omega_n(\eta)} \left(u_{,\eta}^{(n)}\right)^2 dx d\eta, \qquad (5.17)
\end{aligned}$$

where Young's inequality is employed and α_2, α_3 are arbitrary positive constants to be chosen.

Consider the last integral on the right of (5.17). On noting (4.4), integrating by parts, employing (5.16) together with standard inequalities, we obtain

$$\begin{aligned}
\int_0^t \int_{\Omega_n(\eta)} \left(u_{,\eta}^{(n)}\right)^2 dx d\eta &= \int_0^t \int_{\Omega_n(\eta)} u_{,\eta}^{(n)} L\left(u^{(n)}\right) dx d\eta \\
&\leq M_1 \left[\left(a(\Omega) \int_0^t \int_{\Omega_n(\eta)} u_{,i}^{(n)} \kappa_{ij} u_{,j}^{(n)}\, dx d\eta\right)^{1/2} \right. \\
&\quad \left. + \left(b(\Omega) \int_0^t \int_{\Omega_n(\eta)} \left(u_{,\eta}^{(n)}\right)^2 dx d\eta\right)^{1/2}\right],
\end{aligned}$$

where $\Omega \subset \Omega_1$, fixed for all n, is chosen appropriately.

Let α_4 denote an arbitrary positive constant. Young's inequality applied to the last expression leads to

$$\int_0^t \int_{\Omega_n(\eta)} \left(u_{,\eta}^{(n)}\right)^2 dx d\eta \leq M_1^2 \left(\alpha_4 + b(\Omega)\right) + \frac{a(\Omega)}{\alpha_4} \int_0^t \int_{\Omega_n(\eta)} u_{,i}^{(n)} \kappa_{ij} u_{,j}^{(n)}\, dx d\eta. \qquad (5.18)$$

Substitution of (5.18) in (5.17) after rearrangement gives

$$
\begin{aligned}
\left|\int_0^t \int_{\Sigma_n(\eta)} u^{(n+1)} \frac{\partial u^{(n)}}{\partial n}\, dS d\eta\right| \quad \leq \quad & \frac{C^{1/2}}{2}\left(\alpha_2+\alpha_3\right) \int_0^t \int_{\Omega_{n+1}(\eta)\backslash\Omega_n(\eta)} u_{,i}^{(n+1)} \kappa_{ij} u_{,j}^{(n+1)}\, dx d\eta \\
& +\left[\frac{C^{1/2} a}{2\alpha_2}+\frac{C^{1/2} a(\Omega) b}{2\alpha_3\alpha_4}\right] \int_0^t \int_{\Omega_n(\eta)} u_{,i}^{(n)} \kappa_{ij} u_{,j}^{(n)}\, dx d\eta \\
& +M_1^2\left(\alpha_4+b(\Omega)\right) \frac{C^{1/2} b}{2\alpha_3}. \qquad (5.19)
\end{aligned}
$$

5.2 Fundamental inequality (continued)

On returning to (5.8) and using (5.19) to eliminate the surface integral, we obtain

$$
\begin{aligned}
& \frac{1}{\alpha_1} V_{\Omega_{n+1}\backslash\Omega_n}\left(u^{(n+1)}, u^{(n+1)}\right)+\left(1-\frac{1}{\alpha_1}\right) V_{\Omega_{n+1}}\left(u^{(n+1)}, u^{(n+1)}\right) \\
& -\frac{C^{1/2}(\alpha_2+\alpha_3)}{2} \int_0^t \int_{\Omega_{n+1}(\eta)\backslash\Omega_n(\eta)} u_{,i}^{(n+1)} \kappa_{ij} u_{,j}^{(n+1)}\, dx d\eta \\
\leq \quad & \left[\alpha_1-1+\frac{1}{2}\left\{\frac{aC^{1/2}}{\alpha_2}+\frac{bC^{1/2} a(\Omega)}{\alpha_3\alpha_4}\right\}\right] V_{\Omega_n}\left(u^{(n)}, u^{(n)}\right) \\
& +M_1^2\left(\alpha_4+b(\Omega)\right)\left(\frac{bC^{1/2}}{2\alpha_3}\right). \qquad (5.20)
\end{aligned}
$$

Now set

$$
\alpha_1=2, \qquad \alpha_2=\alpha_3=\frac{1}{4C^{1/2}}, \qquad (5.21)
$$

so that (5.20) becomes

$$
\begin{aligned}
\frac{1}{2} V_{\Omega_{n+1}\backslash\Omega_n}\left(u^{(n+1)}, u^{(n+1)}\right) \quad + \quad & \frac{1}{2} V_{\Omega_{n+1}}\left(u^{(n+1)}, u^{(n+1)}\right) \\
& -\frac{1}{4} \int_0^t \int_{\Omega_{n+1}(\eta)\backslash\Omega_n(\eta)} u_{,i}^{(n+1)} \kappa_{ij} u_{,j}^{(n+1)}\, dx d\eta \\
\leq \quad & \frac{q}{2} V_{\Omega_n}\left(u^{(n)}, u^{(n)}\right)+\frac{Q}{2}, \qquad (5.22)
\end{aligned}
$$

where

$$
\begin{aligned}
q &= 2\left[1+2C\left\{a+ba(\Omega)\alpha_4^{-1}\right\}\right], \qquad (5.23) \\
Q &= 4bCM_1^2\left(\alpha_4+b(\Omega)\right). \qquad (5.24)
\end{aligned}
$$

On appealing to definition (5.2), we may finally write (5.22) as

$$
\frac{1}{2} V_{\Omega_{n+1}\backslash\Omega_n}\left(u^{(n+1)}, u^{(n+1)}\right)+V_{\Omega_{n+1}}\left(u^{(n+1)}, u^{(n+1)}\right) \leq qV_{\Omega_n}\left(u^{(n)}, u^{(n)}\right)+Q, \qquad (5.25)
$$

which is the fundamental inequality required subsequently.

6 Monotone sequence

We construct a monotone sequence from inequalities (5.25).

Choose α_4 so that $Q = q > 1$. That is, set

$$\alpha_4 = \frac{H + \sqrt{(H^2 + 4IJ)}}{2J}, \tag{6.1}$$

where

$$\begin{aligned} H &= (1 + 2aC) - 2bCb(\Omega)M_1^2, \qquad I = 2bCa(\Omega), \qquad J = 2bCM_1^2, \\ Q &= q = \left[(1 + 2aC) + 2bCb(\Omega)M_1^2 + \sqrt{(H^2 + 4IJ)}\right]. \end{aligned} \tag{6.2}$$

Recall that both q and Q are independent of n.

With this choice of arbitrary constants, inequality (5.25) becomes

$$\frac{1}{2}V_{\Omega_{n+1}\backslash\Omega_n}(u^{(n+1)}, u^{(n+1)}) + V_{\Omega_{n+1}}(u^{(n+1)}, u^{(n+1)}) \leq q\left[V_{\Omega_n}(u^{(n)}, u^{(n)}) + 1\right]. \tag{6.3}$$

For the moment, the positive-definite first term on the left in (6.3) is discarded, and by recursion, the resulting inequality leads to the sequence

$$\begin{aligned} V_{\Omega_{n+1}}(u^{(n+1)}, u^{(n+1)}) &\leq q\left(V_{\Omega_n}(u^{(n)}, u^{(n)}) + 1\right) && (6.4) \\ &= q\frac{(q^r - 1)}{(q - 1)} + q^r V_{\Omega_{n+1-r}}, \quad r = 1, 2 \ldots n, && (6.5) \end{aligned}$$

where here and subsequently arguments of the respective energies are omitted.

The sequence may be compactly represented on setting

$$a_r^{n+1} = q\frac{(q^r - 1)}{(q - 1)} + q^r V_{\Omega_{n+1-r}}, \qquad r = 0, 1, 2 \ldots n, \tag{6.6}$$

to obtain

$$0 \leq a_r^{n+1} \leq a_s^{n+1}, \qquad 0 \leq r < s \leq n, \quad n = 0, 1, 2, \ldots \tag{6.7}$$

Let M_3 be a specified positive (bounded) constant and suppose for $n = 0, 1, 2, \ldots$ that r lies in the range where

$$0 \leq a_r^{n+1} \leq M_3. \tag{6.8}$$

The lower bound is implied by the positive-definite hypothesis. Upon recalling assumption (5.3), we have also the bound

$$a_r^{n+1} q\left(\frac{q^r - 1}{q - 1}\right) + q^r V_{n+1-r} \leq q\left(\frac{q^r - 1}{q - 1}\right) + q^r M_2. \tag{6.9}$$

Condition (6.8) is consistent with inequality (6.9) provided M_2 and M_3 are selected to satisfy

$$q\left(\frac{q^r - 1}{q - 1}\right) + q^r M_2 \leq M_3.$$

Rearrangement leads to

$$r \leq \ln\left(\frac{M_3 + \frac{q}{q-1}}{M_2 + \frac{q}{q-1}}\right)[\ln q]^{-1}. \tag{6.10}$$

Let $\{x\}$ denote the greatest integer that does not exceed x. Define r_0 by

$$r_0 = \left\{\ln\left(\frac{M_3 + \frac{q}{q-1}}{M_2 + \frac{q}{q-1}}\right)[\ln q]^{-1}\right\}. \tag{6.11}$$

A subsequence, again denoted by a_r^{n+1}, now may be extracted from (6.7) which for $\epsilon > 0$ and sufficiently large n_0, satisfies the condition

$$|a_s^{n+k+1} - a_r^{n+1}| \leq \epsilon, \qquad n \geq n_0, \tag{6.12}$$

for all $k \geq 0$. Here, s, r lie in the interval $[0, r_0]$, and r_0 is given by (6.11). Precise values of r, s are dependent on n and k and together with the particular case $s = r = 0$ require slightly different discussion. By the Bolzano-Weierstrass theorem, as $s \to \infty$ the subsequence converges to a limit $\widetilde{a} \geq 0$ such that

$$|\widetilde{a} - a_s^{n+k+1}| \leq \epsilon, \qquad n \geq n_0, \quad k \geq 0, \tag{6.13}$$

or

$$\widetilde{a} - \epsilon \leq a_s^{n+k+1} \leq \widetilde{a} + \epsilon, \qquad n \geq n_0, \quad k \geq 0. \tag{6.14}$$

We revert to (6.6) to express these bounds in terms of the energies. Subject to the above stated conditions on r, s, and r_0, we have

$$\widetilde{a} - \epsilon + \frac{q}{(q-1)}(1 - q^s) \leq q^s V_{\Omega_{n+k+1-s}} \leq \widetilde{a} + \epsilon + \frac{q}{(q-1)}(1 - q^s), \tag{6.15}$$

which is the basis for the derivation of Saint-Venant's principle.

7 Saint-Venant's principle

The necessary preliminary components have now been assembled for the proof of Zanaboni's version of Saint-Venant's principle which is stated in the following theorem.

Theorem 7.1 *The systems of parabolic initial boundary value problems (4.5)-(4.8) for transient heat conduction subject to the boundedness conditions (5.3) and (4.4) possess solutions which when measured by the energy function (5.2) tend to zero in the accretion regions* $D_0^{(n+1)} = \Omega_{n+1} \backslash \Omega_n$ *as* $n \to \infty$.

Proof
Provided $n \geq n_0$, inequalities (6.15) hold both when $k = 0$ and $k > 0$. Each possibility corresponds, say, to the terms a_r^{n+1} and a_s^{n+k+1} in the convergent

subsequence, and to the respective regions Ω_{n+1-r} and $\Omega_{n+k+1-s}$, where $0 \leq s, r \leq r_0$, and r_0 is specified by (6.11). In what follows, it is assumed without loss that $k > 0$. The regions Ω_n belonging to the original sequence may now be recalibrated by considering a new region composed of $(N+1)$ accretions chosen such that

$$\Omega_{N+1} = \Omega_{n+k+1-s}, \tag{7.1}$$
$$\Omega_N = \Omega_{n+1-r}, \tag{7.2}$$

where

$$s - r < k. \tag{7.3}$$

The difference between r and s is at most r_0, and consequently, condition (7.3) is satisfied for sufficiently large k. At this stage, the ordering of r and s is not assumed, but later the mutually exclusive cases $s \leq r-1$, $r \leq s-1, r = s$, which exhaust all choices, are separately treated.

According to (2.1), condition (7.3) implies

$$\Omega_N = \Omega_{n+1-r} \subset \Omega_{n+k+1-s} = \Omega_{N+1}. \tag{7.4}$$

The thermal energy is uniquely determined in each isolated region and in consequence we have the relations

$$V_{\Omega_{N+1}}\left(u^{(N+1)}, u^{(N+1)}\right) = V_{\Omega_{n+k+1-s}}\left(u^{(n+1+k-s)}, u^{(n+1+k-s)}\right), \tag{7.5}$$
$$V_{\Omega_N}\left(u^{(N)}, u^{(N)}\right) = V_{\Omega_{n+1-r}}\left(u^{(n+1-r)}, u^{(n+1-r)}\right). \tag{7.6}$$

The procedure that established the basic inequality (5.25) may be applied to accretions D_i^{N+1}, defined in (2.2) and (2.3), and leads to

$$\frac{1}{2} V_{D_0^{N+1}}\left(u^{(N+1)}, u^{(N+1)}\right) \leq q\left[V_{\Omega_N}\left(u^{(N)}, u^{(N)}\right) + 1\right] - V_{\Omega_{N+1}}\left(u^{(N+1)}, u^{(N+1)}\right), \tag{7.7}$$

where we recall that $D_0^{N+1} = \Omega_{N+1} \backslash \Omega_N$.

We now assume $s - r \leq -1$ and let $t > 0$ satisfy $s \leq t \leq r-1$. For convenience, we again omit arguments of functions, and multiply inequality (7.7) by q^t and use inequalities (6.15) to obtain

$$\begin{aligned}
\frac{q^t}{2} V_{D_0^{N+1}} &\leq q^{t+1}(V_{\Omega_N} + 1) - q^t V_{\Omega_{N+1}} && (7.8)\\
&\leq q^r V_{\Omega_{n+1-r}} + q^{t+1} - q^s V_{\Omega_{n+k+1-s}}, \qquad s \leq t \leq r-1, &&\\
&= 2\epsilon + \frac{q}{(q-1)}\left[1 - q^r + q^{t+1} - q^t - 1 + q^s\right] && (7.9)\\
&\leq 2\epsilon, && (7.10)
\end{aligned}$$

where the square bracket in (7.9) is non-positive since $1 < q^s \leq q^t$, $q^{t+1} \leq q^r$ by the assumed inequalities $s \leq t \leq r-1$.

But then (7.10) immediately gives

$$V_{D_0^{N+1}} \leq 4q^{-t}\epsilon \leq 4\epsilon, \tag{7.11}$$

since $t \leq r \leq r_0$, and $q > 1$. Inequality (7.11) for $r > s$ represents the desired result for Saint-Venant's principle as formulated by Zanaboni.

To deal with the case $s > r$, select $s_1 \geq 0$ such that

$$\begin{aligned} \Omega_{N+1} &= \Omega_{n+1-r} = \Omega_{n+k+1-s_1}, \\ \Omega_N &= \Omega_{n+k+1-s} \subset \Omega_{n+k+1-s_1} = \Omega_{N+1}, \end{aligned}$$

which are valid subject to

$$s_1 - r = k, \tag{7.12}$$

$$s_1 < s. \tag{7.13}$$

Respective terms in the basic inequality (7.8), but with t now satisfying $r \leq t \leq s-1$, may be treated as follows. Bounds (6.15) are again used to obtain:

$$\begin{aligned} q^t V_{\Omega_{N+1}} &= q^t V_{\Omega_{n+1-r}} \\ &\geq \widetilde{a} - \epsilon + \frac{q}{(q-1)}(1-q^r), \qquad t \geq r, \end{aligned}$$

and

$$\begin{aligned} q^{1+t} V_{\Omega_N} &\leq q^s V_{\Omega n+k+1-s}, \qquad t \leq (s-1), \\ &\leq \widetilde{a} + \epsilon + \frac{q}{(q-1)}(1-q^s). \end{aligned}$$

As indicated, these operations require

$$r \leq t \leq (s-1) \leq (r_0 - 1), \tag{7.14}$$

which implies

$$(r+1) \leq s \leq r_0, \tag{7.15}$$

and shows that $s \neq r$.

Substitution in (7.8) yields

$$\begin{aligned} \frac{q^t}{2} V_{D_0^{N+1}} &\leq \widetilde{a} + \epsilon + \frac{q}{(q-1)}(1-q^s) + q^{(1+t)} - \widetilde{a} + \epsilon - \frac{q}{(q-1)}(1-q^r) \\ &\leq 2\epsilon, \end{aligned} \tag{7.16}$$

by virtue of relations (7.14). Zanaboni's version of Saint-Venant's principle is established for $r < s$.

It remains to consider the case when $r = s$ in the convergent subsequence; that is, when

$$|a_r^{n+k+1} - a_r^{n+1}| \leq \epsilon, \qquad n \geq n_0, \tag{7.17}$$

and inequalities (6.14) and (6.15) are satisfied for $s = r$. Set

$$\begin{aligned} \Omega_{N+1} &= \Omega_{n+k+1-r}, & (7.18)\\ \Omega_N &= \Omega_{n-r} = \Omega_{n+1-(r+1)}. & (7.19) \end{aligned}$$

It follows from (2.1) that

$$\Omega_N \subset \Omega_{N+1}$$

provided

$$(r-k) < (r+1)$$

which is always valid.

The choice of regions implies

$$\begin{aligned} V_{\Omega_{N+1}} &= V_{\Omega_{n+k+1-r}},\\ V_{\Omega_N} &= V_{\Omega_{n-r}}. \end{aligned}$$

Consider inequality (7.8) with $t = r$ and use (6.15) repeatedly to derive the following inequalities

$$\begin{aligned} \frac{q^r}{2} V_{D_0^{N+1}} &\leq \left[q^{(1+r)} V_{\Omega_{n+1-1-r}} + q^{(1+r)} - q^r V_{\Omega_{n+k+1-r}} \right]\\ &\leq \left[\widetilde{a} + \epsilon + \frac{q}{(q-1)}(1 - q^{(1+r)}) + q^{(1+r)} - \widetilde{a} + \epsilon - \frac{q}{(q-1)}(1 - q^r) \right]\\ &= 2\epsilon. \qquad (7.20) \end{aligned}$$

But $0 \leq r \leq r_0 < \infty$, and consequently Saint-Venant's principle is proved when $r = s$.

The special case $r = s = 0$, which may be included in the argument leading to (7.20), presents no difficulty in the derivation of a Saint-Venant principle. In this respect, the constant M_3 appearing in (6.8) may be chosen arbitrarily large or small. A sufficiently small M_3 requires, from (6.11),that $r_0 = 0$, which as just demonstrated may be included in the proof.

8 Concluding remarks

Zanaboni's version of Saint-Venant's principle postulates that in an elongated linear elastic body in the absence of source terms and regardless of the body's shape the strain energy tends to zero in regions increasingly remote from the load surface. This paper extends the result to transient heat conduction subject to bounded thermal energies and a bounded time derivative of the temperature prescribed over the common surface Γ. An advantage of Zanaboni's procedure is its applicability to bodies of general geometry. Such generality, however, is also a weakness. In constrast to the approach based upon, for example, differential inequalities for cylindrical bodies, it is not yet possible to derive precise decay estimates. Nevertheless, the general character of conclusions derived using the Zanaboni argument for the spatial, rather than the temporal, distribution of thermal energy has obvious implications for issues such as domain decomposition in the design of computer programs.

It is well known that the mathematical model adopted here for transient heat conduciton admits an infinite speed of heat propagation. Implications of this property are beyond the intended scope of the present study. Reconciliation, however, is apparently needed since we have shown that the space-time integral of the thermal energy taken over sufficiently remote regions remains small irrespective of time. It would also be of related interest to explore, either by the present or some other method, whether Zanaboni's version of Saint-Venant's principle is valid for the Green-Naghdi [6, 7], Maxwell-Cattaneo [4, 5], or similar hyperbolic theories that admit a finite speed of heat propagation

A further extension of the method to hyperbolic systems, including the wave equation, awaits investigation. By contrast, the treatment of external regions, the half-space, and cone-like regions appears amenable to a direct extension of present methods.

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