On Segre's Lemma of Tangents

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Abstract

Segre's lemma of tangents dates back to the 1950's when he used it in the proof of his "arc is a conic" theorem. Since then it has been used as a tool to prove results about various objects including internal nuclei, Kakeya sets, sets with few odd secants and further results on arcs. Here, we survey some of these results and report on how re-formulations of Segre's lemma of tangents are leading to new results.

Keywords: Kakeya sets, lemma of tangents, sets with no tangents.

1 Introduction

Let $\mathrm{PG}(2,q)$ denote the finite projective plane over the field with q elements.

Let S be a set of $q+2-t$ points, where t can be any integer (not necessarily positive). We say a line is an i-secant to S if it is incident with exactly i points of S. If $i=1$ we say the line is a *tangent* to S.

For each point in $a \in S$, let $f_a(x)$ denote the polynomial one obtains by taking the product of the linear forms whose kernels are tangents to S which are incident with a. Let $g_a(x)$ denote the polynomial one obtains by taking the product of the linear forms whose kernels are j-secants (for all $j \geqslant 3$) to S which are incident with a, taken with multiplicity $j-2$.

Segre published the following lemma in 1967 [11], in the case that $\deg g_a = 0$, and it became known as his lemma of tangents. It has been instrumental in the study of arcs since then and was a generalisation of the approach he took to prove in [10] that an arc of size $q+1$ in $\mathrm{PG}(2,q)$ is a conic when q is odd.

Lemma 1.1 *Let S be a set of $q+2-t$ points in $\mathrm{PG}(2,q)$. If x, y and z are points of S joined by 2-secants then*

$$\frac{f_x(y)f_y(z)f_z(x)}{g_x(y)g_y(z)g_z(x)} = (-1)^{t+1}\frac{f_x(z)f_y(x)f_z(y)}{g_x(z)g_y(x)g_z(y)}.$$

It was not until 2010 in [2] that the coordinate-free version of this lemma was introduced. This has made the lemma far more applicable as it simplifies calculations.

However, there were some important applications of Segre's lemma beforehand. For example, Segre himself used the lemma to prove the following theorem about arcs (a set of points with the property that no three points are collinear) in [11].

Theorem 1.2 *Let S be a set of $q+2-t$ points in $\mathrm{PG}(2,q)$ with the property $t \geqslant 0$ and that no three points are collinear. Then the set $t(q+2-t)$ points, which are dual to the tangents, are contained in an algebraic curve of degree t, if q is even, and degree $2t$, if q is odd.*

Since Theorem 1.2 implies the existence of a curve of small degree containing many points, theorems such as the Hasse-Weil theorem and the Stöhr-Voloch theorem can be used to prove that large arcs are extendable to arcs of maximum size.

Another example is the following theorem from [5]. In fact, it is an immediate consequence of Lemma 1.1, one obtains the contradiction $1 = -1$, assuming that there are three points of S which are incident with only 2-secants.

Theorem 1.3 *Let S be a set of $q+2$ points in $\mathrm{PG}(2, q)$. If q is odd then there are at most two points in S which are incident with only 2-secants.*

Let $\mathrm{AG}(2, q)$ denote the affine plane over the field with q elements.

A *Kakeya set* in $\mathrm{AG}(2, q)$ is a set of points K with the property that for every direction there is a line all of whose points are contained in K. Equivalently, we can consider the set of lines whose points are contained in K (together with the line at infinity) as a set of $q+2$ points S in the dual plane (where the line at infinity corresponds to a point incident with only 2-secants). The Kakeya problem is to determine the smallest Kakeya sets. In the dual plane this is the problem of minimising the number of lines incident with S.

By double counting, one can prove that the number of lines incident with a set of $q+2$ points is

(1) $$\frac{1}{2}(q + 2)(q + 1) + \frac{1}{2}\sum_{i \geqslant 3}(i - 2)(i - 1)\tau_i,$$

where τ_i is the number of lines incident with i points of S. If q is even then there are examples (called *hyperovals*) of such sets where $\tau_i = 0$ for $i \geqslant 3$.

Again Segre's lemma of tangents, Lemma 1.1, was instrumental in the proof of the following theorem from [7].

Theorem 1.4 *Let S be a set of $q + 2$ points in $\mathrm{PG}(2, q)$ with at least one point incident with only 2-secants. If q is odd then there are at least $\frac{1}{2}((q + 2)(q + 1) + (q - 1))$ lines incident with S. Furthemore, if q is odd and there are exactly $\frac{1}{2}((q + 2)(q + 1) + (q - 1))$ lines incident with S then S is a conic and an additional point.*

Theorem 1.4 improved on the bound $\frac{1}{2}(q + 2)(q + 1) + \frac{1}{3}(q - 1)$ from [6], in the case that S has a point incident with only 2-secants. The proof of the bound from [6] also used Segre's lemma of tangents.

The Kakeya problem was the inspiration for the following conjecture from [1].

Conjecture 1.5 *Let S be a set of $q + 2$ points in $\mathrm{PG}(2, q)$. If q is odd then there are at least $2q - 2$ lines incident with S in an odd number of points.*

Again the bound is obtained when S is a conic and an additional point.

An observation in [4] is the following. If we fix a point of $e \in S$ and scale the polynomials f_x and g_x so that

$$f_x(e) = (-1)^{t+1} f_e(x)$$

then Lemma 1.1 simplifies to the following lemma.

Lemma 1.6 *If x and y are points of S joined by a 2-secant and are joined by 2-secants to e then*

$$f_x(y)g_y(x) = (-1)^{t+1} f_y(x)g_x(y).$$

In [4] the focus is again arcs, so g_a can be taken as 1, and the lemma simplifies to

$$f_x(y) = (-1)^{t+1} f_y(x)$$

for all points x and y of S. This leads to the following theorem from [4].

Theorem 1.7 *An arc in $\mathrm{PG}(2, q)$, $q = p^{2h}$, $p \neq 2$, of size at least $q - \sqrt{q} + 3 + \sqrt{q}/p$ is contained in a conic.*

On the other extreme in which S has no tangents, f_a can be taken as 1. In this case, Segre's lemma of tangents was used to prove the following theorem in [8].

Theorem 1.8 *Let S be a set of $q + 2 + t$ points in $\mathrm{PG}(2, q)$. If q is odd and S has no tangents then $t \geq \frac{1}{4}\sqrt{2q}$.*

The following theorem from [9] is a similar theorem for q even, but with the necessary restriction that S has at least some odd-secants.

Theorem 1.9 *Let S be a set of $q + 2 + t$ points in $\mathrm{PG}(2, q)$. If q is even and S has no tangents but some odd-secants then $t \geq \sqrt{q/6} - 1$.*

The following theorem is from [3] and verifies Conjecture 1.5 asymptotically.

Theorem 1.10 *Let S be a set of $q + 2$ points in $\mathrm{PG}(2, q)$. If q is odd then there is a constant c and a q_0, such that for $q \geqslant q_0$, there are at least $2q - c$ lines incident with an odd number of points of S.*

For a set S of $q + 2$ points in $\mathrm{PG}(2, q)$, for each $x \in S$

$$\tau_1(x) = \sum_{i \geqslant 3} (i - 2)\tau_i(x),$$

where $\tau_i(x)$ is the number of i-secants incident with x, which summing over $x \in S$ gives,

$$\tau_1 = \sum_{i \geqslant 3} i(i-2)\tau_i.$$

Therefore, (1) implies that the number of lines incident with a point of a set S of $q+2$ points is at least

$$\frac{1}{2}(q+2)(q+1)+\frac{1}{2}\sum_{i\geqslant 3}((i-2)(i-1)-\frac{1}{2}i(i-2))\tau_i+\frac{1}{4}\tau_1 \geqslant \frac{1}{2}(q+2)(q+1)+\frac{1}{4}o(S),$$

where $o(S)$ is the number of lines incident with an odd number of points of S.

Therefore, Theorem 1.10 implies the following theorem.

Theorem 1.11 *Let S be a set of $q+2$ points in $\mathrm{PG}(2,q)$. If q is odd then there is a constant c and a q_0, such that for $q \geqslant q_0$, there are at least $\frac{1}{2}((q+2)(q+1)+q-c)$ lines incident with S.*

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