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Analysis for the strain gradient theory of porous thermoelasticity

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Abstract

In this paper, we analyze a model involving a strain gradient thermoelastic rod with voids. Existence and uniqueness, as well as an energy decay property, are proved by means of the semigroup arguments. The variational formulation is derived and then, a fully discrete approximation is introduced by using the finite element method to approximate the spatial variable and the implicit Euler scheme to discretize the time derivatives. A stability result and a priori error estimates are obtained, from which the linear convergence of the algorithm is deduced under suitable additional regularity conditions. Finally, some numerical simulations are presented to demonstrate the accuracy of the algorithm and the behaviour of the solution.

Key words: Strain gradient, thermo-elasticity, existence and uniqueness, exponential decay, finite elements, error estimates, numerical simulations.

1 Introduction

Elastic materials with voids are the easiest extension of the classical theory of elasticity. These materials are characterized by an elastic matrix and the interstices are void of the material. In this case the bulk density is the product of two scalar fields, the matrix material density and the volume fraction. The theory of porous materials was established by Cowin and Nunziato [1], Nunziato and Cowin [2] and Cowin [3]. The study of this kind of material is included in the so-called non-classical elasticity. An accurate analysis of them can be found in the book of Ieşan [4] (see also the references therein). This theory has been applied with success to elastic bodies with small voids or vacuo pores which are in the material. In fact, it is currently used to describe engineering and/or biological systems arising in many different areas, such as petroleum industry, material science, ceramics, pressed powders or bones.

The theory of Cowin and Nunziato has been widely accepted and it has been the aim of many research papers (see, for example, [5–13]).

On the other hand, there is the strain gradient theory, characterized by the inclusion of higher gradients of displacement in the basic postulates. The equations of motion, the constitutive equations and the boundary conditions of the strain gradient theory were given in a nonlinear form by Toupin [14,15] and Mindlin [16]. Its linear form was proposed by Green and Rivlin [17] and Mindlin and Eshel [18]. Rymarz [19] and Brulin and Hyalmarss [20] showed that, for the investigation of specific nonlocal phenomena, the second order displacement gradient should be added to the independent constitutive variables. It is commonly accepted that the strain gradient theory of elasticity is suitable to study problems related to the size effects. This theory is also under study and there is a huge amount of contributions referring to it (see, for instance the recent contributions, [21–23]).

In the present work, we consider the theory of thermoelasticity when porous and strain gradient effects are combined. It is worth noting that, recently, another contribution concerning time decay estimates has been presented for the joint combination of these two effects but in the isothermal case [24]. Here, we restrict our attention to the one-dimensional theory and we study several qualitative aspects about the behavior of the solutions.

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We propose two kind of results. On the one side we look for analytic and qualitative results and we prove the existence, uniqueness and exponential decay of the solutions for the one-dimensional strain gradient theory of porous-thermoelasticity. These results extend to the strain gradient theory the one obtained by Casas and Quintanilla [25]. On the other side, we consider numerical results and we propose a finite element method for the problem, proving a stability result, some a priori error estimates and a convergence result. These results continue the research started in [26], where a one-dimensional porous thermoelastic of type II rod is numerically simulated, in [27], where a porous thermoelastic body with microtemperatures is studied from the numerical point of view, and in [28] where a multi-dimensional porous thermoelastic problem is numerically solved. Therefore, this work extends the previous results to the strain gradient theory, whose coupling among the unknown variables is rather different. We want to highlight that properties as the exponential decay imply that after a small period of time the consequence of the perturbations are so small that they can be neglected. Therefore, this kind of results are relevant not only from a mathematical point of view but also from a mechanical point of view.

The outline of this paper is as follows. In Section 2, we describe the mathematical problem. Existence and uniqueness and an energy decay property are proved in Section 3. Moreover, the variational problem is derived in order to be considered in the following sections. Then, fully discrete approximations are introduced in Section 4 by using the finite element method for the spatial approximation and the backward Euler scheme for the discretization of the time derivatives. An error estimate result is proved from which the linear convergence is deduced under suitable regularity assumptions. Finally, in Section 5 some one-dimensional numerical examples are shown to demonstrate the accuracy of the algorithm and the behaviour of the solutions.

2 The mechanical model

In this section, we present a brief description of the model (details can be found in [24]).

Let us denote by $[0, \pi] \times [0, T)$, $T > 0$, the one-dimensional rod of length π and the time interval of interest, respectively. Moreover, let $x \in [0, \pi]$ and $t \in [0, T)$ be the spatial and time variables. In order to simplify the writing, we do not indicate the dependence of the functions on x and t , and the subscript x under a variable represents its spatial derivative. Finally, we denote the time derivative of a variable with a dot over that variable. If the length of the rod is $\ell \neq \pi$, our analysis is still valid and can be adapted to the new situation by making a change of variable. Nevertheless, we do not explicit this study in

this paper.

For the one-dimensional strain gradient theory of porous thermoelasticity the evolution equations are given by

- the equation of motion

$$\rho\ddot{u} = \sigma_x - \mu_{xx}, \quad (1)$$

- the porous evolution equation

$$J\ddot{\phi} = h_x + g, \quad (2)$$

- and the heat equation

$$T_0\dot{\eta} = q_x. \quad (3)$$

In these equations, ρ is the mass density, σ is the stress, μ is the hyperstress, $J = \rho\kappa$ is the product of the mass density by the equilibrated inertia, h is the equilibrated stress, g is the equilibrated body force, T_0 is the temperature in the equilibrium state, η is the entropy and q is the heat flux vector.

We consider two kinds of dissipation: one is present in the equilibrated body force and the other one is the usual thermal type. Then, the constitutive equations are given by

$$\begin{aligned} \sigma &= au_x + b\phi - \beta^*\theta, \\ \mu &= cu_{xx} + d\phi_x, \\ h &= du_{xx} + \beta\phi_x, \\ g &= -\xi - bu_x + m\theta - \tau\dot{\phi}, \\ q &= k\theta_x, \\ \eta &= c^*\theta + \beta^*u_x + m\phi. \end{aligned} \quad (4)$$

In these equations u represents the displacement, ϕ is the volume fraction and θ is the difference of the temperature between the actual state and a reference temperature.

If the constitutive equations are substituted into the evolution equations we obtain the system of the field equations

$$\begin{aligned} \rho\ddot{u} &= -cu_{xxxx} - d\phi_{xxx} + au_{xx} + b\phi_x - \beta^*\theta_x, \\ J\ddot{\phi} &= \beta\phi_{xx} + du_{xxx} - \xi\phi - bu_x + m\theta - \tau\dot{\phi}, \\ c^*\dot{\theta} &= -\beta^*\dot{u}_x - m\dot{\phi} + k^*\theta_{xx}, \end{aligned} \quad (5)$$

where $k^* = \frac{k}{T_0}$.

In order to have a well defined problem we need to impose boundary and initial conditions. One possible family of boundary conditions is given by

$$\begin{aligned} u(0, t) = u(\pi, t) = u_{xx}(0, t) = u_{xx}(\pi, t) = 0, \\ \phi_x(0, t) = \phi_x(\pi, t) = 0, \\ \theta_x(0, t) = \theta_x(\pi, t) = 0. \end{aligned} \tag{6}$$

As initial conditions we impose

$$\begin{aligned} u(x, 0) = u_0(x), \quad \dot{u}(x, 0) = v_0(x), \\ \phi(x, 0) = \phi_0(x), \quad \dot{\phi}(x, 0) = e_0(x), \\ \theta(x, 0) = \theta_0(x). \end{aligned} \tag{7}$$

To guarantee that the solutions of the problem determined by (5)–(7) decay, we should impose that

$$\int_0^\pi \phi_0(x) dx = \int_0^\pi e_0(x) dx = \int_0^\pi \theta_0(x) dx = 0.$$

In the previous system of equations a and c are elastic coefficients, d and b are porosity coefficients, β^* is a thermal expansion coefficient, β and ξ are porosity diffusion coefficients, m is a thermal expansion coefficient, τ is a viscous porosity coefficient, c^* is the heat capacity, k^* represents a thermal diffusion coefficient, and u_0 , v_0 , ϕ_0 , e_0 and θ_0 are given initial conditions.

We assume that the constitutive coefficients satisfy:

$$\begin{aligned} \rho > 0, \quad J > 0, \quad a > 0, \quad c > 0, \quad c^* > 0, \quad k^* > 0, \quad \tau > 0, \\ a\xi - b^2 > 0, \quad c\beta - d^2 > 0. \end{aligned} \tag{8}$$

The mechanical interpretation of the positivity of ρ and J is clear. The positivity of τ and k^* implies that the processes are dissipative. The other assumptions are imposed to guarantee that the internal energy of the system is positive definite, condition that is related with the well-posedness of the problem in the sense of Hadamard.

3 Existence, uniqueness and exponential decay of the solutions

We will prove that the solutions of system (5) with the boundary conditions (6) and the initial conditions (7) decay exponentially, and for this purpose, we will use the contraction semigroup arguments.

If we denote $v = \dot{u}$, $e = \dot{\phi}$ and $D = \frac{\partial}{\partial x}$, we can write system (5) in the following form:

$$\begin{aligned}\dot{u} &= v, \\ \dot{v} &= \frac{1}{\rho} \left(aD^2u + bD\phi - cD^4u - dD^3\phi - \beta^*D\theta \right), \\ \dot{\phi} &= e, \\ \dot{e} &= \frac{1}{J} \left(dD^3u + \beta D^2\phi - \xi\phi - bDu + m\theta - \tau e \right), \\ \dot{\theta} &= \frac{1}{c^*} \left(k^*D^2\theta - \beta^*Dv - me \right).\end{aligned}$$

Next, we define $L_*^2 = \{f \in L^2 : \int_0^\pi f(x)dx = 0\}$, and let $H_*^1 = H^1 \cap L_*^2$.

To formalize the problem, we assume the evolution to be taking place on the Hilbert space

$$\mathcal{H} = \left\{ U = (u, v, \phi, e, \theta) \in \left(H^2 \cap H_0^1 \right) \times L^2 \times H_*^1 \times L_*^2 \times L_*^2, \right. \\ \left. \int_0^\pi \phi(x)dx = \int_0^\pi e(x)dx = \int_0^\pi \theta(x)dx = 0 \right\}$$

driven by the operator

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ \frac{aD^2 - cD^4}{\rho} & 0 & \frac{bD - dD^3}{\rho} & 0 & -\frac{\beta^*D}{\rho} \\ 0 & 0 & 0 & I & 0 \\ \frac{dD^3 - bD}{J} & 0 & \frac{\beta D^2 - \xi}{J} & -\frac{\tau}{J} & \frac{m}{J} \\ 0 & -\frac{\beta^*D}{c^*} & 0 & -\frac{m}{c^*} & \frac{k^*D^2}{c^*} \end{pmatrix},$$

where I denotes the identity operator.

With the aforementioned notation, our initial boundary value problem can be written in abstract form as

$$\frac{dU}{dt} = \mathcal{A}U, \quad U_0 = (u_0, v_0, \phi_0, e_0, \theta_0).$$

It can be proved that the mild solutions of system (5) are given by the semi-group of contractions generated by the operator \mathcal{A} .

We define an inner product in \mathcal{H} . If $U^* = (u^*, v^*, \phi^*, e^*, \theta^*)$, then

$$\begin{aligned} \langle U, U^* \rangle_{\mathcal{H}} = \frac{1}{2} \int_0^\pi & \left(\rho v \bar{v}^* + c^* \theta \bar{\theta}^* + a u_x \bar{u}_x^* + c u_{xx} \bar{u}_{xx}^* + J e \bar{e}^* + \beta \phi_x \bar{\phi}_x^* \right. \\ & \left. + \xi \phi \bar{\phi}^* + d(u_{xx} \bar{\phi}_x^* + \bar{u}_{xx}^* \phi_x) + b(u_x \bar{\phi}^* + \bar{u}_x^* \phi) \right) dx. \end{aligned} \quad (9)$$

Here, a superposed bar denotes the complex conjugation. It should be pointed out that this product is equivalent to the usual product in the Hilbert space \mathcal{H} .

It is worth noting that the squared norm of a vector is given by

$$\begin{aligned} \|U\|^2 = \frac{1}{2} \int_0^\pi & \left(\rho |v|^2 + c^* |\theta|^2 + J |e|^2 + a |u_x|^2 + c |u_{xx}|^2 + \beta |\phi_x|^2 + \xi |\phi|^2 \right. \\ & \left. + 2d \Re(u_{xx} \phi_x) + 2b \Re(u_x \phi) \right) dx. \end{aligned}$$

The domain of \mathcal{A} is the set of $U \in \mathcal{H}$ such that $\mathcal{A}U \in \mathcal{H}$ and $u_{xx}(0) = u_{xx}(\pi) = 0$.

Lemma 1 *The operator \mathcal{A} defined previously is the infinitesimal generator of a C_0 -semigroup of contractions on \mathcal{H} .*

PROOF. First of all, we notice that $\mathcal{D}(\mathcal{A})$ contains a subset that is dense in \mathcal{H} and then $\mathcal{D}(\mathcal{A})$ is also dense in \mathcal{H} (this result comes from the density theorem, see [29], page 9, Theorem 1.4.1). We will show that \mathcal{A} is a dissipative operator and that 0 is in the resolvent set of \mathcal{A} . Using the Lumer-Phillips theorem (see [29], page 3, Theorem 1.2.3), the conclusion will follow.

On the one hand, a direct calculation gives

$$\Re \langle \mathcal{A}U, U \rangle = -\frac{1}{2} \int_0^\pi (k^* |D\theta|^2 + \tau |e|^2) dx \leq 0, \quad (10)$$

and, therefore, the operator \mathcal{A} is dissipative.

On the other hand, for any $\mathcal{F} = (f_1, f_2, f_3, f_4, f_5) \in \mathcal{H}$ we will find a unique

$U \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}U = \mathcal{F}$, or equivalently:

$$\begin{aligned}
v &= f_1, \\
\frac{1}{\rho} \left(aD^2u + bD\phi - cD^4u - dD^3\phi - \beta^*D\theta \right) &= f_2, \\
e &= f_3, \\
\frac{1}{J} \left(dD^3u + \beta D^2\phi - \xi\phi - bDu + m\theta - \tau e \right) &= f_4, \\
\frac{1}{c^*} \left(k^*D^2\theta - \beta^*Dv - me \right) &= f_5.
\end{aligned} \tag{11}$$

The second, fourth and fifth equations can be written in terms of f_1, f_2, f_3, f_4 and f_5 as follows:

$$\begin{aligned}
aD^2u + bD\phi - cD^4u - dD^3\phi - \beta^*D\theta &= \rho f_2, \\
dD^3u + \beta D^2\phi - \xi\phi - bDu + m\theta &= Jf_4 + \tau f_3, \\
k^*D^2\theta &= c^*f_5 + \beta^*Df_1 + mf_3.
\end{aligned} \tag{12}$$

To prove the solvability of this system we develop f_1, f_2, f_3, f_4 and f_5 in Fourier series. The families of $\sin(nx)$ and $\cos(mx)$ are an orthonormal complete system in the Hilbert space L^2 . In particular, we develop f_1 and f_2 in series of sines and f_3, f_4 and f_5 in series of cosines. So, we take $f_1 = \sum_{n=1}^{\infty} a_n \sin(nx)$, $f_2 = \sum_{n=1}^{\infty} b_n \sin(nx)$, $f_3 = \sum_{n=1}^{\infty} c_n \cos(nx)$, $f_4 = \sum_{n=1}^{\infty} d_n \cos(nx)$ and $f_5 = \sum_{n=1}^{\infty} e_n \cos(nx)$. In view of the fact that the averages of f_3, f_4 and f_5 are zero, the developments in the cosines series start from $n = 1$.

We will show that it is possible to find $u = \sum_{n=1}^{\infty} u_n \sin(nx)$, $\phi = \sum_{n=1}^{\infty} \phi_n \cos(nx)$ and $\theta = \sum_{n=1}^{\infty} \theta_n \cos(nx)$ such that $\sum_{n=1}^{\infty} n^4 |u_n|^2 < \infty$, $\sum_{n=1}^{\infty} n^2 |\phi_n|^2 < \infty$ and $\sum_{n=1}^{\infty} |\theta_n|^2 < \infty$. From the assumptions we know that $\sum_{n=1}^{\infty} n^4 |a_n|^2 < \infty$, $\sum_{n=1}^{\infty} |b_n|^2 < \infty$, $\sum_{n=1}^{\infty} n^2 |c_n|^2 < \infty$, $\sum_{n=1}^{\infty} |d_n|^2 < \infty$ and $\sum_{n=1}^{\infty} |e_n|^2 < \infty$.

Substituting the above expressions in system (12) and performing simplifications, we get a linear system for the unknown coefficients u_n, ϕ_n and θ_n for

each n , with the unique solution given by

$$\begin{aligned} u_n &= \frac{u_n^0 + u_n^1 n + u_n^2 n^2 + u_n^3 n^3 + u_n^4 n^4}{kn^3 (b^2 - a\xi) + kn^5 (-a\beta + 2bd - c\xi) + kn^7 (d^2 - \beta c)}, \\ \phi_n &= \frac{\phi_n^0 + \phi_n^1 n + \phi_n^2 n^2 + \phi_n^3 n^3 + \phi_n^4 n^4}{kn^3 (b^2 - a\xi) + kn^5 (-a\beta + 2bd - c\xi) + kn^7 (d^2 - \beta c)}, \\ \theta_n &= -\frac{c^* e_n + mc_n + a_n \beta^* n}{kn^2} \end{aligned}$$

where

$$\begin{aligned} u_n^0 &= -bm(c^* e_n - c_n m), \\ u_n^1 &= b_n k \xi \rho - a_n b \beta^* m, \\ u_n^2 &= -(bk(c_n \tau + d_n J) + dm(c^* e_n + c_n m)), \\ u_n^3 &= \beta b_n k \rho - a_n \beta^* dm, \\ u_n^4 &= -dk(c_n \tau + d_n J) \end{aligned}$$

and

$$\begin{aligned} \phi_n^0 &= ac^* e_n m + ac_n m^2, \\ \phi_n^1 &= aa_n \beta^* m - bb_n k \rho, \\ \phi_n^2 &= k(ac_n \tau + ad_n J) + cc^* e_n m + cc_n m^2, \\ \phi_n^3 &= a_n \beta^* cm - b_n dk \rho, \\ \phi_n^4 &= ck(c_n \tau + d_n J). \end{aligned}$$

Thus, it is clear that u_n , ϕ_n and θ_n satisfy the desired conditions. It is not difficult to see that $\|U\|_{\mathcal{H}} \leq C\|\mathcal{F}\|_{\mathcal{H}}$.

Therefore, 0 is in the resolvent set of \mathcal{A} and the lemma is proved.

Theorem 2 *The problem given by system (5) with boundary conditions (6) and initial conditions (7) in \mathcal{H} has a unique mild solution.*

PROOF. The proof is a direct consequence of the previous lemma.

Remark 3 *It is worth noting that we could impose other boundary conditions as*

$$\begin{aligned} u(0, t) = u(\pi, t) = u_x(0, t) = u_x(\pi, t) &= 0 \quad \text{for } t \in [0, T], \\ \phi(0, t) = \phi(\pi, t) &= 0 \quad \text{for } t \in [0, T], \\ \theta(0, t) = \theta(\pi, t) &= 0 \quad \text{for } t \in [0, T]. \end{aligned} \tag{13}$$

In this situation we should consider the Hilbert space given by $\mathcal{H}^* = H_0^2 \times L^2 \times H_0^1 \times L^2 \times L^2$. The domain of the operator is

$$\mathcal{D}(\mathcal{A}) = \{U \in \mathcal{H}^*, \text{ such that } \mathcal{A}U \in \mathcal{H}^*\},$$

which is a dense subset. In view of the boundary conditions it is also clear that $\Re\langle \mathcal{A}U, U \rangle \leq 0$ for every $U \in \mathcal{D}(\mathcal{A})$.

To prove the existence of a semigroup of contractions it is sufficient to solve system (11). We note that if we substitute the first and third equations into the others we obtain system (12). In this case, to prove the existence of solutions we can use the Lax-Milgram lemma. For this purpose, in the Hilbert space $\mathcal{B} = H_0^2 \times H_0^1 \times H_0^1$ we define the product

$$\begin{aligned} \langle (u, \phi, \theta), (u^*, \phi^*, \theta^*) \rangle = \langle (aD^2u + bD\phi - cD^4u - dD^3\phi - \beta^*D\theta, \\ dD^3u + \beta D^2\phi - \xi\phi - bDu + m\theta, k^*D^2\theta), (u^*, \phi^*, \theta^*) \rangle_{L^2 \times L^2 \times L^2}. \end{aligned}$$

It is clear that this is a bounded and coercive bilinear form. On the other side, $(\rho f_2, Jf_4 + \tau f_3, c^* f_5 + \beta^* Df_1 + mf_3)$ belongs to the dual of \mathcal{B} which is $H^{-2} \times H^{-1} \times H^{-1}$. Therefore, the Lax-Milgram lemma implies the existence of solution. Then, the existence of a semigroup of contractions is proved.

To show the exponential stability, we use a result due to Gearhart, stated in the book of Liu and Zheng (see [29], page 4, Theorem 1.3.2).

Theorem 4 *A semigroup of contractions $\{e^{t\mathcal{A}}\}_{t \geq 0}$ on a Hilbert space \mathcal{H} with norm $\|\cdot\|_{\mathcal{H}}$ is exponentially stable if and only if*

$$\{i\lambda, \lambda \text{ is real}\} \text{ is contained in the resolvent of } \mathcal{A}, \quad (14)$$

and

$$\overline{\lim}_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\| < \infty, \quad (15)$$

where \mathcal{I} denotes the identity operator.

We prove the validness of these conditions in the following two lemmas.

Lemma 5 *Let \mathcal{A} be the operator from Lemma 1. Then condition (14) is satisfied.*

PROOF. The operator $\mathcal{A}^{-1} : \mathcal{H} \rightarrow \mathcal{H}$ is compact. Let us consider a bounded sequence (F_n) in \mathcal{H} and (U_n) the sequence in $\mathcal{D}(\mathcal{A})$ such that $\mathcal{A}U_n = F_n$. Since $\mathcal{A}^{-1} \in \mathcal{L}(\mathcal{H})$, there exists a positive constant C such that

$$\|U_n\|_{\mathcal{H}} + \|\mathcal{A}U_n\|_{\mathcal{H}} \leq C \quad \forall n \in \mathbb{N}.$$

Therefore, (U_n) is bounded in $\mathcal{D}(\mathcal{A})$. Since the embedding of H^m in H^j ($m > j$) is compact, there exists a subsequence (U_{ν}) such that it is convergent to $U = (u, v, \phi, e, \theta)$.

Suppose that there exists $\lambda \in \mathbb{R}$ ($\lambda \neq 0$) such that $i\lambda$ is in the spectrum of \mathcal{A} . As \mathcal{A}^{-1} is compact, $i\lambda$ must be an eigenvalue of \mathcal{A} . Then, there exists a vector $U \neq 0$ such that $(i\lambda\mathcal{I} - \mathcal{A})U = 0$ in \mathcal{H} . Explicitly, this yields

$$\begin{aligned} i\lambda u - v &= 0, \\ i\lambda\rho v - aD^2u + cD^4u + dD^3\phi - bD\phi + \beta^*D\theta &= 0, \\ i\lambda\phi - e &= 0, \\ i\lambda Je - dD^3u + bDu - \beta D^2\phi + \xi\phi - m\theta + \tau e &= 0, \\ i\lambda c^*\theta + \beta^*Dv + me - k^*D^2\theta &= 0. \end{aligned}$$

Since $\langle (i\lambda\mathcal{I} - \mathcal{A})U, U \rangle = 0$, we have $\theta = 0$ and $e = 0$. Therefore, it must be $v = 0$, $u = 0$ and $\phi = 0$. Thus, we have a contradiction and the proof is complete.

Lemma 6 *Let \mathcal{A} be the operator defined above. Then condition (15) holds true.*

PROOF. Given $\lambda \in \mathbb{R}$ and $F = (f, g, p, q, r) \in \mathcal{H}$, there exists a unique $U = (u, v, \phi, e, \theta) \in \mathcal{D}(\mathcal{A})$ such that $(\lambda\mathcal{I} - \mathcal{A})U = F$. If we write this condition term by term we get

$$i\lambda u - v = f, \tag{16}$$

$$i\lambda\rho v - aD^2u + cD^4u + dD^3\phi - bD\phi + \beta^*D\theta = \rho g, \tag{17}$$

$$i\lambda\phi - e = p, \tag{18}$$

$$i\lambda Je - dD^3u + bDu - \beta D^2\phi + \xi\phi - m\theta + \tau e = Jq, \tag{19}$$

$$i\lambda c^*\theta + \beta^*Dv + me - k^*D^2\theta = c^*r. \tag{20}$$

We note that

$$\Re\langle (i\lambda\mathcal{I} - \mathcal{A})U, U \rangle = \int_0^\pi (k^*|D\theta|^2 + \tau|e|^2)dx = \Re\langle F, U \rangle.$$

Then, we obtain that

$$\int_0^\pi (k^*|D\theta|^2 + \tau|e|^2)dx \leq \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

We multiply equation (17) by u and equation (19) by ϕ in L^2 and, using (16) and (18) we obtain

$$\begin{aligned}
& (a - \epsilon_1) \int_0^\pi |Du|^2 dx + (c - \epsilon_1) \int_0^\pi |D^2u|^2 dx + d \int_0^\pi (D^2u D\bar{\phi} + D\phi D^2\bar{u}) dx \\
& \quad + b \int_0^\pi (Du\bar{\phi} + \phi D\bar{u}) dx + (\beta - \epsilon_1) \int_0^\pi |D\phi|^2 dx + (\xi - \epsilon_1) \int_0^\pi |\phi|^2 dx \\
& \leq \rho \int_0^\pi |v|^2 dx + C_2 \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}.
\end{aligned} \tag{21}$$

Here ϵ_1 is a positive constant that we can choose as small as we want and C_2 is a computable positive constant.

The next step is to estimate $|v|^2$ in L^2 . To this end we define η_i , for $i = 1, 2, 3$, and y_j , for $j = 1, 3$, as the solutions to the following problems:

$$\begin{aligned}
D^2\eta_1 &= -v, & D\eta_1(0) &= D\eta_1(\pi) = 0, \\
D^2\eta_2 &= -e, & D\eta_2(0) &= D\eta_2(\pi) = 0, \\
D^2\eta_3 &= -\theta, & D\eta_3(0) &= D\eta_3(\pi) = 0, \\
D^2y_1 &= -g, & y_1(0) &= y_1(\pi) = 0, \\
D^2y_3 &= -r, & y_3(0) &= y_3(\pi) = 0.
\end{aligned}$$

We multiply (20) by $D\eta_1$ in L^2 and we get

$$i\lambda c^*(\theta, D\eta_1) - k^*(D^2\theta, D\eta_1) + \beta^*(Dv, D\eta_1) + m(e, D\eta_1) = c^*(r, D\eta_1). \tag{22}$$

Taking into account the properties of functions η_i , we have

$$\begin{aligned}
i\lambda c^*(\theta, D\eta_1) &= -i\lambda c^*(D^2\eta_3, D\eta_1) = i\lambda c^*(D\eta_3, D^2\eta_1) = c^*(D\eta_3, i\lambda v), \\
k^*(D^2\theta, D\eta_1) &= k^*(D\theta, v), \\
\beta^*(Dv, D\eta_1) &= \beta^*|v|^2, \\
c^*(r, D\eta_1) &= -c^*(D^2y_3, D\eta_1) = c^*(Dy_3, D^2\eta_1) = -c^*(Dy_3, v).
\end{aligned}$$

Moreover,

$$\begin{aligned}
c^*(D\eta_3, i\lambda v) &= \frac{c^*}{\rho} (D\eta_3, aD^2u - cD^4u - dD^3\phi + bD\phi - \beta^*D\theta + \rho g) \\
&= \frac{c^*a}{\rho} (\theta, Du) + \frac{c^*c}{\rho} (D\theta, D^2u) - \frac{c^*d}{\rho} (D\theta, D\phi) - \frac{c^*b}{\rho} (D\theta, \phi) \\
&\quad + \frac{c^*\beta^*}{\rho} |\theta|^2 - c^*(\theta, Dy_1).
\end{aligned}$$

Substituting the above expressions in (22) we obtain:

$$\begin{aligned} \beta^*|v|^2 = & -\frac{c^*a}{\rho}(\theta, Du) + \frac{c^*c}{\rho}(D\theta, D^2u) + \frac{c^*d}{\rho}(D\theta, D\phi) + \frac{c^*b}{\rho}(D\theta, \phi) \\ & -\frac{c^*\beta^*}{\rho}|\theta|^2 + c^*(\theta, Dy_1) - k^*(D\theta, v) - m(e, D\eta_1) - c^*(Dy_3, v). \end{aligned} \quad (23)$$

Suppose that $\beta^* > 0$ (if $\beta^* < 0$, the analysis is similar). We obtain that

$$(\beta^* - \epsilon_2)|v|^2 \leq C_4\|U\|\|F\| + \epsilon_2\left(|Du|^2 + |D^2u|^2 + |\phi|^2 + |D\phi|^2\right), \quad (24)$$

where ϵ_2 is a small positive constant and C_4 is a computable positive constant. In view of (21) and (24) we get that

$$|Du|^2 + |D^2u|^2 + |\phi|^2 + |D\phi|^2 \leq C_5\|U\|\|F\|.$$

Combining these estimates, we conclude that a positive constant C_6 exists such that

$$\|U\| \leq C_6\|F\|,$$

and then the lemma is proved.

Theorem 7 *Let (u, ϕ, θ) be a mild solution of the problem determined by (5), with boundary conditions (6) and initial conditions (7) in \mathcal{H} . Then, (u, ϕ, θ) decays exponentially to zero when time tends to infinity.*

PROOF. The proof is a direct consequence of Lemmas 5 and 6.

Remark 8 *It is also possible to prove the exponential decay of solutions for the problem determined by the boundary conditions given by (13). Nevertheless, to this end we should evaluate several integrals at the boundary of the domain but we will not develop it here. This study could be done in a similar way to the one developed in [30].*

Finally, we provide the variational formulation of Problem (5), (7) and (13). In order to do it, let $Y = L^2(\Omega)$ and denote by (\cdot, \cdot) the scalar product in this space, with corresponding norm $\|\cdot\|$. Moreover, let us define the variational spaces E and V as follows,

$$\begin{aligned} E &= \{r \in H^1(0, \pi); r(0) = r(\pi) = 0\}, \\ V &= \{v \in H^2(0, \pi); v(0) = v_x(0) = v(\pi) = v_x(\pi) = 0\}. \end{aligned}$$

By using integration by parts and boundary conditions (13), we write the variational formulation of Problem (5), (7) and (13) in terms of the velocity $v = \dot{u}$, the porosity speed $e = \dot{\phi}$ and the temperature θ .

Problem VP. Find the velocity field $v : [0, T] \rightarrow V$, the porosity speed field $e : [0, T] \rightarrow E$ and the temperature field $\theta : [0, T] \rightarrow E$ such that $v(0) = v_0$, $e(0) = e_0$, $\theta(0) = \theta_0$ and, for a.e. $t \in (0, T)$,

$$\begin{aligned} \rho(\dot{v}(t), w) + a(u_x(t), w_x) + c(u_{xx}(t), w_{xx}) &= b(\phi_x(t), w) - \beta^*(\theta_x(t), w) \\ &\quad - d(\phi_x(t), w_{xx}), \quad \forall w \in V, \end{aligned} \quad (25)$$

$$\begin{aligned} J(\dot{e}(t), r) + \beta(\phi_x(t), r_x) + \xi(\phi(t), r) &= -d(u_{xx}(t), r_x) - b(u_x(t), r) \\ &\quad + m(\theta(t), r) \quad \forall r \in E, \end{aligned} \quad (26)$$

$$c^*(\dot{\theta}(t), l) + k^*(\theta_x(t), l_x) = -\beta^*(v_x(t), l) - m(e(t), l) \quad \forall l \in E, \quad (27)$$

where the displacement and porosity fields are then recovered from the relations

$$u(t) = \int_0^t v(s) ds + u_0, \quad \phi(t) = \int_0^t e(s) ds + \phi_0. \quad (28)$$

4 Fully discrete approximations: a priori error estimates

In this section, we now consider a fully discrete approximation of Problem VP. This is done in two steps. First, we assume that the interval $[0, \pi]$ is divided into M subintervals $a_0 = 0 < a_1 < \dots < a_M = \pi$ of length $h = a_{i+1} - a_i = \pi/M$, and we consider two finite dimensional spaces $V^h \subset V$ and $E^h \subset E$, approximating the variational spaces V and E , respectively, given by

$$\begin{aligned} E^h &= \{r^h \in C([0, \pi]) ; v_{|[a_i, a_{i+1}]}^h \in P_1([a_i, a_{i+1}]) \quad i = 0, \dots, M-1, \\ &\quad r^h(0) = r^h(\pi) = 0\}, \end{aligned} \quad (29)$$

$$\begin{aligned} V^h &= \{v^h \in C^1([0, \pi]) ; v_{|[a_i, a_{i+1}]}^h \in P_3([a_i, a_{i+1}]) \quad i = 0, \dots, M-1, \\ &\quad v^h(0) = v_x^h(0) = v^h(\pi) = v_x^h(\pi) = 0\}, \end{aligned} \quad (30)$$

where $P_s([a_i, a_{i+1}])$, for $s = 1, 3$, represents the space of polynomials of degree less or equal to s in the subinterval $[a_i, a_{i+1}]$; i.e. the finite element space E^h is composed of continuous and piecewise affine functions, and the finite element space V^h is composed of continuous differentiable and piecewise cubic functions. Here, $h > 0$ denotes the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by u_0^h , v_0^h , ϕ_0^h , e_0^h and θ_0^h , are given by

$$u_0^h = \mathcal{P}^{2h}u_0, \quad v_0^h = \mathcal{P}^{2h}v_0, \quad \phi_0^h = \mathcal{P}^{1h}\phi_0, \quad e_0^h = \mathcal{P}^{1h}e_0, \quad \theta_0^h = \mathcal{P}^{1h}\theta_0, \quad (31)$$

where \mathcal{P}^{1h} is the $L^2(0, \pi)$ -projection operator over E^h and \mathcal{P}^{2h} is the $L^2(0, \pi)$ -projection operator over V^h .

Secondly, we consider a partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \dots < t_N = T$. In this case, we use a uniform partition of the time interval $[0, T]$ with step size $k = T/N$ and nodes $t_n = nk$ for $n = 0, 1, \dots, N$. For a continuous function $z(t)$, we use the notation $z_n = z(t_n)$ and, for the sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/k$ its corresponding divided differences.

Therefore, using the backward Euler scheme, the fully discrete approximations are considered as follows.

Problem VP^{hk}. Find the discrete velocity field $v^{hk} = \{v_n^{hk}\}_{n=0}^N \subset V^h$, the discrete porosity speed field $e^{hk} = \{e_n^{hk}\}_{n=0}^N \subset V^h$ and the discrete temperature field $\theta^{hk} = \{\theta_n^{hk}\}_{n=0}^N \subset V^h$ such that $v_0^{hk} = v_0^h$, $e_0^{hk} = e_0^h$, $\theta_0^{hk} = \theta_0^h$ and, for $n = 1, \dots, N$,

$$\begin{aligned} \rho(\delta v_n^{hk}, w^h) + a((u_n^{hk})_x, w_x^h) + c((u_n^{hk})_{xx}, w_{xx}^h) &= b((\phi_n^{hk})_x, w^h) \\ &\quad - \beta^*((\theta_n^{hk})_x, w^h) - d((\phi_n^{hk})_x, w_{xx}^h), \quad \forall w^h \in V^h, \end{aligned} \quad (32)$$

$$\begin{aligned} J(\delta e_n^{hk}, r^h) + \beta((\phi_n^{hk})_x, r_x^h) + \xi(\phi_n^{hk}, r^h) + \tau(e_n^{hk}, r^h) &= -d((u_n^{hk})_{xx}, r_x^h) \\ &\quad - b((u_n^{hk})_x, r^h) + m(\theta_n^{hk}, r^h) \quad \forall r^h \in E^h, \end{aligned} \quad (33)$$

$$c^*(\delta \theta_n^{hk}, l^h) + k^*((\theta_n^{hk})_x, (l^h)_x) = -\beta^*((v_n^{hk})_x, l^h) - m(e_n^{hk}, l^h) \quad \forall l^h \in E^h, \quad (34)$$

where the discrete displacement field u_n^{hk} and the discrete porosity field w_n^{hk} are now recovered from the relations

$$u_n^{hk} = k \sum_{j=1}^n v_j^{hk} + u_0^h, \quad \phi_n^{hk} = k \sum_{j=1}^n e_j^{hk} + \phi_0^h. \quad (35)$$

The existence and uniqueness of discrete solutions to Problem VP^{hk} is obtained in a straightforward way using the classical Lax-Milgram lemma, for the problem written in the product space $V^h \times E^h \times E^h$, and the positive definitiveness of the resulting matrix under the required conditions (8).

We prove now the following stability result.

Theorem 9 Under the assumptions of Theorem 7, it follows that the sequences $\{u^{hk}, v^{hk}, \phi^{hk}, e^{hk}, \theta^{hk}\}$ generated by Problem VP^{hk} satisfy the stability estimate, for all $n = 1, \dots, N$,

$$\|v_n^{hk}\|^2 + \|(u_n^{hk})_x\|^2 + \|(u_n^{hk})_{xx}\|^2 + \|e_n^{hk}\|^2 + \|(\phi_n^{hk})_x\|^2 + \|\phi_n^{hk}\|^2 + \|\theta_n^{hk}\|^2 \leq C,$$

where C is a positive constant assumed to be independent of the discretization parameters h and k .

PROOF. For the sake of clarity in the writing, we remove the superscripts

h and k in all the variables.

Taking $l^h = \theta_n$ as a test function in discrete variational equation (34) we find that

$$c^*(\delta\theta_n, \theta_n) + k^*((\theta_n)_x, (\theta_n)_x) = -\beta^*((v_n)_x, \theta_n) - m(e_n, \theta_n).$$

Taking into account that

$$(\delta\theta_n, \theta_n) \geq \frac{1}{2k} [\|\theta_n\|^2 - \|\theta_{n-1}\|^2],$$

we have

$$\frac{c^*}{2k} [\|\theta_n\|^2 - \|\theta_{n-1}\|^2] \leq -\beta^*((v_n)_x, \theta_n) - m(e_n, \theta_n). \quad (36)$$

Now, taking $w^h = v_n$ as a test function in discrete variational equation (32) we get

$$\begin{aligned} \rho(\delta v_n, v_n) + a((u_n)_x, (v_n)_x) + c((u_n)_{xx}, (v_n)_{xx}) &= b((\phi_n)_x, v_n) - \beta^*((\theta_n)_x, v_n) \\ &\quad - d((\phi_n)_x, (v_n)_{xx}). \end{aligned}$$

Thus, keeping in mind that

$$\begin{aligned} (\delta v_n, v_n) &\geq \frac{1}{2k} [\|v_n\|^2 - \|v_{n-1}\|^2], \\ ((u_n)_x, (v_n)_x) &= \frac{1}{2k} [\|(u_n)_x\|^2 - \|(u_{n-1})_x\|^2 + \|(u_n - u_{n-1})_x\|^2], \\ ((u_n)_{xx}, (v_n)_{xx}) &= \frac{1}{2k} [\|(u_n)_{xx}\|^2 - \|(u_{n-1})_{xx}\|^2 + \|(u_n - u_{n-1})_{xx}\|^2], \end{aligned}$$

we obtain

$$\begin{aligned} \frac{\rho}{2k} [\|v_n\|^2 - \|v_{n-1}\|^2] + \frac{a}{2k} [\|(u_n)_x\|^2 - \|(u_{n-1})_x\|^2 + \|(u_n - u_{n-1})_x\|^2] \\ + \frac{c}{2k} [\|(u_n)_{xx}\|^2 - \|(u_{n-1})_{xx}\|^2 + \|(u_n - u_{n-1})_{xx}\|^2] \\ \leq b((\phi_n)_x, v_n) - \beta^*((\theta_n)_x, v_n) - d((\phi_n)_x, (v_n)_{xx}). \end{aligned} \quad (37)$$

Finally, taking $r^h = e_n$ as a test function in discrete variational equation (33) it follows that

$$\begin{aligned} J(\delta e_n, e_n) + \beta((\phi_n)_x, (e_n)_x) + \xi(\phi_n, e_n) + \tau(e_n, e_n) &= -d((u_n)_{xx}, (e_n)_x) \\ &\quad - b((u_n)_x, e_n) + m(\theta_n, e_n). \end{aligned}$$

Therefore, taking into account that

$$\begin{aligned}(\delta e_n, e_n) &\geq \frac{1}{2k} [\|e_n\|^2 - \|e_{n-1}\|^2], \\((\phi_n)_x, (e_n)_x) &= \frac{1}{2k} [\|(\phi_n)_x\|^2 - \|(\phi_{n-1})_x\|^2 + \|(\phi_n - \phi_{n-1})_x\|^2], \\(\phi_n, e_n) &= \frac{1}{2k} [\|\phi_n\|^2 - \|\phi_{n-1}\|^2 + \|\phi_n - \phi_{n-1}\|^2],\end{aligned}$$

we find that

$$\begin{aligned}\frac{J}{2k} [\|e_n\|^2 - \|e_{n-1}\|^2] &+ \frac{\beta}{2k} [\|(\phi_n)_x\|^2 - \|(\phi_{n-1})_x\|^2 + \|(\phi_n - \phi_{n-1})_x\|^2] \\&+ \frac{\xi}{2k} [\|\phi_n\|^2 - \|\phi_{n-1}\|^2 + \|\phi_n - \phi_{n-1}\|^2] \\&\leq -d((u_n)_{xx}, (e_n)_x) - b((u_n)_x, e_n) + m(\theta_n, e_n).\end{aligned}\tag{38}$$

Now, combining (36), (37) and (38), and keeping in mind that

$$-\beta^*((v_n)_x, \theta_n) = \beta^*(v_n, (\theta_n)_x),$$

we have

$$\begin{aligned}\frac{c^*}{2k} [\|\theta_n\|^2 - \|\theta_{n-1}\|^2] &+ \frac{a}{2k} [\|(u_n)_x\|^2 - \|(u_{n-1})_x\|^2 + \|(u_n - u_{n-1})_x\|^2] \\&+ \frac{\rho}{2k} [\|v_n\|^2 - \|v_{n-1}\|^2] + \frac{c}{2k} [\|(u_n)_{xx}\|^2 - \|(u_{n-1})_{xx}\|^2 + \|(u_n - u_{n-1})_{xx}\|^2] \\&+ \frac{J}{2k} [\|e_n\|^2 - \|e_{n-1}\|^2] + \frac{\beta}{2k} [\|(\phi_n)_x\|^2 - \|(\phi_{n-1})_x\|^2 + \|(\phi_n - \phi_{n-1})_x\|^2] \\&+ \frac{\xi}{2k} [\|\phi_n\|^2 - \|\phi_{n-1}\|^2 + \|\phi_n - \phi_{n-1}\|^2] \\&\leq b((\phi_n)_x, v_n) - d((\phi_n)_x, (v_n)_{xx}) - d((u_n)_{xx}, (e_n)_x) - b((u_n)_x, e_n).\end{aligned}$$

Keeping in mind that

$$b((\phi_n)_x, v_n) = -b((v_n)_x, \phi_n),$$

observing that

$$\begin{aligned}b((v_n)_x, \phi_n) + b((u_n)_x, \phi_n) &= \frac{b}{k} [((u_n)_x, \phi_n) - ((u_{n-1})_x, \phi_{n-1}) + ((u_n - u_{n-1})_x, \phi_n - \phi_{n-1})], \\d((\phi_n)_x, (v_n)_{xx}) + d((u_n)_{xx}, (e_n)_x) &= \frac{b}{k} [((\phi_n)_x, (u_n)_{xx}) - ((\phi_{n-1})_x, (u_{n-1})_{xx}) \\&\quad + ((\phi_n - \phi_{n-1})_x, (u_n - u_{n-1})_{xx})],\end{aligned}$$

and that, using the assumed conditions $a\xi > b^2$ and $c\beta > d^2$ (see (8)),

$$\begin{aligned} \frac{a}{2k} \|(u_n - u_{n-1})_x\|^2 + \frac{\xi}{2k} \|\phi_n - \phi_{n-1}\|^2 + \frac{b}{k} ((u_n - u_{n-1})_x, \phi_n - \phi_{n-1}) &\geq 0, \\ \frac{c}{2k} \|(u_n - u_{n-1})_{xx}\|^2 + \frac{\beta}{2k} \|(\phi_n - \phi_{n-1})_x\|^2 + \frac{d}{k} ((\phi_n - \phi_{n-1})_x, (u_n - u_{n-1})_{xx}) &\geq 0, \end{aligned}$$

from the previous estimates we find that

$$\begin{aligned} \frac{c^*}{2k} [\|\theta_n\|^2 - \|\theta_{n-1}\|^2] + \frac{a}{2k} [\|(u_n)_x\|^2 - \|(u_{n-1})_x\|^2] \\ + \frac{b}{k} [((u_n)_x, \phi_n) - ((u_{n-1})_x, \phi_{n-1})] + \frac{d}{k} [((\phi_n)_x, (u_n)_{xx}) - ((\phi_{n-1})_x, (u_{n-1})_{xx})] \\ + \frac{\rho}{2k} [\|v_n\|^2 - \|v_{n-1}\|^2] + \frac{c}{2k} [\|(u_n)_{xx}\|^2 - \|(u_{n-1})_{xx}\|^2] \\ + \frac{J}{2k} [\|e_n\|^2 - \|e_{n-1}\|^2] + \frac{\beta}{2k} [\|(\phi_n)_x\|^2 - \|(\phi_{n-1})_x\|^2] + \frac{\xi}{2k} [\|\phi_n\|^2 - \|\phi_{n-1}\|^2] \leq 0. \end{aligned}$$

Summing up to n the previous estimates and multiplying it by $2k$, it follows that

$$\begin{aligned} c^* \|\theta_n\|^2 + a \|(u_n)_x\|^2 + 2b((u_n)_x, \phi_n) + 2d((\phi_n)_x, (u_n)_{xx}) + \rho \|v_n\|^2 \\ + c \|(u_n)_{xx}\|^2 + J \|e_n\|^2 + \beta \|(\phi_n)_x\|^2 + \xi \|\phi_n\|^2 \\ \leq C (\|\theta_0\|^2 + \|(u_0)_x\|^2 + \|v_0\|^2 + \|(u_0)_{xx}\|^2 + \|e_0\|^2 + \|(\phi_0)_x\|^2 + \|\phi_0\|^2). \end{aligned}$$

Now, using again conditions (8), we can choose $\zeta_1, \zeta_2 > 0$ such that

$$b/\xi < \zeta_1 < a/b, \quad d/\beta < \zeta_2 < c/d,$$

leading to the estimates

$$\begin{aligned} a \|(u_n)_x\|^2 + 2b((u_n)_x, \phi_n) + \xi \|\phi_n\|^2 &\geq \left(a - \frac{b}{\zeta_1}\right) \|(u_n)_x\|^2 + (\xi - \zeta_1 b) \|\phi_n\|^2, \\ c \|(u_n)_{xx}\|^2 + 2d((\phi_n)_x, (u_n)_{xx}) + \beta \|(\phi_n)_x\|^2 &\geq \left(c - \frac{d}{\zeta_2}\right) \|(u_n)_{xx}\|^2 + (\beta - \zeta_2 d) \|(\phi_n)_x\|^2. \end{aligned}$$

Therefore, we obtain the desired stability property.

From Theorem 9, we obtain the following discrete version of the energy decay property.

Corollary 10 *If we define the discrete energy at time $t = t_n$, E_n^{hk} , as follows*

$$\begin{aligned} E_n^{hk} = \rho \|v_n^{hk}\|^2 + J \|e_n^{hk}\|^2 + c^* \|\theta_n^{hk}\|^2 + \alpha \|(u_n^{hk})_x\|^2 + c \|(u_n^{hk})_{xx}\|^2 + \xi \|\phi_n^{hk}\|^2 \\ + \beta \|(\phi_n^{hk})_x\|^2 + 2b((u_n^{hk})_x, \phi_n^{hk}) + 2d((\phi_n^{hk})_x, (u_n^{hk})_{xx}), \end{aligned} \quad (39)$$

then we have

$$\frac{E_n^{hk} - E_{n-1}^{hk}}{k} \leq 0.$$

Now, we will obtain some a priori error estimates for the numerical errors $u_n - u_n^{hk}$, $v_n - v_n^{hk}$, $\phi_n - \phi_n^{hk}$, $e_n - e_n^{hk}$ and $\theta_n - \theta_n^{hk}$.

We have the following theorem which gives some a priori error estimates.

Theorem 11 *Under the assumptions of Theorem 2, if we denote by (v, e, θ) the solution to problem VP and by $(v^{hk}, e^{hk}, \theta^{hk})$ the solution to problem VP^{hk}, then we have the following a priori error estimates, for all $z^h = \{w_j^h\}_{j=0}^N \subset V^h$, $r^h = \{r_j^h\}_{j=0}^N \subset E^h$ and $l^h = \{l_j^h\}_{j=0}^N \subset E^h$,*

$$\begin{aligned} & \max_{0 \leq n \leq N} \left\{ \|\theta_n - \theta_n^{hk}\|^2 + \|v_n - v_n^{hk}\|^2 + \|(u_n - u_n^{hk})_x\|^2 + \|(u_n - u_n^{hk})_{xx}\|^2 \right. \\ & \quad \left. + \|e_n - e_n^{hk}\|^2 + \|(\phi_n - \phi_n^{hk})_x\|^2 + \|\phi_n - \phi_n^{hk}\|^2 \right\} \\ & \leq Ck \sum_{j=1}^N \left(\|\dot{\theta}_j - \delta\theta_j\|^2 + \|\theta_j - l_j^h\|^2 + \|(\theta_j - l_j^h)_x\|^2 + \|\dot{v}_j - \delta v_j\|^2 \right. \\ & \quad + \|v_j - w_j^h\|^2 + \|(v_j - w_j^h)_x\|^2 + \|(\dot{u}_j - \delta u_j)_x\|^2 + \|e_j - r_j^h\|^2 \\ & \quad + \|(\dot{u}_j - \delta u_j)_{xx}\|^2 + \|\dot{e}_j - \delta e_j\|^2 + \|(\dot{\phi}_j - \delta\phi_j)_x\|^2 + \|(e_j - r_j^h)_x\|^2 \\ & \quad \left. + \|\dot{\phi}_j - \delta\phi_j\|^2 \right) + C \max_{0 \leq n \leq N} \left\{ \|v_n - z_n^h\|^2 + \|e_n - r_n^h\|^2 + \|\theta_n - l_n^h\|^2 \right\} \\ & \quad + \frac{C}{k} \sum_{j=1}^{N-1} \|v_j - z_j^h - (v_{j+1} - z_{j+1}^h)\|^2 + \frac{C}{k} \sum_{j=1}^{N-1} \|e_j - r_j^h - (e_{j+1} - r_{j+1}^h)\|^2 \\ & \quad + \frac{C}{k} \sum_{j=1}^{N-1} \|\theta_j - l_j^h - (\theta_{j+1} - l_{j+1}^h)\|^2 \\ & \quad + C \left(\|\theta_0 - \theta_0^h\|^2 + \|v_0 - v_0^h\|^2 + \|(u_0 - u_0^h)_x\|^2 + \|(u_0 - u_0^h)_{xx}\|^2 \right. \\ & \quad \left. + \|e_0 - e_0^h\|^2 + \|(\phi_0 - \phi_0^h)_x\|^2 + \|\phi_0 - \phi_0^h\|^2 \right), \end{aligned} \tag{40}$$

where $C > 0$ is a positive constant assumed to be independent of the discretization parameters h and k , but depending on the continuous solution, and $\delta\theta_j = (\theta_j - \theta_{j-1})/k$, $\delta v_j = (v_j - v_{j-1})/k$, $\delta u_j = (u_j - u_{j-1})/k$, $\delta\phi_j = (\phi_j - \phi_{j-1})/k$ and $\delta e_j = (e_j - e_{j-1})/k$.

PROOF. First, we obtain some estimates for the temperature field. We subtract variational equation (27) at time $t = t_n$ for a test function $l = l^h \in E^h \subset E$ and discrete variational equation (34) to obtain, for all $l^h \in E^h$,

$$c^*(\dot{\theta}_n - \delta\theta_n^{hk}, l^h) + k^*((\theta_n - \theta_n^{hk})_x, l_x^h) = -\beta^*((v_n - v_n^{hk})_x, l^h) - m(e_n - e_n^{hk}, l^h),$$

and therefore we have, for all $l^h \in E^h$,

$$\begin{aligned}
& c^*(\dot{\theta}_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk}) + k^*((\theta_n - \theta_n^{hk})_x, (\theta_n - \theta_n^{hk})_x) \\
& \quad + \beta^*((v_n - v_n^{hk})_x, \theta_n - \theta_n^{hk}) + m(e_n - e_n^{hk}, \theta_n - \theta_n^{hk}) \\
& = c^*(\dot{\theta}_n - \delta\theta_n^{hk}, \theta_n - l^h) + k^*((\theta_n - \theta_n^{hk})_x, (\theta_n - l^h)_x) \\
& \quad + \beta^*((v_n - v_n^{hk})_x, \theta_n - l^h) + m(e_n - e_n^{hk}, \theta_n - l^h).
\end{aligned}$$

Taking into account that

$$\begin{aligned}
& (\dot{\theta}_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk}) \geq (\dot{\theta}_n - \delta\theta_n, \theta_n - \theta_n^{hk}) \\
& \quad + \frac{1}{2k} \left\{ \|\theta_n - \theta_n^{hk}\|^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|^2 \right\}, \\
& \beta^*((v_n - v_n^{hk})_x, \theta_n - l^h) = -\beta^*(v_n - v_n^{hk}, (\theta_n - l^h)_x), \\
& \beta^*((v_n - v_n^{hk})_x, \theta_n - \theta_n^{hk}) = -\beta^*(v_n - v_n^{hk}, (\theta_n - \theta_n^{hk})_x),
\end{aligned}$$

where $\delta\theta_n = (\theta_n - \theta_{n-1})/k$, using Cauchy-Schwarz inequality and the inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2, \quad a, b, \epsilon \in \mathbb{R}, \quad \epsilon > 0, \quad (41)$$

it follows that

$$\begin{aligned}
\frac{c^*}{2k} \left\{ \|\theta_n - \theta_n^{hk}\|^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|^2 \right\} & \leq C \left(\|v_n - v_n^{hk}\|^2 + \|e_n - e_n^{hk}\|^2 \right. \\
& \quad + \|\theta_n - l^h\|^2 + \|(\theta_n - l^h)_x\|^2 + \|\dot{\theta}_n - \delta\theta_n\|^2 + \|\theta_n - \theta_n^{hk}\|^2 \\
& \quad \left. + (\delta\theta_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk}) \right) \quad \forall l^h \in E^h. \quad (42)
\end{aligned}$$

Now, we obtain some estimates for the velocity field. Then, we subtract variational equation (25) at time $t = t_n$ for a test function $w = w^h \in V^h \subset V$ and discrete variational equation (32) to obtain

$$\begin{aligned}
\rho(\dot{v}_n - \delta v_n^{hk}, w^h) + a((u_n - u_n^{hk})_x, w_x^h) + c((u_n - u_n^{hk})_{xx}, w_{xx}^h) - b((\phi_n - \phi_n^{hk})_x, w^h) \\
+ \beta^*((\theta_n - \theta_n^{hk})_x, w^h) + d((\phi_n - \phi_n^{hk})_x, w_{xx}^h) = 0 \quad \forall w^h \in V^h,
\end{aligned}$$

and so, we find that, for all $w^h \in V^h$,

$$\begin{aligned}
\rho(\dot{v}_n - \delta v_n^{hk}, v_n - v_n^{hk}) + a((u_n - u_n^{hk})_x, (v_n - v_n^{hk})_x) \\
+ c((u_n - u_n^{hk})_{xx}, (v_n - v_n^{hk})_{xx}) - b((\phi_n - \phi_n^{hk})_x, v_n - v_n^{hk}) \\
+ \beta^*((\theta_n - \theta_n^{hk})_x, v_n - v_n^{hk}) + d((\phi_n - \phi_n^{hk})_x, (v_n - v_n^{hk})_{xx})
\end{aligned}$$

$$\begin{aligned}
&= \rho(\dot{v}_n - \delta v_n^{hk}, v_n - w^h) + a((u_n - u_n^{hk})_x, (v_n - w^h)_x) \\
&\quad + c((u_n - u_n^{hk})_{xx}, (v_n - w^h)_{xx}) - b((\phi_n - \phi_n^{hk})_x, v_n - w^h) \\
&\quad + \beta^*((\theta_n - \theta_n^{hk})_x, v_n - w^h) + d((\phi_n - \phi_n^{hk})_x, (v_n - w^h)_{xx}).
\end{aligned}$$

Keeping in mind that

$$\begin{aligned}
(\dot{v}_n - \delta v_n^{hk}, v_n - v_n^{hk}) &\geq (\dot{v}_n - \delta v_n, v_n - v_n^{hk}) + \frac{1}{2k} [\|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2], \\
((u_n - u_n^{hk})_x, (v_n - v_n^{hk})_x) &= ((u_n - u_n^{hk})_x, (\dot{u}_n - \delta u_n)_x) \\
&\quad + \frac{1}{2k} [\|(u_n - u_n^{hk})_x\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_x\|^2 + \|(u_n - u_n^{hk})_x - (u_{n-1} - u_{n-1}^{hk})_x\|^2], \\
((u_n - u_n^{hk})_{xx}, (v_n - v_n^{hk})_{xx}) &\geq ((u_n - u_n^{hk})_{xx}, (\dot{u}_n - \delta u_n)_{xx}) \\
&\quad + \frac{1}{2k} [\|(u_n - u_n^{hk})_{xx}\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_{xx}\|^2 + \|(u_n - u_n^{hk})_{xx} - (u_{n-1} - u_{n-1}^{hk})_{xx}\|^2], \\
((\theta_n - \theta_n^{hk})_x, v_n - z^h) &= -(\theta_n - \theta_n^{hk}, (v_n - z^h)_x),
\end{aligned}$$

where $\delta u_n = (u_n - u_{n-1})/k$, $\delta v_n = (v_n - v_{n-1})/k$ and we recall that $v_n^{hk} = \delta u_n^{hk} = (u_n^{hk} - u_{n-1}^{hk})/k$, using Cauchy-Schwarz inequality and inequality (41) we have, for all $w^h \in V^h$,

$$\begin{aligned}
&\frac{\rho}{2k} [\|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2] + \frac{a}{2k} [\|(u_n - u_n^{hk})_x\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_x\|^2 \\
&\quad + \|(u_n - u_n^{hk})_x - (u_{n-1} - u_{n-1}^{hk})_x\|^2] + b(\phi_n - \phi_n^{hk}, (v_n - v_n^{hk})_x) \\
&\quad + \frac{c}{2k} [\|(u_n - u_n^{hk})_{xx}\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_{xx}\|^2 \\
&\quad + \|(u_n - u_n^{hk})_{xx} - (u_{n-1} - u_{n-1}^{hk})_{xx}\|^2] + d((\phi_n - \phi_n^{hk})_x, (v_n - v_n^{hk})_{xx}) \\
&\leq C \left(\|v_n - v_n^{hk}\|^2 + \|\dot{v}_n - \delta v_n\|^2 + \|(u_n - u_n^{hk})_x\|^2 + \|(u_n - u_n^{hk})_{xx}\|^2 \right. \\
&\quad + \|(\phi_n - \phi_n^{hk})_x\|^2 + \|v_n - w^h\|^2 + \|(v_n - w^h)_x\|^2 + \|(\dot{u}_n - \delta u_n)_{xx}\|^2 \\
&\quad + \|(\dot{u}_n - \delta u_n)_x\|^2 + (\delta v_n - \delta v_n^{hk}, v_n - z^h) + \|\theta_n - \theta_n^{hk}\|^2 \\
&\quad \left. + \|(v_n - w^h)_{xx}\|^2 \right). \tag{43}
\end{aligned}$$

Finally, we obtain some estimates for the porosity speed field. Thus, subtracting variational equation (26) at time $t = t_n$ for a test function $r = r^h \in E^h \subset E$ and discrete variational equation (33) we have, for all $r^h \in E^h$,

$$\begin{aligned}
&J(\dot{e}_n - \delta e_n^{hk}, r^h) + \beta((\phi_n - \phi_n^{hk})_x, r_x^h) + \xi(\phi_n - \phi_n^{hk}, r^h) + \tau(e_n - e_n^{hk}, e_n - e_n^{hk}) \\
&\quad + d((u_n - u_n^{hk})_{xx}, r_x^h) + b((u_n - u_n^{hk})_x, r^h) + m(\theta_n - \theta_n^{hk}, r^h) = 0,
\end{aligned}$$

and therefore, we find that, for all $r^h \in E^h$,

$$\begin{aligned}
& J(\dot{e}_n - \delta e_n^{hk}, e_n - e_n^{hk}) + \beta((\phi_n - \phi_n^{hk})_x, (e_n - e_n^{hk})_x) + \xi(\phi_n - \phi_n^{hk}, e_n - e_n^{hk}) \\
& \quad + d((u_n - u_n^{hk})_{xx}, (e_n - e_n^{hk})_x) + b((u_n - u_n^{hk})_x, e_n - e_n^{hk}) \\
& \quad + \tau(e_n - e_n^{hk}, e_n - e_n^{hk}) + m(\theta_n - \theta_n^{hk}, e_n - e_n^{hk}) \\
& = J(\dot{e}_n - \delta e_n^{hk}, e_n - r^h) + \beta((\phi_n - \phi_n^{hk})_x, (e_n - r^h)_x) + \xi(\phi_n - \phi_n^{hk}, e_n - r^h) \\
& \quad + d((u_n - u_n^{hk})_{xx}, (e_n - r^h)_x) + b((u_n - u_n^{hk})_x, e_n - r^h) \\
& \quad + \tau(e_n - e_n^{hk}, e_n - r^h) + m(\theta_n - \theta_n^{hk}, e_n - r^h).
\end{aligned}$$

Keeping in mind that

$$\begin{aligned}
(\dot{e}_n - \delta e_n^{hk}, e_n - e_n^{hk}) & \geq (\dot{e}_n - \delta e_n, e_n - e_n^{hk}) + \frac{1}{2k} \left[\|e_n - e_n^{hk}\|^2 - \|e_{n-1} - e_{n-1}^{hk}\|^2 \right], \\
((\phi_n - \phi_n^{hk})_x, (e_n - e_n^{hk})_x) & = ((\phi_n - \phi_n^{hk})_x, (\dot{\phi}_n - \delta \phi_n)_x) \\
& \quad + \frac{1}{2k} \left[\|(\phi_n - \phi_n^{hk})_x\|^2 - \|(\phi_{n-1} - \phi_{n-1}^{hk})_x\|^2 \right. \\
& \quad \left. + \|(\phi_n - \phi_n^{hk})_x - (\phi_{n-1} - \phi_{n-1}^{hk})_x\|^2 \right], \\
(\phi_n - \phi_n^{hk}, e_n - e_n^{hk}) & = (\phi_n - \phi_n^{hk}, \dot{\phi}_n - \delta \phi_n)_x + \frac{1}{2k} \left[\|\phi_n - \phi_n^{hk}\|^2 - \|\phi_{n-1} - \phi_{n-1}^{hk}\|^2 \right. \\
& \quad \left. + \|\phi_n - \phi_n^{hk} - \phi_{n-1} - \phi_{n-1}^{hk}\|^2 \right],
\end{aligned}$$

where $\delta \phi_n = (\phi_n - \phi_{n-1})/k$, $\delta e_n = (e_n - e_{n-1})/k$ and we recall that $e_n^{hk} = \delta \phi_n^{hk} = (\phi_n^{hk} - \phi_{n-1}^{hk})/k$, using Cauchy-Schwarz inequality and inequality (41) we obtain, for all $r^h \in E^h$,

$$\begin{aligned}
& \frac{J}{2k} \left[\|e_n - e_n^{hk}\|^2 - \|e_{n-1} - e_{n-1}^{hk}\|^2 \right] + \frac{\beta}{2k} \left[\|(\phi_n - \phi_n^{hk})_x\|^2 - \|(\phi_{n-1} - \phi_{n-1}^{hk})_x\|^2 \right. \\
& \quad \left. + \|(\phi_n - \phi_n^{hk})_x - (\phi_{n-1} - \phi_{n-1}^{hk})_x\|^2 \right] + m(\theta_n - \theta_n^{hk}, e_n - e_n^{hk}) \\
& \quad + \frac{\xi}{2k} \left[\|\phi_n - \phi_n^{hk}\|^2 - \|\phi_{n-1} - \phi_{n-1}^{hk}\|^2 + \|\phi_n - \phi_n^{hk} - \phi_{n-1} - \phi_{n-1}^{hk}\|^2 \right] \\
& \quad + d((u_n - u_n^{hk})_{xx}, (\phi_n - \phi_n^{hk})_x) + b((u_n - u_n^{hk})_x, e_n - e_n^{hk}) \\
& \leq C \left(\|\dot{e}_n - \delta e_n\|^2 + \|\dot{\phi}_n - \delta \phi_n\|^2 + \|(\dot{\phi}_n - \delta \phi_n)_x\|^2 + \|e_n - r^h\|^2 \right. \\
& \quad + \|(e_n - r^h)_x\|^2 + (\delta e_n - \delta e_n^{hk}, e_n - r^h) + \|e_n - e_n^{hk}\|^2 + \|\phi_n - \phi_n^{hk}\|^2 \\
& \quad \left. + \|\theta_n - \theta_n^{hk}\|^2 + \|(u_n - u_n^{hk})_{xx}\|^2 + \|(u_n - u_n^{hk})_x\|^2 \right). \tag{44}
\end{aligned}$$

Combining estimates (42), (43) and (44) we find that, for all $w^h \in V^h$, $r^h \in E^h$

and $l^h \in E^h$,

$$\begin{aligned}
& \frac{c^*}{2k} \left\{ \|\theta_n - \theta_n^{hk}\|^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|^2 \right\} + \frac{\rho}{2k} \left[\|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2 \right] \\
& + \frac{a}{2k} \left[\|(u_n - u_n^{hk})_x\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_x\|^2 \right. \\
& + \|(u_n - u_n^{hk})_x - (u_{n-1} - u_{n-1}^{hk})_x\|^2 \left. \right] + b(\phi_n - \phi_n^{hk}, (v_n - v_n^{hk})_x) \\
& + \frac{c}{2k} \left[\|(u_n - u_n^{hk})_{xx}\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_{xx}\|^2 \right. \\
& + \|(u_n - u_n^{hk})_{xx} - (u_{n-1} - u_{n-1}^{hk})_{xx}\|^2 \left. \right] + d((\phi_n - \phi_n^{hk})_x, (v_n - v_n^{hk})_{xx}) \\
& + \frac{J}{2k} \left[\|e_n - e_n^{hk}\|^2 - \|e_{n-1} - e_{n-1}^{hk}\|^2 \right] + \frac{\beta}{2k} \left[\|(\phi_n - \phi_n^{hk})_x\|^2 - \|(\phi_{n-1} - \phi_{n-1}^{hk})_x\|^2 \right. \\
& + \|(\phi_n - \phi_n^{hk})_x - (\phi_{n-1} - \phi_{n-1}^{hk})_x\|^2 \left. \right] \\
& + \frac{\xi}{2k} \left[\|\phi_n - \phi_n^{hk}\|^2 - \|\phi_{n-1} - \phi_{n-1}^{hk}\|^2 + \|\phi_n - \phi_n^{hk} - \phi_{n-1} - \phi_{n-1}^{hk}\|^2 \right] \\
& + d((u_n - u_n^{hk})_{xx}, (\phi_n - \phi_n^{hk})_x) + b((u_n - u_n^{hk})_x, e_n - e_n^{hk}) \\
\leq & C \left(\|v_n - v_n^{hk}\|^2 + \|e_n - e_n^{hk}\|^2 + \|\theta_n - l^h\|^2 + \|(\theta_n - l^h)_x\|^2 + \|\dot{\theta}_n - \delta\theta_n\|^2 \right. \\
& + (\delta\theta_n - \delta\theta_n^{hk}, \theta_n - l^h) + \|\dot{v}_n - \delta v_n\|^2 + \|(u_n - u_n^{hk})_x\|^2 + \|(u_n - u_n^{hk})_{xx}\|^2 \\
& + \|(\phi_n - \phi_n^{hk})_x\|^2 + \|v_n - w^h\|^2 + \|(v_n - w^h)_x\|^2 + \|(\dot{u}_n - \delta u_n)_{xx}\|^2 \\
& + \|(\dot{u}_n - \delta u_n)_x\|^2 + (\delta v_n - \delta v_n^{hk}, v_n - w^h) + \|\theta_n - \theta_n^{hk}\|^2 \\
& + \|(v_n - w^h)_{xx}\|^2 + \|\dot{e}_n - \delta e_n\|^2 + \|\dot{\phi}_n - \delta\phi_n\|^2 + \|(\dot{\phi}_n - \delta\phi_n)_x\|^2 + \|e_n - r^h\|^2 \\
& \left. + \|(e_n - r^h)_x\|^2 + (\delta e_n - \delta e_n^{hk}, e_n - r^h) + \|\phi_n - \phi_n^{hk}\|^2 \right).
\end{aligned}$$

Observing that

$$\begin{aligned}
& b((v_n - v_n^{hk})_x, \phi_n - \phi_n^{hk}) + b((u_n - u_n^{hk})_x, \phi_n - \phi_n^{hk}) \\
& = b((\dot{u}_n - \delta u_n)_x, \phi_n - \phi_n^{hk}) + b((v_n - v_n^{hk})_x, \dot{\phi}_n - \delta\phi_n) \\
& + \frac{b}{k} \left[((u_n - u_n^{hk})_x, \phi_n - \phi_n^{hk}) - ((u_{n-1} - u_{n-1}^{hk})_x, \phi_{n-1} - \phi_{n-1}^{hk}) \right. \\
& \left. + ((u_n - u_n^{hk})_x - (u_{n-1} - u_{n-1}^{hk})_x, \phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk})) \right], \\
& d((\phi_n - \phi_n^{hk})_x, (v_n - v_n^{hk})_{xx}) + d((u_n - u_n^{hk})_{xx}, (e_n - e_n^{hk})_x) \\
& = d((\phi_n - \phi_n^{hk})_x, (\dot{u}_n - \delta u_n)_{xx}) + d((u_n - u_n^{hk})_{xx}, (\dot{\phi}_n - \delta\phi_n)_x) \\
& + \frac{d}{k} \left[((\phi_n - \phi_n^{hk})_x, (u_n - u_n^{hk})_{xx}) - ((\phi_{n-1} - \phi_{n-1}^{hk})_x, (u_{n-1} - u_{n-1}^{hk})_{xx}) \right. \\
& \left. + ((\phi_n - \phi_n^{hk})_x - (\phi_{n-1} - \phi_{n-1}^{hk})_x, (u_n - u_n^{hk})_{xx} - (u_{n-1} - u_{n-1}^{hk})_{xx}) \right],
\end{aligned}$$

and that, thanks to conditions (8),

$$\begin{aligned}
& \frac{a}{2k} \|(u_n - u_n^{hk})_x - (u_{n-1} - u_{n-1}^{hk})_x\|^2 + \frac{\xi}{2k} \|\phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk})\|^2 \\
& \quad + \frac{b}{k} ((u_n - u_n^{hk})_x - (u_{n-1} - u_{n-1}^{hk})_x, \phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk})) \geq 0, \\
& \frac{c}{2k} \|(u_n - u_n^{hk})_{xx} - (u_{n-1} - u_{n-1}^{hk})_{xx}\|^2 + \frac{\beta}{2k} \|(\phi_n - \phi_n^{hk})_x - (\phi_{n-1} - \phi_{n-1}^{hk})_x\|^2 \\
& \quad + \frac{d}{k} ((u_n - u_n^{hk})_{xx} - (u_{n-1} - u_{n-1}^{hk})_{xx}, (\phi_n - \phi_n^{hk})_x - (\phi_{n-1} - \phi_{n-1}^{hk})_x) \geq 0,
\end{aligned}$$

and the previous estimates imply that, for all $v^h \in V^h$ and $r^h, l^h \in E^h$,

$$\begin{aligned}
& \frac{c^*}{2k} \left\{ \|\theta_n - \theta_n^{hk}\|^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|^2 \right\} + \frac{\rho}{2k} \left[\|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2 \right] \\
& \quad + \frac{a}{2k} \left[\|(u_n - u_n^{hk})_x\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_x\|^2 \right] \\
& \quad + \frac{b}{k} \left[(\phi_n - \phi_n^{hk}, (u_n - u_n^{hk})_x) - (\phi_{n-1} - \phi_{n-1}^{hk}, (u_{n-1} - u_{n-1}^{hk})_x) \right] \\
& \quad + \frac{c}{2k} \left[\|(u_n - u_n^{hk})_{xx}\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_{xx}\|^2 \right] \\
& \quad + \frac{d}{k} \left[((\phi_n - \phi_n^{hk})_x, (u_n - u_n^{hk})_{xx}) - ((\phi_{n-1} - \phi_{n-1}^{hk})_x, (u_{n-1} - u_{n-1}^{hk})_{xx}) \right] \\
& \quad + \frac{J}{2k} \left[\|e_n - e_n^{hk}\|^2 - \|e_{n-1} - e_{n-1}^{hk}\|^2 \right] + \frac{\xi}{2k} \left[\|\phi_n - \phi_n^{hk}\|^2 - \|\phi_{n-1} - \phi_{n-1}^{hk}\|^2 \right] \\
& \quad + \frac{\beta}{2k} \left[\|(\phi_n - \phi_n^{hk})_x\|^2 - \|(\phi_{n-1} - \phi_{n-1}^{hk})_x\|^2 \right] \\
& \leq C \left(\|v_n - v_n^{hk}\|^2 + \|e_n - e_n^{hk}\|^2 + \|\theta_n - l^h\|^2 + \|(\theta_n - l^h)_x\|^2 + \|\dot{\theta}_n - \delta\theta_n\|^2 \right. \\
& \quad + (\delta\theta_n - \delta\theta_n^{hk}, \theta_n - l^h) + \|\dot{v}_n - \delta v_n\|^2 + \|(u_n - u_n^{hk})_x\|^2 + \|(u_n - u_n^{hk})_{xx}\|^2 \\
& \quad + \|(\phi_n - \phi_n^{hk})_x\|^2 + \|v_n - w^h\|^2 + \|(v_n - w^h)_x\|^2 + \|(\dot{u}_n - \delta u_n)_{xx}\|^2 \\
& \quad + \|(\dot{u}_n - \delta u_n)_x\|^2 + (\delta v_n - \delta v_n^{hk}, v_n - w^h) + \|\theta_n - \theta_n^{hk}\|^2 \\
& \quad + \|(v_n - w^h)_{xx}\|^2 + \|\dot{e}_n - \delta e_n\|^2 + \|\dot{\phi}_n - \delta\phi_n\|^2 + \|(\dot{\phi}_n - \delta\phi_n)_x\|^2 \\
& \quad \left. + \|e_n - r^h\|^2 + \|(e_n - r^h)_x\|^2 + (\delta e_n - \delta e_n^{hk}, e_n - r^h) + \|\phi_n - \phi_n^{hk}\|^2 \right).
\end{aligned}$$

Choosing $\zeta_1, \zeta_2 > 0$ such that

$$b/\xi < \zeta_1 < a/b, \quad d/\beta < \zeta_2 < c/d,$$

as we did in the proof of Theorem 9, we find that

$$\begin{aligned}
& a\|(u_n - u_n^{hk})_x\|^2 + \xi\|\phi_n - \phi_n^{hk}\|^2 + 2b(\phi_n - \phi_n^{hk}, (u_n - u_n^{hk})_x) \\
& \geq \left(a - \frac{\zeta_1}{b}\right)\|(u_n - u_n^{hk})_x\|^2 + (\xi - \zeta_1 b)\|\phi_n - \phi_n^{hk}\|^2, \\
& c\|(u_n - u_n^{hk})_{xx}\|^2 + \beta\|(\phi_n - \phi_n^{hk})_x\|^2 + 2d((\phi_n - \phi_n^{hk})_x, (u_n - u_n^{hk})_{xx}) \\
& \geq \left(c - \frac{\zeta_2}{d}\right)\|(u_n - u_n^{hk})_{xx}\|^2 + (\beta - \zeta_2 d)\|(\phi_n - \phi_n^{hk})_x\|^2.
\end{aligned}$$

Therefore, summing up to n and multiplying it by $2k$, it follows that

$$\begin{aligned}
& \|\theta_n - \theta_n^{hk}\|^2 + \|v_n - v_n^{hk}\|^2 + \|(u_n - u_n^{hk})_x\|^2 + \|(u_n - u_n^{hk})_{xx}\|^2 \\
& + \|e_n - e_n^{hk}\|^2 + \|(\phi_n - \phi_n^{hk})_x\|^2 + \|\phi_n - \phi_n^{hk}\|^2 \\
& \leq Ck \sum_{j=1}^n \left(\|v_j - v_j^{hk}\|^2 + \|e_j - e_j^{hk}\|^2 + \|\theta_j - l_j^h\|^2 + \|(\theta_n - l_j^h)_x\|^2 + \|\dot{\theta}_j - \delta\theta_j\|^2 \right. \\
& + (\delta\theta_j - \delta\theta_j^{hk}, \theta_j - l_j^h) + \|\dot{v}_j - \delta v_j\|^2 + \|(u_j - u_j^{hk})_x\|^2 + \|(u_j - u_j^{hk})_{xx}\|^2 \\
& + \|(\phi_j - \phi_j^{hk})_x\|^2 + \|v_j - w_j^h\|^2 + \|(v_j - w_j^h)_x\|^2 + \|(\dot{u}_j - \delta u_j)_{xx}\|^2 \\
& + \|(\dot{u}_j - \delta u_j)_x\|^2 + (\delta v_j - \delta v_j^{hk}, v_j - w_j^h) + \|\theta_j - \theta_j^{hk}\|^2 + \|e_j - r_j^h\|^2 \\
& + \|(v_j - w_j^h)_{xx}\|^2 + \|\dot{e}_j - \delta e_j\|^2 + \|\dot{\phi}_j - \delta\phi_j\|^2 + \|(\dot{\phi}_j - \delta\phi_j)_x\|^2 \\
& \left. + \|(e_j - r_j^h)_x\|^2 + (\delta e_j - \delta e_j^{hk}, e_j - r_j^h) + \|\phi_j - \phi_j^{hk}\|^2 \right) + C \left(\|\theta_0 - \theta_0^h\|^2 \right. \\
& + \|v_0 - v_0^h\|^2 + \|(u_0 - u_0^h)_x\|^2 + \|(u_0 - u_0^h)_{xx}\|^2 + \|e_0 - e_0^h\|^2 + \|(\phi_0 - \phi_0^h)_x\|^2 \\
& \left. + \|\phi_0 - \phi_0^h\|^2 \right) \quad \forall v^h = \{v_j^{hk}\}_{j=0}^n \subset V^h, \quad r^h = \{r_j^h\}_{j=0}^n, \quad l^h = \{l_j^h\}_{j=0}^n \subset E^h.
\end{aligned}$$

Keeping in mind that

$$\begin{aligned}
& k \sum_{j=1}^n (\delta v_j - \delta v_j^{hk}, v_j - w_j^h) = \sum_{j=1}^n (v_j - v_j^{hk} - (v_{j-1} - v_{j-1}^{hk}), v_j - w_j^h) \\
& = (v_n - v_n^{hk}, v_n - w_n^h) + (v_0^h - v_0, v_1 - w_1^h) + \sum_{j=1}^{n-1} (v_j - v_j^{hk}, v_j - w_j^h - (v_{j+1} - w_{j+1}^h)), \\
& k \sum_{j=1}^n (\delta\theta_j - \delta\theta_j^{hk}, \theta_j - l_j^h) = \sum_{j=1}^n (\theta_j - \theta_j^{hk} - (\theta_{j-1} - \theta_{j-1}^{hk}), \theta_j - l_j^h) \\
& = (\theta_n - \theta_n^{hk}, \theta_n - l_n^h) + (\theta_0^h - \theta_0, \theta_1 - l_1^h) + \sum_{j=1}^{n-1} (\theta_j - \theta_j^{hk}, \theta_j - l_j^h - (\theta_{j+1} - l_{j+1}^h)), \\
& k \sum_{j=1}^n (\delta e_j - \delta e_j^{hk}, e_j - r_j^h) = \sum_{j=1}^n (e_j - e_j^{hk} - (e_{j-1} - e_{j-1}^{hk}), e_j - r_j^h) \\
& = (e_n - e_n^{hk}, e_n - r_n^h) + (e_0^h - e_0, e_1 - r_1^h) + \sum_{j=1}^{n-1} (e_j - e_j^{hk}, e_j - r_j^h - (e_{j+1} - r_{j+1}^h)),
\end{aligned}$$

using a discrete version of Gronwall's inequality (see, for instance, [31]), we have the a priori error estimates (40).

We note that estimates (40) are the basis to derive the convergence order under suitable regularity conditions. Thus, as an example, we have the following result which states the linear convergence of the algorithm.

Corollary 12 *Let the assumptions of Theorem 11 still hold. If we assume that the solution to Problem VP has the additional regularity,*

$$\begin{aligned} u &\in H^2(0, T; H^2(0, \pi)) \cap H^3(0, T; Y) \cap C^1([0, T]; H^3(0, \pi)), \\ \phi &\in H^2(0, T; H^1(0, \pi)) \cap H^3(0, T; Y) \cap C^1([0, T]; H^2(0, \pi)), \\ \theta &\in H^1(0, T; H^1(0, \pi)) \cap H^2(0, T; Y) \cap C([0, T]; H^2(0, \pi)), \end{aligned} \quad (45)$$

and we use the finite element spaces V^h and E^h defined in (30) and (29), respectively, and the discrete initial conditions u_0^h , v_0^h , ϕ_0^h , e_0^h and θ_0^h given in (31), the linear convergence of the algorithm is deduced; i.e. there exists a positive constant $C > 0$, independent of the discretization parameters h and k , such that

$$\max_{0 \leq n \leq N} \left\{ \|\theta_n - \theta_n^{hk}\| + \|v_n - v_n^{hk}\| + \|(u_n - u_n^{hk})_x\| + \|(u_n - u_n^{hk})_{xx}\| + \|e_n - e_n^{hk}\| + \|(\phi_n - \phi_n^{hk})_x\| + \|\phi_n - \phi_n^{hk}\| \right\} \leq C(h + k).$$

The proof of this result is based on some well-known results concerning the approximation by the finite element method (see, for instance, [32]), the discretization of the time derivatives and the following result (see [33,31] for details),

$$\begin{aligned} &\frac{1}{k} \sum_{j=1}^{N-1} \|v_j - w_j^h - (v_{j+1} - w_{j+1}^h)\|^2 + \frac{1}{k} \sum_{j=1}^{N-1} \|e_j - r_j^h - (e_{j+1} - r_{j+1}^h)\|^2 \\ &\quad + \frac{1}{k} \sum_{j=1}^{N-1} \|\theta_j - l_j^h - (\theta_{j+1} - l_{j+1}^h)\|^2 \\ &\leq Ch^2 \left(\|u\|_{H^2(0, T; H^1(0, \pi))}^2 + \|\phi\|_{H^2(0, T; H^1(0, \pi))}^2 + \|\theta\|_{H^1(0, T; H^1(0, \pi))}^2 \right). \end{aligned}$$

5 Numerical results

Here, we present the numerical scheme which we have implemented in MATLAB in order to obtain the solutions to Problem VP^{hk} and then, we show

some numerical examples to demonstrate its accuracy and the behaviour of the solutions.

Considering the finite element spaces defined in (30) and (29), for $n = 1, 2, \dots, N$ and given $u_{n-1}^{hk}, v_{n-1}^{hk}, \phi_{n-1}^{hk}, e_{n-1}^{hk}$ and θ_{n-1}^{hk} , the discrete velocity field v_n^{hk} , the discrete porosity speed e_n^{hk} and the discrete temperature field θ_n^{hk} , at time $t = t_n$, are then calculated solving the following coupled system of discrete variational equations, for all $w^h \in V^h, r^h, l^h \in E^h$,

$$\begin{aligned}
& \rho(v_n^{hk}, w^h) + a k^2((v_n^{hk})_x, w_x^h) + c k^2((v_n^{hk})_{xx}, w_{xx}^h) - b k^2((e_n^{hk})_x, w^h) \\
& \quad + d k^2((e_n^{hk})_x, w_{xx}^h) + \beta^* k((\theta_n^{hk})_x, w^h) \\
& = \rho(v_{n-1}^{hk}, w^h) - a k((u_{n-1}^{hk})_x, w_x^h) - c k((u_{n-1}^{hk})_{xx}, w_{xx}^h) \\
& \quad + b k((\phi_{n-1}^{hk})_x, w^h) - d k((\phi_{n-1}^{hk})_x, w_{xx}^h), \\
& J(e_n^{hk}, r^h) + \beta k^2((e_n^{hk})_x, r_x^h) + \xi k^2(e_n^{hk}, r^h) + d k^2((v_n^{hk})_{xx}, r_x^h) \\
& \quad + b k^2((v_n^{hk})_x, r^h) - m k(\theta_n^{hk}, r^h) \\
& = J(e_{n-1}^{hk}, r^h) - \beta k((\phi_{n-1}^{hk})_x, r_x^h) - \xi k(\phi_{n-1}^{hk}, r^h) \\
& \quad - d k((u_{n-1}^{hk})_{xx}, r_x^h) - b k((u_{n-1}^{hk})_x, r^h), \\
& c^*(\theta_n^{hk}, l^h) + k^* k((\theta_n^{hk})_x, (l^h)_x) + \beta^* k((v_n^{hk})_x, l^h) + m k(e_n^{hk}, l^h) = c^*(\theta_{n-1}^{hk}, l^h),
\end{aligned}$$

where we recall that the discrete displacement field u_n^{hk} and the discrete porosity field ϕ_n^{hk} are now recovered from the relations

$$u_n^{hk} = k v_n^{hk} + u_{n-1}^{hk}, \quad \phi_n^{hk} = k e_n^{hk} + \phi_{n-1}^{hk}.$$

This problem leads to a linear system for a variable U in an adequate product space which is solved by using classical Cholesky's method. This numerical scheme was implemented on a 3.2 Ghz PC using MATLAB, and a typical run ($h = k = 0.01$) took about 0.622 seconds of CPU time.

5.1 First example: numerical convergence

As an academical example, in order to show the accuracy of the approximations we consider the following simpler problem.

Problem \mathbf{P}^{ex} . Find the displacement field $u : [0, \pi] \times [0, 1] \rightarrow \mathbb{R}$, the porosity field $\phi : [0, \pi] \times [0, 1] \rightarrow \mathbb{R}$ and the temperature field $\theta : [0, \pi] \times [0, 1] \rightarrow \mathbb{R}$ such that

$$\begin{aligned}
\ddot{u} &= -2u_{xxxx} - \dot{\phi}_{xxx} + 2u_{xx} + \dot{\phi}_x - \dot{\theta}_x + F_1 \quad \text{in } (0, \pi) \times (0, 1), \\
\ddot{\phi} &= 2\dot{\phi}_{xx} + u_{xxx} - 2\dot{\phi} - u_x + \theta - \dot{\phi} + F_2 \quad \text{in } (0, \pi) \times (0, 1), \\
\dot{\theta} &= -\dot{u}_x - \dot{\phi} + \theta_{xx} + F_3 \quad \text{in } (0, \pi) \times (0, 1), \\
u(0, t) &= u(\pi, t) = u_x(0, t) = u_x(\pi, t) = 0 \quad \text{for a.e. } t \in (0, 1), \\
\theta(0, t) &= \theta(\pi, t) = 0, \quad \phi(0, t) = \phi(\pi, t) = 0 \quad \text{for a.e. } t \in (0, 1). \\
u(x, 0) &= x^2(x-1)^2, \quad u_t(x, 0) = x^2(x-1)^2, \quad \text{for a.e. } x \in (0, \pi), \\
\phi(x, 0) &= x^2(x-1), \quad \phi_t(x, 0) = x^2(x-1) \quad \text{for a.e. } x \in (0, \pi). \\
\theta(x, 0) &= x^2(x-1) \quad \text{for a.e. } x \in (0, \pi),
\end{aligned}$$

where the artificial volume forces F_1 , F_2 and F_3 are given by

$$\begin{aligned}
F_1(x, t) &= e^t [x^2(x-\pi)^2 + 54 - 2(12x^2 - 12\pi x + 2\pi^2)], \\
F_2(x, t) &= e^t [3x^2(x-\pi) - 36x + 16\pi + (4x^3 - 6\pi x^2 + 2\pi^2 x)], \\
F_3(x, t) &= e^t [2x^2(x-\pi) + 4x^3 - 6\pi x^2 + 2\pi^2 x - 6x + 2\pi].
\end{aligned}$$

We point out that Problem P^{ex} corresponds to Problem P with the following data:

$$\begin{aligned}
T &= 1, \quad \rho = 1, \quad c = 2, \quad d = 1, \quad a = 2, \quad b = 1, \quad \beta^* = 1, \\
J &= 1, \quad \beta = 2, \quad \xi = 2, \quad m = 1, \quad \tau = 1, \quad c^* = 1, \quad k^* = 1,
\end{aligned}$$

and initial conditions

$$u_0 = v_0 = x^2(x-\pi)^2, \quad \phi_0 = e_0 = \theta_0 = x^2(x-\pi).$$

The exact solution to Problem P^{ex} can be easily calculated and it has the form, for $(x, t) \in (0, \pi) \times (0, 1)$,

$$u(x, t) = e^t x^2(x-\pi)^2, \quad \phi(x, t) = e^t x^2(x-\pi), \quad \theta(x, t) = e^t x^2(x-\pi).$$

Our aim, in this example, is to show the numerical convergence of the algorithm and its asymptotic behaviour.

The numerical errors, given by

$$\max_{0 \leq n \leq N} \left\{ \|\theta_n - \theta_n^{hk}\| + \|v_n - v_n^{hk}\| + \|(u_n - u_n^{hk})_x\| + \|(u_n - u_n^{hk})_{xx}\| \right. \\
\left. + \|e_n - e_n^{hk}\| + \|(\phi_n - \phi_n^{hk})_x\| + \|\phi_n - \phi_n^{hk}\| \right\},$$

and obtained for different discretization parameters h and k , are depicted (multiplied by 10) in Table 1. Moreover, the evolution of the error depending

on the parameter $h + k$ is plotted in Fig. 1. We notice that the convergence of the algorithm is clearly observed, and the linear convergence, stated in Corollary 12, is achieved.

$h \downarrow k \rightarrow$	0.1	0.05	0.01	0.005	0.001	0.0005
0.1	3.310250	2.775993	2.537746	2.535947	2.573062	2.582896
0.05	2.465241	1.705253	1.265408	1.229777	1.225896	1.229056
0.02	2.146094	1.223928	0.576248	0.518426	0.481190	0.479294
0.01	2.092496	1.128041	0.367718	0.292247	0.243944	0.240014
0.005	2.078592	1.101476	0.282937	0.187597	0.127343	0.122199
0.002	2.074654	1.093781	0.250549	0.138633	0.058656	0.052171
0.001	2.074084	1.092653	0.244959	0.129861	0.037985	0.030148
0.0005	2.074324	1.092445	0.239953	0.135056	0.045327	0.014494

Table 1

Example 1: Numerical errors ($\times 10$) for some h and k .

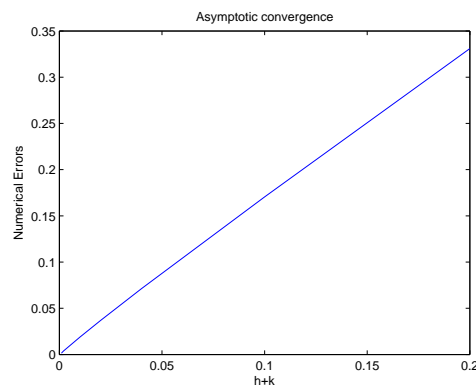


Fig. 1. Example 1: Asymptotic constant error.

If we assume now that there are not volume forces, and we use the final time $T = 5$ and the remaining data the same than in the previous simulation, taking the discretization parameters $h = k = 10^{-3}$ the evolution in time of the discrete energy E^{hk} , defined in (39), is plotted in Fig. 2. As can be seen, it converges to zero and an exponential decay seems to be achieved.

5.2 Second example: Application of a surface force

As a second example we consider a rod of length 1 clamped at its left end $x = 0$, we use the same data as in the previous simulation but with the initial conditions

$$u_0 = u_0 = v_{x_0} = \phi_0 = e_0 = \theta_0 = 0,$$

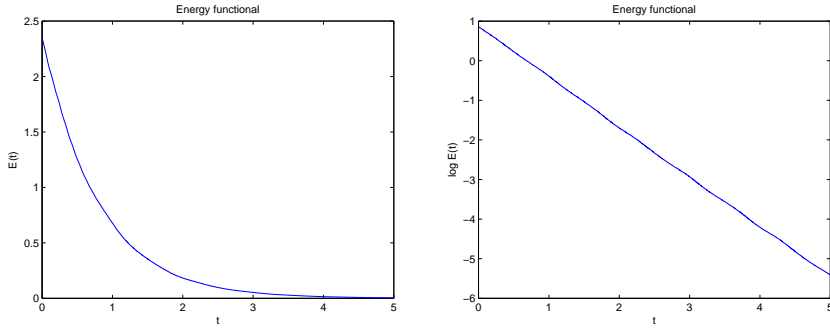


Fig. 2. Example 1: Evolution of the discrete energy in natural and semi-log scales.

and we apply a Neumann condition on the right end for the displacement field; that is,

$$-(5.1 u_x(1, t) + \phi(1, t) - \theta(1, t)) = g(t),$$

where the surface force $g(t)$ is given by

$$g(t) = 0.005 t^2.$$

For the porosity and temperature fields homogeneous Neumann conditions are applied on the right end.

By using the discretization parameters $h = k = 0.001$, in Fig. 3 the displacement field, the porosity field and the temperature field are plotted at final time. As expected, the displacements increase with respect to the spatial variable. Moreover, the porosity and the temperature, generated by these displacements, decrease.

Furthermore, in Fig. 4 the temporal evolution of the displacement field, the porosity field and the temperature field at the right end point $x = 1$ are plotted.

5.3 Third example: Influence of the temperature

As a third example we aim to study the influence of the temperature and to analyze how an initial temperature generates displacements and porosities. Moreover, we try to show the diffusion effect in the temperature. Therefore, we consider again a rod of length 1, we use the same data as in the first example with the initial conditions

$$u_0 = v_0 = \phi_0 = e_0 = 0, \quad \theta_0 = 270x^3(x - 1)^2,$$

and we apply homogeneous Dirichlet boundary conditions for all variables at both ends.

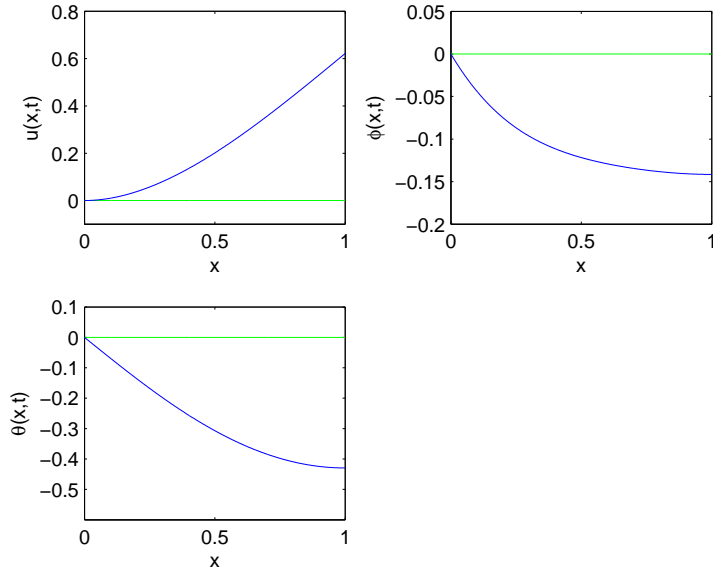


Fig. 3. Example 2: Displacements, porosity and temperature (plotted in green at initial time and in blue at final time).

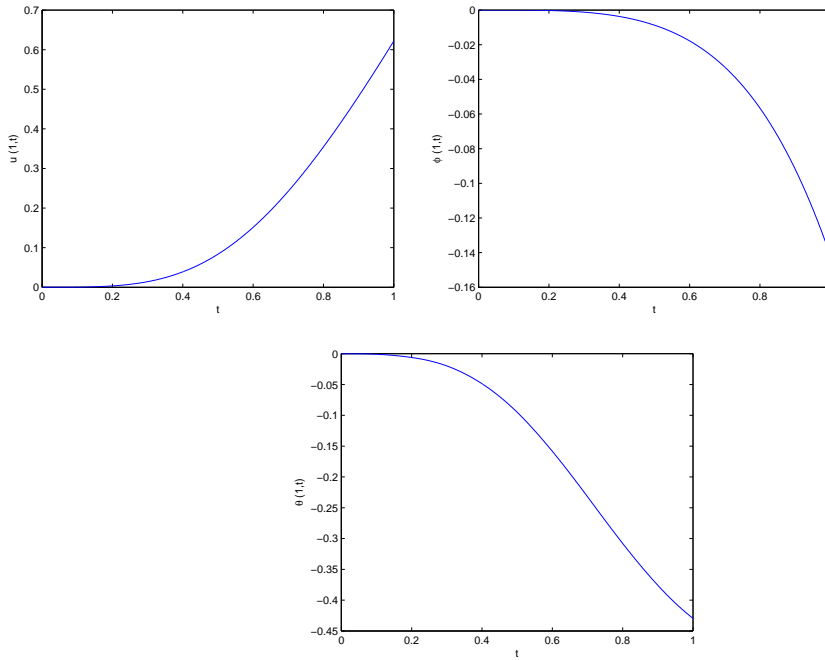


Fig. 4. Example 2: Temporal evolution of the displacements, porosity and temperature at the right end point $x = 1$.

Taking $h = k = 0.001$ as the discretization parameters, in Fig. 5 the displacement, porosity and temperature fields are plotted at final time. As can be seen, the initial temperature generates displacements and porosities, and the temperature seems to decrease to zero. Finally, in Fig. 6 the evolution in time

of the temperature is plotted to show the effect of the diffusion. As expected, the temperature seems to converge to zero.

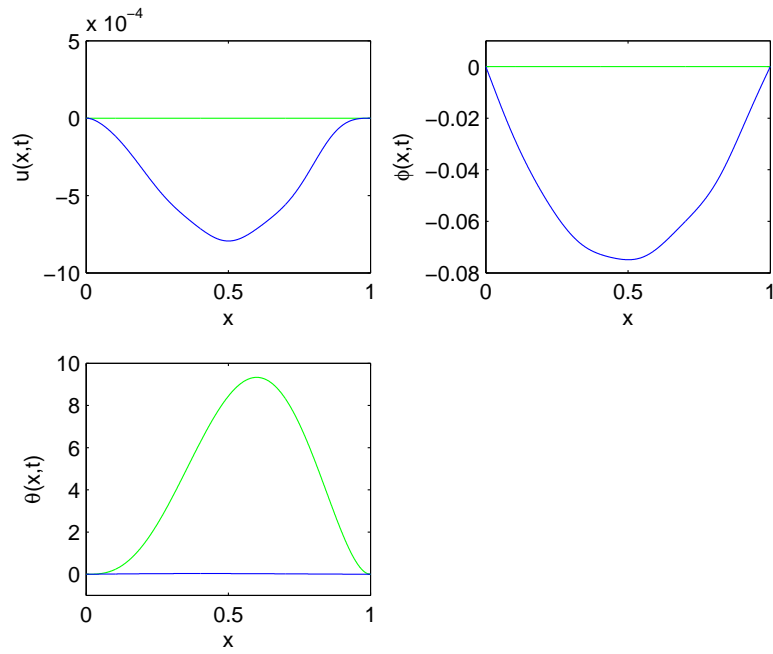


Fig. 5. Example 3: Displacements, porosity and temperature (plotted in green at initial time and in blue at final time).

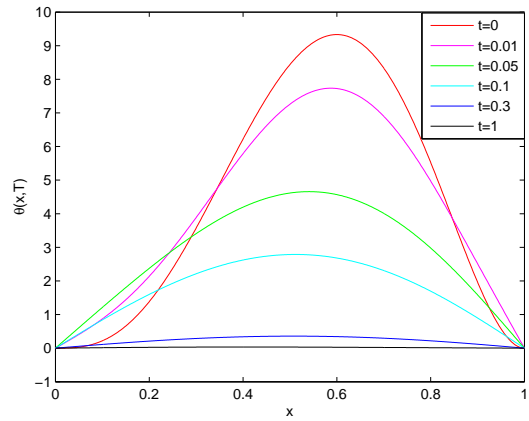


Fig. 6. Example 3: Evolution of the temperature. Diffusion effect.

6 Conclusions

In this paper we have studied the porous strain-gradient thermoelasticity problem from two different points of view: analytical and numerical. With respect to the first one, we have seen

- existence and uniqueness of solutions, under suitable constitutive assumptions, by means of the semigroup theory.
- The combination of the thermal effects with viscoporous effects is strong enough to guarantee the exponential stability of the solutions. Therefore from a physical point of view we can conclude that the thermomechanical perturbations are damped so fast that they can be neglected after a small period of time.

With respect to the numerical problem,

- we have introduced a fully discrete scheme to approximate the solutions of the variational problem using the finite element method and the implicit Euler scheme.
- We have proved a discrete stability property, from which the decay of the discrete energy has been derived.
- We have obtained some a priori error estimates, which have led to the linear convergence of the algorithm under suitable additional regularity conditions.
- We have provided some numerical simulations which have shown the numerical convergence of the approximations and the behaviour of the solution.

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