arXiv:1806.07855v1 [math.CO] 20 Jun 2018

Structure and enumeration of K_4-minor-free links and link-diagrams

Juanjo Rué* Dimitrios M. Thilikos†‡ Vasiliki Velona§

June 21, 2018

Abstract

We study the class $\mathcal{L}$ of link-types that admit a K_4-minor-free diagram, i.e., they can be projected on the plane so that the resulting graph does not contain any subdivision of K_4. We prove that $\mathcal{L}$ is the closure of a subclass of torus links under the operation of connected sum. Using this structural result, we enumerate $\mathcal{L}$ and subclasses of it, with respect to the minimum number of crossings or edges in a projection of $L \in \mathcal{L}$. Further, we obtain counting formulas and asymptotic estimates for the connected K_4-minor-free link-diagrams, minimal K_4-minor-free link-diagrams, and K_4-minor-free diagrams of the unknot.

1 Introduction

The exhaustive generation of knots and links according to their crossing number is a well-established problem in low dimensional geometry. For an account, see [25, Chapter 5]. In the last decades, there has also been interest in properties of random knots and links and their models, as well as random generation of them; see for instance [11], [8], [14], [17], [9, Chapter 25]. In parallel, various combinatorial and algorithmic questions of more deterministic nature have been addressed, for example in [1], [10], [23].

However, it appears that there are very few enumerative results of knots and links in the combinatorics literature. In fact, they are relatively recent and related to the enumeration of prime alternating links, such as [30] and [22]. Moreover, it seems that there are no known results connecting graph-theoretic classes with link classes. The present paper makes contribution in this direction. We present both enumerative and structural results, the latter relating in a precise

*Departament de Matemàtiques, Universitat Politècnica de Catalunya and Barcelona Graduate School of Mathematics, Barcelona, Spain. Partially supported by the the Spanish MICINN projects MTM2014-54745-P, MTM2017-82166-P and the María de Maetzu research grant MDM- 2014-0445 (BGSMath). Email: juan.jose.rue@upc.edu

†AlGCo project-team, LIRMM, Université de Montpellier, CNRS, Montpellier, France. Supported by the projects "ESIGMA" (ANR-17-CE40-0028) and "DEMOGRAPH" (ANR-16-CE40-0028). Email: sedthilk@thilikos.info

‡Department of Mathematics, National and Kapodistrian University of Athens, Athens, Greece.

§Department of Economics, Universitat Pompeu Fabra, and Departament de Matemàtiques, Universitat Politècnica de Catalunya, and Barcelona Graduate School of Mathematics, Spain. Supported by the Spanish Ministry of Economy and Competitiveness, Grant MTM2015-67304-P and FEDER, EU. Partially supported by the MINECO projects MTM2014-54745-P, MTM2017-82166-P, and the María de Maetzu research grant MDM- 2014-0445 (BGSMath). Email: vasiliki.velona@upf.edu

way a fundamental class of links, torus links, with the family of *series-parallel* graphs[1] and, more generally, graphs that exclude K_4 as a minor (K_4-minor-free graphs). The latter is an extensively studied graph class. For instance, it is known that they are exactly the graphs with treewidth at most 2, while a graph is K_4-minor free if and only if all its non-trivial biconnected[2] components are *series-parallel* graphs [6],[7]. Enumerative results for series-parallel graphs are available in [5].

Before stating the results, let us give some definitions. A *knot* is a smooth embedding of the 1-dimensional sphere $\mathbb{S}^1$ in $\mathbb{R}^3$. A *link* is a finite disjoint union of knots. A standard way to associate links to graphs is to represent them via *link-diagrams* that are their projections to the plane. That way, link-diagrams are seen as 4-regular maps, where each vertex corresponds to a crossing of the link with itself and where we mark the pair of opposite edges that is overcrossing the other. Notice that link-diagrams may contain vertex-less edges. Clearly, even the simplest link, that is the unknot link (equivalent to a cycle), may have arbitrarily many different link-diagrams. A *minimal* link-diagram is one with the minimum possible number of vertices, for the link L it represents. This number is called *crossing number* of L.

Our first result is a complete structural characterisation of K_4-minor-free links, i.e., links that admit some K_4-minor-free link-diagram, via a decomposition theorem (Theorem 4) derived after a series of graph-theoretic lemmata. Using this decomposition and analytic techniques of generating functions, we are able to deal with a series of enumeration problems.

Denote by $\mathcal{L}$ the set of all K_4-minor-free link-types. Among them, we distinguish the subset $\bar{\mathcal{L}}$ of the non-split links (i.e., those without disconnected link-diagrams), the subset $\hat{\mathcal{L}}$ of the links without trivial disjoint components (i.e., those without link-diagrams with vertex-less edges), and the subset $\mathcal{K}$ of the knots in $\mathcal{L}$. For each object in a set of links, we denote by n (resp. m) the number of vertices (rep. edges) of a minimal diagram and we define the combinatorial classes $(\mathcal{L}, m)$, $(\bar{\mathcal{L}}, n)$, $(\hat{\mathcal{L}}, n)$, and $(\mathcal{K}, n)$.

Our enumerative results on link-types are the following. Both classes $(\bar{\mathcal{L}}, n)$ and $(\hat{\mathcal{L}}, n)$ have an asymptotic growth of the form

$$Cm^{-3/2}\rho^{-m}, \tag{1}$$

where in both cases $\rho \approx 0.44074$, and $C \approx 5.04342$ for $(\bar{\mathcal{L}}, n)$, $c \approx 12.53228$ for $(\hat{\mathcal{L}}, n)$ (Theorem 5). For the class $(\mathcal{L}, m)$, we have to distinguish between even and odd m. In both cases the type of growth is the same, that is the same ρ, but C changes. For even m, $C \approx 63.38145$, and for odd m, $C \approx 42.07788$ (Corollary 1). The class $(\mathcal{K}, n)$ follows an estimate of the form

$$Cn^{-7/4}\exp(\beta n^{-1/2}), \tag{2}$$

where $C \approx 0.26275$ and $\beta \approx 2.56509$ (Theorem 6). The latter follows by Meinardus Theorem, which generalises the Hardy-Ramanujan estimates for integer partitions.

Our next set of results concerns the enumeration of link-diagrams. Let $\mathcal{M}$ be the set of all connected K_4-minor-free link-diagrams, $\mathcal{M}_1$ be the set all minimal connected K_4-minor-free link-diagrams, and let $\mathcal{M}_2$ be the set of all connected K_4-minor-free link-diagrams of the unknot. We

[1] A graph is series-parallel if it can be obtained from an double edge after a series of subdivisions or edge duplications.

[2] A graph is *biconnected* if it does not have cut-vertices (i.e., vertices whose removal increase the number of connected components). A *biconneted component* is a subgraph-maximal biconnected subgraph or a bridge, that is en edge whose removal increases the number of connected components. We say that the bridges are the *trivial* biconnected components.

define the combinatorial classes $(\mathcal{M}, m)$, $(\mathcal{M}_1, m)$, and $(\mathcal{M}_2, m)$. We obtain that all these three combinatorial classes follow an asymptotic growth of the form

$$\frac{1}{2m} C m^{-3/2} \rho^{-m}, \tag{3}$$

where the constants can be found in Table 1.

Family of diagrams	ρ	C
All	0.15592	5.27587
Minimal	0.41456	8.35785
Unknot	0.23188	7.58884

Our strategy to get these results relies first on adapting and refining the equations given in [27] for 4-regular graphs in the rooted map context. This way, we obtain a defining polynomial system for the rooted analogues of the aforementioned classes and analyse the corresponding asymptotic behaviour. Later, by using techniques from [29] and [3], we are able to transfer these results to the unrooted map classes under study.

Structure of the paper. In Section 2.2 we introduce all topological notions and definitions in knot theory that we will use in the rest of the paper. Similarly, in Section 2.1 we state the preliminaries needed for combinatorial enumeration, and in Section 2.3 we resume most of the analytic tools needed to provide asymptotic estimates. Later, in Section 3 we prove our structural result for K_4-free links, and in Section 4 their enumeration, both exact (by means of generating functions) and asymptotic. Later, in Section 5.2 we provide enumerative formulas for different kinds of link-diagrams, using tools from map enumeration. The paper concludes with Section 6, where an unrooting argument for maps is proven in our particular setting.

2 Preliminaries

2.1 Graph-theoretic Preliminaries

Given a graph $G = (V, E)$ and $v \in V$, we denote by $N_G(v) \subseteq V$ the set of neighbours of v. Also, for a vertex subset $A \subseteq V$ we denote by $G - A$ the graph obtained from G by removing the vertices in A and all edges incident with elements in A. Similarly, for a set $B \subseteq E$, we denote by $G - B$ the graph obtained from G by removing the edges in B and all vertices incident with elements in B.

A graph G is *k-vertex connected* (or shortly, *k-connected*) if it has more than k vertices and, if A is a subset of V of size strictly smaller than k, then $G - A$ is always connected. Similarly, a graph G is *k-edge connected* if it has more than k edges and, if B is a subset of E of size strictly smaller than k, then $G - B$ is always connected.

We say that a graph H is a *subdivision* of a graph G if G can be obtained from G after replacing some of its edges by paths with the same endpoints. Given two graphs H and G, we say that H is a *topological minor* of G if it contains as a subgraph some subdivision of H. If G does not contain H as a topological minor, then we say that G is *H-topological minor free.* We also say that H is a *minor* of G if H can be obtained from some subgraph of G after contracting edges. It is easy to see that K_4 is contained as a minor if and only if it is contained as a topological minor. Therefore, K_4-topological minor free graphs are exactly the K_4-minor free graphs. are *series-parallel* graphs [7].

A graph is *outerplanar* if it can be embedded on the plane in such a way that all vertices lie on the outer face. Equivalently, it does not contain a subdivision of K_4 or $K_{2,3}$. (see [19]).

For every $n \geq 3$, we denote by $\hat{C}_n$ the graph obtained if in a cycle of n vertices we replace all edges by double edges. We extend this definition so that $\hat{C}_2$ is the graph consisting of two vertices connected with an edge of multiplicity 4, $\hat{C}_1$ is a vertex with a double loop, and $\hat{C}_0$ is the vertex-less edge (that is the edge without endpoints).

2.2 Preliminaries for knots and links

A *knot* K is a smooth embedding of the 1-dimensional sphere $\mathbb{S}^1$ in $\mathbb{R}^3$. A *link* is a finite disjoint union of knots $L = K_1 \cup \cdots \cup K_\mu$. In this situation, each knot K_i is called a *component* of the link L. Note that there are alternative formulations in the literature [12, Ch. 1], either using polygonal knots or the notion of local flatness, which are equivalent to the previous one.

Two links L_1 and L_2 are said to be *ambient isotopic* (or equivalent) if there is a continuous map $h : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$, such that, for all $t \in [0,1]$, $h(x,t)$ is a homeomorphism and $h(L_1, 0) = L_1$, $h(L_1, 1) = L_2$. We then say that L_1 and L_2 have the same *type* and write $L_1 \equiv L_2$. Note that ambient isotopies preserve the orientation of $\mathbb{R}^3$.

A link equivalent to a set of disjoint circles in the plane is called a *trivial link*. Likewise, a knot equivalent to a circle is called the *trivial knot* or the *unknot*. Two components C_1, C_2 of a link L will be called *equivalent* if there is an ambient isotopy that maps L to itself and C_1 to C_2. The latter is an equivalence relation on the components of the link.

Decomposition of links. Given two links L_1, L_2, their *disjoint sum* is obtained by embedding L_1 in the interior of a standard sphere and L_2 in the exterior. We denote the resulting link by $L_1 \uplus L_2$ and call each L_1, L_2 a *disjoint component* of L. A link – and, accordingly, all members of its equivalence class – is *split* if it is the disjoint sum of two links.

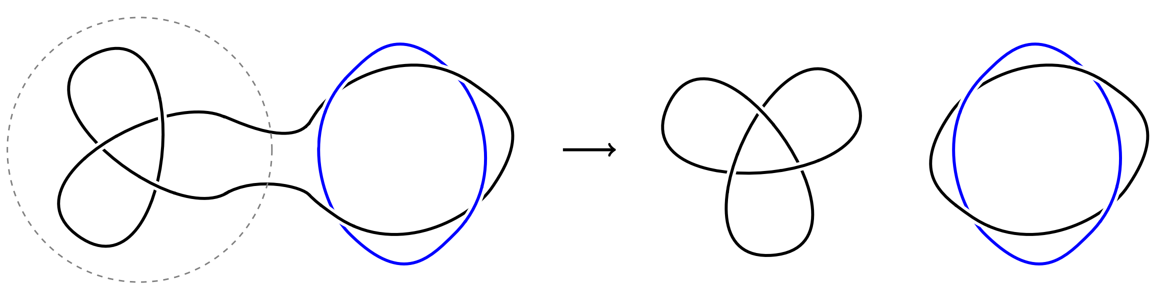

Figure 1: A connected sum, corresponding to the links $T(2,3)$ and $T(2,-4)$.

Consider a link L and the sphere $\mathbb{S}^2$ embedded in such a way that it meets the link transversely in exactly two points P_1 and P_2. Then we discern two different links L_1, L_2, when connecting P_1 and P_2. The first corresponds to the part of L in the interior of the sphere and the second to the part in the exterior. We then say that L is a *connected sum* with *factors* L_1, L_2, denoted $L_1 \# L_2$ (see Figure 1 for an example). A factor of a link is a *proper factor* if it is not the trivial knot and is not equivalent to the link itself. A link with proper factors is called *composite*. Otherwise, it is called *locally trivial*. Finally, a link is called *prime* if it is non-trivial, non-split, and locally trivial. The following two theorems are well known in Knot Theory:

Theorem 1. *[20, Theorem 3.2.1] Let L be a link such that $L = L_1 \# L_2$ for two links L_1 and L_2. Then L is trivial if and only if both links L_1 and L_2 are trivial.*

Theorem 2. *[20, Theorem 3.2.6] A non-split link can be decomposed into finitely many prime links with respect to the connected sum. Moreover, the decomposition is unique in the following sense: If $L_1 \# L_2 \# \cdots \# L_m \equiv L_1^{'} \# L_2^{'} \# \cdots \# L_n^{'}$ for prime links $L_i\,(i = 1, 2, \ldots, m)$ and $L_j^{'}\,(j = 1, 2, \ldots, n)$, then we have $m = n$ and, for each $i \in [n]$, $L_i \equiv L_{\sigma(i)}^{'}$ for some permutation σ of $[n]$.*

Note that the connected sum of two given knots is only well-defined for oriented knots. However, if they are *invertible*, i.e., are equivalent to themselves with opposite orientation, then it is well defined (see the relevant discussion in [12, Ch. 4.6]). In this case, the connected sum between links is also well-defined, if one specifies the equivalence classes of the components that get connected.

Definition 1. *Let $\mathcal{L}$ a family of links. We denote by $\mathbf{dcl}(\mathcal{L})$ the set of finite disjoint sums of links in $\mathcal{L}$. By $\mathbf{ccl}(\mathcal{L})$, we denote the set of finite connected sums of links in $\mathcal{L}$ that are non-split.*

Maps and link-diagrams. All graphs in this paper are multi-graphs, i.e., they may have loops of multiple edges. In particular we use the term *maps* for graphs that are embedded in the sphere and we say that they are *4-regular* when each vertex is incident to 4 edges. We also permit 4-regular graphs to contain vertex-less-edges.

Given a map G, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. Let G a 4-regular map and let $v \in V(G)$. We denote by $\overline{e}$ be the set of points of the plane corresponding to an edge $e \in E(G)$ and we pick a point $x_e \in \overline{e}$. We call the two connected components of $\overline{e} \setminus \{x_e\}$ *half-edges of G* corresponding to the edge e. We also use the notation $\hat{E}(G)$ to denote the set of half-edges of the embedding of G. For every $v \in V(G)$ we denote by $\hat{E}_v$ the set of half-edges containing v in their boundary. Notice that $\hat{E}_v$ is cyclically ordered as indicated by the embedding of G. Two half-edges in $\hat{E}_v$ are called *opposite* if they are non-consecutive in this cyclic ordering. Clearly, $\hat{E}_v$ contains two pairs of opposite half-edges. A *corner* on a map is the region between two consecutive half-edges around a given vertex.

Two maps are considered to be the same if the first is obtained from the second by an homeomorphism of the sphere which preserves its orientation. For enumerative purposes, we consider *rooted* maps: a rooted map is a map with a marked corner; the incident vertex is called the *root vertex*, and the edge following the marked corner in clockwise order around the root vertex is called the *root edge*. Finally, the face that contains the marked corner is the *root face* of the map. Equivalently, a rooted map is defined by orienting an edge in the map (the root vertex corresponds to the initial vertex of the edge) and choosing the root face as the one on the left of the rooted edge.

A *link-diagram* is a triple $D = (V, E, \sigma)$, where $G = (V, E)$ is a connected 4-regular map and $\sigma : V(G) \to \binom{\hat{E}(G)}{2}$, such that for every $v \in V(G)$, $\sigma(v)$ is a set of two opposite half-edges of the embedding of G. A link-diagram (V, E, σ) is *reduced* if the graph $G = (V, E)$ does not contain any cut-vertex.

Notice that each link-diagram $D = (V, E, \sigma)$ corresponds to a link-type which we denote by $L(D)$. The link-diagram D is obtained from $L(D)$ by projecting it on the sphere (or equivalently, on the plane). Moreover, it is known [12, Ch. 3] that for each link-type L there is at least one link-diagram L where $L(D) = L$. Given a link-type L, we denote by $\mathcal{D}_L$ the set containing every diagram D such that $L(D) = L$. Let L a link-type and D a diagram of L with the minimum number of vertices, n, over all the link-diagrams in $\mathcal{D}_L$. D is called a *minimal diagram* and n is called the *crossing number* of L.

Finally, we can apply certain local moves on link-diagrams, called *Reidemeister moves*, that do not alter the type of the link, as depicted in Figure 2. It is known that, given two link-diagrams that correspond to the same knot, one can be obtained from the other by a sequence of Reidemeister moves [28]. In Figure 2, there is a depiction of these moves.

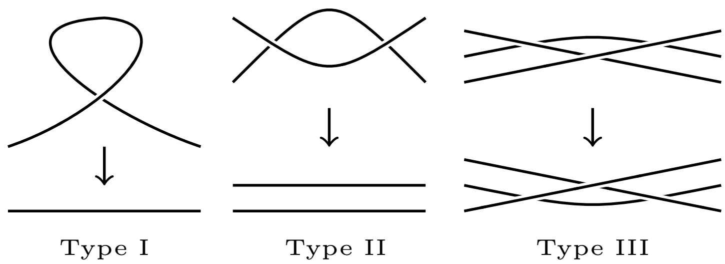

Figure 2: The 3 types of Reidemeister moves.

Torus links. Torus links are links that can be embedded on the standard torus. They are denoted by $T(p, q)$, $p, q \in \mathbb{Z}$. These are invertible links, with crossing number equal to $\min\{|p|(|q|-1), |q|(|p|-1)\}$ and number of components equal to $\gcd(p, q)$. We will be interested in types $T(\pm 2, q)$, equivalently $T(2, q)$. When $q = \pm 1$ or $q = 0$, $T(2, q)$ is the unknot. Otherwise, it is a prime link and is distinct from $T(2, -q)$ when $|q| > 2$. $T(2, q)$ is a knot if and only if q is odd. Intuitively, links of type $T(2, q)$ cross the meridian cycle 2 times and the longitude cycle $|q|$ times, and the sign of q determines the two different ways in which the crossings occur (see the links $T(2, 3)$ and $T(2, -4)$ in Figure 1). For more details on the properties of torus links, we refer to [26] and [12]. Note that for every q there is a link-diagram of $T(2, q)$ with graph $\hat{C}_{|q|}$. Finally, the crossing number of connected sums of torus links is additive [13].

2.3 Analytic tools for combinatorial enumeration

Most of the preliminaries in this section can be extensively found in the reference book [18].

Symbolic Method. A *combinatorial class* is a pair $(\mathcal{A}, |\cdot|)$, where $\mathcal{A}$ is a set of objects and $|\cdot|$ is the *size* of the object. In our setting, the objects will be graphs or maps, and the size will be typically the number of vertices or the number of edges. The latter will also be called the *atoms* of an object. We restrict ourselves to the case where the number of elements in $(\mathcal{A}, |\cdot|)$ with a prescribed size is finite. Under this assumption, we define the formal power series $A(z) = \sum_{a \in \mathcal{A}} z^{|a|} = \sum_{n=0}^{\infty} a_n z^n$, and conversely, $[z^n]A(z) = a_n$. We say that $A(z)$ is the *generating function* (or shortly the *GF*) associated to the combinatorial class $(\mathcal{A}, |\cdot|)$. We will usually not write the size function whenever the size of an object is clear. We will also write $\mathcal{A}_n$ for the set of elements in $\mathcal{A}$ wit size n, and $|\mathcal{A}_n| = a_n$.

The *Symbolic Method* provides a systematic tool to translate set conditions between combinatorial classes into algebraic conditions between GFs. The basic constructions are the following:

The *(disjoint) union* $\mathcal{A} \cup \mathcal{B}$ of two classes $\mathcal{A}$ and $\mathcal{B}$ refers to the disjoint union of the classes (and the corresponding induced size). The cartesian product $\mathcal{A} \times \mathcal{B}$ of two classes $\mathcal{A}$ and $\mathcal{B}$ is the set of pairs (a, b) where $a \in \mathcal{A}$ and $b \in \mathcal{B}$. The size of (a, b) is the sum of the sizes of a and b. We can define then the sequence and the multiset construction of a set $\mathcal{A}$, defined as the set of sequences (resp. multisets) of elements in $\mathcal{A}$. Finally, the composition of combinatorial classes corresponds to substitution of combinatorial objects of one of the classes into atoms of the elements of the second class. In Table 1 we include all the encodings into generating functions. Note that, in order for the GF encoding of the composition to work, one needs to assume that the atoms are distinguishable.

Construction		Generating function
Union	$\mathcal{A} \cup \mathcal{B}$	$A(z) + B(z)$
Product	$\mathcal{A} \times \mathcal{B}$	$A(z) \cdot B(z)$
Sequence	$\mathrm{Seq}(\mathcal{A})$	$(1 - A(z))^{-1}$
Multiset	$\mathrm{Mset}(\mathcal{A})$	$\prod_{a\in\mathcal{A}}(1 - z^{\lvert a\rvert})^{-1} = \exp\left(\sum_{r=1}^{\infty} \frac{1}{r} A(z^r)\right)$
Composition	$\mathcal{A} \circ \mathcal{B}$	$A(B(z))$
Pointing	$\mathcal{A}^\bullet$	$\partial_z A(z)$

Table 1: Table of combinatorial relations, and its generating function counterpart.

Complex analysis and generating functions. We apply singularity analysis over generating functions to obtain asymptotic estimates. The main reference here is again [18]. We say that a domain in $\mathbb{C}$ is *dented* at a value $\rho > 0$ if it is a set of the form

$$\Delta(\theta, R) = \{z \in \mathbb{C} : |z| < R,\ \arg(z - \rho) \notin [-\theta, \theta]\}$$

for some real number $R > \rho$ and some positive angle $0 < \theta < \pi/2$. Let $f(z)$ be a generating function which is analytic in a dented domain at $z = \rho$. The singular expansions we encounter in this paper are always of the form

$$f(z) = f_0 + f_1 Z + f_2 Z^2 + f_3 Z^3 + f_4 Z^4 + \cdots + f_{2k} Z^{2k} + f_{2k+1} Z^{2k+1} + O\left(Z^{2k+2}\right),$$

where $Z = \sqrt{1 - z/\rho}$ and $k = 0$ or $k = 1$. That is, $2k + 1$ is the smallest odd integer i such that $f_i \neq 0$.The even powers of Z are analytic functions and do not contribute to the asymptotic of $[z^n]f(z)$. The number $\alpha = (2k + 1)/2$ is called the *singular exponent*. If there is no other complex value of the same modulus on which such an expansion holds, we can apply the Transfer Theorem [18, Corollary VI.1] and obtain the estimate

$$[z^n]f(z) \sim c \cdot n^{\alpha-1} \rho^{-n}, \tag{4}$$

where $c = f_{2k+1}/\Gamma(-\alpha)$, and Γ is the classical Gamma function. If there is a a finite number of such values, the same estimates apply and the contributions are added [18, Theorem VI.5].

Meinardus Theorem. We will use the following result due to Meinardus [24] which generalizes the classical asymptotic estimate for integer partitions due to Hardy and Ramanujan. For convenience we use the version stated in [18, Section VIII.6] (see also [2, Theorem 6.2]).

Theorem 3 (Meinardus Theorem). *Let $A = (a_n)_{n\geq 1}$ a sequence of real positive numbers and*

$$F_A(z) = \prod_{n\geq 1} \frac{1}{(1 - z^n)^{a_n}}.$$

Let $\zeta_A(s) = \sum_{n\geq 1} \frac{a_n}{n^s}$ and $g_A(z) = \sum_{n\geq 1} a_n z^n$. Assume also that:

(M1) There exists a positive constant C_0 such that $\zeta_A(s)$ is continuable to a meromorphic function in $\mathrm{Re}(s) \geq -C_0$, and this meromorphic function has a single pole at $s = s_0$ with residue $\mathrm{Res}(\zeta_A, s_0)$,

(M2) There exists a positive constant C_1 such that $\zeta_A(s) = O(|s|^{C_1})$ whenever $|s|$ tends to infinity with $\mathrm{Re}(s) \geq -C_0$,

(M3) For each $t > 0$, y real numbers such that $|y| \leq 1/2$ and $\mathrm{Arg}(t+2\pi iy) > \frac{\pi}{4}$, there exist constants $C_2, \varepsilon > 0$ such that $\mathrm{Re}\left(g_A\left(e^{-t-2\pi yi}\right)\right) - g_A\left(e^{-t}\right) \leq -C_2 t^{-\varepsilon}$.

We write $\zeta(s)$ and $\Gamma(z)$ to denote the Riemann zeta function and the Gamma function, respectively. Then, the following asymptotic estimate holds:

$$[z^n]F_A(z) = Cn^{\alpha}\exp\left(\beta n^{\frac{s_0}{s_0+1}}\right), \tag{5}$$

where

$$\begin{aligned} \alpha &= \frac{\zeta_A(0) - 1 - \frac{1}{2}s_0}{1+s_0}, \\ \beta &= (1+s_0^{-1})\left(\mathrm{Res}(\zeta_A, s_0)\Gamma(1+s_0)\zeta(1+s_0)\right)^{\frac{1}{1+s_0}}, \\ C &= e^{\zeta_A'(0)}\left(2\pi(1+s_0)\right)^{-1/2}\left(\mathrm{Res}(\zeta_A, s_0)\Gamma(1+s_0)\zeta(1+s_0)\right)^{\frac{1-2\zeta_A(0)}{2+2s_0}}. \end{aligned}$$

3 Structure of K_4-minor free link-diagrams

We say that a link-type L is *K_4-minor free* if some of the diagrams in $\mathcal{D}_L$ is K_4-minor free (recall that $\mathcal{D}_L$ denotes all possible diagrams arising from L). Given an $i \in \mathbb{N}$, we denote by $\mathcal{D}_{\geq i}$ the set of all link-diagrams whose graph is $\hat{C}_j$ for some $j \geq i$.

Let $D_i = (V_i, E_i, \sigma_i), i \in [2]$ be two diagrams, where $V_i \neq \emptyset$. We say that a diagram $D(V, E, \sigma)$ is a *2-edge-sum* of D_1 and D_2 if D can be created from D_1 and D_2 as follows: we pick two edges $e_1 \in E_1$ and $e_2 \in E_2$, we remove them, and add two edges f_1 and f_2 such that both f_1, f_2 have endpoints from both e_1 and e_2, and such that the resulting embedding remains plane. The σ function is preserved, i.e., for all $v \in V_i$, $\sigma(v) \cap \sigma_i(v) \neq \emptyset$.

Let $\mathcal{D}$ be a set of all link-diagrams. We define the *closure* of $\mathcal{D}$, denoted by $\mathbf{cl}(\mathcal{D})$ with respect to 2-edge sums as the set containing every diagram D such that

- either $D \in \mathcal{D}$ or
- there exists $D_1, D_2 \in \mathcal{D}$ such that D is a 2-edge sum of D_1 and D_2.

From now on, we denote by $\mathcal{D}$ the set of all link-diagrams whose graph is K_4-minor-free.

Lemma 1. *Let $G = (V, E)$ be a 4-regular, K_4-minor-free and 3-edge-connected graph. Then G is outerplanar.*

Proof. Suppose to the contrary that G is not outerplanar, and hence it contains as a subgraph some subdivision H of $K_{2,3}$. Let v_1 and v_2 the vertices of H that have degree 3. Let also P_1, P_2, and P_3 the paths that are the connected components of $H - \{v_1, v_2\}$.

Let $G^- = G - \{v_1, v_2\}$. We first observe that none of the connected components of G^- contains more than one of the paths in $\{P_1, P_2, P_3\}$, as this would imply the existence of a path between vertices of these two paths in the connected component that contains them. This would imply the existence of a copy of K_4 as a topological minor, a contradiction.

Using the above claim, we deduce that G^- has at least 3 connected components C_1, C_2, C_3, where P_i is a subgraph of $C_i, i \in [3]$. Let $F \subseteq E(G)$ be the set of edges that are incident to either v_1 or v_2. Clearly, by the 4-regularity of G, F contains 7 or 8 edges, depending on whether v_1 and v_2 are adjacent or not. Moreover, because of the 3-edge-connectivity of G, for each $i \in [3]$ there are at least 3 edges in F that are incident to vertices in C_i and this implies that $|F| \geq 9$, a contradiction. □

Lemma 2. *Every 2-connected, 4-regular, K_4-minor free, and 3-edge-connected graph on $n \geq 1$ vertices is isomorphic to $\hat{C}_n$.*

Proof. We examine the non-trivial case where $n \geq 3$. From Lemma 1, G is $K_{2,3}$-free, therefore it is outerplanar and can be embedded in the plane so that all its vertices lay on its unbounded face F. Let E_{out} be the set of the edges of G that are incident to F. For each edge e in E_{out}, we denote by F_e the face that is incident to e and is different to F.

We next claim that for every $e \in E_{\text{out}}$, F_e is incident to exactly two edges. Suppose to the contrary that this is not correct for some $e \in E_{\text{out}}$ with end-vertices x and y. Let z be a vertex incident to F_e that is different to x and y. Notice that z is a cut-vertex in the graph $G^- = G - \{e\}$, that places x, y in different connected components in $G^- - \{z\}$. Let us call them C_x and C_y. Since z has degree 4, for one of them it holds that $|V(C_j) \cap N_{G^-}(z)| \leq 2$, say for C_x.

Let $S = V(C_j) \cap N_{G^-}(z)$ and observe that $\{\{z, w\} \mid w \in S\} \cup \{e\}$ is an edge separator of G of size ≤ 3 (an edge separator is a set of edges whose removal increases the number of connected components). As G is connected and every edge separator of a 4-regular graph contains an even number of edges, we obtain that $|S| = 2$, a contradiction to the 3-edge connectivity of G.

We just proved that G contains $\hat{C}_n$ as a spanning subgraph (i.e., a subgraph with the same set of vertices). The fact that G does not contain more edges than $\hat{C}_n$ follows from the fact that $\hat{C}_n$ is already 4-regular. □

Lemma 3. $\mathbf{cl}(\mathcal{D}_{\geq 1})$ *is the set of all reduced and connected K_4-minor free link-diagrams.*

Proof. We set $\mathcal{C} = \mathbf{cl}(\mathcal{D}_{\geq 1})$. Suppose that there exists a $D = (V, E, \sigma)$ that is a reduced K_4-minor free link-diagram and does not belong in $\mathcal{C}$. Let D be such a diagram where $|V|$ is minimized. If $G = (V, E)$ is 3-edge-connected then, by Lemma 2, G is isomorphic to $\hat{C}_n \in \mathcal{D}_{\geq 1} \subseteq \mathcal{C}$, a contradiction. Therefore G has an edge-cut consisting of two edges $e_1 = \{x_1, x_2\}$ and $e_2 = \{y_1, y_2\}$. As D is reduced, G has no cut-vertices, therefore x_1, x_2, y_2, y_2 are pairwise distinct. Let G_1^- and G_2^- be the connected components of $G - \{e_1, e_2\}$ and without loss of generality, we assume that $x_i, y_i \in V(G_i^-), i \in [2]$. Let G_i be the graph obtained from G_i^- after adding the edge $\{x_i, y_i\}$, $i \in [2]$. We also set $\sigma_i = \sigma|_{V(G_i)}, i \in [2]$. Observe that D is a 2-edge sum of $D_1 = (G_1, \sigma_1)$ and $D_2 = (G_1, \sigma_2)$. Moreover both G_1 and G_2 are 2-connected, K_4-minor free, and 4-regular. As G_1 and G_2 have both less vertices than G, by the minimality of the choice of D, we have that $D_1, D_2 \in \mathcal{C}$, therefore $D \in \mathcal{C}$, a contradiction.

Suppose there exists a diagram $D \in \mathcal{C}$ that either is not reduced or contains K_4 as a minor. We again choose such a $D = (V, E, \sigma)$ where $|V|$ is minimized. This cannot be of the form of $\hat{C}_n$, as all such diagrams are biconnected and K_4-minor free. If $D \notin \mathcal{D}_{\geq 1}$, then there are $D_1, D_2 \in \mathcal{C}$ with smaller vertex set, such that D is the 2-edge sum of D_1 and D_2. The latter diagrams are reduced and K_4-minor-free, because of the minimality of D. Consequently, D is also K_4-minor-free, since the 2-edge sum operation does not create any new K_4 in D. Moreover, the 2-edge sum operation maintains 2-connectivity. The two last facts imply a contradiction to the choice of D. □

Let $\mathcal{T}_2 = \mathbf{ccl}(\bigcup_{q\in\mathbb{N}}\{T(2,q)\})$. Let $\mathcal{L}$ be the class of links that have a K_4-minor-free link-diagram, namely, $\mathcal{L} = \{L \mid \mathcal{D}_L \cap \mathcal{D} \neq \emptyset\}$. We then have the following theorem giving a structural decomposition of links in $\mathcal{L}$:

Theorem 4. $\mathcal{L} = \mathbf{dcl}(\mathcal{T}_2)$.

Proof. It is clear that both classes are closed under disjoint sums, so it is enough to prove the Theorem for non-split links in $\mathcal{L}$, $\mathcal{L}'$.

We first prove that $\mathcal{L}' \subseteq \mathbf{ccl}(\mathcal{T}_2)$. Let $L \in \mathbf{ccl}(\mathcal{T}_2)$ and non-split. Then it has a diagram that is K_4-minor-free. Let us pick a diagram D_L of minimal $|V|$. This is also reduced, so, from Lemma 3, we have $D_L \in \mathbf{cl}(\mathcal{D}_{\geq 1})$. Then D_L is either some $\hat{C}_i$ or a series of consecutive 2-edge sums between $\hat{C}_i$ objects. The operation of 2-edge sums can be translated to the operation of connected sum in the corresponding links. Thus, either L is a torus link $T(2,q)$, $q \in \mathbb{Z} \setminus \{0\}$, or the result of a series of connected sums of such torus links, i.e. $L \in \mathbf{ccl}(\mathcal{T}_2)$.

We now prove that $\mathbf{ccl}(\mathcal{T}_2) \subseteq \mathcal{L}'$. Let $T \in \mathbf{ccl}(\mathcal{T}_2)$ and non-split, i.e. $T = T_1\#...\#T_n$, where $T_i \in \mathcal{T}_2$ and prime. The claim is shown by induction on n. If $n = 1$, i.e. T is prime, then the claim is true. Suppose that the claim is true for $n < k$ and let $T = T_1\#...\#T_k$, $T' = T_1\#...\#T_{k-1}$, and C the component of T on which T_k is connected. Then T' belongs in $\mathcal{L}$ by the induction hypothesis, thus it has a K_4-minor-free link-diagram D. We know there is an i such that $\hat{C}_i$ is a diagram of T_k with these properties. We embed $\hat{C}_i$ in a face adjacent to an edge of C and perform a 2-edge sum operation. The resulting diagram remains K_4-free and represents the link T: the way the half-edges were connected does not matter, since the class $\mathcal{T}_2$ is a class of reversible links. □

4 Enumeration of knots and links

Recall that $\mathcal{L}$ is the set of link-types that have a K_4-minor-free link-diagram. Let $\mathcal{K}, \bar{\mathcal{K}}$ be, respectively, the set of knot types in $\mathcal{L}$ and the set of prime knot types in $\mathcal{L}$. We denote by $\bar{\mathcal{L}}$ the set of non-split link-types in $\mathcal{L}$, and $\hat{\mathcal{L}}$ the set of the link-types in $\mathcal{L}$ with no trivial disjoint components.

4.1 Enumeration of $\mathcal{L}$

In this section, we enumerate the combinatorial classes $(\mathcal{L}, m)$, $(\bar{\mathcal{L}}, n)$, $(\hat{\mathcal{L}}, n)$, $(\bar{\mathcal{K}}, n)$, $(\mathcal{K}, n)$, where m is the number of edges in a minimal diagram of a link and n is the crossing number. We denote by $L(z)$, $\bar{L}(z)$, $\hat{L}(z)$, $\bar{K}(z)$, and $K(z)$ the corresponding generating functions. Notice that it is not possible to enumerate $\mathcal{L}$ with respect to crossing number; the number of links with a given crossing number is infinite, since the disjoint sum of any such link and a trivial link of arbitrarily many components has the same crossing number.

Let $\mathcal{G}$ be the combinatorial class of unrooted, unlabelled trees, with size equal to the number of vertices. Consider the sets $A = \{2\nu + 1 : \nu \in \mathbb{Z}\} \setminus \{1, -1\}$, $B = \{2\nu : \nu \in \mathbb{Z}\} \setminus \{0, -2\}$. For $T \in \mathcal{G}$, consider all possible labelings of T, such that the vertices are labeled with a multiset of A or 1, and each edge of T is labeled with a number in B. We consider two such labelled trees equivalent if there is a graph automorphism of the first that identifies them as trees and also identifies their labels. Let $\mathcal{T}$ be the set of the resulting equivalence classes. We define the size of a label $i = i_1, ..., i_k$ to be the sum of the absolute values $|i_j|$, and the size of a tree in $\mathcal{T}$ to be the sum of all labels. These labels will be used to encode crossing numbers. See Figure 3 for an example of an object in $\mathcal{T}$, of size 68.

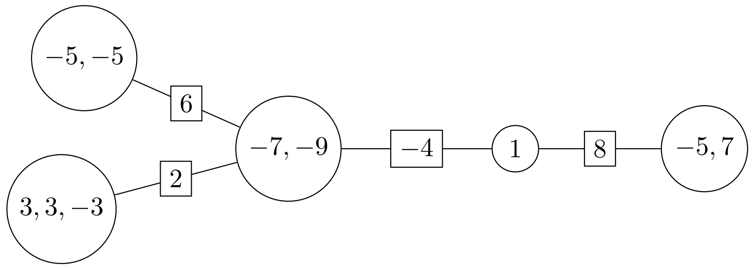

Figure 3: An element of $\mathcal{T}$. Labels in the edges are drawn inside a square.

Proposition 1. *There is a size-preserving bijection between $\bar{\mathcal{L}}$ and $\mathcal{T}$, hence $\bar{\mathcal{L}} \cong \mathcal{T}$.*

Proof. Let $L \in \bar{\mathcal{L}}$. By Theorem 4, there exist prime torus links $T_j := T(2, q_j)$, such that

$$L \equiv T_1 \# ... \# T_r, \tag{6}$$

and $L_j := T_1 \# ... \# T_j$ is non-split for any j.

Let $C_1, ..., C_l$ be the components of the links $T_1, ..., T_r$, with some arbitrary numbering. Notice that for every $i \in [r]$, T_i contains one or two of the components C_j. For every such component, we write $L(C_j) := T_i$. Each time a connected sum is realised between L_k and T_{k+1}, one component of T_{k+1} is identified with a component of L_k. Consider the corresponding equivalence relation, i.e., two components C_i, C_j are in the same equivalence class if they are identified in L. Let $I_1, ..., I_m$ be the equivalence classes. We define $P(I_j)$ as the multiset of prime torus knots that belong to I_j, formally,

$$P(I_j) := \{(i, q) |\ \text{for exactly } i \text{ components } C \in I_j,\ \text{it holds}\, L(C) = T(2, q), |q| \geq 3,\ \text{odd}\}.$$

Let $G_L(V, E)$ be the graph, where $V = \{I_1, ..., I_m\}$ and there is an edge $I_i I_j$ if and only if there is a link $T_l \equiv T(2, q_l)$ such that one of its components belongs in I_i and the other belongs in I_j. Notice that such a link is unique when it exists, hence we can refer to q_l as q_{ij}. Let T_L be the graph G_L, where the vertices I_i have the label $P(I_i)$ and the edges $I_i I_j$ have the label q_{ij}. Then, $T_L \in \mathcal{T}$ and we define $\phi : \bar{\mathcal{L}} \to \mathcal{T}$ such that $\phi(L) = T_L$.

We first show that ϕ is well defined. Suppose that $L_1 = T_1 \# ... \# T_r \equiv T'_1 \# ... \# T'_r = L_2$. Let G^1_i, G^2_j be the components of L_1 and L_2, corresponding to the associated equivalence classes I^1_i, I^2_j. Since $L_1 \equiv L_2$, there is a permutation σ of $[n]$, such that there is an ambient isotopy of $\mathbb{R}^3$ that identifies G^1_i with $G^2_{\sigma(i)}$ for all i. Then, the labels on the vertices $I^1_i, I^2_{\sigma(i)}$ are the same, because of the uniqueness of factorisation in knots (Theorem 2). Moreover, an edge $I^1_i I^1_j$ exists if and only if $I^2_{\sigma(i)} I^2_{\sigma(j)}$ exists, and the label on it is the same: otherwise, it holds that

$$L_1 \setminus \{ \bigcup_{h \in [m] \setminus \{i,j\}} G_h \} \not\equiv L_2 \setminus \{ \bigcup_{h \in [m] \setminus \{\sigma(i), \sigma(j)\}} G_h \}$$

for some i, j, a contradiction.

Now we prove that ϕ is a bijection. Given $T \in \mathcal{T}$, consider the following link L_T : for some v in T, consider a trivial knot K and perform all the connected sums indicated by its edges and $p(I)$. For each one of the new components, do the same as indicated by $T - v$. By construction, $\phi(L_T) = T$, hence ϕ is surjective. Notice that if $\phi(L) = T_L$, then any complete exploration of T_L corresponds to a connected sum decomposition of L: each new vertex or edge that is encountered corresponds to connected sums indicated by the labels. Then, L is of the same type as T_L, because of the uniqueness of factorisation in links. Hence, ϕ is injective. By the additive property of crossing numbers in torus links, ϕ also preserves size. □

We now aim to obtain functional equations that define uniquely the generating functions under study. Let us start with $\bar{K}(z)$, the generating function associated to $\bar{\mathcal{K}}$ (prime torus knots $T(2, 2i+1)$, $i \in \mathbb{Z}\backslash\{0, 1, -1\}$), where z marks crossings. Observe that

$$\bar{K}(z) = 2 \cdot \sum_{i \geq 1} z^{2i+1} = \frac{2z^3}{1 - z^2}.$$

Moreover, every object in $\mathcal{K}$ is defined uniquely by a multiset of prime torus knots, therefore $\mathcal{K} = \mathrm{Mset}(\bar{\mathcal{K}})$ and hence

$$K(z) = \exp\left(\sum_{k \geq 1} \frac{1}{k} \bar{K}(z^k)\right) = \exp\left(\sum_{k \geq 1} \frac{1}{k} \frac{2z^{3k}}{1 - z^{2k}}\right). \tag{7}$$

The first terms of $K(z)$ are the following:

$$K(z) = 1 + 2\,z^3 + 2\,z^5 + 3\,z^6 + 2\,z^7 + 4\,z^8 + 6\,z^9 + 7\,z^{10} + 8\,z^{11} + 13\,z^{12} + 14\,z^{13} + 19\,z^{14} + 26\,z^{15} + \cdots$$

We denote by $\mathcal{E}$ the combinatorial class of all possible edge labels. Then, $E(z) = z^2 + \frac{2z^4}{1-z^2}$. The following proposition combines all previous counting formulas, in order to obtain the generating function associated to $\bar{\mathcal{L}}$, which we denote by $\bar{L}(z)$:

Proposition 2. *Let $\mathcal{F} = \mathcal{G}^\bullet \circ (\mathcal{E} \times \mathcal{K})$, where $\mathcal{G}$ is the class of unrooted, unlabelled trees (counted according to vertices), and denote by $F(z)$ the generating function associated to $\mathcal{F}$. Then,*

$$\bar{L}(z) = \frac{F(z)}{E(z)} + \frac{E(z)}{2}\left(- \frac{F(z)^2}{E(z)^2} + \frac{F(z^2)}{E(z^2)}\right). \tag{8}$$

Proof. Since unlabelled trees have vertices that are equivalent, the substitution must be performed using cycle index sums[3]. The cycle index sum of $\mathcal{G}^\bullet$ is known to satisfy the following functional equation in infinitely many variables (see [4, Chapter 4.1]):

$$\mathcal{Z}_{\mathcal{G}^\bullet}(s_1, s_2, ...) = s_1 \exp\left(\sum_{k \geq 1} \frac{1}{k} \mathcal{Z}_{\mathcal{G}^\bullet}(s_k, s_{2k}, ...)\right). \tag{9}$$

[3] The cycle index series of a combinatorial structure $\mathcal{F}$ is the formal power series (in an infinite number of variables) $Z_F(x_1, x_2, x_3, ...) = \sum_{n \geq 0} \frac{1}{n!}\big(\sum_{\sigma \in \mathcal{S}_n} \mathrm{fix}F[\sigma] x_1^{\sigma_1} x_2^{\sigma_2} x_3^{\sigma_3} \cdots\big)$, where $\mathcal{S}_n$ denotes the group of permutations of $[n]$, σ_i is the number of cycles of length i in σ, and $\mathrm{fix}F[\sigma]$ is the number of objects in $\mathcal{F}$ for which σ is an automorphism.

We can now obtain the ordinary generating function of $\mathcal{F} = \mathcal{G}^\bullet \circ (\mathcal{E} \times \mathcal{K})$. By Pólya's Enumeration Theorem (see for instance [16, Theorem 2.8]), the latter satisfies the equation $F(z) = \mathcal{Z}_{\mathcal{G}^\bullet}(f(z), f(z^2), ...)$, where $f(z) = E(z)K(z)$.

A $T \in \mathcal{F}$ is equivalent to a tree in $\mathcal{T}^\bullet$ (pointing on a vertex), such that all labels are on the vertices, an label on an edge $e = \{v_i, v_j\}$ is on the vertex in e that is closer to the root, and the root-vertex has an extra edge label. We eliminate the extra label from the enumeration, dividing $F(z)$ by $E(z)$. We obtain $T^\bullet(z) = \frac{F(z)}{E(z)}$.

We can obtain an expression for $T(z)$ using $T^\bullet(z)$, by an unrooting argument. By the Dissymmetry Theorem for trees, we can express unrooted families in terms of rooted ones. More precisely, given a family of trees $\mathscr{T}$ denote by $\mathscr{T}^\bullet$, $\mathscr{T}^{\bullet-\bullet}$, and $\mathscr{T}^{\bullet\to\bullet}$ the same family with a rooted vertex, a rooted edge and a rooted and oriented edge. Let $T(z)$, $T^\bullet(z)$, $T^{\bullet-\bullet}(z)$, and $T^{\bullet\to\bullet}(z)$ the corresponding generating functions. Then, the Dissymmetry Theorem for trees states that

$$T(z) = T^\bullet(z) + T^{\bullet-\bullet}(z) - T^{\bullet\to\bullet}(z). \tag{10}$$

In $\mathcal{T}$, it holds that $T^\bullet(z) = \frac{F(z)}{E(z)}$, $T^{\bullet\to\bullet}(z) = \frac{E(z)F(z)^2}{E(z)^2}$, and

$$T^{\bullet-\bullet}(z) = \frac{E(z)}{2}\left(\frac{F(z)^2}{E(z)^2} - \frac{F(z^2)}{E(z^2)}\right) + \frac{E(z)F(z^2)}{E(z^2)},$$

where the common factor $E(z)$ encodes the label of the marked edge. Substituting these expressions in Equation 10 and using Proposition 1, we obtain the indicated relation for $T(z)$ and then for $\bar{L}(z)$. $\square$

The first terms of $\bar{L}(z)$ are the following:

$$\bar{L}(z) = 1 + z^2 + 2\,z^3 + 3\,z^4 + 4\,z^5 + 9\,z^6 + 12\,z^7 + 26\,z^8 + 40\,z^9 + 82\,z^{10} + 136\,z^{11} + 280\,z^{12} + \dots$$

Lemma 4. *Let $L^*(z) = \bar{L}(z) - 1$. Then, $\hat{L}(z) = \exp\Big(L^*(z)\Big)\exp\Big(\sum_{k\geq 2}\frac{1}{k}L^*(z^k)\Big)$.*

Proof. Immediate, since links in $\hat{\mathcal{L}}$ are multisets of links in $\bar{\mathcal{L}}$, excluding the trivial knot. $\square$

The first terms of $\hat{L}(z)$ are the following:

$$\hat{L}(z) = 1 + z^2 + 2\,z^3 + 4\,z^4 + 6\,z^5 + 16\,z^6 + 24\,z^7 + 56\,z^8 + 98\,z^9 + 208\,z^{10} + 382\,z^{11} + 805\,z^{12} + \dots$$

We would like to study K_4-free link-types by the number of edges of a minimal diagram, so as to account also for trivial components. We obtain the following lemma.

Lemma 5. *For the combinatorial class $\mathcal{L}$ with size equal to the number of edges in a minimal diagram, it holds that*

$$L(z) = \frac{\hat{L}(z^2)}{1-z}.$$

Proof. Immediate, since a link-diagram of n vertices has $2n$ edges and there is one choice for the number of trivial components that are added. $\square$

The first terms of $L(z)$ are the following:

$$L(z) = 1 + z + z^2 + z^3 + 2\,z^4 + 2\,z^5 + 4\,z^6 + 4\,z^7 + 8\,z^8 + 8\,z^9 + 14\,z^{10} + 14\,z^{11} + 30\,z^{12} + 30\,z^{13} + \cdots$$

4.2 Asymptotic analysis

It is well known that the the exponential generating function for rooted labelled trees, $G(z)$, is defined by the functional equation $G(z) = z\exp(G(z))$. Additionally, when dealing with $G(z)$ as an analytic function, it is known that it has a unique minimal singularity at $z = e^{-1}$ of square root type, with singular expansion $1 - \sqrt{2}Z + O(Z^2)$, where $Z = (1 - ze)^{1/2}$ (see [18] for all details).

The following theorem determines the asymptotic growth of $[z^n]\bar{L}(z)$. The analysis of the multiset operator is based on the analysis of Otter trees (see [18, Chapter VII.5]).

Theorem 5. *The following asymptotic estimates hold:*

$$[z^n]\bar{L}(z) \sim \frac{c_1}{\Gamma(-1/2)}\, n^{-3/2}\, \rho^{-n},\ [z^n]\hat{L}(z) \sim \frac{c_2}{\Gamma(-1/2)}\, n^{-3/2}\, \rho^{-n},$$

where $\rho \approx 0.44074$ *(*$\rho^{-1} \approx 2.26891$*),* $c_1 \approx 1.45557$*,* $c_2 \approx 3.61691$*, and* Γ *is the Gamma function.*

Proof. Recall that $f(z) = E(z)K(z)$. Observe that, due to the cycle index sum relation (9) and that $F(z) = \mathcal{Z}_{\mathcal{G}^\bullet}(f(z), f(z^2), \ldots)$, $F(z)$ satisfies the implicit equation

$$F(z) = f(z)\exp\left(\sum_{k\geq 1}\frac{1}{k}F(z^k)\right).$$

Let $\xi(z) = f(z)\exp\left(\sum_{k\geq 2}\frac{1}{k}F(z^k)\right)$ and ρ_F, ρ_ξ be the smallest positive singularities of $F(z)$ and $\xi(z)$, respectively. Notice that $\rho_F < 1$.

We first show that $\xi(z)$ is analytic in $|z| \leq \rho_F$. The function $f(z)$ has radius of convergence equal to 1, while for $|z| < 1$ it holds that

$$\left|\exp\left(\sum_{k\geq 2}\frac{1}{k}F(z^k)\right)\right| \leq \exp\left(\sum_{k\geq 2}\frac{1}{k}F(|z|^k)\right) < \exp\left(F(|z|^2)\sum_{k\geq 0}|z|^k\right).$$

The last inequality follows due the following argument: let $k \geq 3$, and let $z < 1$ be a positive real number and write $F(z) = \sum_{n\geq 1} f_n z^n$. Then,

$$F(z^k) = \sum_{n\geq 1} f_n z^{nk} = \ \sum\nolimits_{n\geq 1} f_n z^{2n+n(k-2)} < z^{k-2}\sum\nolimits_{n\geq 1} f_n z^{2n} = z^{k-2}F(z^2)$$

Hence, for $z < 1$ positive number

$$\exp\left(\frac{1}{2}F(z^2) + \sum_{k\geq 3}\frac{1}{k}F(z^k)\right) < \exp\left(F(z^2) + \sum_{k\geq 3}F(z^k)\right) <$$
$$\exp\left(F(z^2) + \sum_{k\geq 3}z^{k-2}F(z^2)\right) = \exp\left(F(z^2)\sum_{k\geq 0}z^k\right).$$

The radius of convergence of $F(z^2)$ is equal to $\sqrt{\rho_f} > \rho_F$, hence indeed $\xi(z)$ is analytic in $|z| \leq \rho_F$.

Observe that $F(z) \equiv G(\xi(z))$, since $F(z)$ is defined by $F(z) = \xi(z)\exp(F(z))$ and $G(\xi(z))$ by the same relation $G\xi(z) = \xi(z)\exp(G\xi(z))$. Since $\xi(z)$ is analytic in $|z| \leq \rho_F$, it holds that $F(z)$ is singular on $\rho > 0$, such that $\xi(\rho) = e^{-1}$. Moreover, $G(\xi(z))$ (equivalently, $F(z)$) has a singular expansion of square-root type on ρ, which can be recovered by composing the singular expansion of G at e^{-1} with the regular expansion of $\xi(z)$ at ρ.

More precisely, writing $Z = (1 - z/\rho)^{1/2}$, we obtain a singular expansion for $F(z)$ in a dented domain around ρ of the form $F_0 - F_1 Z + O\left(Z^2\right)$, where $F_1 = \sqrt{2e\xi'(\rho)\rho} \neq 0$. The function $E(z)^{-1}F(z)$ has the same singular expansion at ρ, but divided by $E(\rho)$. One can obtain the singular expansion of $\bar{L}(z)$, after applying the dissymmetry relation (8) to the singular expansion of $E(z)^{-1}F(z)$. The expansion is again of the square root type and the coefficient of Z can be computed as indicated.

By Lemma 4, $\hat{L}(z)$ also has a unique minimal singularity of square root type at ρ and singular expansion

$$(1 + L_1^* Z) \exp\left(L_0^* + \sum_{k\geq 2} \frac{1}{k} L^*\left(z^k\right)\right) + O(Z^2),$$

where $L_0^* = \bar{L}_0 - 1$ and $L_1^* = \bar{L}_1$. Then, $\hat{L}_1 = \bar{L}_1 \exp\left(\bar{L}_0 - 1 + \sum_{k\geq 2} \frac{1}{k} L^*\left(z^k\right)\right)$.

To conclude, the stated asymptotic estimates are obtained by the transfer Theorem of singularity analysis stated in Equation (4). □

Corollary 1. *The coefficients of $L(z)$ have asymptotic growth of the form:*

$$[z^n]L(z) \sim \frac{c}{\Gamma(-1/2)} n^{-3/2} \rho^{-n},$$

where $\rho \approx 0.44074$ and $c \approx 18.29238$ or $c \approx 12.14400$, when n is even or odd respectively, and Γ is the Gamma function.

Proof. By Lemma 5, the generating function $L(z)$ is equal to $\hat{L}(z^2)\frac{1}{1-z}$. Then, $\hat{L}(z^2)$ is symmetric and has two singularities at $\pm\sqrt{\rho}$ that induce two singular expansions of the square-root type. In particular, the coefficient of Z in both is equal to $\sqrt{2}\hat{L}_1$. $L(z)$ has the same singular expansions on $\sqrt{\rho}$ and $-\sqrt{\rho}$, with an extra factor $c_k = \frac{1}{1-k\sqrt{\rho}}$, $k \in \{+,-\}$, respectively on these two points. The transfer principles of singularity analysis yield an asymptotic growth of the form $\sqrt{2}\hat{L}_1\rho^n(c_-(-1)^n + c_+)$. □

Theorem 6. *The coefficients of $K(z)$ have asymptotic growth of the form:*

$$[z^n]K(z) = cn^{\alpha} \exp(\beta n^{1/2}),$$

where $c = \mathrm{e}^{2\log(2)\zeta(0)} \left(\Gamma(2)\,\zeta(2)\right)^{5/4}/(2\sqrt{\pi}) \approx 0.26275$, $\alpha = -7/4$, $\beta = 2\sqrt{\Gamma(2)\,\zeta(2)} \approx 2.56509$.

Proof. This estimate will follow from an application of Meinardus Theorem 3. The expression given in Equation (7) for $K(z)$ can be rewritten as an infinite product as

$$\prod_{n\geq 1} \frac{1}{(1 - z^n)^{a_n}},$$

where $a_n = 2$ if $n \geq 3$ and odd, $a_n = 0$ otherwise. Consider $\zeta_A(s) = \sum_{n\geq 1} \frac{a_n}{n^s}$. This Dirichlet series can be written in terms of the Riemann zeta function $\zeta(s)$, observing that:

$$\zeta_A(s) = 2\left(-1 + \sum_{n\,\mathrm{odd}} \frac{1}{n^s}\right) = -2 + 2\left(\zeta(s) - \frac{1}{2^s}\zeta(s)\right) = -2 + 2\zeta(s)\left(1 - \frac{1}{2^s}\right).$$

Recall that $\zeta(s)$ can be extended to a meromorphic function in the whole complex plane, with a single pole at $s = 1$ with residue $\text{Res}(\zeta, 1) = 1$. As the function $f(s) = 1 - 2^{-s}$ is an entire function, it follows that $\zeta_A(s)$ is a meromorphic function on $\mathbb{C}$, with a single pole at $s = 1$ with residue $\text{Res}(\zeta_A, 1) = 1$. Hence Condition (M1) from Theorem 3 is satisfied. Condition (M2) is also satisfied due to the fact that $\zeta(s)$ satisfies (M2), and that for $\text{Re}(s) \geq -1$ (for instance), $(1 - 2^{-s})$ is bounded. For Condition (M3), observe that $g_A(z) = 2\sum_{n\geq 1} z^{2n+1}$ and hence $\text{Re}(g_A(\exp(-t - 2\pi yi))) - g_A(\exp(-t))$ equals

$$2\sum_{n\geq 1} e^{-t(2n+1)} \cos(2\pi y(2n+1)) - e^{-t(2n+1)} = 2\sum_{n\geq 1} e^{-t(2n+1)} \left(\cos(2\pi y(2n+1)) - 1\right).$$

This term can be easily bounded using that for each y the term $(\cos(2\pi y(2n+1)) - 1)$ is negative (or 0), and using trivial lower bound estimates for

$$\left|\sum_{n\geq 1} e^{-t(2n+1)} \cos(2\pi y(2n+1)) - e^{-t(2n+1)}\right|.$$

Thus, Condition (M3) holds as well. The computation of the constants is obtained by using the relation of $\zeta_A(s)$ with $\zeta(s)$, joint with the fact that $\zeta(0) = \frac{-1}{2}$, $\zeta(2) = \frac{\pi^2}{6}$, and $\zeta'(0) = -\frac{1}{2}\log(2\pi)$. □

5 Enumeration of link-diagrams

In this section, we enumerate different kinds of connected link-diagrams (from now on, we refer to them plainly as link-diagrams). We start with link-diagrams without local conditions (Subsection 5.1), in which we show the main decomposition technique used in the forthcoming subsections. Later, as application of our method, we obtain combinatorial formulas for minimal link-diagrams (Subsection 5.2) and link-diagrams arising from the unknot (Subsection 5.3).

In all this section, we will deal with *rooted* planar maps. In Section 8, we will apply an unrooting argument to get asymptotic estimates for the unrooted maps. To that end, we will use the counting formulas deduced in the following subsections.

5.1 Enumeration of K_4-minor-free link-diagrams

We denote by $\mathcal{M}$ the class of K_4-minor-free link-diagrams, with size being the number of edges. Enumerating $\mathcal{M}$ is equivalent to enumerate K_4-minor-free 4-regular maps. We first give a combinatorial decomposition for the rooted version of $\mathcal{M}$, denoted by $\overrightarrow{\mathcal{M}}$, where the root-edge has size zero (recall the definition of rooted maps in Section 2).

The decomposition is done by adapting the construction of 4-regular graphs in [27]. Let us mention that the main simplification compared to [27] is that in our situation we do not obtain 3-connected components. For completeness, and because this decomposition is critical to understand the following subsections, we write it in full and recall all the needed definitions and arguments.

For a map $R \in \overrightarrow{\mathcal{M}}$, where st is the root-edge with initial and final vertex s and t, respectively, we write R^- for the map $R - st$ (this is what is said a *network* in map enumeration). Consider the following subclasses of $\mathcal{M}$:

1. $\mathcal{L}$ corresponds to maps $R \in \overrightarrow{\mathcal{M}}$, where $s = t$ (*loop composition*).
2. $\mathcal{S}$ corresponds to maps $R \in \overrightarrow{\mathcal{M}}$, where R^- is connected and has a bridge (*series composition*).
3. $\mathcal{P}$ corresponds to maps $R \in \overrightarrow{\mathcal{M}}$, where R^- is 2-edge-connected and either $R^- - \{s,t\}$ is disconnected or s,t are connected with at least three edges in M (*parallel composition*).
4. $\mathcal{F}$ corresponds to maps $R \in \overrightarrow{\mathcal{M}}$, where R^- is 2-edge-connected, $R^- - \{s,t\}$ is connected, and s,t are connected with exactly 2 edges in M.

See Figure 4 for a pictorial representation of these classes. When an object is dotted, its existence is optional; otherwise, it is mandatory.

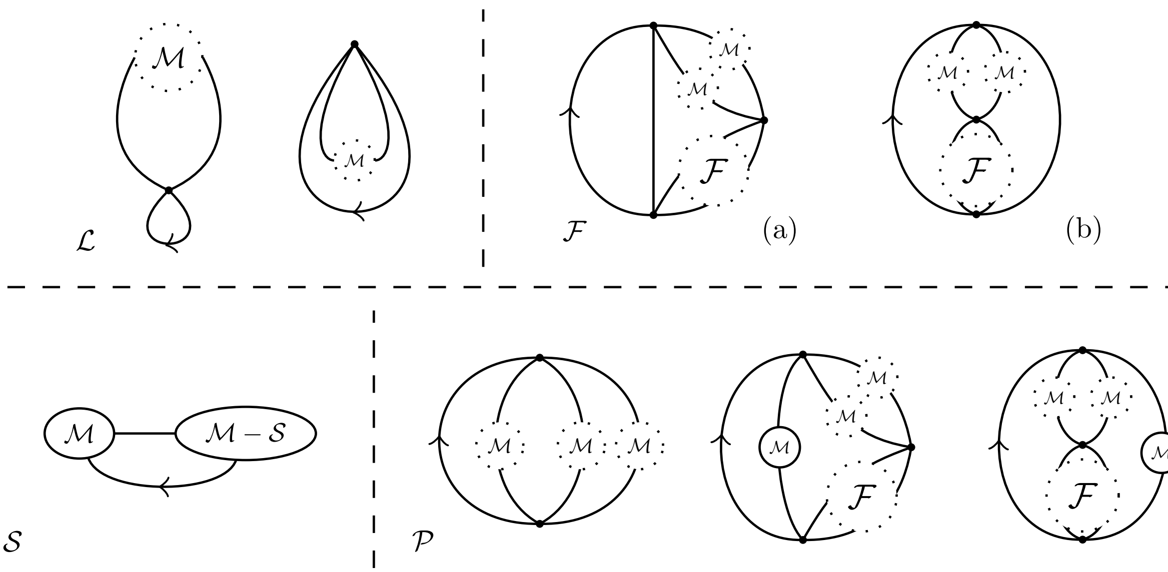

Figure 4: The decomposition of rooted 4-regular maps.

We denote by $\overrightarrow{M}(z)$ the generating function of rooted K_4-minor-free link-diagrams, where z marks vertices. Similarly, we denote by L, S, P, and F the corresponding generating functions of the classes $\mathcal{L}, \mathcal{S}, \mathcal{P}, \mathcal{F}$. The following Proposition relates all these generating functions in a system of equations:

Proposition 3. *The generating function of rooted K_4-minor-free link-diagrams, $\overrightarrow{M}(z) := \overrightarrow{M}$, satisfies the following system of equations:*

$$\begin{aligned}
\overrightarrow{M} &= L + S + P + F \\
L &= 2z^2\overrightarrow{M} + 2z \\
S &= z(M - S)\overrightarrow{M} \\
P &= z^3(1 + z\overrightarrow{M})^3 + zF\overrightarrow{M} \\
F &= (z + z^2\overrightarrow{M})^2\big(F + 2z(z + z^2\overrightarrow{M})^2\big)
\end{aligned}$$

Proof. (Stepping on the proof of [27, Lemma 5.1]) The classes $\mathcal{L}, \mathcal{S}, \mathcal{P}, \mathcal{F}$ are by definition disjoint. Moreover, it is not possible that $R^- - \{s,t\}$ is connected and s,t are connected with exactly one edge in M, since this would force the existence of a K_4 minor. Also, it is not possible that R^- is disconnected, since this would imply that st is a bridge and would contradict the 4-regularity. Hence, $\overrightarrow{\mathcal{M}}$ is partitioned as $\overrightarrow{\mathcal{M}} = \mathcal{L} \cup \mathcal{S} \cup \mathcal{P} \cup \mathcal{F}$.

For $\mathcal{L}$, there are two different maps of size one. Any other map in $R \in \mathcal{L}$ can be decomposed uniquely into a map of size one and another map J that is pasted on its non-root edge in the canonical way with respect to the root edge. The latter means that the non-root edge st is subdivided into $svv't$, vv' is removed, and the endpoints of J's root-edge are identified with v, v', respecting the orientation induced by R's root-edge. For $R \in \mathcal{S}$ notice that R is uniquely decomposed into two maps R_1 and R_2, where $R_1 \notin \mathcal{S}$ and $R_2 \in \overrightarrow{\mathcal{M}}$. The bridge between them is counted by z.

If $R \in \mathcal{F}$, it can be decomposed uniquely to a single edge and a series of double edges, on each of which there may be pasted other maps from $\overrightarrow{\mathcal{M}}$, in a canonical way. The factor $(z + z^2\overrightarrow{M})^2$ corresponds to the first pair of edges and the factor $F + 2z(z+z^2\overrightarrow{M})^2$ to the rest of the double-edges. In the latter, the factor z corresponds to the single edge and the factor 2 counts its two possible positions with respect to the root-edge.

For $\mathcal{P}$ there are two cases: Either each of the connected components in R^- is connected with one edge to each of the s,t, or there is a component connected with two edges to each of the s,t. In the second case we have an object in $\mathcal{F}$, where now an object from $\overrightarrow{\mathcal{M}}$ is pasted on the single edge. ☐

We can now analyze this system of equations by means of asymptotic techniques.

Theorem 7. *The class of rooted K_4-free link-diagrams $\overrightarrow{\mathcal{M}}$ grows asymptotically as:*

$$[z^n]\overrightarrow{M}(z) \sim \frac{c}{\Gamma(-1/2)} n^{-3/2} 2^n \gamma^{-n},$$

where $\gamma \approx 0.31184$, $c \approx 1.52265$, and $\frac{2}{\gamma} \approx 6.41337$.

Proof. By Proposition 3, $\overrightarrow{M}(z)$ satisfies a polynomial system of equations. By algebraic elimination we obtain the following polynomial $P_{\overrightarrow{M}}(x,z)$, which satisfies that $p_{\overrightarrow{M}}(\overrightarrow{M}(z), z) = 0$:

$$\begin{aligned} p_{\overrightarrow{M}}(x,z) \;=\;& x^6 z^{11} + 6\,x^5 z^{10} + 15\,x^4 z^9 - x^4 z^7 + 20\,x^3 z^8 - 4\,x^3 z^6 + 15\,x^2 z^7 + x^3 z^4 - \\ & -6\,x^2 z^5 + 6\,x z^6 + 4\,x^2 z^3 - 4\,x z^4 + z^5 + 5\,x z^2 - z^3 - x + 2\,z. \end{aligned}$$

The singularities of $\overrightarrow{M}(z)$ belong to the exceptional set of $p_{\overrightarrow{M}}(x,z)$, defined as the common roots of $\{p_{\overrightarrow{M}}(x,z), \frac{\partial}{\partial x} p_{\overrightarrow{M}}(x,z)\}$ (see [18, Chapter VII. 7.1.]). Again, by algebraic elimination we obtain that the exceptional set satisfies the following equation:

$$30976\,z^8 - 33152\,z^6 + 10904\,z^4 - 1053\,z^2 + 27 = 0.$$

Now, one can decide which roots are the smallest singularities of $M(z)$ by computing the Puiseux series of $p_{\overrightarrow{M}}$ on each of these points (see again [18, Chapter VII. 7.1.]). In our case, $\overrightarrow{M}(z)$ has a unique positive singularity and admits a Puiseux expansion of the form

$$a + b\sqrt{1 - z/\gamma} + O(1 - z/\gamma), \quad \text{where} \quad \gamma \approx 0.31184, \quad b \approx -4.88269.$$

Then, by the principles of singularity analysis, $[z^n]\overrightarrow{M}(z)$ grows asymptotically as $\frac{bn^{-3/2}}{\Gamma(-1/2)}\gamma^{-n}$. $z\overrightarrow{M}(z)$ does so as well, with $b' = \gamma b \approx -1.52265$. Finally, a factor 2^n accounts for all the possible undercrossings and overcrossings. $\square$

The first terms of the series are as follows:

$$\overrightarrow{M} = 2\,z^2 + 9\,z^4 + 54\,z^6 + 374\,z^8 + 2816\,z^{10} + 22384\,z^{12} + 184820\,z^{14} + 1569598\,z^{16} + 13622592\,z^{18}.$$

5.2 Minimal diagrams

Let $\mathcal{M}_1$ be the class of all minimal link-diagrams in $\mathcal{M}$, counting by the number of edges. Let $\overrightarrow{\mathcal{M}_1}$ be the rooted version of $\mathcal{M}_1$ and $\overrightarrow{\mathcal{M}_1}(z)$ the corresponding generating function.

In order to assure minimality in the maps, one must remember the crossing pattern of each map that is being pasted in the construction of Proposition 3. To this end, we first define the subclasses $\mathcal{M}_1, \mathcal{S}_1, \mathcal{P}_1, \mathcal{F}_1$ of the classes $\mathcal{M}, \mathcal{S}, \mathcal{P}, \mathcal{F}$, such that each contains all minimal diagrams of its respective superclass. We then partition each of these classes into four smaller, $\mathcal{M}_i^j, \mathcal{S}_i^j, \mathcal{P}_i^j, \mathcal{F}_i^j$, where $i, j \in \{-, +\}$. The subscript indicates whether the tail of the root-edge is overcrossing or not and, accordingly, the superscript indicates whether the head of the root-edge is overcrossing or not. See Figure 5 for all possible root-edge types, depending on the overcrossing pattern. We denote by $M_i^j, S_i^j, P_i^j, F_i^j$, where $i, j \in \{-, +\}$, the corresponding generating functions (as in the previous Section, all rooted classes are counted by the number of edges, excluding the root-edge).

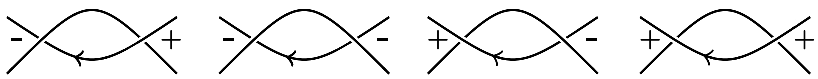

Figure 5: The possible root-edge types.

Proposition 4. *The generating function of minimal, rooted, K_4-minor-free link-diagrams, $\overrightarrow{M_1}(z) := \overrightarrow{M_1}$, satisfies the following polynomial system of equations:*

$$\begin{aligned}
\overrightarrow{M_1} &= M_+^+ + M_+^- + M_-^+ + M_-^- \\
M_-^+ &= S_-^+ + P_-^+ + F_-^+ \\
M_+^- &= S_+^- + P_+^- + F_+^- \\
M_+^+ &= S_+^+ \\
M_-^- &= S_-^- \\
P_-^+ &= z^3(1 + z\overrightarrow{M_1})^3 + zF_-^+\overrightarrow{M_1} \\
P_+^- &= z^3(1 + z\overrightarrow{M_1})^3 + zF_+^-\overrightarrow{M_1}
\end{aligned}
\qquad
\begin{aligned}
S_+^+ &= z(M_-^+ + M_+^+)(M_+^+ + M_+^- - S_+^+ - S_+^-) \\
S_-^- &= z(M_-^- + M_+^-)(M_-^- + M_-^+ - S_-^- - S_-^+) \\
S_-^+ &= z(M_+^+ + M_-^+)(M_-^- + M_-^+ - S_-^- - S_-^+) \\
S_+^- &= z(M_-^- + M_+^-)(M_+^+ + M_+^- - S_+^+ - S_+^-) \\
F_-^+ &= (z + z^2\overrightarrow{M_1})^2(F_-^+ + 2z(z + z^2\overrightarrow{M_1})^2) \\
F_+^- &= (z + z^2\overrightarrow{M_1})^2(F_+^- + 2z(z + z^2\overrightarrow{M_1})^2)
\end{aligned}$$

Proof. The defining equations for $\overrightarrow{M_1}, M_-^+, M_+^-$ are straightforward. Observe that $\mathcal{P}_+^+$, $\mathcal{P}_-^-$, $\mathcal{F}_+^+$, $\mathcal{F}_-^-$ are empty, since they can be transformed to diagrams with less crossings with a Type II move (in the case of parallel networks, this could require first an ambient isotopy of the link that allows this move). Hence, the defining equations for M_+^+ and M_-^- are also justified. For the classes S_i^j, recall that a series map is decomposed into another map R_1 and a non-series map R_2, joined together with an edge. Then, the head of its root-edge must agree (with respect to overcrossing or undercrossing) with the head of R_1, and the tail of the its root-edge must agree with the tail of R_2. This suffices for minimality, since the crossing number in our link classes is additive. In fact, whenever a pasting of an object occurs in this construction, it corresponds to a connected sum and, by additivity, minimality is not affected. Thus follow the equations for P_+^- and P_-^+.

Recall that each object in $\mathcal{F}$, thus also in $\mathcal{F}_i^j$, is associated to a series of double edges. The corresponding crossings are now uniquely defined by i, j and they must alternate. Suppose $R_2 \in \mathcal{F}_1$ is used in the recursive construction of $R_1 \in \mathcal{F}_i^j$. Then, there are two case for R_2. Either the crossings of its root edge agree with i, j and it is of the type (b) in Figure 4, or the crossings of its root edge do not agree with i, j and it is of the type (a). Otherwise, the diagram can be simplified by a Type II move (after a suitable ambient isotopy of the link that allows this move). Observe that each such series of k double edges constitutes a minimal link-diagram of the torus link $T(2, k)$, thus cannot be further simplified. Since the sum of the objects in these two cases is equal to $(\mathcal{F}_i^j)_n$ for every n, we can use the GF F_i^j. Finally, the objects pasted on the double edges contribute to the crossing number additively. $\square$

Theorem 8. *The class of K_4-free minimal rooted link-diagrams, $\overrightarrow{\mathcal{M}_1}$, grows asymptotically as:*

$$[z^n]\overrightarrow{\mathcal{M}_1}(z) \sim \frac{c}{\Gamma(-1/2)} n^{-3/2} \rho^{-n},$$

where $\rho \approx 0.41456$, $\rho^{-1} \approx 2.41214$, and $c \approx 0.81415$.

Proof. The proof is similar to the one in Theorem 7. Here, the defining polynomial of $\overrightarrow{\mathcal{M}}(z)$ is

$$\begin{aligned} p_{\overrightarrow{\mathcal{M}}} \;\; := \;\; & 2\,x^6 z^{11} + 12\,x^5 z^{10} + 30\,x^4 z^9 + 2\,x^4 z^7 + 40\,x^3 z^8 + 8\,x^3 z^6 + 30\,x^2 z^7 + x^3 z^4 + \\ & + 12\,x^2 z^5 + 12\,x z^6 + 2\,x^2 z^3 + 8\,x z^4 + 2\,z^5 + x z^2 + 2\,z^3 - x \end{aligned}$$

and after algebraic elimination between $p_{\overrightarrow{M}}, \partial_x p_{\overrightarrow{M}}$ we obtain that the exceptional set satisfies the equation

$$320000\,z^8 + 148800\,z^6 + 5103\,z^4 - 7054\,z^2 + 27.$$

Finally, $\overrightarrow{M_1}(z)$ has a unique minimal singularity of square type, giving the expansion

$$a + b\sqrt{1 - z/\rho} + O(1 - z/\rho), \quad \text{where} \quad \rho \approx 0.41456, \quad b \approx -1.96385.$$

We then multiply b by ρ to obtain the corresponding constant ≈ 0.81415 for $z\overrightarrow{\mathcal{M}}(z)$, which has a same kind of singular expansion. The asymptotic result follows from the transfer properties of singularity analysis. $\square$

The first terms of the series are as follows:

$$\overrightarrow{M_1} = 2\,z^4 + 4\,z^6 + 20\,z^8 + 84\,z^{10} + 372\,z^{12} + 1796\,z^{14} + 8516\,z^{16} + 42340\,z^{18} + 211332\,z^{20}.$$

5.3 Link-diagrams of the unknot

Let $\overrightarrow{\mathcal{M}_2}$ be the class of rooted link-diagrams of the unknot and $\mathcal{M}_2$ the corresponding unrooted class. We define the subclasses $\mathcal{L}_2, \mathcal{S}_2, \mathcal{P}_2, \mathcal{F}_2$ of $\mathcal{L}, \mathcal{S}, \mathcal{P}, \mathcal{F}$, such that each contains all diagrams of the unknot in its respective superclass. We then partition each of these classes into four smaller combinatorial classes, which we denote with the same symbols as in the previous subsection for simplicity, i.e., $\mathcal{M}_i^j, \mathcal{L}_i^j, \mathcal{S}_i^j, \mathcal{P}_i^j, \mathcal{F}_i^j$, where $i, j \in \{-,+\}$. We denote by $M_i^j, L_i^j, S_i^j, P_i^j, F_i^j$, where $i, j \in \{-,+\}$, the corresponding generating functions (keeping the same convention, all rooted classes are counted by the number of edges, excluding the root-edge).

We also need the classes $\mathcal{T}_r$, $r \in \{1,3\}$, that correspond to all possible ways to split a sequence of $2n+r$ points into two groups of size n and $n+r$. Then,

$$T_r(z) = z^r \sum_{n\geq 0} \binom{2n+r}{n} z^{2n}$$

Observe that

$$\begin{aligned}
\sum_{n\geq 0} \binom{2n+r}{n} z^n &= 1 + \sum_{n\geq 1} \left(\binom{2n+r-1}{n-1} + \binom{2n+r-1}{n} \right) z^n \\
&= \sum_{n\geq 1} \binom{2n+r-1}{n-1} z^n + \sum_{n\geq 0} \binom{2n+r-1}{n} z^n \\
&= \frac{2z}{r} \sum_{n\geq 0} \frac{nr}{2n+r} \binom{2n+r}{n} z^{n-1} + \sum_{n\geq 0} \frac{r}{2n+r} \binom{2n+r}{n} z^n \\
&= \frac{2z}{r} [B_2(z)^r]' + B_2(z)^r,
\end{aligned}$$

where $B_k(z)$ are known as *generalised binomial series* and it holds that

$$B_t(z)^r = \sum_{n\geq 0} \binom{tn+r}{n} \frac{r}{tn+r} z^n$$

(see [21, Ch. 5.4]). In particular, $B_2(z)$ is the series of the Catalan numbers, i.e., $B_2(z) = \frac{1-\sqrt{1-4z}}{2z}$. Then,

$$T_r(z) = z^r \left[\frac{2z}{r} [B_2(z)^r]' + B_2(z)^r \right] \Bigg|_{z=z^2}$$

Proposition 5. *The generating function of rooted, K_4-minor-free link-diagrams of the unknot, $\overrightarrow{M_2}(z) := \overrightarrow{M_2}$, satisfies the following polynomial system of equations:*

$$
\begin{array}{rclrcl}
\overrightarrow{M_2} & = & M_+^+ + M_-^+ + M_+^- + M_-^- & L^+ & = & 2z + z^2\overrightarrow{M_2} \\
M_+^+ & = & S_+^+ + P_+^+ + F_+^+ & L^- & = & 2z + z^2\overrightarrow{M_2} \\
M_-^+ & = & S_-^+ + P_-^+ + F_-^+ + L^+ & S_+^+ & = & z(M_-^+ + M_+^+)(M_+^+ + M_+^- - S_+^+ - S_+^-) \\
M_+^- & = & S_+^- + P_+^- + F_+^- + L^- & S_-^+ & = & z(M_+^+ + M_-^+)(M_-^- + M_-^+ - S_-^- - S_-^+) \\
M_-^- & = & S_-^- + P_-^- + F_-^- & S_+^- & = & z(M_-^- + M_+^-)(M_+^+ + M_+^- - S_+^+ - S_+^-) \\
 & & & S_-^- & = & z(M_-^- + M_+^-)(M_-^- + M_-^+ - S_-^- - S_-^+) \\
P_+^+ & = & zF_+^+\overrightarrow{M_2} & F_-^+ & = & 4z(z + z^2\overrightarrow{M_2})^2 T_1((z + z^2\overrightarrow{M_2})^2) \\
P_-^+ & = & zF_-^+\overrightarrow{M_2} & F_+^- & = & 4z(z + z^2\overrightarrow{M_2})^2 T_1((z + z^2\overrightarrow{M_2})^2) \\
P_+^- & = & zF_+^-\overrightarrow{M_2} & F_+^+ & = & 2z(z + z^2\overrightarrow{M_2})^2\big(T_1((z + z^2\overrightarrow{M_2})^2) + T_3((z + z^2\overrightarrow{M_2})^2)\big) \\
P_-^- & = & zF_-^-\overrightarrow{M_2} & F_-^- & = & 2z(z + z^2\overrightarrow{M_2})^2\big(T_1((z + z^2\overrightarrow{M_2})^2) + T_3((z + z^2\overrightarrow{M_2})^2)\big)
\end{array}
$$

Proof. The defining equations for $\overrightarrow{M_2}, M_j^i, S_j^i$ can be justified in the same way as in Proposition 3 and Proposition 4. Let $R \in \mathcal{P}_i^j$. If $R^- - \{s,t\}$ is empty or disconnected, then R has two components and does not represent the unknot. Thus, $P_i^j = zF_i^j\overrightarrow{M_2}$ (recall the construction in Proposition 3).

The equations for the classes $\mathcal{F}_i^j$ need to change substantially. Let $R \in \mathcal{F}_i^j$. Recall that R is decomposed into the root-edge e_1, an edge e_2 parallel to it (either to the left or to the right face that is adjacent to e_1), and a chain of double edges, C, on which other objects of $\overrightarrow{\mathcal{M}_2}$ may be pasted.

Traversing the knot in the direction of the root-edge, we can associate on each point of the knot a tangent arrow. Consider the corresponding arrows on the link-diagram and notice that each crossing point has two such arrows. Moreover, there is a unique face of the diagram that is adjacent to both arrows, let us call it F. On each crossing point, we associate a plus sign or a minus sign, according to whether the left or the right arrow is overcrossing, with respect to the joint direction of the two arrow heads on F. Observe that if two consecutive vertices on C bear different signs, the diagram can be reduced by a move of Type II. Hence, in order to obtain a trivial knot, the sum s of the signs should be either $+1$ or -1: otherwise, either we have more than one components, or the diagram corresponds to a non-trivial knot. The sum of the signs of the root vertices can be 0 or ± 2.

We use the generating functions T_1 and T_3 that encode all the possibilities, so that the total sum of the signs on C equals ± 1. In particular, when the sum on the root vertices is zero, we use the GF T_1 twice, since we distinguish on whether the total sum is -1 or 1. When the sum of the root vertices is 2 or -2, we use the functions T_1, T_3 that account likewise for both cases. We substitute each atom on T_1, T_3 by a double edge that may or not have other objects pasted, and obtain $T_1((z+z^2\overrightarrow{M_2})^2), T_3((z+z^2\overrightarrow{M_2})^2)$. The extra factor $(z+z^2\overrightarrow{M_2})^2$ accounts for the first double edge after the head of the root.

□

Theorem 9. *The class of K_4-free link-diagrams of the unknot, $\overrightarrow{\mathcal{M}_2}$, grows asymptotically as:*

$$[z^n]\overrightarrow{M_2}(z) \sim \frac{cn^{-3/2}}{\Gamma(-1/2)}\rho^{-n},$$

where $\rho \approx 0.23188$, $\rho^{-1} \approx 4.31246$, $c \approx 2.19020$, and Γ is the classical Gamma function.

Proof. The proof is similar to the one in Theorem 7. We first obtain the defining polynomial of $\overrightarrow{M_2}(z)$ with respect to z, x, t_1,t_3, denoted by $p_{\overrightarrow{M_2}}$, by means of algebraic elimination:

$$\begin{aligned} p_{\overrightarrow{M_2}} := \; & 12\,x^4z^7t_1 + 4\,x^4z^7t_3 + 48\,x^3z^6t_1 + 16\,x^3z^6t_3 + 72\,x^2z^5t_1 + \\ & +24\,x^2z^5t_3 + 48\,x^4t_1 + 16\,x^4t_3 + 4\,x^2z^3 + 12\,z^3t_1 + 4\,z^3t_3 + \\ & +8\,x^2 - x + 4\,z. \end{aligned}$$

Then, we substitute t_1 and t_3 by the closed forms of $T_1(z)$ and $T_3(z)$, where z substituted by $(z + z^2x)^2$. We solve the characteristic system of the resulting equation, $\{p_{\overrightarrow{M_2}}, \partial_x p_{\overrightarrow{M_2}}\}$, and obtain $z \approx 0.23188$ and $x \approx 3.43141$. Then, by a Theorem of Drmota [15, Proposition 1, Lemma 1], $\overrightarrow{\mathcal{M}_2}(z)$ has a unique positive singularity, on which we obtain a square root singular expansion

$$a + b\sqrt{1 - z/\rho} + O(1 - z/\rho), \quad \text{where} \quad \rho \approx 0.23188, \quad b \approx -9.44515.$$

We multiply b by ρ to obtain the corresponding constant ≈ 2.19020 for $z\overrightarrow{M_2}(z)$. The asymptotic estimate follows by the transfer properties of singularity analysis. □

The first terms of the series are as follows:

$$\overrightarrow{M_1} = 4\,z^2 + 32\,z^4 + 332\,z^6 + 3968\,z^8 + 51688\,z^{10} + 712416\,z^{12} + 10214604\,z^{14} + 150776064\,z^{16}.$$

6 The unrooting argument

In this section, we develop an unrooting argument for the families of maps we have enumerated, using results from [29] and [3].

Recall that a map in a certain class is *symmetric* if it has a non-trivial (graph) automorphism. In particular, in this section we prove that the proportion of objects in $\mathcal{M}_n$, $(\mathcal{M}_1)_n$, $(\mathcal{M}_2)_n$ that are symmetric is exponentially small. From this result, we can deduce asymptotic estimates for $|\mathcal{M}_n|$, $|(\mathcal{M}_1)_n|$, $|(\mathcal{M}_2)_n|$.

We recall some definitions from [29] and adapt them to our maps: a *submap* R' of a map R is a map such that R' is a set of faces of R and their boundary edges and vertices, and R' is continuous. Since our maps have an extra information about the crossings on each vertex, we consider that a submap has this information too, i.e., R' contains all the semi-edges of its vertices in R and each overcrossing pair is marked. We call $R \setminus R'$ the map obtained after removing the faces of R'. All the semi-edges are preserved in $R \setminus R'$, as well. We say that two maps are *glued* when we identify their outer faces, which have the same degree, in a compatible way to the existing crossings.

A map R' is called *outercyclic* if the edges of its unbounded face induce a cycle with no repeated vertices. It is called *free* if in all its occurrences as a submap in maps R, all maps resulting by gluing R' to $R \setminus R'$, on the face where R' initially belonged, belong in the same class of maps as R, $\mathcal{R}$. It is called *ubiquitous* if for small enough $c > 0$, there is a positive $d < 1$ such that the proportion of objects in $\mathcal{R}$ that do not contain at least cn copies of R' is at most d^n for large enough n. Two maps have disjoint appearances when they do not share a face.

By the main Theorem in [29], in order to prove that symmetric maps in a map class $\mathcal{R}$ are exponentially few, it is enough to find an outercyclic map R with the following properties:

($u1$) R has no reflection symmetry,

($u2$) the appearances of R as submap are pairwise disjoint in $\overrightarrow{\mathcal{R}}$,

($u3$) R is free and ubiquitous in $\overrightarrow{\mathcal{R}}$.

We observe in the proof of the Theorem that one can relax the requirement of freeness and demand a number of different gluings that is at least two. For the classes $\mathcal{M}, \mathcal{M}_1$, we will use the map R_A in Figure 6. For the class $\mathcal{M}_2$, we will use the map R_B. Notice that they both have size equal to 39.

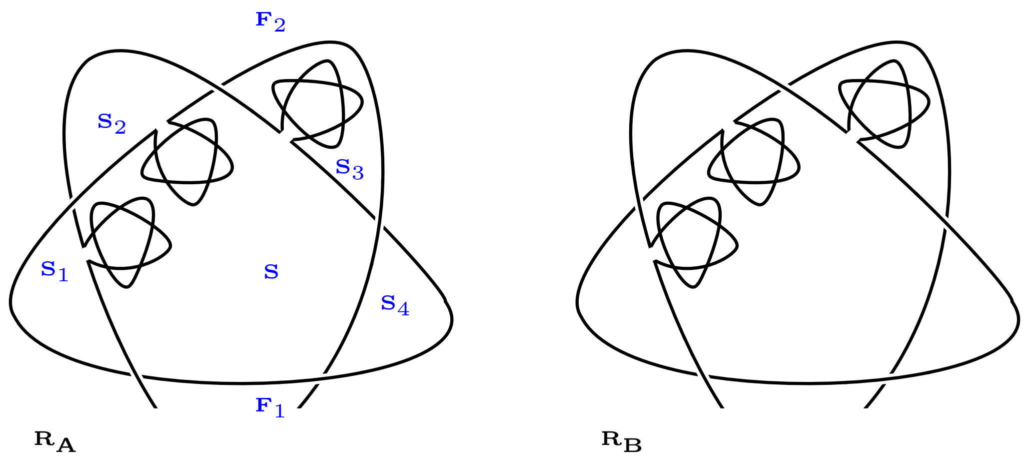

Figure 6: An asymmetric and ubiquitous submap in $\overrightarrow{\mathcal{M}}$ and $\overrightarrow{\mathcal{M}_1}$ (R_A), and in $\overrightarrow{\mathcal{M}_2}$ (R_B). The missing crossings follow the pattern of the existing crossings.

We start by showing property ($u2$):

Lemma 6. *The appearances of R_A as submap in $\overrightarrow{\mathcal{M}}$ or $\overrightarrow{\mathcal{M}_1}$ are disjoint. The same holds for the appearances of R_B in $\overrightarrow{\mathcal{M}_2}$.*

Proof. It is enough to prove the claim for R_A and $\mathcal{M}$. We denote by S, S_i, F_i faces and neighbouring faces of R_A, respectively, as shown in Figure 6. Suppose $R \in \mathcal{M}$ and two submaps of R, called R_A^1, R_A^2, such that $R_A^1, R_A^2 \cong R_A$. Then, there is a homeomorphism of the sphere ϕ, such that the faces of R_A^1 are mapped to faces of R_A^2 of the same degree. Observe that if $S \in R_A^2$, then $R_A^2 = R_A^1$. So, we can suppose that $S \notin R_A^2$.

Suppose that R_A^1, R_A^2 share a face. Then, at least one of the remaining border faces S_i must belong to R_A^2. S_1 and S_2 are the only ones with degree 4 in R, while S_3 is the only face with degree 7. Notice that S_4 cannot be mapped to the inner faces of degree 2. Hence, in any case these faces are mapped either to some other border face or to themselves and they are all adjacent to S. In that case, the only face that could be mapped to S is F_2. This is not possible, since F_2 is adjacent to at least two faces of degree 4 with only one edge to each, while S does not, regardless of F_1. □

The following lemma is a direct consequence of [3, Cor. 1], but we mention the main argument for the sake of a cleaner exposition.

Lemma 7. *There is a $c > 0$ small enough such that the proportion of objects in $\overrightarrow{\mathcal{M}_n}$, $(\overrightarrow{\mathcal{M}_1})_n$, (resp. $(\overrightarrow{\mathcal{M}_2})_n$) that do not contain at least cn copies of R_A (resp. R_B) is exponentially small.*

Proof. Let $\mathcal{H}$ be the class of objects in $\mathcal{Z} \times \overrightarrow{\mathcal{M}}$ that contain less than cn copies of R_A. Let $\mathcal{G}$ be the class of objects made by elements in $\mathcal{H}$, where on each non-root edge one choses to paste or not M_A in the canonical way. Then, $\mathcal{G} \subseteq \mathcal{M}$ and the copies of R_A are always disjoint, by Lemma 6. Let $G \in \mathcal{G}$. By the recursive decomposition of $\mathcal{M}$, one can detect all submaps M_A that were possibly added to a map G' to create G. Conversely, given these submaps, G' is uniquely defined. Hence, one can apply the counting argument in [3, Cor. 1] and obtain $r(H)/r(M) \geq (1 + r(G)^{39})^{1/2}(c/e)^c > 1$ for c sufficiently small. The cases of $\overrightarrow{\mathcal{M}_1}$ and $\overrightarrow{\mathcal{M}_2}$ follow accordingly. □

Theorem 10. *The proportion of objects in $\mathcal{M}_n$, $(\mathcal{M}_1)_n$, $(\mathcal{M}_2)_n$ that is symmetric is exponentially small.*

Proof. The maps R_A, R_B have no reflective symmetry in the plane. Also, they always appear disjointly in their respective classes, by Lemma 6. There are exactly two distinct gluings of them, the identity and the reflection around the vertical axis, because of the 4-regularity. The maps are ubiquitous by Lemma 7. Hence, the Theorem follows, by [29]. □

We can now get the final asymptotic result for unroote link-diagrams in the previous studied families:

Corollary 2. *The class of connected K_4-free link-diagrams $\mathcal{M}$ satisfies:*

$$[z^n]M(z) \sim \frac{1}{2n}\frac{cn^{-3/2}}{\Gamma(-1/2)}\rho^{-n}2^n, \ \rho \approx 0.31184, \, c \approx 1.52265$$

The class of connected K_4-free minimal link-diagrams, $\mathcal{M}_1$, and the class of K_4-free link-diagrams of the unknot, $\mathcal{M}_2$, satisfy:

$$[z^n]M_1(z) \sim \frac{1}{2n}\frac{c_1 n^{-3/2}}{\Gamma(-1/2)}\rho_1^{-n}, \ \rho_1 \approx 0.41456, \, c_1 \approx 0.81415,$$

$$[z^n]M_2(z) \sim \frac{1}{2n}\frac{c_2 n^{-3/2}}{\Gamma(-1/2)}\rho_2^{-n}, \ \rho_2 \approx 0.23188, \, c_2 \approx 2.19020.$$

Acknowledgement: The last author wish to thank Anastasios Sidiropoulos for early discussions on knots, during MFO Workshop 1542 on Computational Geometric and Algebraic Topology, that inspired this research.

References

[1] C. Adams, T. Crawford, B. DeMeo, M. Landry, A. T. Lin, M. Montee, S. Park, S. Venkatesh, and F. Yhee. Knot projections with a single multi-crossing. *Journal of Knot Theory and Its Ramifications*, 24(03):1550011, 2015.

[2] G. E. Andrews. *The Theory of Partitions.* In *Enciclopedia of of Mathematics and its applications.* Addison-Wesley Publishing Company, 1976.

[3] E. A. Bender, Z.-C. Gao, and L. B. Richmond. Submaps of maps. i. general 0–1 laws. *Journal of Combinatorial Theory, Series B*, 55(1):104–117, 1992.

[4] F. Bergeron, G. Labelle, and P. Leroux. *Combinatorial species and tree-like structures*, volume 67. Cambridge University Press, 1998.

[5] M. Bodirsky, O. Giménez, M. Kang, and M. Noy. Enumeration and limit laws for series–parallel graphs. *European Journal of Combinatorics*, 28(8):2091–2105, 2007.

[6] H. L. Bodlaender. A partial k-arboretum of graphs with bounded treewidth. *Theoretical computer science*, 209(1-2):1–45, 1998.

[7] A. Brandstadt, J. P. Spinrad, et al. *Graph classes: a survey.* Siam, 1999.

[8] G. R. Buck. Random knots and energy: Elementary considerations. *Journal of Knot Theory and its Ramifications*, 3(03):355–363, 1994.

[9] J. A. Calvo. *Physical and numerical models in knot theory: including applications to the life sciences*, volume 36. World Scientific, 2005.

[10] H.-C. Chang and J. Erickson. Untangling planar curves. *Discrete & Computational Geometry*, 58(4):889–920, 2017.

[11] H. Chapman. Asymptotic laws for random knot diagrams. *Journal of Physics A: Mathematical and Theoretical*, 50(22):225001, 2017.

[12] P. R. Cromwell. *Knots and links.* Cambridge University Press, 2004.

[13] Y. Diao. The additivity of crossing numbers. *Journal of knot theory and its Ramifications*, 13(07):857–866, 2004.

[14] Y. Diao, N. Pippenger, and D. W. Sumners. On random knots. In *Random knotting and linking*, pages 187–197. World Scientific, 1994.

[15] M. Drmota. Systems of functional equations. *Random Structures and Algorithms*, 10(1-2):103–124, 1997.

[16] M. Drmota. *Random trees: an interplay between combinatorics and probability.* Springer Science & Business Media, 2009.

[17] C. Even-Zohar, J. Hass, N. Linial, and T. Nowik. Invariants of random knots and links. *Discrete & Computational Geometry*, 56(2):274–314, 2016.

[18] P. Flajolet and R. Sedgewick. *Analytic combinatorics.* Cambridge University press, 2009.

[19] F. Harary. *Graph theory.* Addison-Wesley, 1969.

[20] A. Kawauchi. *A survey of knot theory.* Birkhäuser, 2012.

[21] D. E. Knuth, R. L. Graham, and O. Patashnik. Concrete mathematics. *Adison Wesley*, 1989.

[22] S. Kunz-Jacques and G. Schaeffer. The asymptotic number of prime alternating links. In *Formal Power Series and Algebraic Combinatorics-FPSAC'2001*, pages 10–p. Arizona State University, 2001.

[23] C. Medina, J. Ramírez-Alfonsín, and G. Salazar. On the number of unknot diagrams. *arXiv preprint arXiv:1710.06470*, 2017.

[24] G. Meinardus. Asymptotische Aussagen über Partitionen. *Math. Z.*, 59:388–398, 1954.

[25] W. Menasco and M. Thistlethwaite. *Handbook of knot theory*. Elsevier, 2005.

[26] K. Murasugi. *Knot theory and its applications*. Springer Science & Business Media, 2007.

[27] M. Noy, C. Requilé, and J. Rué. Enumeration of labeled 4-regular planar graphs. Submitted. Available on-line at `arXiv:1709.04678`.

[28] K. Reidemeister. Elementare begründung der knotentheorie. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 5(1):24–32, Dec 1927.

[29] L. B. Richmond and N. C. Wormald. Almost all maps are asymmetric. *Journal of Combinatorial Theory, Series B*, 63(1):1–7, 1995.

[30] C. Sundberg and M. Thistlethwaite. The rate of growth of the number of prime alternating links and tangles. *Pacific journal of mathematics*, 182(2):329–358, 1998.