Equivariant classification of b^m-symplectic surfaces

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Abstract—Inspired by Arnold's classification of local Poisson structures [AR1] in the plane using the hierarchy of singularities of smooth functions, we consider the problem of global classification of Poisson structures on surfaces. Among the wide class of Poisson structures, we consider the class of b^m-Poisson structures which can be also visualized using differential forms with singularities as b^m-symplectic structures. In this paper we extend the classification scheme in [S] for b^m-symplectic surfaces to the equivariant setting. When the compact group is the group of deck-transformations of an orientable covering, this yields the classification of these objects for non-orientable surfaces. The paper also includes recipes to construct b^m-symplectic structures on surfaces. Feasibility of such constructions depends on orientability and on the colorability of an associated graph. The desingularization technique in [GMW] is revisited for surfaces and the compatibility with this classification scheme is analyzed in detail.

1. Introduction

The topological classification of closed surfaces is determined by orientability and genus. The geometrical classification of symplectic surfaces was established by Moser [M]. Moser proved that any two closed symplectic surfaces with symplectic forms lying in the same de Rham cohomology class are equivalent in the sense that there exists a diffeomorphism taking one symplectic structure to the other.

Poisson structures show up naturally in this scenario as a generalization of symplectic structures where the non-degeneracy condition is relaxed. The first examples of Poisson structures are symplectic manifolds and manifolds with the zero Poisson structure. In-between these two extreme examples there is a wide variety of Poisson manifolds. Poisson structures with dense symplectic leaves and controlled singularities have been the object of study of several recent articles (see for instance [GMP], [GMP2], [GMPS], [GMW], [GL], [MO2]). The classification of these objects in dimension 2 is given by a suitable cohomological condition. In the extreme case of symplectic manifolds this cohomology coincides with de Rham cohomology and, as explained above, this classification was already known to Moser [M]. For orientable b-symplectic manifolds, the classification can be formulated in terms of b-cohomology (see [GMP]) which reinterprets former classification invariants by Radko [R].

It is possible to consider other classes of Poisson manifolds with simple singularities like b^m-symplectic manifolds [S] or more general singularities [MS] by relaxing the transversality condition for b-symplectic manifolds. These structures have relevance in mechanics: most of the examples are found naturally in the study of celestial mechanics (see [KM], [KMS], [DKM]). In the same way, b^m-symplectic structures are classified in terms of b^m-cohomology [S]. The recent papers [MO2] and [FMM] have renewed interest in the non-orientable counterparts of these structures.

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In this article we focus our attention on b^m-symplectic surfaces and we prove an equivariant Radko-type [R] classification theorem (Theorem 8). Some new applications of similar ideas to the context of Nambu structures can be found in [MP]. When the group considered is the group of deck-transformations of an orientable covering, this yields the classification of non-orientable compact surfaces in the b^m-case (Corollary 1). Such a classification was missing in the literature. We also examine the compatibility of this classification scheme and a desingularization procedure described in [GMW] proving that, for surfaces, equivalent b^{2k}-symplectic structures get mapped to equivalent symplectic structures under the desingularization procedure (Theorem 13) though non-equivalent b^{2k}-symplectic structures might get mapped to equivalent symplectic structures via this procedure (Remark 11 and Example 3).

Organization of the paper: In Section 2 we include the necessary preliminaries of b^m-structures. In Section 3 we present some examples of b^m-symplectic surfaces on orientable and non-orientable manifolds. In Section 4 we present an equivariant b^m-Moser theorem and use it to classify non-orientable b^m-symplectic surfaces. In Section 5 we give explicit constructions of b^m-symplectic structures with prescribed critical set depending on orientability and colorability of an associated graph. In Section 6 we analyze the behavior of this classification under the desingularization procedure described in [GMW].

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2. Preliminaries

Let M be a smooth manifold, a **Poisson structure** on M is a bilinear map $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ which is skew symmetric and satisfies both the Jacobi identity and the Leibniz rule. It is possible to express $\{f,g\}$ in terms of a bivector field via the following equality $\{f,g\} = \Pi(df \wedge dg)$ with Π a section of $\Lambda^2(TM)$. Π is the associated **Poisson bivector**. We will use indistinctively the terminology of Poisson structure when referring to the bracket or the Poisson bivector.

A b-*Poisson bivector field* on a manifold M^{2n} is a Poisson bivector such that the map

$$F : M \to \bigwedge^{2n} TM : p \mapsto (\Pi(p))^n \tag{2.1}$$

is transverse to the zero section. Then, a pair (M, Π) is called a b-**Poisson manifold** and the vanishing set Z of F is called the **critical hypersurface**. Observe that Z is an embedded hypersurface.

This class of Poisson structures was studied by Radko [R] in dimension two and considered in numerous papers in the last years: [GMP], [GMP2], [GMPS], [GMW], [MO2] and [GLPR] among others.

2.1. b-Poisson manifolds

Next, we recall Radko's classification theorem and the cohomological re-statement presented in [GMP2].

In what follows, (M, Π) will be a closed smooth surface with a b-Poisson structure on it, and Z its critical hypersurface.

Let h be the distance function to Z as in [MO2][1].

Definition 1. *The **Liouville volume of** (M, Π) is the following limit:* $V(\Pi) := \lim_{\epsilon \to 0} \int_{|h| > \epsilon} \omega^n$[2].

The previous limit exists and it is independent of the choice of the defining function h of Z (see [R] for the proof).

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[1] Notice the difference with [R] where h is assumed to be a global defining function.
[2] For surfaces $n = 1$.

Definition 2. *For any (M, Π) oriented Poisson manifold, let Ω be a volume form on it, and let u_f denote the Hamiltonian vector field of a smooth function $f : M \to \mathbb{R}$. The **modular vector field** X^Ω is the derivation defined as follows:*

$$f \mapsto \frac{\mathcal{L}_{u_f} \Omega}{\Omega}.$$

Definition 3. *Given γ a connected component of the critical set $Z(\Pi)$ of a closed b-Poisson manifold (M, Π), the **modular period** of Π around γ is defined as:*

$$T_\gamma(\Pi) := period\ of\ X^\Omega|_\gamma.$$

Remark 1. The modular vector field X^Ω of the b-Poisson manifold (M, Z) does not depend on the choice of Ω because for different choices for volume form the difference of modular vector fields is a Hamiltonian vector field. Observe that this Hamiltonian vector field vanishes on the critical set as Π vanishes there too.

Definition 4. *Let $\mathcal{M}_n(M) = \mathcal{C}_n(M)/\sim$ where $\mathcal{C}_n(M)$ is the space of disjoint oriented curves and $\sim$ identifies two sets of curves if there is an orientation-preserving diffeomophism mapping the first one to the second one and preserving the orientations of the curves.*

The following theorem classifies b-symplectic structures on surfaces using these invariants:

Theorem 1 (Radko [R]). *Consider two b-Poisson structures Π, Π' on a closed orientable surface M. Denote its critical hypersurfaces by Z and Z'. These two b-Poisson structures are globally equivalent (there exists a global orientation preserving diffeomorphism sending Π to Π') if and only if the following coincide:*

1. *the equivalence classes of $[Z]$ and $[Z'] \in \mathcal{M}_n(M)$,*

2. *their modular periods around the connected components of Z and Z',*

3. *their Liouville volume.*

An appropriate formalism to deal with these structures was introduced in [GMP].

Definition 5. *A **b-manifold**[3] is a pair (M, Z) of a manifold and an embedded hypersurface.*

In this way the concept of b-manifold previously introduced by Melrose is generalized.

Definition 6. *A b-**vector field** on a b-manifold (M, Z) is a vector field tangent to the hypersurface Z at every point $p \in Z$.*

Definition 7. *A b-**map** from (M, Z) to (M', Z') is a smooth map $\phi : M \to M'$ such that $\phi^{-1}(Z') = Z$ and ϕ is transverse to Z'.*

Observe that if x is a local defining function for Z and $(x, x_1, \ldots, x_{n-1})$ are local coordinates in a neighborhood of $p \in Z$ then the $C^\infty(M)$-module of b-vector fields has the following local basis

$$\{x \frac{\partial}{\partial x}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}\}. \tag{2.2}$$

In contrast with [GMP], in this paper we are not requiring the existence of a global defining function for Z and orientability of M but we require the existence of a defining function in a neighborhood of each point of Z. By relaxing this condition the normal bundle of Z need not be trivial.

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[3]The 'b' of b-manifolds stands for 'boundary', as initially considered by Melrose (Chapter 2 of [Me]) for the study of pseudo-differential operators on manifolds with boundary.

Given (M, Z) a b-manifold, [GMP] shows that there exists a vector bundle, denoted by ^bTM whose smooth sections are b-vector fields. This bundle is called the b-**tangent bundle** of (M, Z).

The b-**cotangent bundle** $^bT^*M$ is defined using duality. A b-**form** is a section of the b-cotangent bundle. Around a point $p \in Z$ the $C^\infty(M)$-module of these sections has the following local basis:

$$\{\frac{1}{x}dx, dx_1, \ldots, dx_{n-1}\}. \tag{2.3}$$

In the same way we define a b-**form of degree** k to be a section of the bundle $\bigwedge^k(^bT^*M)$, the set of these forms is denoted $^b\Omega^k(M)$. Denoting by f the distance function[4] to the critical hypersurface Z, we may write the following decomposition as in [GMP] for any $\omega \in^b \Omega^k(M)$:

$$\omega = \alpha \wedge \frac{df}{f} + \beta, \text{ with } \alpha \in \Omega^{k-1}(M) \text{ and } \beta \in \Omega^k(M). \tag{2.4}$$

This decomposition allows to extend the differential of the de Rham complex d to $^b\Omega(M)$ by setting $d\omega = d\alpha \wedge \frac{df}{f} + d\beta$. The associated cohomology is called b-**cohomology** and it is denoted by $^bH^*(M)$.

Definition 8. *A b-**symplectic** form on a b-manifold (M^{2n}, Z) is defined as a non-degenerate closed b-form of degree 2 (i.e., ω_p is of maximal rank as an element of $\Lambda^2(^bT_p^*M)$ for all $p \in M$).*

The notion of b-symplectic forms is dual to the notion of b-Poisson structures. The advantage of using forms is that symplectic tools can be 'easily' exported.

Radko's classification theorem [R] can be translated into this language. This translation was already formulated in [GMP]:

Theorem 2 (Radko's theorem in b-cohomological language, [GMP2]). *Let S be a closed orientable surface and let ω_0 and ω_1 be two b-symplectic forms on (S, Z) defining the same b-cohomology class (i.e.,$[\omega_0] = [\omega_1]$). Then there exists a diffeomorphism $\phi : S \to S$ such that $\phi^*\omega_1 = \omega_0$.*

2.2. b^m-Symplectic manifolds

By relaxing the transversality condition allowing higher order singularities ([AR1] and [AR2]) we may consider other symplectic structures with singularities as done by Scott [S] with b^m-symplectic structures. Let m be a positive integer a b^m-**manifold** is a b-manifold (M, Z) together with a b^m-tangent bundle attached to it. The b^m-tangent bundle is (by Serre-Swan theorem [Sw]) a vector bundle, $^{b^m}TM$ whose sections are given by,

$$\Gamma(^{b^m}TM) = \{v \in \Gamma(TM) : v(x) \quad \text{vanishes to order } m \text{ at } Z\},$$

where x is a defining function for the critical set Z in a neighborhood of each connected component of Z and can be defined as $x : M \setminus Z \to (0, \infty), x \in C^\infty(M)$ such that:

1. $x(p) = d(p)$ a distance function from p to Z for $p : d(p) \le 1/2$

2. $x(p) = 1$ on $M \setminus \{p \in M \text{ such that } d(p) < 1\}$.[5]

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[4]Originally in [GMP] f stands for a global function, but for non-orientable manifolds we may use the distance function instead.
[5]Then a b^m-manifold will be a triple (M, Z, x), but for the sake of simplicity we refer to it as a pair (M, Z) and we tacitly assume the function x is fixed.

(This definition of x allows us to extend the construction in [S] to the non-orientable case as in [MO2].) We may define the notion of a b^m-map as a map in this category (see [S]). The sections of this bundle as referred to as b^m-**vector fields** and their flows define b^m-maps. In local coordinates the sections of the b^m-tangent bundle are generated by:

$$\{x^m \frac{\partial}{\partial x}, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{n-1}}\}. \tag{2.5}$$

Proceeding *mutatis mutandis* as in the b-case one defines the b^m-cotangent bundle ($^{b^m}T^*M$), the b^m-de Rham complex and the b^m-symplectic structures.

A **Laurent Series** of a closed b^m-form ω is a decomposition of ω in a tubular neighborhood U of Z of the form

$$\omega = \frac{dx}{x^m} \wedge \left(\sum_{i=0}^{m-1} \pi^*(\alpha_i) x^i\right) + \beta \tag{2.6}$$

with $\pi : U \to Z$ the projection of the tubular neighborhood onto Z, α_i a closed smooth de Rham form on Z and β a de Rham form on M.

In [S] it is proved that in a neighborhood of Z, every closed b^m-form ω can be written in a Laurent form of type (2.6) having fixed a (semi)local defining function.

b^m-Cohomology is related to de Rham cohomology via the following theorem:

Theorem 3 (b^m-Mazzeo-Melrose, [S]). *Let (M, Z) be a b^m-manifold, then:*

$$^{b^m}H^p(M) \cong H^p(M) \oplus (H^{p-1}(Z))^m. \tag{2.7}$$

The isomorphism constructed in the proof of the theorem above is non-canonical (see [S]).

The Moser path method can be generalized to b^m-symplectic structures:

Theorem 4 (Moser path method, [S]). *Let ω_0, ω_1 be two b^m-symplectic forms defining the same b^m-cohomology class $[\omega_0] = [\omega_1]$ on (M^{2n}, Z) with M^{2n} closed and orientable then there exist a b^m-symplectomorphism $\varphi : (M^{2n}, Z) \longrightarrow (M^{2n}, Z)$ such that $\varphi^*(\omega_1) = \omega_0$.*

An outstanding consequence of Moser path method is a global classification of closed orientable b^m-symplectic surfaces à la Radko in terms of b^m-cohomology classes.

Theorem 5 (Classification of closed orientable b^m-surfaces, [S]). *Let ω_0 and ω_1 be two b^m-symplectic forms on a closed orientable connected b^m-surface (S, Z). Then, the following conditions are equivalent:*

1. *their b^m-cohomology classes coincide $[\omega_0] = [\omega_1]$,*

2. *the surfaces are globally b^m-symplectomorphic,*

3. *the Liouville volumes of ω_0 and ω_1 and the numbers*

$$\int_\gamma \alpha_i$$

   for all connected components $\gamma \subseteq Z$ and all $1 \leq i \leq m$ coincide (where α_i are the one-forms appearing in the Laurent decomposition of the two b^m-forms of degree 2, ω_0 and ω_1).

A relative version of Moser path method is proved in [GMW] as a corollary we obtain the following local description of a b^m-symplectic manifold:

Theorem 6 (b^m-Darboux theorem, [GMW]). *Let ω be a b^m-symplectic form on (M, Z) and $p \in Z$. Then we can find a coordinate chart $(U, x_1, y_1, \ldots, x_n, y_n)$ centered at p such that on U the hypersurface Z is locally defined by $x_1 = 0$ and*

$$\omega = \frac{dx_1}{x_1^m} \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

Remark 2. For the sake of simplicity sometimes we will omit describing Z and we will talk directly about b^m-symplectic structures on manifolds M implicitly assuming that Z is the vanishing locus of Π^n where Π is the Poisson vector field dual to the b^m-symplectic form.

2.3. Desingularizing b^m-Poisson manifolds

In [GMW] Guillemin, Miranda and Weitsman presented a desingularization procedure for b^m-symplectic manifolds proving that we may associate a family of folded symplectic or symplectic forms to a given b^m-symplectic structure depending on the parity of m. Namely,

Theorem 7 (Guillemin-Miranda-Weitsman, [GMW]). *Let ω be a b^m-symplectic structure on a closed orientable manifold M and let Z be its critical hypersurface.*

- *If $m = 2k$, there exists a family of symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighborhood of Z and for which the family of bivector fields $(\omega_\epsilon)^{-1}$ converges in the C^{2k-1}-topology to the Poisson structure ω^{-1} as $\epsilon \to 0$.*

- *If $m = 2k + 1$, there exists a family of folded symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighborhood of Z.*

As a consequence of Theorem 7, any closed orientable manifold that supports a b^{2k}-symplectic structure necessarily supports a symplectic structure.

In [GMW] explicit formulae are given for even and odd cases. Let us refer here to the even dimensional case as these formulae will be used later on.

Let us briefly recall how the desingularization is defined and the main result in [GMW]. Recall that we can express the b^{2k}-form as:

$$\omega = \frac{dx}{x^{2k}} \wedge \left(\sum_{i=0}^{2k-1} x^i \alpha_i \right) + \beta. \tag{2.8}$$

This expression holds on a ϵ-tubular neighborhood of a given connected component of Z. This expression comes directly from equation 2.6, to see a proof of this result we refer to [S].

Definition 9. *Let (S, Z, x), be a b^{2k}-manifold, where S is a closed orientable manifold and let ω be a b^{2k}-symplectic form. Consider the decomposition given by the expression (2.8) on an ϵ-tubular neighborhood U_ϵ of a connected component of Z.*

Let $f \in \mathcal{C}^\infty(\mathbb{R})$ be an odd smooth function satisfying $f'(x) > 0$ for all $x \in [-1, 1]$ and satisfying outside that

$$f(x) = \begin{cases} \frac{-1}{(2k-1)x^{2k-1}} - 2 & for \quad x < -1, \\ \frac{-1}{(2k-1)x^{2k-1}} + 2 & for \quad x > 1. \end{cases} \tag{2.9}$$

Let $f_\epsilon(x)$ be defined as $\epsilon^{-(2k-1)} f(x/\epsilon)$.

*The f_ϵ-**desingularization** ω_ϵ is a form that is defined on U_ϵ by the following expression:*

$$\omega_\epsilon = df_\epsilon \wedge \left(\sum_{i=0}^{2k-1} x^i \alpha_i \right) + \beta.$$

This desingularization procedure is also known as **deblogging** in the literature.

Remark 3. Though there are infinitely many choices for f, we will assume that we choose one, and assume it fixed through the rest of the discussion. It would be interesting to discuss the existence of an isotopy of forms under a change of function f.

Remark 4. Because ω_ϵ can be trivially extended to the whole S in such a way that it agrees with ω (see [GMW]) outside a neighborhood of Z, we can talk about the f_ϵ-desingularization of ω as a form on S.
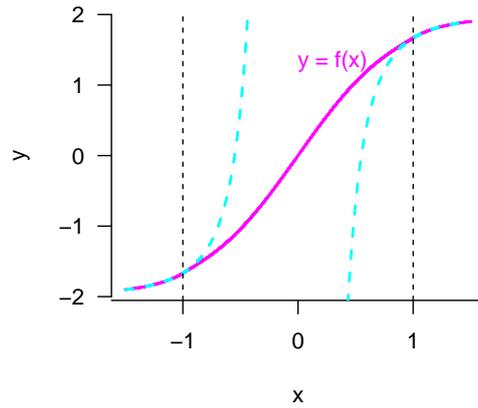
Fig. 1. Smooth and odd extension of f inside the interval $[-1, 1]$ such that $f' > 0$.

3. Toy examples of b^m-symplectic surfaces

In this section we describe some examples of orientable and non-orientable b^m-symplectic surfaces.

1. **A b^m-symplectic structure on the sphere:** Consider the sphere $S^2 \subset \mathbb{R}^3$ with the equator $Z = \{(x_1, x_2, x_3) \in S^2 | x_3 = 0\}$ as critical set. Let $h = x_3$ the height function. Then (S^2, Z, h) is a b^m-manifold for any m. Consider $\omega = \frac{1}{h^m} dh \wedge d\theta$, where θ stands for the angular coordinate. This form is a b^m-symplectic form.

2. **A b^m-symplectic structure on the torus:** Consider $\mathbb{T}^2$ as quotient of the plane ($\mathbb{T}^2 = \{(x, y) \in (\mathbb{R}/\mathbb{Z})^2\}$). Let $\omega = \frac{1}{(\sin 2\pi x)^m} dx \wedge dy$ be a b^m-symplectic structure on $\mathbb{R}^2$. The action of $\mathbb{Z}^2$ leaves this form invariant and therefore this b^m-form descends to the quotient. Observe that this b^m-form defines $Z = \{x \in \{0, \frac{1}{2}\}\}$.
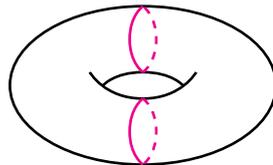

Fig. 2. Example: b^m-symplectic structure in the torus.

3. **A b^{2k+1}-symplectic structure on the projective space:** Consider Example (1) and consider the quotient of S^2 by the antipodal action. Because this action leaves the critical set invariant, the b^m-manifold structure (S^2, Z) descends to $(\mathbb{RP}^2, \hat{Z})$ and gives a non-orientable b^m manifold. $\hat{Z}$ is the equator modulo the antipodal identification (thus diffeomorphic to $\mathbb{RP}^1 \cong S^1$). Moreover a neighborhood of Z is diffeomorphic to the Moebius band. Observe that ω is invariant by the action for $m = 2k + 1$, yielding a b^{2k+1}-symplectic form in $\mathbb{RP}^2$ with critical set $\hat{Z}$.

4. **A b^{2k+1}-symplectic structure on a Klein bottle:** Consider the torus with the structure given in Example (2).
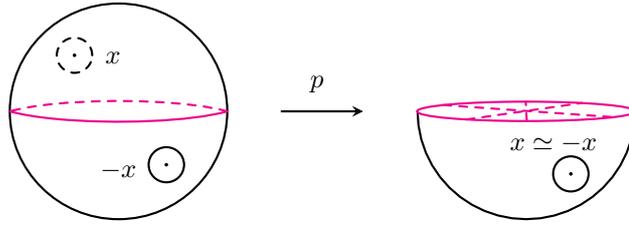
Fig. 3. The b^{2k+1}-symplectic structure on the sphere S^2 that vanishes at the equator induces a b^{2k+1}-symplectic structure on the projective space $\mathbb{RP}^2$.

Consider $\mathbb{Z}/2\mathbb{Z}$ acting on $(x,y) \in \mathbb{T}^2$ by -Id $\cdot (x,y) = (1-x, y+1/2 \pmod 1)$. The orbit space by this action is the Klein bottle $\mathbb{K}$. Then the b^m-manifold $(\mathbb{R}^2, Z)$, descends to $(\mathbb{K}, \hat{Z})$ where $\hat{Z}$ is the quotient of Z by the action. It is easy to see that $\hat{Z} \cong S^1 \sqcup S^1$. Moreover the tubular neighborhood of each S^1 is isomorphic to the Moebius band.

Thus, the b^m-symplectic form $\omega = \frac{1}{(\sin 2\pi x)^m} dx \wedge dy$ induces a b^m-symplectic structure in T if ω is invariant by the action of the group. It is easy to check that ω is invariant if and only if m is odd, in this case one obtains a b^m-symplectic structure in the Klein bottle.

Remark 5. The previous examples only exhibit b^{2k+1}-symplectic structures on non-orientable surfaces. As we will see in Section 5 only orientable surfaces can admit b^{2k}-symplectic structures.

4. Equivariant classification of b^m-surfaces. Non-orientable b^m-surfaces.

In this section we give an equivariant Moser theorem for b^m-symplectic manifolds. This yields the classification of non-orientable surfaces thus extending the classification theorems of Radko and Scott for orientable surfaces (see Theorem 9).

We now extend the classification result (Theorem 4) for manifolds admitting a compact Lie group action leaving the b^m-symplectic structure invariant. The following theorem is a simple consequence of applying the equivariant tools to the Moser path method. We include the detail of the proof for the sake of completeness. Other applications of the equivariant tools in b-geometry can be found in [GMPS] and [GMW2].

Theorem 8 (Equivariant b^m-Moser theorem for surfaces). *Suppose that S is a closed surface, let Z be a union of non-intersecting embedded curves. Consider the b^m-manifold given by (S, Z). Fix $m \in \mathbb{N}$ and let ω_0 and ω_1 be two b^m-symplectic structures on (S, Z) which are invariant under the action of a compact Lie group $\rho : G \times (S, Z) \longrightarrow (S, Z)$ and defining the same b^m-cohomology class, $[\omega_0] = [\omega_1]$. Then, there exists an equivariant b^m-diffeomorphism $\xi_1 : (S, Z) \to (S, Z)$, such that $\xi_1^* \omega_1 = \omega_0$.*

Proof 1. Denote by ρ_g the induced diffeomorphism for a fixed $g \in G$. i.e., $\rho_g(x) := \rho(g, x)$. Consider the linear family of b^m-forms $\omega_s = s\omega_1 + (1-s)\omega_0$. Since the manifold is a surface, the fact that ω_0 and ω_1 are non-degenerate b^m-forms and of the same sign on $S \setminus Z^6$ (thus non-vanishing sections of $\Lambda^2(^bT^*(S))$) implies that the linear path is non-degenerate too. We will prove that there exists a family $\xi_s : S \to S$, with $0 \le s \le 1$ such that

$$\xi_s^* \omega_s = \omega_0. \tag{4.1}$$

We want to construct ξ_1 as the time-1 flow of a time-dependent Hamiltonian vector field X_s (as in the standard Moser trick).

Since the cohomology class of both forms coincide, $\omega_1 - \omega_0 = d\alpha$ for α a b^m-form of degree 1.

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[6]This is a consequence of Mazzeo-Melrose theorem and the determination of the Liouville volume from it.

Therefore Moser's equation reads

$$\iota_{X_s}\omega_s = -\alpha. \tag{4.2}$$

This equation has a unique solution X_s because ω_s is b^m-symplectic and therefore it is non-degenerate. X_s depends smoothly on s because ω_s depends smoothly on s and ω_s defines a non-degenerate pairing between b^m-vector fields and b^m-forms. Furthermore, the solution is a b^m-vector field but this solution may not be compatible with the group action. From this solution we will construct an equivariant solution such that its s-dependent flow gives an equivariant diffeomorphism.

Since the forms ω_0 and ω_1 are G-invariant, we can find a G-invariant primitive $\tilde{\alpha}$ by averaging with respect to a Haar measure the initial form α: $\tilde{\alpha} = \int_G \rho_g^*(\alpha)d\mu$ and therefore the invariant vector field, $X_s^G = \int_G \rho_{g*}(X_s)d\mu$ is a solution of the equation,

$$\iota_{X_s^G}\omega_s = -\tilde{\alpha}. \tag{4.3}$$

We can get an equivariant ξ_s^G by integrating X_s^G. This family satisfies $\xi_t^{G*}\omega_t = \omega_0$ and it is equivariant. Also observe that since X_t^G is a b^m-vector field ξ_t^G is a b^m-diffeomorphism of (S, Z).

A non-orientable manifold can be seen as a pair $(\tilde{M}, \rho)$ with $\tilde{M}$ the orientable covering and ρ the action given by deck-transformations of $\mathbb{Z}/2\mathbb{Z}$ on $\tilde{M}$. This perspective is very convenient for classification issues because equivariant mappings on the orientable covering yield actual diffeomorphisms on the non-orientable manifolds. We adopt this point of view to provide a classification theorem for non-orientable b^m-surfaces in cohomological terms.

Remark 6. Observe that the b^m-Mazzeo-Melrose allows us to determine whether a given b^m-cohomology of degree 2 is non-zero by reducing this question to de Rham cohomology.

Corollary 1. *Let (S, Z) be a non-orientable b^m-manifold where Z is its critical set and let ω_1 and ω_2 be two b^m-symplectic forms such that $[\omega_1] = [\omega_2]$ in b^m-cohomology then (S, ω_1) is equivalent to (S, ω_2), i.e., there exists a b^m-diffeomorphism $\varphi : (S, Z) \to (S, Z)$ such that $\varphi^*\omega_2 = \omega_1$.*

Proof 2. Consider m fixed and assume $[\omega_1] = [\omega_2]$ in b^m-symplectic cohomology. Let $p : \tilde{S} \to S$ be a covering map, and $\tilde{S}$ the orientation double cover. $(\hat{S}, p^{-1}(Z))$ is a b^m-manifold and $p^*(\omega_1)$, $p^*(\omega_2)$ are b^m-symplectic structures on $(\hat{S}, p^{-1}(Z))$. By construction the previous two forms are invariant under the action by deck transformations of $\mathbb{Z}/2\mathbb{Z}$. The defining function of the critical set in the double cover is the pullback by p of the defining function in (S, Z). Since $[\omega_1] = [\omega_2]$ then $[p^*(\omega_1)] = [p^*(\omega_2)]$. By Theorem 8, there exists a $\mathbb{Z}/2\mathbb{Z}$-equivariant b^m-diffeomorhism $\tilde{\varphi} : (\hat{S}, p^{-1}(Z)) \to (\hat{S}, p^{-1}(Z))$ such that $\tilde{\varphi}^*p^*\omega_2 = p^*\omega_1$. Since $\tilde{\varphi}$ is $\mathbb{Z}/2\mathbb{Z}$-equivariant it descends to a map $\varphi : S \to S$. Moreover, because $\tilde{\varphi}(p^{-1}(Z)) = p^{-1}(Z)$, it follows that $\varphi(Z) = Z$. Since $\tilde{\varphi}$ is smooth and p is a submersion, then ϕ is smooth, (the same argument shows φ^{-1} is smooth). It follows that φ is a diffeomorphism and because $\varphi(Z) = Z$ it is also a b^m-diffeomorphism. Moreover, by construction, the condition $\tilde{\varphi}^*p^*\omega_2 = p^*\omega_1$ implies that $\varphi^*\omega_2 = \omega_1$.

A similar equivariant b^m-Moser theorem as theorem 8 holds for higher dimensions. In that case we need to require that there exists a path ω_t of b^m-symplectic structures connecting ω_0 and ω_1, which is not true in general [MD]. The proof follows the same lines as Theorem 8. Such a result was already proved for b-symplectic manifolds (see Theorem 8 in [GMPS]).

Theorem 9 (Equivariant b^m-Moser theorem). *Let (M, Z) be a closed b^m-manifold with m a fixed natural number and let ω_t for $0 \le t \le 1$ be a smooth family of b^m-symplectic forms on (M, Z) such that the b^m-cohomology class $[\omega_t]$ does not depend on t.*

Assume that the family of b^m-symplectic structures is invariant by the action of a compact Lie group G on M, then, there exists a family of equivariant b^m-diffeomorphisms $\phi_t : (M, Z) \to (M, Z)$, with $0 \le t \le 1$ such that $\phi_t^*\omega_t = \omega_0$.

5. Constructions and classification of b^m-symplectic structures

In this section we describe constructions of b^m-symplectic structures on closed surfaces. We obtain topological constraints on b^{2k}-symplectic surfaces as we will prove that the underlying closed surface needs to be orientable, see Theorem 10. Then we characterize the existence of b^m-symplectic forms depending on the parity of m and the colorability of an associated graph. We also obtain a result about non-orientable surfaces: if $m = 2k + 1$ we find necessary and sufficient conditions for a non-orientable b^m-surface to admit a b^m-symplectic structure (see Theorem 12).

5.1. b^{2k}-symplectic orientable surfaces

We start by proving that only orientable surfaces admit b^{2k}-symplectic structures:

Theorem 10. *If a closed surface admits a b^{2k}-symplectic structure then it is orientable.*

Proof 3. The proof consists in building a collar of b^{2k}-Darboux neighborhoods with compatible orientations (the local orientations on the complement of the critical hypersurface induced by the b^{2k}-Darboux charts agree) in a neighborhood of each connected component of Z. Indeed the proof does more, it constructs a symplectic structure in a neighborhood of Z which can be extended to S. This in particular will give an orientation on S.

Let (S, Z) be a closed b^{2k}-surface and let ω denote a b^{2k}-symplectic structure on (S, Z). Pick $(\tilde{S}, \tilde{Z})$ an orientable double cover of the b^{2k}-surface (S, Z), with $\rho : \mathbb{Z}/2\mathbb{Z} \times \tilde{S} \to \tilde{S}$ the action by deck transformations. For each point $q \in \tilde{Z}$, using Theorem 6, we can find a b^{2k}-Darboux neighborhood U_q (by shrinking the neighborhood if necessary) which does not contain other points identified by ρ ($\rho(U_q) \cong U_q$). Let us define $V_q := p(U_q)$, where p is the projection from $\tilde{S}$ to S. With the previous construction we have $\omega|_{U_q} = \frac{1}{x^{2k}} dx \wedge dy$.

Now we can use the desingularization formulae in Theorem 7 and Definition 9 in each U_q (because every U_q is orientable) to obtain a symplectic form $\omega_{\epsilon q}$ on each U_q. All these symplectic structures and hence the orientations on each U_q glue in a compatible manner because the function x is globally defined.

Since $\tilde{Z}$ is compact we can take a finite subcovering for U_q to define a collar U of symplectic and compatible orientations. Furthermore we can assume this covering to be symmetric as we can shrink further the neighborhoods and add the pre-images of all of them -for each U_q the image $\rho(U_q)$ is included in the covering.

Since ρ preserves ω, and the defining function is invariant by ρ, it also preserves the deblogged symplectic forms $\omega_{\epsilon q}$ and the compatible orientations and indeed the deblogged symplectic form descends to S, thus defining a symplectic form and an orientation on $V = p(U)$. Using the standard techniques of Radko [R] the symplectic structures on $V \setminus Z$ can be glued to define a compatible symplectic structure on the whole S. When Z has more than one connected component we may proceed in the same way by isolating collar neighborhoods of each component. Thus proving that S admits a symplectic structure and in particular it is oriented.

5.2. Associated graph of a b-manifold.

Let us introduce some definitions that will be needed in the next subsection.

Definition 10. *Let (M, Z) be a closed b-manifold. The associated graph $\Gamma(M, Z)$ to this b-manifold is defined as follows:*

1. *The set of vertices is in one-to-one correspondence with the connected components $(U_1, \ldots, U_n)$ of $M \setminus Z$.*

2. *Let $(Z_1, \ldots, Z_n)$ be the connected components of Z. Two vertices (v_i, v_j), (represented by (U_i, U_j)) are connected by an edge if and only if for any tubular neighborhood of some Z_k, it intersects both U_i and U_j.*
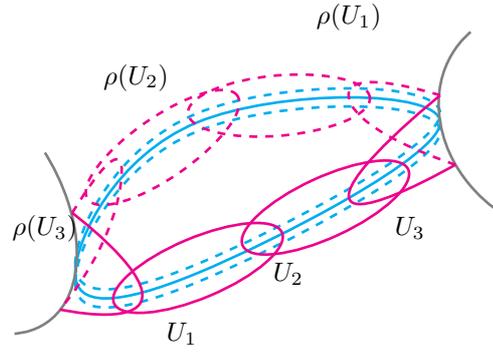
Fig. 4. A collar of compatible neighborhoods.

(a) Example of a non-colorable associated graph.   (b) Example of a colorable associated graph.
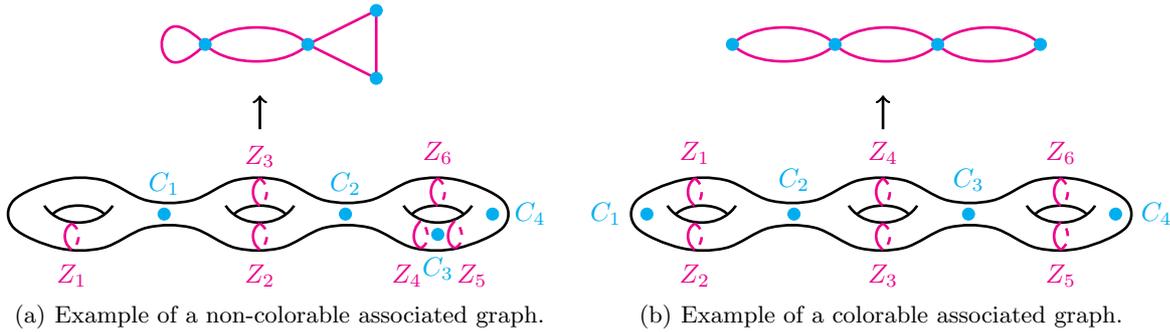
Fig. 5. Examples of associated graphs.

Remark 7. As observed in [S] (Section 3.2), associated to any b^m-manifold there is a canonical b-manifold, obtained by forgetting the distance function. The latter is henceforth said to be underlying the former. Using definition 10, the graph associated to a b^m-manifold is the graph associated to its underlying b-manifold.

Remark 8. Given an oriented closed b^m-manifold (M, Z), a b^m-symplectic structure induces a standard orientation on each connected component of $M \setminus Z$. Comparing this orientation with the fixed one determines a sign that can be attached to each vertex of $\Gamma(M, Z)$. A natural question to ask is whether adjacent vertices possess equal or opposite signs, thus yielding the following notions.

Definition 11. *A* 2-***coloring of a graph*** *is a labeling (with only two labels) of the vertices of the graph such that no two adjacent vertices share the same label.*

Definition 12. *Since not every graph admits a* 2*-coloring, a graph is called* 2-***colorable*** *if it admits a* 2*-coloring.*

5.3. b^{2k+1}-symplectic orientable surfaces

Theorem 11. *Given a b^m-manifold (S, Z) (fixed m) with S closed and orientable, there exists a b^m-symplectic structure whenever:*

1. $m = 2k$,

2. $m = 2k + 1$ *if only if the associated graph $\Gamma(S, Z)$ is 2-colorable.*

Proof 4. (of Theorem 11)
    Let $C_1, \ldots, C_r$ be the connected components of $S \setminus Z$, let $Z_1, \ldots, Z_s$ the connected components of Z and let $\mathcal{U}(Z_1), \ldots, \mathcal{U}(Z_s)$ tubular neighborhoods of the connected components. Moreover, we denote the union of $\mathcal{U}(Z_1), \ldots, \mathcal{U}(Z_s)$ by $\mathcal{U}(Z)$.

We assume there is an orientation defined by some symplectic form in S, that allows us to define a sign criterion.

The proof consists in 3-steps:

1. **Using Weinstein normal form theorem.** Fix $i \in \{1, \ldots, s\}$, where s is the number of connected components. By virtue of Weinstein's normal form theorem for Lagrangian submanifolds (Corollary 6.2 in [We2]) each tubular neighborhood $\mathcal{U}(Z_i)$ can be identified with the zero section of the cotangent bundle of Z_i. Now replace, the cotangent bundle of Z_i by the b^m-cotangent bundle of Z_i[7]. In this way the neighborhood of the zero section of the b^m-cotangent bundle has a b^m-symplectic structure that we will denote $\omega_{\mathcal{U}(Z_i)}$.

2. **Constructing compatible orientation using the graph.** For any $i = 1, \ldots, s$, $\mathcal{U}(Z_i) \setminus Z_i$ has two connected components (as S is orientable); to each such component, we assign the sign of the restriction of the b^m-symplectic form $\omega_{\mathcal{U}(Z_i)}$. Note that the sign does not change for m even, but it changes for k odd. Observe that we can apply Moser's trick to glue two rings that share some C_j (as done in Radko [R] to extend a symplectic form between the two rings) if and only if the sign of the two rings match on this component.

   Now, let us consider separately the odd and even cases:

   (a) For b^{2k} the color of adjacent vertices must coincide. And hence we have no additional constraint on the topology of the graph.

   (b) In the b^{2k+1} case the sign of two adjacent vertices must be different. Then, we have to impose the associated graph to be 2-colorable.

   These two conditions are necessary for the existence of the $b^{2k}-$ and $b^{2k+1}-$forms respectively.

3. **Gluing.** Now we may glue back this neighborhood to $S \setminus \mathcal{U}(Z)$ in such a way that the symplectic structures fit on the boundary (again using the standard techniques used in Radko [R] to extend with a symplectic form between the two rings), using the Moser's path method.

Given a b^{2k+1}-symplectic structure ω on a b^{2k+1}-surface (S, Z) (where S is closed oriented) one can obtain a 2-coloring of the associated graph (by the local expression given by the b^m-Darboux theorem -see Theorem 6-, the sign has to change every time we cross a component Z_i) by assigning to each connected component C_i of $S \setminus Z$ the 'color' $\mathrm{sign}(\int_{C_i} \omega)$.

Remark 9. Observe that any given 2-coloring has to be equivalent to the 2-coloring induced by a b^{2k+1}-symplectic form. This is due to the fact that there exist only 2 possible 2-colorings of a graph (when it is 2-colorable). The difference between the two 2-colorings is only re-labeling of the signs. Then, if the 2-coloring induced by the b^{2k+1}-symplectic form does not correspond to the prescribed 2-coloring, it can be matched by changing the orientation of the underlying manifold.

Another way to construct b^{2k}-structures on a surface is to use decomposition theorem as connected sum of b^{2k}-spheres (1) and b^{2k}-torus (2). The drawback of this construction is that it is harder to adapt having fixed a prescribed Z.

_____

[7]This can be done after fixing a point in Z to define a b^m-structure on Z.

5.4. b^{2k+1}-symplectic non-orientable surfaces

Definition 13. *Let (S, Z) be a closed orientable b^{2k+1}-surface and $\Gamma(S, Z)$ its associated graph. Fix the 2-coloring on $\Gamma(S, Z)$ given by by $\text{sign}(\int_{C_i} \omega)$. We say that a b^{2k+1}-map φ **inverts colors** of the associated graph if $\text{sign}(\int_{C_i} \omega) = -\text{sign}(\int_{\varphi(C_i)} \omega)$.*

Theorem 12. *Let (S, Z) be a closed non-orientable b^{2k+1}-surface. Then, (S, Z) admits a b^{2k+1}-symplectic structure if and only if the following two conditions hold:*

1. *the graph of the covering $(\tilde{S}, \tilde{Z})$, $G(\tilde{S}, \tilde{Z})$ is 2-colorable and*

2. *the non-trivial deck transformation inverts colors of the graph obtained in the covering[8].*

Proof 5. Let us assume the two conditions on the statement of the theorem hold. Apply Theorem 11 to endow the covering $(\tilde{S}, \tilde{Z})$ with a b^{2k+1}-symplectic structure, if the form obtained is invariant by the deck transformations, then it descends to the quotient, thus obtaining a b^{2k+1}-symplectic structure on $(\tilde{S}, \tilde{Z})$ and then we are done.

Now, let us assume that the b^{2k+1}-form ω obtained via theorem 11 is not invariant by deck transformations. We will note the deck transformation induced by $-\text{Id}$ as ρ. Observe that

$$\text{sign}\left(\int_{C_i} \rho^* \omega\right) = -\text{sign}\left(\int_{\rho(C_i)} \omega\right) = +\text{sign}\left(\int_{C_i} \omega\right). \qquad (5.1)$$

The first equality is due to ρ changing orientations and the second one is due to ρ inverting colors. Then the pullback of ω has the same sign as ω, and hence $\omega + \rho^*(\omega)$ is a non-degenerate b^{2k+1}-form that is invariant under the action of ρ, and it descends to the quotient. Hence a b^{2k+1}-symplectic structure is obtained on (S, Z).

The other implication is easier. If we have a b^{2k+1}-symplectic form on (S, Z) we can pull it back to the double cover by means of the projection. Then we obtain a b^{2k+1}-form on the double cover, that induces a 2-coloring defined by the orientations. And since the b^{2k+1}-form on the double cover has to be invariant by the deck transformation, the deck transformation has to invert colors.

Example 1. Let us illustrate what is happening in the previous proof with an example. Take the sphere having the equator as critical set and endowed with the b-symplectic form $\omega = \frac{1}{h} dh \wedge d\theta$. Let us call the north hemisphere C_1 and the south hemisphere C_2, and let ρ be the antipodal map. Look at the coloring of the graph (a path graph of length two):

$$\text{sign}(C_1) = \text{sign}\left(\int_{C_1} \omega\right) = \text{sign}\left(\lim_{\epsilon \to 0} \int_{-\pi}^{\pi} \int_{\epsilon}^{1} \frac{1}{h} dh \wedge d\theta\right) = \text{sign}(\lim_{\epsilon \to 0} -2\pi \log |\epsilon|) \qquad (5.2)$$

which is positive. And

$$\text{sign}(C_2) = \text{sign}\left(\int_{C_2} \omega\right) = \text{sign}\left(\lim_{\epsilon \to 0} \int_{-\pi}^{\pi} \int_{-1}^{-\epsilon} \frac{1}{h} dh \wedge d\theta\right) = \text{sign}(\lim_{\epsilon \to 0} 2\pi \log |\epsilon|) \qquad (5.3)$$

which is negative. Then,

$$\int_{C_1} \rho^* \omega = \lim_{\epsilon \to 0} \int_{-\pi}^{\pi} \int_{\epsilon}^{1} \rho^* \left(\frac{1}{h} dh \wedge d\theta\right) = \lim_{\epsilon \to 0} \int_{-\pi}^{\pi} \int_{\epsilon}^{1} \frac{1}{-h} d(-h) \wedge d\theta = \int_{C_1} \omega. \qquad (5.4)$$

In this case ω was already invariant, but one can observe that if ρ inverts colors then the signs of the form and the pullback are the same.

---

[8]Observe that if a transformation inverts colors for a given coloring, then it inverts colors for all of them (there is only 2 possible 2-colorings when a graph is 2-colorable, and they correspond with the possible choices of orientation).

Example 2. One may ask why the condition of inverting colors is necessary. Next we provide an example where b^{2k+1}-structures can be exhibited on the double cover but cannot be projected to induce a b^{2k+1}-structure on the non-orientable surface.

Consider the Example 3 in Section 3. If one translates the critical set in the h direction in the projective space, the double cover is still the sphere, but instead of Z being the equator, Z consists of different meridians $\{h = h_0\}$ and $\{h = -h_0\}$.

Observe that the associated graph of this double cover is a path graph of length 3, that can be easily 2-colored. Take a generic b^{2k+1}-form $\omega = f(h,\theta)dh \wedge d\theta$, and look at the poles N, S. $\text{sign}(f(N)) = \text{sign}(f(S))$ because of the 2-coloring of the graph. But $\rho^*(\omega)|_N = f(\rho(N))d(-h) \wedge d\theta = -f(S)dh \wedge d\theta$. Then $\text{sign}(\omega) \neq \text{sign}(\rho^*(\omega))$, and hence ω can not be invariant for ρ.

6. Desingularization of closed b^{2k}-symplectic surfaces

In this section we only refer to the desingularization of b^{2k}-symplectic structures, because as we explained in section 2.3 the desingularization procedure, associates folded symplectic structures to b^{2k+1}-symplectic structures instead of symplectic structures. The goal of this section is to compare the classification schemes in the b^m-symplectic and symplectic realms.

The aim of this section is to use the desingularization formulas described in section 2.3 in the case of closed orientable surfaces. The main result of this section (Theorem 13) is that if $[\omega_1] = [\omega_2]$ (where ω_1 and ω_2 are two b^{2k}-symplectic forms on a b^{2k}-surface (S, Z)) in b^{2k}-cohomology, then the desingularization of the two forms also is in the same class $[\omega_{1\epsilon}] = [\omega_{2\epsilon}]$. But the converse is not true: it is possible to find different classes of b^{2k}-forms that go the same class when desingularized.

Next we apply our classification scheme and see how it behaves under the desingularization procedure.

Theorem 13. *Let (S, Z, x), be a b^{2k}-manifold, where S is a closed orientable surface and let ω_1 and ω_2 be two b^{2k}-symplectic forms. Also let $\omega_{1\epsilon}$ and $\omega_{2\epsilon}$ be the f_ϵ-desingularizations of ω_1 and ω_2 respectively. If $[\omega_1] = [\omega_2]$ in b^{2k}-cohomology then $[\omega_{1\epsilon}] = [\omega_{2\epsilon}]$ in de Rham cohomology for any fixed ϵ.*

Before proceeding to proving the theorem we will state some definitions in [S] that are necessary for the proof.

Definition 14. *Let (M, Z, x) be an n-dimensional b^m-manifold. Given ω a b^m-form of top degree with compact support, $\epsilon > 0$ small, and let U_ϵ an ϵ-tubular neighborhood[9], then $\text{vol}_\epsilon(\omega)$ is defined as:*

$$\text{vol}_\epsilon(\omega) = \int_{M \setminus U_\epsilon} \omega$$

Theorem 14 (Theorem 4.3 in [S]). *For a fixed $[\omega]$ the b^m-cohomology class of a b^m-form ω, on a b^m-manifold (M, Z, x) with Z compact , there is a polynomial $P_{[\omega]}(t)$ for which*

$$\lim_{\epsilon \to 0}(P_{[\omega]}(1/\epsilon) - \text{vol}_\epsilon(\omega)) = 0$$

for any ω representing $[\omega]$.

Definition 15. *The polynomial $P_{[\omega]}$ described in Theorem 14 is the **volume polynomial** of $[\omega]$. Its constant term $P_{[\omega]}(0)$ is the **Liouville volume** of $[\omega]$.*

---

[9] the ϵ-tubular neighborhood is defined using the x from the b^m-manifold

Remark 10. Let $U = [-1, 1] \times Z$ be a tubular neighborhood of Z containing U_ε. From the definition of the Liouville volume we may write:

$$P_{[\omega]}(0) = \left(\int_{M \setminus U} \omega + \int_U \beta + \sum_{i=1}^k \left(\frac{-2}{2i-1} \right) \int_Z \alpha_{2i} \right). \tag{6.1}$$

Observe that in the proof of Theorem 5.3 in [S] the term $\int_{M \setminus U} \omega$ does not appear. This is because in [S] M is assumed to be U for the sake of simplicity. Adding this term is the way to extend this expression when $U \subsetneq M$.

Proof 6. (of Theorem 13) Our strategy for the proof is to show that the cohomology class of a desingularization of a b^{2k}-symplectic structure on a closed orientable surface (which is the cohomology class of a symplectic structure and hence it can be encoded by its signed area, i.e. the integral of itself over S), only depends on the b^{2k}-cohomology of the b^{2k}-symplectic structure (which, in its turn, can be encoded by the integral of the forms appearing in its Laurent series and its Liouville volume -Theorem 5-).

In order to compute the class of the desingularization we calculate the integral of the desingularized form over the whole manifold. We are going to proceed in two steps. Firstly we are going to compute the integral of the desingularization inside the ϵ-neighborhood U_ϵ of Z, and then we compute it outside.

Using the expression of ω_ϵ we compute:

$$
\begin{aligned}
\int_{U_\epsilon} \omega_\epsilon &= \int_{U_\epsilon} df_\epsilon \wedge \left(\sum_{i=0}^{2k-1} x^i \alpha_i \right) + \int_{U_\epsilon} \beta \\
&= \epsilon^{-(2k-1)} \int_{U_\epsilon} df(x/\epsilon) \wedge \left(\sum_{i=0}^{2k-1} x^i \alpha_i \right) + \int_{U_\epsilon} \beta \\
&= \epsilon^{-2k} \int_{U_\epsilon} \frac{df(x/\epsilon)}{dx} dx \wedge \left(\sum_{i=0}^{2k-1} x^i \alpha_i \right) + \int_{U_\epsilon} \beta \\
&= \epsilon^{-2k} \sum_{i=0}^{2k-1} \int_{-\epsilon}^{+\epsilon} \frac{df(x/\epsilon)}{dx} x^i dx \int_Z \alpha_i + \int_{U_\epsilon} \beta.
\end{aligned}
$$

Then, because f is an odd function, $df(x/\epsilon)/dx$ is even and hence the integral $\int_{-\epsilon}^{+\epsilon} \frac{df(x/\epsilon)}{dx} x^i dx$ is going to be different from 0 if i is even. Thus,

$$\int_{U_\epsilon} \omega_\epsilon = \epsilon^{-2k} \sum_{i=1}^{k-1} \int_{-\epsilon}^{+\epsilon} \frac{df(x/\epsilon)}{dx} x^{2i} dx \int_Z \alpha_{2i} + \int_{U_\epsilon} \beta.$$

Recall that outside the ϵ-neighborhood the desingularization ω_ϵ coincides with the b^{2k}-symplectic form ω. Moreover, let us define U a tubular neighborhood of Z containing U_ϵ, (assume $U = [-1, 1] \times Z$). Following the computations in [S] we obtain,

$$
\begin{aligned}
\int_{M \setminus U_\epsilon} \omega_\epsilon &= \int_{M \setminus U_\epsilon} \omega \\
&= \int_{M \setminus U} \omega + \int_{U \setminus U_\epsilon} \omega \\
&= \int_{M \setminus U} \omega + \left(\int_{U \setminus U_\epsilon} \beta + \sum_{i=1}^k \frac{-2}{2i-1} \int_Z \alpha_{2i} \right) + \sum_{i=1}^k \left(\frac{2}{2i-1} \int_Z \alpha_{2i} \right) \epsilon^{2i-1}.
\end{aligned}
$$

Now we may add the two terms in order to compute the integral over the whole surface M:

$$\int_M \omega_\epsilon = \epsilon^{-2k} \sum_{i=1}^{k-1} \int_{-\epsilon}^{+\epsilon} \frac{df(x/\epsilon)}{dx} x^{2i} dx \int_Z \alpha_{2i} + \int_{U_\epsilon} \beta$$

$$+ \int_{M\setminus U} \omega + \left(\int_{U\setminus U_\epsilon} \beta + \sum_{i=1}^{k} \frac{-2}{2i-1} \int_Z \alpha_{2i} \right) + \sum_{i=1}^{k} \left(\frac{2}{2i-1} \int_Z \alpha_{2i} \right) \epsilon^{2i-1}$$

$$= \epsilon^{-2k} \sum_{i=1}^{k-1} \int_{-\epsilon}^{+\epsilon} \frac{df(x/\epsilon)}{dx} x^{2i} dx \int_Z \alpha_{2i}$$

$$+ \underbrace{\int_{M\setminus U} \omega + \left(\int_U \beta + \sum_{i=1}^{k} \frac{-2}{2i-1} \int_Z \alpha_{2i} \right)}_{=P_{[\omega]}(0) \text{ by the expression 6.1}} + \sum_{i=1}^{k} \left(\frac{2}{2i-1} \int_Z \alpha_{2i} \right) \epsilon^{2i-1}.$$

In a more compact way:

$$\int_M \omega_\epsilon = \sum_{i=1}^{k-1} a_i(\epsilon) \int_Z \alpha_{2i} + P_{[\omega]}(0) + \sum_{i=1}^{k} b_i(\epsilon) \int_Z \alpha_{2i} \tag{6.2}$$

This integral only depends on the classes $[\alpha_i]$ and the Liouville Volume $P_{[\omega]}(0)$, which are determined by (and determine) the class of $[\omega]$. So, two b^{2k}-forms on the same cohomology class, determine the same cohomology class when desingularized.

Remark 11. This previous theorem asserts that, for b^{2k}-surfaces (S, Z) with S closed and orientable and f and ϵ fixed, equivalent b^{2k}-symplectic structures get mapped to equivalent symplectic structures under the desingularization procedure. Non-equivalent b^{2k}-symplectic structures might get mapped to equivalent symplectic structures via deblogging. It is easy to see that there are different classes of b^{2k}-forms that desingularize to the same class by looking at expression (6.2). We only have terms $[\alpha_i]$ with i even. As a consequence, if two forms differ only in the odd terms, they have the same desingularized forms (assuming the auxiliary function f in the desingularization process is the same). We compute a particular example below.

Example 3. Consider S^2 with coordinates (h, θ). Consider the b^2-manifold given by $(S^2, \{h = 0\}, h)$ with the following two b^2-symplectic structures:

$$\omega_1 = \frac{1}{h^2} dh \wedge d\theta, \qquad \omega_2 = \left(\frac{1}{h} + \frac{1}{h^2} \right) dh \wedge d\theta = \frac{1}{h^2} dh \wedge (h d\theta + d\theta). \tag{6.3}$$

As before, assume f and ϵ fixed. Observe that for ω_1, the forms in the Laurent series are $\alpha_0^1 = d\theta$ and $\alpha_1^1 = 0$, while for ω_2 they are $\alpha_0^2 = d\theta$ and $\alpha_1^2 = d\theta$. Then $\int_Z \alpha_1^1 = 0 \neq \int_Z \alpha_1^2 = 2\pi$, and hence $[\alpha_1^1] \neq [\alpha_1^2]$ and $[\omega_1] \neq [\omega_2]$. The desingularized expressions of those forms are given by:

$$\omega_{1\epsilon} = \begin{cases} \dfrac{df_\epsilon(h)}{dh} dh \wedge d\theta & \text{if } |h| \leq \epsilon, \\ \omega_1 & \text{otherwise,} \end{cases} \quad \text{and} \quad \omega_{2\epsilon} = \begin{cases} \dfrac{df_\epsilon(h)}{dh} dh \wedge (h d\theta + d\theta) & \text{if } |h| \leq \epsilon, \\ \omega_2 & \text{otherwise.} \end{cases} \tag{6.4}$$

Let us compute the classes of $\omega_{1\epsilon}$ and $\omega_{2\epsilon}$.

$$\int_{S^2} \omega_{2\epsilon} = \int_{S^2\setminus U_\epsilon} \omega_2 + \int_{U_\epsilon} \frac{df_\epsilon(h)}{dh} (h d\theta + d\theta)$$

$$= \int_{S^2\setminus U_\epsilon} \frac{1}{h^2} dh \wedge (h d\theta + d\theta) + \int_{U_\epsilon} \frac{df_\epsilon(h)}{dh} (d\theta) + \underbrace{\int_{U_\epsilon} \frac{df_\epsilon(h)}{dh} (h d\theta)}_{=0}$$

$$= \int_{S^2\setminus U_\epsilon} \frac{1}{h^2} dh \wedge d\theta + \underbrace{\int_{S^2\setminus U_\epsilon} \frac{1}{h} dh \wedge d\theta}_{=0} + \int_{U_\epsilon} \frac{df_\epsilon(h)}{dh} (d\theta)$$

$$= \int_{S^2} \omega_{1\epsilon}.$$

Let us consider the action of S^1 over S^2 given by $\phi : S^1 \times S^2 \to S^2 : (t, (h, \theta)) \mapsto (h, \theta + t)$. Observe that both ω_1 and ω_2 are invariant under the previous action. Moreover, their desingularizations are also invariant.

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