BACHELOR'S THESIS

Degree in Mathematics

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1 Introduction

Group theory is one of the most important branches of modern mathematics. It has countless aplications, from number theory to chemistry and physics. Since the beginning, classifying groups has been a big issue and still nowadays it continues to be a significant area of investigation.

When you deepen in this study, there naturally arises a big question: given two groups, G and K, how many groups E there are that have K as a normal subgroup such that $G = E/K$? This question induces the group extension problem: given G and K, we want to find all groups E that satisfy that properties and classify them. The main objective of this thesis is describe the solution of this problem when K is abelian.

The classification of extensions when K is abelian is obtained using group cohomology in the following form: there is a natural bijection the second cohomology group $H^2(G, K)$ and the equivalence classes of extensions of G by K. The study of group extensions using group cohomology was introduced by Eilenberg and MacLane in the two fundamental papers [6] and [7].

This thesis is divided in two noticeably different sections:

In the first section we make a basic study of group cohomology, since it is the main tool we have to deepen in group extensions.

The second section is the main body of this thesis, in which we study group extensions themselves. At first we study splitting extensions, which are the simplest extensions we can construct. Secondly, we will prove the Main Theorem of this thesis, which gives us the relation between the second cohomology group and extensions. Since all this work will be constructive, we will show its utility with a detailed example. Moreover, using the Main Theorem we will give a proof to the Schur-Zassenhaus theorem, an important theorem about Hall subgroups of a group. Finally, we will see a generalization of the Main Theorem when K is non-abelian.

2 Preliminaries

Definition 2.1. *Let G, K be groups. A (left) group action α of G on K is a map from $G \times K$ to K, denoted as $(x, k) \longrightarrow {}^x k$ which satisfies ${}^1 k = k$, ${}^{xy}k = {}^x({}^y k)$ and also it satisfies ${}^x(kk') ={}^x k\, {}^x k'$, $\forall k, k' \in K$, $\forall x, y \in G$. This is equivalent to have a homomorphism*

$$\alpha : G \longrightarrow \mathrm{Aut}(K) : \; x \longrightarrow \alpha_x$$

putting $\alpha_x(k) = {}^x k$.

Example 2.1. *We can always define the trivial action of G on K with ${}^x k = k$ for all $k \in K$, $x \in G$.*

Example 2.2. *Let $G = \mathbb{Z}$, and K an abelian group. We can define the inverse action as ${}^{(x)}k = k^{(-1)^x}$, in other words, it sends even integers to* id *and odd integers to the inversion.*

Definition 2.2. *Let G_1, G_2 be abelian groups. We denote the tensor product of G_1 and G_2 as $G_1 \otimes G_2$. We define $G_1 \otimes G_2$ to be the quotient of the free abelian group on the product $G_1 \times G_2$ by the relations*

$$(a, g_2)(b, g_2) \sim (ab, g_2)$$
$$(g_1, c)(g_1, d) \sim (g_1, cd)$$

for all $a, b, g_1 \in G_1$ and $c, d, g_2 \in G_2$.
If G_1 and G_2 are arbitrary groups, then we define the tensor product of G_1 and G_2 by $G_1 \otimes G_2 = G_1/G_1' \otimes G_2/G_2'$.

Remark 2.1. *Recall that the commutator or derived subgroup G' of a group G is the subgroup of G generated by the commutators $xyx^{-1}y^{-1}$ with $x, y \in G$. Moreover, $G/G' := G_{ab}$ is the abelianized of G, which is characterized by the fact that any normal subgroup N of G has abelian quotient G/N if and only if $G' \leq N$.*

Example 2.3. $\mathbb{Z}_n \otimes \mathbb{Z}_m \cong \mathbb{Z}_{(n,m)}$.

Proposition 2.1. *Let G_1, G_2 be finite abelian groups with $(|G_1|, |G_2|) = 1$. Then*

$$G_1 \otimes G_2 = 1$$

Proof. Let $n = |G_1|$, $m = |G_2|$, $a \in G_1$, $b \in G_2$. Since n and m are coprime, we can find $k \in \mathbb{Z}$ such that $nk \equiv 1 \mod m$. We have that $a^n = 1$ and $b^m = 1$, and then, by te properties of the tensor product

$$a \otimes b = a^{n+1} \otimes b = a \otimes b^{n+1} = a \otimes b^{(m-1)kn+1} = a \otimes b^m = a \otimes 1 = 1 \otimes 1$$

□

The following definitions and results will be useful in section 4.3:

Definition 2.3. *Let K be a subgroup of a group E. A subgroup $G \leq E$ is a complement of K in E if $K \cap G = 1$ and $KG = E$.*

Proposition 2.2. *Given K a subgroup of E, then the following statements are equivalents:*

1. *K has a complement in E.*
2. *There's a subgroup G of E such that every element $e \in E$ has a unique expression $e = kx$ with $k \in K$ and $x \in G$.*

Proof. Take $e \in E$, then we have to see that the expression $e = kx,\ k \in K,\ x \in G$ is unique. Suppose we have another expression $e = k'x',\ k' \in K,\ x' \in G \Longrightarrow kx = k'x' \Longrightarrow (k')^{-1}k = x'x^{-1} \Longrightarrow (k')^{-1}k = 1,\ x'x^{-1} = 1 \Longrightarrow k' = k,\ x' = x$.
Now suppose we have a unique expression $e = kx$ for every element of E. We only have to see that $K \cap G = 1$. Suppose there exists $e \in K \cap G$ with $e \neq 1$, then $e^{-1} \in K \cap G$ and $1 = 1 \cdot 1 = ee^{-1}$ has two representations in the form $1 = kx$, and so $K \cap G = 1$. □

Definition 2.4. *Let K be a subgroup of a finite group E. K is a Hall subgroup if its order and index are relatively prime.*

Example 2.4. *Any p-Sylow subgroup of E is a Hall subgroup.*

Example 2.5. $\mathfrak{A}_4$ *is a Hall subgroup of* $\mathfrak{A}_5$ *since* $|\mathfrak{A}_4| = 12$ *and* $|\mathfrak{A}_5| = 60$*, so the index of* $\mathfrak{A}_4$ *is* 5*, relatively prime with* 12.

Proposition 2.3. *If K is a Hall subgroup of a finite group E, with $|K| = m$ and $|E| = mn$. Then a subgroup $G \leq E$ is a complement of K if and only if $|G| = n$.*

Proof. Obviously a complement of K has order n. Now suppose G has order n. We have to see that it is a complement of K. Since $(m, n) = 1$, we have that $K \cap G = 1$. Moreover, if their intersection is trivial, if we take $k, k' \in K$ and $x, x' \in G$, $kx = k'x' \Rightarrow (k')^{-1}k = x'x^{-1} \Rightarrow k'k^{-1} \in G,\ x'x^{-1} \in K$ so $k'k^{-1} \in K \cap G,\ x'x^{-1} \in K \cap G \Rightarrow k = k',\ x = x'$. Then $|GK| = |G||K| = |E| \Rightarrow GK = E$. □

Proposition 2.4. *If K is a normal subgroup of a finite group E, and P a p-Sylow subgroup of K, then*

$$E = KN_E(P)$$

Proof. Take $e \in E$. Since K is normal, $ePe^{-1} \leq eKe^{-1} = K$, so ePe^{-1} is a p-Sylow subgroup of K, and there exists $k \in K$ such that $ePe^{-1} = kPk^{-1}$. Hence, $P = (k^{-1}e)P(k^{-1}e)^{-1} \Rightarrow k^{-1}e \in N_E(P)$, and we can express $e = k(k^{-1}e),\ k \in K,\ (k^{-1}e) \in N_E(P)$. □

Definition 2.5. *A subgroup K of a group E is called characteristic in E if $\varphi(K) = K$ for every automorphism $\varphi : E \to E$. It is denoted K char E.*

Lemma 2.1. *1. If H char K and $K \lhd E$, then $H \lhd E$.*

2. The centre of a group K is characteristic in K.

Proof. 1. Let $x \in E$ and let $c : E \longrightarrow E$ be the conjugation by x. We know that, since K is normal, $c|_K \in \mathrm{Aut}(K)$ and then $c(H) = c|_K(H) = H$ since H char K.

2. Take $\varphi \in \mathrm{Aut}(K)$ and $x \in Z(K)$, then, for all $y \in K$, $\varphi(x)\varphi(y) = \varphi(xy) = \varphi(yx) = \varphi(y)\varphi(x)$, so, since φ is a bijection, $\varphi(x)$ commutes with every element of K and so it is in the center. To see the other inclusion, apply φ^{-1} in every side of the inclusion $\varphi(Z(K)) \subseteq Z(K) \Rightarrow Z(K) \subseteq \varphi^{-1}(Z(K))$ and, since φ^{-1} is also an automorphism, we have both inclusions for φ^{-1} and so for φ.

□

Corollary 2.1. *If K is a normal p-subgroup of a group E, for some prime p, and there is no proper nontrivial subgroup of K that is normal in E, then K is abelian.*

Proof. By the lemma we have that $Z(K)$ char K, and $K \lhd E$; then the first part of the lemma implies that $Z(K) \lhd E$. But we know that the center of a p-group is not trivial, so we have $Z(K) \lhd E$, $Z(K) \neq 1$ an then, by the property of K, $Z(K) = K$. □

3 Group cohomology

As we will see later, cohomology of groups appears naturally in the study of group extensions, and it is the most important tool that we have to work with extensions, so it is essential to understand well it before we begin to study group extensions. Moreover, group cohomology has other aplications in group theory.
In this chapter we suppose G, K to be groups, with K abelian, written both multiplicatively, and we also suppose that G acts on K with the action α.

3.1 Definition of $H^n(G,K)$

Definition 3.1. *For $n \in \mathbb{N}_0$, let $C^n(G,K)$ denote the abelian group of functions $f : G^n \to K$ under the multiplication $(fg)(x_1,\ldots,x_n) = f(x_1,\ldots,x_n)g(x_1,\ldots,x_n)$, for $f, g \in C^n(G,K)$ and $x_1,\ldots,x_n \in G$. If $n = 0$, then by definition $G^0 := 1$. We call the elements of $C^n(G,K)$ n-cochains. For each $n \in \mathbb{N}_0$, we define the coboundary homomorphisms*

$$\delta^n_\alpha := \delta^n : C^n(G,K) \longrightarrow C^{n+1}(G,K)$$

such that given $f \in C^n(G,K)$ and $(x_1,\ldots,x_{n+1}) \in G^{n+1}$

$$\delta^n(f)(x_1,\ldots,x_{n+1}) = \\ = {}^{x_1}f(x_2,\ldots,x_{n+1}) \left(\prod_{i=1}^{n} f(x_1,\ldots,x_i x_{i+1},\ldots,x_{n+1})^{(-1)^i} \right) f(x_1,\ldots,x_n)^{(-1)^{n+1}}$$

We will write simply δf and we call δf the coboundary of f.

Proposition 3.1. *$\delta^{n+1} \circ \delta^n = 1$ for each $n \in \mathbb{N}_0$. Then, we have that $\operatorname{im}(\delta^n) \leq \ker(\delta^{n+1}) \leq C^{n+1}(G,K)$. We will write $\delta \circ \delta = 1$.*

Proof. We don't prove this proposition since it is only a large sequence of computations. It can be found in all the works which study cohomology, like Michael Tsiang's article [3], p.12-14. □

Definition 3.2. *We define*

$$B^n(G,K) := \operatorname{im}(\delta^{n-1})$$

$$Z^n(G,K) := \ker(\delta^n)$$

and we set $B^0(G,K) = 1$. The elements of $B^n(G,K)$ are called n-coboundaries (or just coboundaries) and the elements of $Z^n(G,K)$ are called n-cocycles (or just cocycles). By *the last proposition, $B^n(G,K)$ is a subgroup of $Z^n(G,K)$. We can now define the n-th cohomology group of G with coefficients in K (under the action α):*

$$H^n(G,K) := Z^n(G,K)/B^n(G,K)$$

The elements of $H^n(G,K)$ are classes of cocycles modulo coboundaries. If $f \in Z^n(G,K)$, then we denote its cohomology class by $\bar{f} \in H^n(G,K)$, and two cocycles with the same cohomology class are said to be cohomologous. Since the cohomology group is associated to an action α, we will write $H^n(G,K)_\alpha$ if we need to emphasize that the action is α.

3.2 The cases $n = 0,\ 1,\ 2,\ 3$

- $n = 0$

  A 0-cochain $f \in C^0(G, K)$ is by definition an element $k \in K$. Since $\delta f(g) = {}^x k k^{-1}$ we have that k is a cocycle if and only if ${}^x k = k$. By definition $B^0(G, K) = 1$ so $H^0(G, K)$ is the subgroup of K of those elements on which G operates trivially (${}^x k = k,\ \forall x \in G$), in other words, the fixed points of the action.

- $n = 1$

  A 1-cochain $f \in C^1(G, K)$ is a function $f : G \longrightarrow K$. It is a 1-cocycle if

  $$\delta(f)(x, y) = {}^x f(y) f(xy)^{-1} f(x) = 1 \Longleftrightarrow f(xy) = {}^x f(y) f(x),\ \forall x, y \in G$$

  and f is called a crossed homomorphism of G to K.

  A 1-coboundary $f \in B^1(G, K)$ is a function such that $f(x) = {}^x k k^{-1}$ for some $k \in K$ and it is called a principal homomorphism. An important fact is that, if G acts trivially on K, then $B^1(G, K) = 1$ and $H^1(G, K)$ is the group $\mathrm{Hom}(G, K)$ of all homomorphisms of G into K.

- $n = 2$

  This is the most important case in our study.

  A 2-cochain $f \in C^2(G, K)$ is a function $f : G \times G \longrightarrow K$. It is a 2-cocycle if

  $$\delta(f)(x, y, z) = 1 \Longleftrightarrow f(x, y) f(xy, z) = {}^x f(y, z) f(x, yz),\ \forall x, y, z \in G$$

  A 2-coboundary $f \in B^2(G, K)$ is a 2-cocycle such that

  $$f(x, y) = {}^x \phi(y) \phi(x) \phi(xy)^{-1}$$

  for some function $\phi : G \longrightarrow K$.

  Suppose G acts trivially on K. Then a 2-cocycle is symmetric if $f(x, y) = f(y, x)$ for all $x, y \in G$. We will denote the subgroup of $H^2(G, K)$ of classes of symmetric cocycles as $H^2(G, K)_s$. This subgroup will be important later, so we will focus in it together with the second cohomology group.

- $n = 3$

  A 3-cochain $f \in C^3(G, K)$ is a function $f : G \times G \times G \longrightarrow K$. It is a 3-cocycle if

  $$f(xy, z, w) f(x, y, zw) = {}^x f(y, z, w) f(x, yz, w) f(x, y, z),\ \forall x, y, z, w \in G$$

  A 3-coboundary $f \in B^3(G, K)$ is a 3-cocycle such that

  $$f(x, y, z) = {}^x \phi(y, z) \phi(xy, z)^{-1} \phi(x, yz) \phi(x, y)^{-1}$$

  for some function $\phi : G \times G \longrightarrow K$.

3.3 Normalized cochains

In this section we want to simplify the calculations in the following section, defining a subgroup of $C^n(G,K)$ where we can restrict our study of the cohomology groups. It could be also useful for future work.

Definition 3.3. *A cochain $f \in C^n(G,K)$ is said to be normalized if $f(x_1,\dots,x_n) = 1$ if $x_i = 1$ for some $i = 1\dots n$.*

With this definition, we have an obvious consequence:

Proposition 3.2. *The normalized cochains form a subgroup of $C^n(G,K)$, the normalized cocycles form a subgroup of $Z^n(G,K)$ and the normalized coboundaries form a subgroup of $B^n(G,K)$.*

All the work in tis section will be to prove the following theorem, which is all we want about normalized cochains:

Theorem 3.1. *Every cocycle is cohomologous to a normalized cocycle and every normalized coboundary is the coboundary of a normalized cochain.*

To prove it, we will need some previous definitions and lemmas:

Definition 3.4. *A cochain f will be said i-normalized if $f(x_1,\dots,x_n) = 1$ when one of the first i variables $x_1,\dots,x_i$ is 1. Obviously every cochain is 0-normalized and normalized cochains are by definition n-normalized cochains.*

Lemma 3.1. *A coboundary of an i-normalized cochain is i-normalized.*

Proof. Suppose, f is k-normalized, and take $(x_1,\dots,x_n,x_{n+1}) \in G^{n+1}$ with $x_j = 1$ for some $j \le k$, then if $j = 1$:

$$\begin{aligned}\delta(f)(1,\dots,x_{n+1}) &= {}^1f(x_2,\dots,x_{n+1})\left(\prod_{i=1}^{n} f(1,\dots,x_ix_{i+1},\dots,x_{n+1})^{(-1)^i}\right) f(1,\dots,x_n)^{(-1)^{n+1}} \\ &= f(x_2,\dots,x_{n+1})f(1x_2,\dots,x_{n+1})^{-1} = 1\end{aligned}$$

if $2 \le j \le n$:

$$\begin{aligned}\delta(f)(x_1,\dots,x_{n+1}) &= \prod_{i=1}^{n} f(x_2,\dots,x_ix_{i+1},\dots,x_{n+1})^{(-1)^i} \\ &= f(x_2,\dots,x_{j+1},\dots,x_{n+1})^{(-1)^{j-1}} f(x_2,\dots,x_{j+1},\dots,x_{n+1})^{(-1)^j} = 1\end{aligned}$$

and if $j = n+1$ and $k = n+1$:

$$\delta(f)(x_1, \dots, 1) = {}^{x_1}f(x_2, \dots, 1) \left(\prod_{i=1}^{n} f(x_1, \dots, x_i x_{i+1}, \dots, 1)^{(-1)^i} \right) f(x_1, \dots, x_n)^{(-1)^{n+1}}$$
$$= f(x_1, \dots, 1x_n)^{(-1)^n} f(x_1, \dots, x_n)^{(-1)^{n+1}} = 1$$

□

Definition 3.5. *Given $f \in C^n(G, K)$ we define $f_i \in C^n(G, K)$ and $g_i \in C^n(G, K)$ for $i = 1, \dots, n$ recursively:*

$$f_0 = f, \;\; f_i = f_{i-1} \cdot (\delta g_i)^{-1}$$

with $g_i(x_1, ..., x_{n-1}) = (f_{i-1}(x_1, \dots, x_{i-1}, 1, x_i, \dots, x_{n-1}))^{(-1)^{i-1}}$. It follows that $\delta f = \delta f_i$ for all i because $\delta \circ \delta = 1$.

The idea of this procedure is the following lemma:

Lemma 3.2. *If δf is normalized, then f_i is i-normalized, for $i = 0, \dots, n$.*

Proof. We will prove it by induction on i. For $i = 0$ is trivial. Then we suppose it is true for i and prove it for $i+1$:

We have that f_i is i-normalized, but then g_{i+1} is i-normalized directly by its formula. So $f_{i+1} = f_i \cdot (\delta g_{i+1})^{-1}$ is normalized, because f_i is normalized and the coboundary of a normalized cochain (like g_{i+1}) is normalized. Then, to prove that f_{i+1} is $(i+1)$-normalized, we have to prove that $f_{i+1}(x_1, \dots, x_i, 1, x_{i+2}, \dots, x_n) = 1$ using that g_{i+1} is i-normalized and $\delta f_i = \delta f$ is normalized, it's only matter of computation:

$$f_{i+1}(x_1, \dots, x_i, 1, \dots, x_n) = f_i(x_1, \dots, x_i, 1, \dots, x_n)(\delta g_i(x_1, \dots, x_i, 1, \dots, x_n))^{-1}$$
$$= f_i(x_1, \dots, x_i, 1, \dots, x_n) g(x_1, \dots, x_{i-1}, 1x_i, \dots, x_n)^{(-1)^{i-1}} g(x_1, \dots, x_i, 1, \dots, x_n)^{(-1)^i} \cdot$$
$$\cdot \left(\prod_{j=i+2}^{n-1} g_{i+1}(x_1, \dots, x_i, 1, \dots, x_j x_{j+1}, \dots, x_n)^{(-1)^i} \right) g_{i+1}(x_1, \dots, x_i, 1, \dots, x_{n-1})^{(-1)^{n-1}}$$
$$= f_i(x_1, \dots, x_i, 1, \dots, x_n) \left(\prod_{j=i+2}^{n-1} f_i(x_1, \dots, x_i, 1, 1, \dots, x_j x_{j+1}, \dots, x_n)^{(-1)^{i+j-1}} \right) \cdot$$
$$\cdot f_i(x_1, \dots, x_i, 1, 1, \dots, x_n)^{(-1)^{i+n-1}} = \delta f_i(x_1, \dots, x_i, 1, 1, x_{i+2}, \dots, x_n)^{(-1)^i}$$

Where in the last equality we have used that f_i is i-normalized, so the first terms in δf_i vanish. Since δf_i is i-normalized, the result follows directly. □

With this we can prove the theorem as a direct corollary:

Corollary 3.1. *Every cocycle is cohomologous to a normalized cocycle and every normalized coboundary is the coboundary of a normalized cochain.*

Proof. Suppose f is a cocycle, and again we have f_n is normalized because $\delta f = 1$ is normalized. But, by construction, $f_n = f \cdot (\delta g_1)^{-1} \cdots (\delta g_n)^{-1}$ so f is cohomologous to f_n.
Now take f with δf a normalized coboundary. Then, we have that f_n is normalized, and $\delta f = \delta f_n$ so δf is the coboundary of the normalized cochain f_n.

□

With this result, in the rest of all this study about group cohomology we will deal only with normalized cochains and normalized cocycles, since we are only interested in $H^n(G, K)$. And so, during the rest of this section $C^n(G, K)$, $Z^n(G, K)$, $B^n(G, K)$ will refer to normalized cochains, normalized cocycles and normalized coboundaries respectively.

3.4 The first reduction theorem

In this section we will see that $H^{n+1}(G,K)_\alpha$ depends on the $H^n(G,C^1(G,K))_\sigma$ with G operating on $C^1(G,K)$ with a certain action σ. Since the most important aplication of cohomology in group extensions is the second cohomology group, this theorem will show that it makes sense to study the other cohomology groups, not only the second one.
Our objective is to stablish isomorphisms

$$\sigma_n : H^n(G,C^1(G,K))_\sigma \longrightarrow H^{n+1}(G,K)_\alpha$$

We define the action σ in the following way; given $x, y \in G$ and $f \in C^1(G,K)$:

$$({}^x f)(y) = f(yx)({}^y f(x))^{-1}$$

The isomorphism between the cohomology groups will will be induced by a homomorphism

$$\sigma_n : C^n(G,C^1(G,K)) \longrightarrow C^{n+1}(G,K)$$

defined by

$$(\sigma_n f)(x_1,\dots,x_{n+1}) := (\sigma f)(x_1,\dots,x_{n+1}) = (f(x_2,\dots,x_{n+1})(x_1))^{(-1)^n}$$

Lemma 3.3. *σ_n is an isomorphism between $C^n(G,C^1(G,K))$ and $C^{n+1}(G,K)$.*

Proof. It is a homomorphism since

$$\begin{aligned}(\sigma_n(fg))(x_1,\dots,x_{n+1}) &= ((fg)(x_2,\dots,x_{n+1})(x_1))^{(-1)^n}\\ &= (f(x_2,\dots,x_{n+1})(x_1)g(x_2,\dots,x_{n+1})(x_1))^{(-1)^n}\\ &= (\sigma_n f)(x_1,\dots,x_{n+1})(\sigma_n g)(x_1,\dots,x_{n+1})\end{aligned}$$

Secondly if $\sigma = 1$ then $\sigma = 1$ inmediately. Finally, if $g \in C^{n+1}(G,K)$ we take $f(x_1,\dots,x_n)(x_{n+1}) = g(x_2,x_1,\dots,x_{n+1})^{(-1)^n}$ and it is clear that $\sigma f = g$.

□

Proposition 3.3. *Suppose $n > 0$. Then $\delta\sigma = \sigma\delta$.*

Proof. For $f \in C^n(G,C^1(G,K))$ we must show that

$$(\sigma(\delta f))(x_1,\dots,x_{n+2}) = (\delta(\sigma f))(x_1,\dots,x_{n+2})$$

so it is matter of computation, the first term results in:

$$(\sigma(\delta f))(x_1,\dots,x_{n+2}) = ([{}^{x_2} f(x_3,\dots,x_{n+2})](x_1))^{(-1)^{n+1}}((\delta f)(x_2,\dots,x_{n+2}))(x_1))^{(-1)^{n+1}}$$

and the second term

$$(\delta(\sigma f))(x_1,\ldots,x_{n+2}) = ([^{x_1}f(x_3,\ldots,x_{n+2})](x_2))^{(-1)^n}(f(x_3,\ldots,x_{n+2})(x_1x_2))^{(-1)^{n+1}}\cdot$$
$$\cdot((\delta f)(x_2,\ldots,x_{n+2}))(x_1))^{(-1)^{n+1}} = ([^{x_2}f(x_3,\ldots,x_{n+2})](x_1))^{(-1)^{n+1}}((\delta f)(x_2,\ldots,x_{n+2}))(x_1))^{(-1)^{n+1}}$$
$$= (\sigma(\delta f))(x_1,\ldots,x_{n+2})$$

Finally, we have used the definition of the new action of G on $C^1(G,K)$:

$$([^{x_2}f(x_3,\ldots,x_{n+2})](x_1))^{(-1)^{n+1}} = (^{x_1}[f(x_3,\ldots,x_{n+2})](x_2))^{(-1)^n}(f(x_3,\ldots,x_{n+2})(x_1x_2))^{(-1)^{n+1}}$$

□

Theorem 3.2. *(First Reduction Theorem) Suppose $n>0$. Then σ_n induces an isomorphism between $H^n(G,C^1(G,K))_\sigma$ and $H^{n+1}(G,K)_\alpha$.*

Proof. Since $\delta\sigma = \sigma\delta$, σ_n carries n-cocycles and n-coboundaries into $(n+1)$-cocycles and $(n+1)$-coboundaries respectively, so it induces an isomorphism between the corresponding cohomology groups. □

3.5 The second cohomology group

In this section and the following, we will now focus in the second cohomology group. First, we begin with an important fact:

Proposition 3.4. *Let G act on two abelian groups K_1, K_2, and define $K = K_1 \times K_2$ with the action of G on K defined by ${}^x(k_1, k_2) = ({}^x k_1, {}^x k_2)$. Then*

$$H^2(G, K) \cong H^2(G, K_1) \times H^2(G, K_2)$$

Proof. Given $\overline{f_1} \in H^2(G, K_1)$ and $\overline{f_2} \in H^2(G, K_2)$ we define the cocycle $f = f_1 \times f_2$ to be $f(x, y) = (f_1(x, y), f_2(x, y))$ and this induces directly a well defined isomorphism between $H^2(G, K_1) \times H^2(G, K_2)$ and $H^2(G, K)$. □

As a direct consequence we have:

Corollary 3.2. *Suppose G acts trivially on K_1 and K_2. Then*

$$H^2(G, K)_s \cong H^2(G, K_1)_s \times H^2(G, K_2)_s$$

Now we will study how the second cohomology group behaves respect to the product on G when we have the trivial action, in other words, given G_1, G_2 arbitrary groups acting on K trivially, if we consider $G = G_1 \times G_2$ acting trivially on K, we want to see how $H^2(G, K)$ depends on $H^2(G_1, K)$ and $H^2(G_2, K)$. This case is more complicated, and we have to introduce new concepts:

Definition 3.6. *A map $f : G_1 \times G_2 \longrightarrow K$ is called a pairing of G_1 and G_2 into K (or a pairing of G into K if $G_1 = G_2 = G$) if*

$$f(x_1 x_2, y) = f(x_1, y) f(x_2, y)$$
$$f(x, y_1 y_2) = f(x, y_1) f(x, y_2)$$

We denote $P(G_1, G_2, K)$ the set of all pairings of G_1 and G_2 into K. It is an abelian group together with the multiplication defined by

$$(f_1 f_2)(x, y) = f_1(x, y) f_2(x, y), \qquad for\ x \in G_1\ and\ y \in G_2$$

If $G_1 = G_2 = G$ we will write $P(G, K)$ instead of $P(G_1, G_2, K)$.

Remark 3.1. *This definition of tensor product of groups is analogous to the tensor product of vector spaces with multilinear maps.*

Lemma 3.4.

$$P(G_1, G_2, K) \cong \mathrm{Hom}(G_1 \otimes G_2, K)$$

Proof. Let $f \in P(G_1, G_2, K)$ be a pairing and $g_1 \in G_1,\ g_2 \in G_2$. Then directly from the definition, we have two homomorphisms $f_1 : G_1 \longrightarrow K,\ f_2 : G_2 \longrightarrow K$ with $f_1(a) = f(a, g_2)$ and $f_2(b) = f(g_1, b)$. Then, if $xG_1' = x'G_1'$ and $yG_2' = y'G_2'$, we have that $f(x, y) = f(x', y')$, so we can assume that both G_1 and G_2 are abelian. With this, we can define a homomorphism $\varphi_f : G_1 \otimes G_2 \longrightarrow K$ with $\varphi_f(x \otimes y) = f(x, y)$. One can easily check that the map $f \longrightarrow \varphi_f$ is an isomorphism by the definition of the tensor product and the pairings. □

Lemma 3.5. *Let G_1, G_2 be subgroups of G with $xy = yx$ for all $x \in G_1$ and $y \in G_2$. Given $\alpha \in Z^2(G, K)$, we define $\beta : G_1 \times G_2 \longrightarrow K$ by*

$$\beta(x, y) = \alpha(x, y)\alpha(y, x)^{-1}$$

Then $\beta \in P(G_1, G_2, K)$ and β depends only on the cohomology class $\overline{\alpha} = c$. Moreover, the induced map between $H^2(G, K)$ with trivial action and $P(G_1, G_2, K)$ is a homomorphism. Given $c \in H^2(G, K)$ we will denote β as β_c.

Proof. Take $x, y, z \in G$. Since $\alpha \in Z^2(G, K)$ by definition

$$(\delta\alpha)(x, y, z) = \alpha(y, z)\alpha(x, yz)\alpha(xy, z)^{-1}\alpha(x, y)^{-1} = 1$$

Then, if we choose $x \in G_1$ and $y, z \in G_2$ we have

$$\begin{aligned}
\beta(x, z)^{-1}\beta(x, yz)\beta(x, y)^{-1} &= \alpha(z, x)\alpha(x, z)^{-1}\alpha(x, yz)\alpha(yz, x)^{-1}\alpha(y, x)\alpha(x, y)^{-1} \\
&= \alpha(z, x)\alpha(y, xz)\alpha(yz, x)^{-1}\alpha(y, z)^{-1} \\
&\cdot \alpha(y, z)\alpha(x, yz)\alpha(yx, z)^{-1}\alpha(x, y)^{-1} \\
&\cdot \alpha(x, z)^{-1}\alpha(y, xz)^{-1}\alpha(yx, z)\alpha(y, x) \\
&= (\delta\alpha)(y, z, x)(\delta\alpha)(x, y, z)(\delta\alpha)(y, x, z)^{-1} = 1
\end{aligned}$$

so $\beta(x, yz) = \beta(x, y)\beta(x, z)$. In the same way, we can prove that $\beta(xy, z) = \beta(x, z)\beta(y, z)$ for $x, y \in G_1$ and $z \in G_2$. Now take $\alpha' \in Z^2(G, K)$ such that $\overline{\alpha'} = \overline{\alpha}$ in $H^2(G, K)$. By definition, there exists $t \in C^1(G, K)$ such that $\alpha'(x, y) = \alpha(x, y)t(x)t(y)t(xy)^{-1}$. Then, take $x \in G_1$ and $y \in G_2$:

$$\begin{aligned}
\alpha'(x, y)\alpha'(x, y)^{-1} &= \alpha(x, y)t(x)t(y)t(xy)^{-1}t(yx)t(y)^{-1}t(x)^{-1}\alpha(y, x)^{-1} \\
&= \alpha(x, y)t(xy)^{-1}t(xy)\alpha(y, x)^{-1} = \alpha(x, y)\alpha(y, x)^{-1} = \beta(x, y)
\end{aligned}$$

and we have a induced map in $H^2(G, K)$. Finally, it is a homomorphism directly by the definition. □

Theorem 3.3. *Let $G_1,\ G_2,\ G_1 \times G_2$ be groups acting trivially on K. Then*

$$H^2(G_1 \times G_2, K) \cong H^2(G_1, K) \times H^2(G_2, K) \times \mathrm{Hom}(G_1 \otimes G_2, K)$$

Proof. We define $G = G_1 \times G_2$ and we will denote an element (x_1, x_2) of G as x_1x_2. If $c \in H^2(G,K)$ we will denote c_1 and c_2 the respective restrictions of c to $H^2(G_1,K)$ and $H^2(G_2,K)$. Then by the last lemma, the map

$$\begin{aligned} \varphi : H^2(G,K) &\longrightarrow H^2(G_1,K) \times H^2(G_2,K) \times \mathrm{Hom}(G_1 \otimes G_2, K) \\ c &\longmapsto (c_1, c_2, \beta_c) \end{aligned}$$

is a homomorphism. If we show that it is an isomorphism, we will have finished using Lemma 3.4.

First, if we take $\alpha_1, \in Z^2(G_1,K)$, $\alpha_2 \in Z^2(G_2,K)$ and $\beta \in P(G_1,G_2,K)$ we can define $\alpha \in C^2(G,K)$ by

$$\alpha(x_1x_2, y_1y_2) = \alpha_1(x_1,y_1)\alpha_2(x_2,y_2)\beta(x_1,y_2)$$

Then, it is only matter of computation to prove that $\alpha \in Z^2(G,K)$ and it is obvious that $\pi_1(\alpha) = \alpha_1$, $\pi_2(\alpha) = \alpha_2$ and $\beta_{\overline{\alpha}} = \beta$, so the homomorphism φ is surjective.

To prove injectivity, take $\alpha \in Z^2(G,K)$ such that $\alpha_1 \in B^2(G_1,K)$, $\alpha_2 \in B^2(G_2,K)$ and the corresponding β is trivial; we want to see that α is a coboundary. We have $\alpha_i = \delta(t_i)$ for some $t_i \in C^1(G_i,K)$ and define $t \in C^1(G,K)$ by $t(x_1,x_2) = t_1(x_1)t_2(x_2)$; then $\pi_1(\alpha \cdot (\delta t)^{-1}) = \pi_2(\alpha \cdot (\delta t)^{-1}) = 1$ so we can suppose that α_1 and α_2 are both trivial.

Now take $x_1, y_1 \in G_1$ and $x_2, y_2 \in G_2$. Since β is trivial, we have two equalities:

$$\begin{aligned} \beta(x_1y_1, x_2) &= \alpha(x_1y_1, x_2)\alpha(x_2, x_1y_1)^{-1} = 1 \\ \beta(x_1, x_2)^{-1} &= \alpha(x_1, x_2)^{-1}\alpha(x_2, x_1) = 1 \end{aligned}$$

Finally, since $\delta\alpha = 1$ we have 3 more equalities:

$$\begin{aligned} (\delta\alpha)(x_1x_2, y_1, y_2) &= \alpha(y_1,y_2)\alpha(x_1x_2y_1,y_2)^{-1}\alpha(x_1x_2,y_1y_2)\alpha(x_1x_2,y_1)^{-1} = 1 \\ (\delta\alpha)(x_1y_1, x_2, y_2) &= \alpha(x_2,y_2)\alpha(x_1y_1x_2,y_2)^{-1}\alpha(x_1y_1,x_2y_2)\alpha(x_1y_1,x_2)^{-1} = 1 \\ (\delta\alpha)(x_2, x_1, y_1) &= \alpha(x_1,y_1)\alpha(x_1x_2,y_1)^{-1}\alpha(x_2,x_1y_1)\alpha(x_2,x_1)^{-1} = 1 \end{aligned}$$

All of these equalities imply that

$$\alpha(x_1x_2, y_1y_2)^{-1}\alpha(y_1,y_2)^{-1}\alpha(x_1y_1,x_2y_2)\alpha(x_1,x_2)^{-1} = 1$$

so α is a coboundary since

$$\alpha(x_1y_1, x_2y_2) = \alpha(x_1x_2, y_1y_2)\alpha(y_1,y_2)\alpha(x_1,x_2)$$

for all $x_1x_2, y_1y_2 \in G$. □

Iterating the last theorem and using that $\mathrm{Hom}(\prod_{i\in I} G_i, K) \cong \prod_{i\in I} \mathrm{Hom}(G_i,K)$, we can compute the second cohomology group of a finite product of groups:

Corollary 3.3. *Let $G_1, \dots, G_n$ be arbitrary groups acting trivially on K. Then*

$$H^2(\prod_{i=1}^{n} G_i, K) \cong \prod_{i=1}^{n} H^2(G_i, K) \times \prod_{1 \le j < k \le n} \mathrm{Hom}(G_j \otimes G_k, K)$$

Corollary 3.4. *Let $G_1, \dots, G_n$ and $\prod_{i=1}^{n} G_i$ be finite groups acting trivially on K with $(|G_i/G_i'|, |G_i/G_i'|) = 1$ for all $i \neq j$. Then*

$$H^2(\prod_{i=1}^{n} G_i, K) \cong \prod_{i=1}^{n} H^2(G_i, K)$$

Proof. It follows from the last corollary and Proposition 2.1. □

Corollary 3.5. *Suppose $G_1, \dots, G_n$ and $G = \prod_{i=1}^{n} G_i$ are abelian groups acting trivially on K. Then*

$$H^2(G, K)_s \cong \prod_{i=1}^{n} H^2(G_i, K)_s$$

Proof. We will prove the result for $n = 2$ and the rest will follow it by induction. Let $c \in H^2(G, K)$. We observe that c is symmetric if and only if c_1, c_2 are symmetric and $\beta_c = 1$. Then by the isomorphism of Theorem 3.3, $H^2(G, K)_s \cong H^2(G_1, K)_s \times H^2(G_2, K)_s \times \{1\}$ and we have finished. □

This results will be complemented with the next subsection, where there are some examples.

3.6 Cohomology of cyclic and finite abelian groups

We want to study cohomology of cyclic groups with trivial action. Then, we suppose G to be $\mathbb{Z}_n$ acting trivially on K and let g be a generator of G. All we need is this useful theorem:

Theorem 3.4. *Suppose $\mathbb{Z}_n$ acts trivially on K. Then*

$$H^2(\mathbb{Z}_n, K) \cong K/K^n$$

Proof. We will divide the proof in the following steps:

a) Let $m \in K$. If we define $f_m \in C^2(G, K)$ by $f_m(g^i, g^j) = 1$ if $i + j < n$ and $f_m(g^i, g^j) = m$ if $i + j \geq n$ then $f_m \in Z^2(G, K)$.

b) Any cocycle $f \in Z^2(G, K)$ is cohomologous to f_m where $m = \prod_{i=0}^{n-1} f(g^i, g)$.

c) The map $m \longrightarrow f_m$ induces an isomorphism $K/K^n \cong H^2(G, K)$.

a) We won't do this step. It is only matter of computations.

b) Given a cocycle f we will define $b(g^i, g^j) = k_i k_j k_{i+j}^{-1}$ a coboundary and we want $f(g^i, g^j) = b(g^i, g) f_m(g^i, g^j)$. It is easy to check that we only have to prove it when $j = 1$, and then all we need is $k_{i+1} = k_i k_1 f(g^i, g) f_m(g^i, g)$. So we will need $k_0 = 1$, $k_1 = f(1, g) = 1$, $k_{i+1} = k_i f(g^i, g)$ for $i + 1 < n$ and $1 = k_0 = k_n = k_{n-1} f(g^{n-1}, g) f_m(g^{n-1}, g)^{-1}$. Then we observe that all this system of equations has a simple solution, because the only non-trivial equality is the last equality, but it is true that $k_n = 1$ since $k_{n-1} f(g^{n-1}, g) = \prod_{i=0}^{n-1} f(g^i, g) = f_m(g^{n-1}, g)$.

c) By b) this map is surjective, then by the First Isomorphism Theorem we only need to show that the kernel of this map is K^n. In other words, we have to prove that $f_m \in B^2(G, K)$ if and only if $m \in K^n$. First, we note that $f_m \in B^2(G, K)$ if and only if there exist $k_i \in K$ such that $k_i k_j = k_{i+j}$ for $i + j < n$ and $k_i k_j = m k_{i+j}$ for $i + j \geq n$. We observe that we can find k_i if and only if $k_0 = k_n = 1$; by this relations, taking $i = n - 1,\ j = 1$ we have

$$m = m k_n = k_{n-1} k_1 = k_1^n$$

so we can find k_i if and only if $m = k_1^n$ and so $m \in K^n$.

□

This result together with the following lemma is all we need to know about cohomology of cyclic groups with trivial action:

Lemma 3.6.

$$\mathrm{Hom}(\mathbb{Z}_n, K) \cong K[n] := \{k \in K|\ k^n = 1\}$$

Proof. Let $f \in \mathrm{Hom}(\mathbb{Z}_n, K)$. It is clear that $f(g) \in K[n]$. Now we define the homomorphism $\phi : \mathrm{Hom}(\mathbb{Z}_n, K) \longrightarrow K[n]$ with $\phi(f) = f(g)$. Since ϕ is injective by the definition and for every $k \in K[n]$ we can define $f \in \mathrm{Hom}(\mathbb{Z}_n, K)$ by $f(g) = k$ (because g generates $\mathbb{Z}_n$), ϕ is also surjective and so an isomorphism. □

Corollary 3.6. *Suppose $\mathbb{Z}_n$ acts trivially on K. Then*

$$H^2(\mathbb{Z}_n, K) \cong \mathbb{Z}_n \otimes K \cong K/K^n$$

Proof. Let g be a generator of $\mathbb{Z}_n$. We will show that $\varphi : K/K^n \longrightarrow \mathbb{Z}_n \otimes K$ with

$$\varphi(kK^n) = g \otimes k$$

is an isomorphism. First, it is a well defined homomorphism since $g \otimes k^n = g^n \otimes k = 1$. By the properties of the tensor product, a generator of $\mathbb{Z}_n \otimes K$ is any $g^m \otimes k$, and so we can construct an inverse φ^{-1} in the following way:

$$\varphi^{-1}(g^m \otimes k) = k^m K$$

and it is clear that it is a homomorphism, and it is the inverse of φ since

$$\varphi(\varphi^{-1}(g^m \otimes k)) = \varphi(k^m K) = g \otimes k^m = g^m \otimes k$$

□

Corollary 3.7. *Let G be a finite abelian group, with a representation as the product of cyclic groups*

$$G \cong \mathbb{Z}_{t_1} \times \cdots \times \mathbb{Z}_{t_n}$$

and define $d_{jk} = (t_j, t_k)$. Then

$$H^2(G, K) \cong \prod_{i=1}^{n} (K/K^{t_i}) \times \prod_{1 \leq j < k \leq n} K[d_{jk}]$$

Proof. The first result is the consequence of Corollary 3.6, Corollary 3.3 and Lemma 3.6. The second is the consequence of Corollary 3.6 and Corollary 3.5. □

Now we will give some examples. All the actions are suposed to be trivial:

Example 3.1. $H^2(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_n \otimes \mathbb{Z}_m \cong \mathbb{Z}_{(n,m)}$.

Proof. We have that $\mathbb{Z}_m/\mathbb{Z}_m^n \cong \mathbb{Z}_m[n] \cong \mathbb{Z}_{(m,n)}$, since they are cyclic groups of order (m,n): First, we call $d = (m,n)$. Then $\langle d \rangle = \mathbb{Z}_m^n$ so it has order m/d, and the quocient $\mathbb{Z}_m/\mathbb{Z}_m^n$ has order $m/(m/d) = d$. The group $\mathbb{Z}[n]$ is generated by m/d so it has order d, and $\mathbb{Z}_{(m,n)}$ is a cyclic group of order d by definition.
Finally using our results we have finished; by the last corollary $H^2(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_m/\mathbb{Z}_m^n$ and, by the lemma, $\mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_m[n]$. □

Example 3.2. $H^2(\prod_k \mathbb{Z}_n, \mathbb{Z}_m) \cong (\mathbb{Z}_{(n,m)})^{k+k(k-1)/2}$

Example 3.3. $H^2(\prod_k \mathbb{Z}_n, \prod_t \mathbb{Z}_m) \cong (H^2(\prod_k \mathbb{Z}_n, \mathbb{Z}_m))^t \cong (\mathbb{Z}_{(n,m)})^{t(k+k(k-1)/2)}$

4 Group Extensions

Definition 4.1. *A group extension of G by K is a short exact sequence*

$$1 \longrightarrow K \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1$$

We will say that K is the kernel group of the extension, G is que quotient group of the extension and E is the group in the middle of the extension.

The definition means that i is injective, $\text{im}(i) = \ker(\pi)$, and π is surjective. By the First Isomorphism Theorem, E has a normal subgroup $i(K) \cong K$ with factor group $E/i(K) = E/\ker(\pi) \cong G$.
We must observe the reciprocal is also true; if E is a group with normal subgroup $\tilde{K}$ with $\tilde{K} \cong K$ and $E/\tilde{K} \cong G$, then $1 \longrightarrow K \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1$ is a group extension of G by K defining $i = i \circ \psi$ with ψ the isomorphism between K and $\tilde{K}$, i the inclusion map of $\tilde{K}$ in E, and $\pi = \pi' \circ \varphi$ with π' the canonical projection of E to $E/\tilde{K}$ and φ the isomorphism between $E/\tilde{K}$ and G:

$$1 \longrightarrow K \overset{\psi}{\cong} \tilde{K} \xrightarrow{i} E \xrightarrow{\pi'} E/\tilde{K} \overset{\varphi}{\cong} G \longrightarrow 1$$

From now on, we will say that E is an extension of G by K to express that there exists an extension of G by K with E as the group in the middle and we will identify K with $i(K)$.

Example 4.1. *$\mathbb{Z}_6$ is an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_3$, because the subgroup $2\mathbb{Z}_6 \cong \mathbb{Z}_3$ and $\mathbb{Z}_6/2\mathbb{Z}_6 \cong \mathbb{Z}_2$. $\mathfrak{S}_3$ is also an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_3$, defining $i(\overline{a}) = (123)^a$ and $\pi(\sigma) = \text{sgn}(\sigma) \in \{\pm 1\} \cong \mathbb{Z}_2$.*

Example 4.2. *$\mathfrak{S}_n$ is an extension of $\mathbb{Z}_2$ by $\mathfrak{A}_n$ since $\mathfrak{S}_n/\mathfrak{A}_n \cong \mathbb{Z}_2$.*

Example 4.3. *Given $n \in \mathbb{N}$, the sequence $1 \longrightarrow \mathbb{Z}_n \xrightarrow{i} \mathbb{Z}_{2n} \xrightarrow{\pi} \mathbb{Z}_2 \longrightarrow 1$ is an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_n$, with $i(\overline{a}) = \overline{2a}$ and $\pi(\overline{k}) = \overline{k}$. Moreover, take the Dihedral group D_n (for $n \geq 3$), generated by the rotation r and the reflection s. Then $\langle r \rangle$ is isomorphic to $\mathbb{Z}_n$ and $D_n/\langle r \rangle \cong \mathbb{Z}_2$ so D_n is also an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_n$.*
Note that D_n is not isomorphic to $\mathbb{Z}_{2n}$, but both are extensions of $\mathbb{Z}_2$ by $\mathbb{Z}_n$, furthermore, $\mathbb{Z}_2$ and $\mathbb{Z}_n$ are abelian but D_n isn't.

Example 4.4. *Let G, K be two arbitrary groups. Define E as the direct product of K and G ($E = K \times G$ with the operation $(k, x)(h, y) = (kh, xy)$). Then $1 \longrightarrow K \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1$ is an extension of G by K defining $i(k) = (k, 1)$ and $\pi(k, x) = x$, so we can always construct an extension given G and K.*

We need to know when two extensions are different, in a certain way, to classify them. The first idea that comes to mind is to consider two extensions equivalent if their groups in the middle are isomorphic, but there is a more natural notion of equivalence:

Definition 4.2. *Two group extensions of G by K*

$$1 \longrightarrow K \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1$$

$$1 \longrightarrow K \xrightarrow{i'} E' \xrightarrow{\pi'} G \longrightarrow 1$$

are equivalent if there exists a homomorphism $\varphi : E \longrightarrow E'$ such that the diagram below commutes.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \xrightarrow{i} & E & \xrightarrow{\pi} & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle id_K} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle id_G} & & \\ 1 & \longrightarrow & K & \xrightarrow{i'} & E' & \xrightarrow{\pi'} & G & \longrightarrow & 1 \end{array}$$

This definition of equivalent extensions creates an equivalence relation on extensions. The set of equivalence classes of extensions of G by K will be denoted as $\mathrm{ext}(G, K)$.

Proposition 4.1. *If φ makes the diagram above commute, then φ has to be an isomorphism.*

Proof. Given $e \in E$ such that $\varphi(e) = 1$, we apply π' in the equality and using that φ commutes the diagram we have $1 = \pi'(\varphi(e)) = \pi(e)$ so $e \in K$. But $\varphi|_K = id$, so $e = 1$ and φ is injective.
Given $e' \in E'$, consider $\pi'(e') \in G$. Since π is surjective, there exists $f \in E$ such that $\pi(f) = \pi'(e')$. Since the diagram commutes, $\pi'(e) = \pi(f) = \pi'(\varphi(f))$. Now we consider $e'^{-1}\varphi(f)$, $\pi(e'^{-1}\varphi(f)) = \pi'(e^{-1})\pi'(e) = 1$ so $e'^{-1}\varphi(f) = k \in K$. Defining $e = fk^{-1}$ we are done, because using that $\varphi|_K = id$:

$$\varphi(e) = \varphi(f)k^{-1} = \varphi(f)\varphi(f)^{-1}e' = e'$$

and we conclude that φ is surjective. $\square$

We remark that if $1 \longrightarrow K \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1$ and $1 \longrightarrow K \xrightarrow{\tilde{i}} \tilde{E} \xrightarrow{\tilde{\pi}} G \longrightarrow 1$ are equivalent, then $E \cong \tilde{E}$. But, if $E \cong \tilde{E}$, then the extensions may be inequivalent. Here is an example:

Example 4.5. *There are two inequivalent extensions of $\mathbb{Z}_3$ by $\mathbb{Z}_3$ with group in the middle $\mathbb{Z}_9$.*

Proof. $\mathbb{Z}_9$ is an extension of $\mathbb{Z}_3$ by $\mathbb{Z}_3$, but we can define the surjective homomorphism $\pi : \mathbb{Z}_9 \to \mathbb{Z}_3$ in two different ways, which we will call π_1 and π_2. Let a be a generator of $\mathbb{Z}_9$, for $j = 1, 2$ we define $\pi_j(a) = \overline{j}$ and we have 2 extensions

$$1 \longrightarrow \langle a^3 \rangle \xrightarrow{i} \mathbb{Z}_9 \xrightarrow{\pi_j} \mathbb{Z}_3 \longrightarrow 1$$

which we will show they are not equivalent by contradiction. Suppose we have an isomorphism $\varphi : \mathbb{Z}_9 \to \mathbb{Z}_9$ with $\varphi|_{\langle a^3 \rangle} = id$ and $\pi_2 \circ \varphi = \pi_1$. Since it is an isomorphism, $\varphi(a) = a^k$ with $(k, 9) = 1$, $a^3 = \varphi(a^3) = a^{3k}$ and so $3k \equiv 3 \pmod 9 \Rightarrow k \equiv 1 \pmod 3$. On the other hand, $\pi_2(\varphi(a)) = \overline{2k} = \pi_1(a) = \overline{1}$ and this is impossible because $k \equiv 1 \pmod 3 \Rightarrow 2k \equiv 2 \pmod 3$. □

Definition 4.3. *If π is surjective, a section of π is a map $s : G \longrightarrow E$ with $s(x) := s_x$ such that $\pi(s_x) = x$. Hereinafter, we will always refer to a section of π as s.*

Remark 4.1. *In this paper a section will be such a map, and if we want to say that it is an homomorphism, we will refer to it as a section homomorphism. This is not a universal notation; in other works they refer to sections which are homomorphisms just as sections.*

In the following chapters, it will be useful an important lemma:

Lemma 4.1. *Given an extension of G by K and s, then $\forall e \in E$ there exist unique $k \in K$, $x \in G$ such that $e = ks_x$.*

Proof. Let $e \in E$ and $x := \pi(e)$. Then taking es_x^{-1}, we have that $\pi(es_x^{-1}) = \pi(e)\pi(s_x)^{-1} = xx^{-1} = 1$ so it belongs to $\ker(\pi) \Rightarrow es_x^{-1} = k \in K$ and we are done because we have $e = ks_x$. To prove uniqueness, suppose $e = ks_x = hs_y \Longrightarrow x = \pi(ks_x) = \pi(hs_y) = y \Longrightarrow x = y \Longrightarrow k = h$. □

A consequence of this lemma is that s is determined only up to multiplication by an element of K.

4.1 Splitting extensions and semidirect product

In this section we will see splitting extensions and their relation with the semidirect product of groups, which is the simplest way to to build an extension. Moreover, it will give us an idea of how to continue our study, specifically in the construction of extensions with K abelian.

Definition 4.4. *An extension* $1 \longrightarrow K \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1$ *splits if there exists a section homomorphism of* π.

Definition 4.5. *Suppose that* G *acts on* K *with the action* $\alpha : G \longrightarrow \mathrm{Aut}(K)$. *Then, the semidirect product of* K *by* G *defined by the action* α *is the set of all ordered pairs* $(k, x) \in K \times G$ *together with the operation*

$$(k, x)(h, y) = (k\,{}^{x}h, xy)$$

It will be denoted as $K \rtimes G$.

Proposition 4.2. *The semidirect product of* G *by* K *is a group with identity element* $(1, 1)$ *and inverse element* $(k, x)^{-1} = (({}^{x^{-1}}k)^{-1}, x^{-1}), \quad k \in K, \ x \in G$.

Proof.

- Associativity:

$$((k, x)(h, y))(f, z) = (k\,{}^{x}h, xy)(f, z) = (k\,{}^{x}h\,{}^{xy}f, xyz)$$

$$(k, x)((h, y)(f, z)) = (k, x)(h\,{}^{z}f, yz) = (k\,{}^{x}(h\,{}^{y}f), xyz) = (k\,{}^{x}h\,{}^{xy}f, xyz)$$

- Identity element and inverse element:

$$(1, 1)(k, x) = ({}^{1}k, x) = (k, x)$$

$$(({}^{x^{-1}}k)^{-1}, x^{-1})(k, x) = (({}^{x^{-1}}k)^{-1}({}^{x^{-1}}k), 1) = (1, 1)$$

□

Example 4.6. *The direct product of* K *and* G, $E = K \times G$ *is the semidirect product defined by the trivial action.*

Proposition 4.3. *Let* E *be a group. Then the following statements are equivalent:*

1. *There is an extension* $1 \longrightarrow K \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1$ *which splits.*
2. E *is a semidirect product of* K *by* G.
3. *There is a normal subgroup* $\tilde{K}$ *in* E, *with* $\tilde{K} \cong K$ *and* $\tilde{K}$ *has a complement* $\tilde{G} \cong G$ *in* E.

Proof.

- 1. ⇒ 2.
  Suppose we have $1 \longrightarrow K \stackrel{i}{\longrightarrow} E \stackrel{\pi}{\longrightarrow} G \longrightarrow 1$ which splits, and take a section s. Then, we define an action α of G on K in this way:

  $$^{x}k = s_x k s_x^{-1}$$

  This is clearly an action since K is normal. Then the map $f : E \longrightarrow K \rtimes G,\ f(e) = (k, x)$ is a bijection since the element x associated to an element e is unique, an it is a homomorphism:
  Given $e = ks_x$ and $f = hs_y$ and their unique expression of the Lemma 4.1: $k, h \in K,\ x, y \in G$ then $ef = ks_x hs_y = (ks_x hs_x^{-1})(s_x s_y) = (ks_x hs_x^{-1})(s_{xy})$ is the expression of the Lemma 4.1 with $ks_x hs_x^{-1} = k\ {}^{x}h \in K$ and $xy \in G$.
  So E is the semidirect product of K by G defined by this action.

- 2. ⇒ 3.
  Define the subgroup in $\tilde{K} \leq E$ of all the pairs $(k, 1),\ k \in K$. It is a subgroup ($(k, 1)(h, 1) = (kh, 1),\ \forall k, h \in K$ and $(1, 1) \in \tilde{K}$) isomorphic to K, and it is normal, let $k, h \in K$ and $x \in G$:

  $$(k, x)(h, 1)(k, x)^{-1} = (k\ {}^{x}h, x)(k, x)^{-1} = (k\ {}^{x}h, x)((^{x^{-1}}k)^{-1}, x^{-1}) = (k', 1) \in \tilde{K}$$

  Also, we define $\tilde{G} \leq E$ as all the pairs $(1, x),\ x \in G$. It is obviously a subgroup, $\tilde{K} \cap \tilde{G} = (1, 1)$ and $\tilde{K}\tilde{G} = E$.

- 3. ⇒ 1.
  First of all we identify $\tilde{K}$ with K and $\tilde{G}$ with G. By Proposition 2.2, there's a unique expression for every element $e \in E$ in the form $E = kx$, with $k \in K$ and $x \in G$.
  Furthermore, since K is normal in E, given $Ke \in E/K \Longrightarrow Ke = Kkx = Kx,\ x \in G$. Then, we define $s : E/K \cong G \longrightarrow E$ with $s(Ke) = x$. It's a well defined monomorphism, and so we have $E/K \cong Im(s) = G$:

  - Well defined. Given $e, e' \in E$ with $Ke = Ke'$, we have:

    $$e = kx,\ e = k'x' \Rightarrow Kx = Kx' \Rightarrow Kx'x^{-1} = K \Rightarrow x'x^{-1} \in K \cap G \Rightarrow x'x^{-1} = 1$$

    So $x' = x$ and $s(Ke) = s(Ke')$.We have used that, since the representation $e = kx$ is unique, if $e \in K \cap G \Longrightarrow e = k1 = 1x \Longrightarrow e = k = x = 1$ so $K \cap G = 1$.
  - Homomorphism. $e, e' \in E$ then $s(Kee') = s(Kxk'x') = s(KxKx') = s(Kxx') = xx' = s(Ke)s(Ke')$ and $l(K1) = 1$.
  - Injective. Given $e, e' \in E$, by uniqueness of the representation:

    $$s(Ke) = s(Ke') \Longrightarrow x = x' \Longrightarrow e = e'$$

Furthermore, we have the extension $1 \longrightarrow K \xrightarrow{i} E \xrightarrow{\pi} E/K \cong G \longrightarrow 1$ and it is clear that s is a section of π.

$\square$

Example 4.7. *Consider the alternate group $\mathfrak{A}_4$, the Klein-four group $V_4 = \{id, (12)(34), (13)(24), (14)(23)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ normal in $\mathfrak{A}_4$, and $G = \langle (123) \rangle \cong \mathbb{Z}_3$. Since $(|V_4|, |G|) = 1$ and $|V_4||G| = |\mathfrak{A}_4|$, $\mathfrak{A}_4$ is a semidirect product of V_4 by G but not the direct product, because it is not abelian.*

Example 4.8. *D_n is a semidirect product of $\mathbb{Z}_n$ by $\mathbb{Z}_2$ since $\langle r \rangle \lhd D_n$, $\langle r \rangle \cap \langle s \rangle = 1$ and $\langle r, s \rangle = D_n$. Moreover, we can determine the action since $srs = r^{-1}$, so ${}^{s^k}(r^i) = (r^i)^{(-1)^k}$. Also we know that $\mathbb{Z}_{2n}$ is the direct product of $\mathbb{Z}_n$ and $\mathbb{Z}_2$ when n is odd, but for n even the direct product $\mathbb{Z}_n \times \mathbb{Z}_n$ is not cyclic.*

Example 4.9. *$\mathbb{Z}_8$ is an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_4$ but it is not a semidirect product since we cannot find subgroups $K \cong \mathbb{Z}_4$ and $G \cong \mathbb{Z}_2$ such that $K \cap G = 1$.*

4.2 Extensions with abelian kernel

In this section, we will study extensions of G by K with K abelian. We will see how group cohomology naturally appears with the fact that we have a bijection between equivalent extensions of G by K and the second cohomology group in a certain way.

4.2.1 Action and cocycle associated to an extension

We have seen in the last proof that given a section s we can define an action α of G on K such that $\alpha_x(k) := {}^x k = s_x k s_x^{-1}$ (we note that we don't need s to be a homomorphism). Later we will see that for extensions with non-abelian kernel we can define generalisations of action and cocycle, but this new elements won't necessarily define a group.

Lemma 4.2. *Let s, s' be sections of π. Then $s_x k s_x^{-1} = s'_x k s'^{-1}_x$, $\forall k \in K$ and so the action α is well determined by the extension.*

Proof. Take $k \in K$. By definition $\pi(s_x) = \pi(s'_x)$, then $\pi(s_x^{-1} s'_x) = 1 \Longrightarrow s_x^{-1} s'_x \in K$, but K is abelian, so $(s_x^{-1} s'_x)k = k(s_x^{-1} s'_x) \Longrightarrow s_x k s_x^{-1} = s'_x k s'^{-1}_x$. □

Hereinafter, we will take s such that $s_1 = 1$. With this, we can define the semidirect product of K by G, but this is only one of the possible extensions of G by K so we do need more information. At this point, there naturally appears the 2-cocycle f associated to our extension and s_x:

Definition 4.6. *The 2-cocycle f associated to an extension and a section s is:*

$$f : G \times G \longrightarrow K$$
$$(x, y) \longrightarrow f(x, y) = s_x s_y s_{xy}^{-1}$$

We observe that this is equivalent to define $f(x, y)$ such that $s_x s_y = f(x, y) s_{xy}$ so, s is a homomorphism if and only if f is the trivial cocycle.

Proposition 4.4. *The 2-cocycle f associated to our extension and s is a normalized 2-cocycle if $s_1 = 1$.*

Proof. Since $s_1 = 1$, $f(1, x) = s_1 s_x s_x^{-1} = 1 = s_x s_1 s_x^{-1} = f(x, 1)$. Using that E is associative, we have $(s_x s_y)s_z = s_x(s_y s_z)$ and from this follows the 2-cocycle condition:

$$\begin{aligned}(s_x s_y)s_z &= f(x, y)s_{xy}s_z = f(x, y)f(xy, z)e_{xyz} \\ s_x(s_y s_z) &= s_x(f(y, z)s_{yz}) = s_x f(x, y)s_x^{-1} s_x s_{yz} = {}^x f(y, z)s_x s_{yz} \\ &= {}^x f(x, yz)s_{xyz} = f(x, y)f(xy, z)s_{xyz}\end{aligned}$$

□

As we have seen in the proof, the 2-cocycle condition follows from the associativity of E and the choice of f depends on the section s, so it's not unique.

4.2.2 The relation between $H^2(G, K)$ and extensions

We have seen that we can associate a cocycle f to the extension, given an choice of s. Now we will see that, given an action α of G on K, there's a bijection between $H^2(G, K)$ and the classes of extensions of G by K associated to α, so the first step is to see that the class of f in $H^2(G, K)$ does not depend on the choice of s.

Proposition 4.5. *Let s, s' be two sections. Suppose that $s'_x = k_x s_x$ for some $k_x \in K$ (we can suppose this because of Lemma 4.1), and f, f' the 2-cocycles associated to s and s'. Then f and f' are cohomologous.*

Proof.

$$\begin{aligned} f'(x, y) &= (k_x s_x)(k_y s_y)(k_{xy} s_{xy})^{-1} = k_x s_x k_y s_x^{-1} s_x s_y s_{xy}^{-1} k_{xy}^{-1} \\ &= k_x s_x k_y s_x^{-1} f(x, y) k_{xy}^{-1} = k_x\, {}^x k_y f(x, y) k_{xy}^{-1} = {}^x k_y k_x k_{xy}^{-1} f(x, y) \end{aligned}$$

Defining $b(x, y) = {}^x k_y k_x k_{xy}^{-1}$, it is obviously a 2-coboundary by definition and we have $f'(x, y) = b(x, y) f(x, y)$. $\square$

From this immediately follows the first important result of our study:

Corollary 4.1. *Let $1 \longrightarrow K \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1$ be an extension of G by K. Given a section s, an action α of G on K defined by ${}^x k = s_x k s_x^{-1}$ and the 2-cocycle f defined by $f(x, y) = s_x s_y s_{xy}^{-1}$, then the class of f in $H^2(G, K)$ does not depend on the choice of s.*

As a consequence, each extension whose action is α has a cohomology class in $H^2(G, K)$ associated to it. Furthermore, this is conserved by the equivalence of extensions:

Proposition 4.6. *Suppose we have two equivalent extensions*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \xrightarrow{i} & E & \xrightarrow{\pi} & G & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle id_K} & & \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle id_G} & & \\ 1 & \longrightarrow & K & \xrightarrow{\tilde{i}} & \tilde{E} & \xrightarrow{\tilde{\pi}} & G & \longrightarrow & 1 \end{array}$$

then the action and the cohomology classes associated to both extensions are the same.

Proof. First, as always we suppose K is a subgroup of E and $\tilde{E}$. Given s, define $s'_x := \varphi(s_x)$. Since the diagram commutes, $\tilde{\pi}(s'_x) = \pi(s_x) = x$, so s' is also a section of $\tilde{\pi}$. With this, we have that the action associated to the second extension is $\alpha'_x = s'_x k s'^{-1}_x$ and the cocycle class of the extension is the class of $f'(x, y) = s'_x s'_y s'^{-1}_{xy}$. But, since the diagram commutes, $\varphi(k) = k, \forall k \in K$:

$$s_x k s_x^{-1} = \varphi(s_x k s_x^{-1}) = \varphi(s_x)\varphi(k)\varphi(s'_x)^{-1} = s'_x k s'^{-1}_x$$

$$f'(x,y) = \varphi(s_x s_y s_{xy}^{-1}) = s_x s_y s_{xy}^{-1}$$

and so the action and the cocycle class are the same. □

We will denote the set of equivalent extensions whose action is α by $\text{ext}(G,K)_\alpha$.

What follows is to construct the inverse relation. In other words, given an action α of G on K and a 2-cocycle $f \in Z^2(G,K)$ associated to α, we have to construct an extension of G by K. This lemma gives us the idea to do it:

Lemma 4.3. *Given $ks_x, hs_y \in E$ with $k,h \in K$ then their product in this form is:*

$$(ks_x)(hs_y) = k\, {}^x h f(x,y) s_{xy}$$

with $k^x hf(x,y) \in K$.

Proof.

$$(ks_x)(hs_y) = ks_x h s_x^{-1} s_x s_y = k\, {}^x h s_x s_y = k\, {}^x h f(x,y) s_{xy}$$

□

Example 4.10. *E is the direct product of G and K if and only if the action and the cocycle class associated to it are trivial.*

Example 4.11. *Consider $\mathbb{Z}_6$ as an extension of $\mathbb{Z}_2 \cong \mathbb{Z}_6/\langle a^2 \rangle$ by $\mathbb{Z}_3 \cong \langle a^2 \rangle$. Observe that, if E is abelian, then the action associated to the extension is trivial. Now take s_x as $s_1 = 1$, $s_{\overline{a}} = a$. Then the cocycle associated to it is $f(1,1) = f(1,\overline{a}) = f(\overline{a},1) = 1$, $f(\overline{a},\overline{a}) = s_{\overline{a}} s_{\overline{a}} 1 = a^2$, which is a coboundary (the cohomology class of f is 1), since $\mathbb{Z}_6$ is the direct product of $\mathbb{Z}_2$ and $\mathbb{Z}_3$.*

Example 4.12. *Now consider $\mathfrak{S}_3$ as an extension of $\mathbb{Z}_2 \cong \mathbb{Z}_6/\langle(123)\rangle$ by $\mathbb{Z}_3 \cong \langle(123)\rangle$. Take $s_1 = id$, $s_{\overline{a}} = (12)$. The action of $\mathbb{Z}_2$ in $\mathbb{Z}_3$ defined by this extension is ${}^{\overline{a}}k = k^{-1}$ the inverse action, because $(12)(123)(12) = (132) = (123)^{-1}$. The cocycle class is trivial, since $f(1,\overline{a}) = f(\overline{a},1) = f(1,1) = 1$, $f(\overline{a},\overline{a}) = (12)(12)1 = id$.*

Now we he have an idea of how to construct an extension given an action and a 2-cocycle. It will be very similar to the semidirect product, but adding the information of the cocycle, which will give us the associativity in E.

Definition 4.7. *Given an action α of G on K and a cocycle $f \in Z^2(G,K)$ associated to α (which we can suppose that is normalized), we define E_f, the group associated to the cocycle f. As a set, $E_f := K \times G$, and the operation in E_f is defined by*

$$(k,x)(h,y) = (k\, {}^x h f(x,y), xy)$$

Proposition 4.7. *Let E_f be the construction in the previous definition. Then, it has the following properties:*

1. *E_f is a group, with identity $(1,1)$ and inverse element $(k,x)^{-1} = (f(x^{-1},x)^{-1}({}^{x^{-1}}k)^{-1}, x^{-1})$.*
2. *We have that $1 \longrightarrow K \xrightarrow{i} E_f \xrightarrow{\pi} G \longrightarrow 1$ is an extension of G by K with $i(k) = (k,1)$ and $\pi(k,x) = x$.*
3. *The action of this extension is α and the cocycle class is the class of f.*

Proof. 1. • Identity element.

$$(k,x)(1,1) = (k\,{}^{1}f(1,1), x) = (k,x)$$

• Inverse element.

$$\begin{aligned}(f(x^{-1},x)^{-1}({}^{x^{-1}}k)^{-1}, x^{-1})(k,x) &= (f(x^{-1},x)^{-1}({}^{x^{-1}}k)^{-1}({}^{x^{-1}}k)f(x^{-1},x), 1) \\ &= (f(x^{-1},x)^{-1}f(x^{-1},x), 1) = (1,1)\end{aligned}$$

• Associativity

$$\begin{aligned}((k,x)(h,y))(t,z) &= (k\,{}^{x}hf(x,y), xy)(t,z) = (k\,{}^{x}hf(x,y)^{xy}tf(xy,z), xyz) \\ (k,x)((h,y)(t,z)) &= (k,x)(h\,{}^{y}tf(y,z), yz) = k\,{}^{x}(h\,{}^{y}tf(y,z))f(x,yz) \\ &= (k\,{}^{x}h\,{}^{xy}t\,{}^{x}f(y,z)f(x,yz), xyz) = (k\,{}^{x}h\,{}^{xy}tf(x,y)f(xy,z), xyz) \\ &= (k\,{}^{x}hf(x,y)\,{}^{xy}tf(xy,z), xyz) = ((k,x)(h,y))(t,z)\end{aligned}$$

2. Trivially i and π are injective and surjective respectively, and $\ker(\pi) = K$.

3. First, we set $s_x := (1,x)$. Hence, the action associated to the extension is:

$$\begin{aligned}\alpha'_x((k,1)) &= s_x(k,1)s_x^{-1} = (1,x)(k,1)({}^{x^{-1}}(f(x^{-1},x))^{-1}, x^{-1}) \\ &= ({}^{x}k\,{}^{x}({}^{x^{-1}}(f(x^{-1},x))^{-1})f(x,x^{-1}), 1) \\ &= ({}^{x}kf(x^{-1},x)^{-1}f(x,x^{-1}), 1) = ({}^{x}k, 1) = (\alpha_x(k), 1)\end{aligned}$$

The cocycle associated to E_f is:

$$\begin{aligned}f'(x,y) = s_xs_ys_{xy}^{-1} &= (f(x,y), xy)({}^{(xy)^{-1}}(f((xy)^{-1}, xy))^{-1}, (xy)^{-1}) \\ &= (f(x,y)\,{}^{xy}({}^{(xy)^{-1}}(f((xy)^{-1}, xy))^{-1})f(xy,(xy)^{-1}), 1) = (f(x,y), 1)\end{aligned}$$

□

We will use all of these results to achieve our objective: to prove the Main Theorem of this thesis.

Theorem 4.1. *Given an action $\alpha : G \longrightarrow \mathrm{Aut}(K)$, there's a bijection between* $\mathrm{ext}(G, K)_\alpha$ *and* $H^2(G, K)_\alpha$.

Proof. Given an extension $1 \longrightarrow K \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1$ of G by K we have constructed an action α and a cocycle class in $H^2(G, K)$ associated to it, and we have seen that equivalent extensions give us the same action and the same cocycle class. Thus, we have a well-defined map from $\mathrm{ext}(G, K)_\alpha$ to $H^2(G, K)$. We will show that this map is a bijection. Given $f \in Z^2(G, K)$, we have constructed an extension $1 \longrightarrow K \xrightarrow{i} E_f \xrightarrow{\pi} G \longrightarrow 1$ whose action is α and whose associated cocycle class is the class of f, so this proves surjectivity.
To prove injectivity, suppose we have two extensions $1 \longrightarrow K \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1$ and $1 \longrightarrow K \xrightarrow{i'} E' \xrightarrow{\pi'} G \longrightarrow 1$ whose action is α and whose associated cocycle classes are the same in $H^2(G, K)$, and we will see they are equivalent. First, we can suppose that they are represented by the same cocycle f. So we have that there are two choices s and s' such that $f(x, y) = s_x s_y s_{xy}^{-1} = s'_x s'_y s'^{-1}_{xy}$. With this, as the elements in E have a unique representation in the form ks_x and the elements in E' in the form ks'_x, we can define a map $\phi : E \to E'$ with $\phi(ks_x) = ks'_x$. It is a bijection because the representation is unique, and it is a group homomorphism (and so an isomorphism) due to Lemma 4.3 (since the cocycle and the action in both extensions are the same, the operation has the same form $(ks_x)(hs_y) = k^x h f(x, y) s_{xy}$ and $(ks'_x)(hs'_y) = k^x h f(x, y) s'_{xy}$). At last, $\pi'(\phi(ks_x)) = \pi'(ks'_x) = x = \pi(ks_x)$, and $\phi|_K = id$, this shows that both extensions are equivalent. □

Remark 4.2. *With this bijection, we can define a group structure in ext$(G, K)_\alpha$. So, given two extensions of G by K, we can define the extension of G by K which is the product of both extensions. With the non-trivial example which we will study later, we will see how this works.*

Moreover, with the construction of the proof we can directly conclude two important results:

Corollary 4.2. *Given an action $\alpha : G \longrightarrow \mathrm{Aut}(K)$, then $H^2(G, K) = 1$ if and only if the unique extension associated to α is the semidirect product $K \rtimes G$.*

Proof. If $H^2(G, K) = 1$, we can only construct the extension associated to the trivial cocycle, so we will have that any section s is a homomorphism, then it is an splitting extension, and so a semidirect product.
If we have that the unique extension of G by K is the semidirect product, then $|H^2(G, K)| = 1$ so $H^2(G, K) = 1$. □

Corollary 4.3. *If K, G are finite and $(|K|, |G|) = 1$ all the extensions of G by K are semidirect products.*

Example 4.13. *All the extensions of $\mathbb{Z}_2$ by $\mathbb{Z}_3$ are the direct product $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathfrak{S}_3$ (we can only make them in one way, so we can identify in this case the middle group with the extension), since we only have two possible extensions associated to the two actions of $\mathbb{Z}_2$ in $\mathbb{Z}_3$, because $(2, 3) = 1$.*

Example 4.14. *Extensions of $\mathbb{Z}_p$ by $\mathbb{Z}_p$, with p prime. The only action of $\alpha : \mathbb{Z}_p \longrightarrow \mathrm{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}$ is the trivial action. Also, we know that, with this action, $H^2(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_{(p,p)} \cong \mathbb{Z}_p$ so there are p (inequivalent) extensions of $\mathbb{Z}_p$ by $\mathbb{Z}_p$. But the only groups that have cardinal p^2 are the direct product $\mathbb{Z}_p \times \mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$; there is one extension corresponding to the direct product and $p - 1$ corresponding to $\mathbb{Z}_{p^2}$.*

Example 4.15. *Extensions of $\mathbb{Z}_p$ by $\mathbb{Z}_q$, with $p < q$ primes such that $p|(q - 1)$. We know there is a nontrivial action of $\mathbb{Z}_p$ in $\mathbb{Z}_q$, and with this action $H^2(\mathbb{Z}_p, \mathbb{Z}_q) = 1$. With the trivial action, $H^2(\mathbb{Z}_p, \mathbb{Z}_q) = 1$ too, so all the extensions will be semidirect products.*

Example 4.16. *If G is a free grop, then $H^n(G, K) = 1$ for all $n \geq 2$. Particularly, all the extensions of G by K are semidirect products.*

Proof. We will prove it for $n = 2$ and the rest will follow trivially by induction using Theorem 3.2.

Take any cocycle $f \in Z^2(G, K)$ and construct the group extension E associated to it. For every generator $x_i \in G$ take an arbitrary s_{x_i}, and for any element $x = x_1^{n_1} \dots x_k^{n_k} \in G$ choose the corresponding s_x as

$$s_x = s_{x_1}^{n_1} \dots s_{x_k}^{n_k}$$

With this choice of s_x it is clear that s is a homomorphism so the cocycle class corresponding to the extension is trivial, then $H^2(G, K) = 1$. This is, in fact, a consequence of the universal property of free groups. □

4.2.3 Abelian extensions

Now we will see the conditions we need so that E is an abelian group. First, if E is abelian, we observe that K and G are necessarily abelian, because K is a subgroup of E and $G \cong E/K$. But this is not a sufficient condition, as we have seen in some examples.

Lemma 4.4. *If E is abelian, then the action associated to the extension is trivial.*

Proof. We have that ${}^x k = s_x k s_x^{-1} = s_x s_x^{-1} k = k$ and it is trivial. □

Proposition 4.8. *Let G be abelian, and $1 \longrightarrow K \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1$ an extension of G by K with trivial action. Then E is abelian if and only if any cocycle f associated to the extension is symmetric.*

Proof. Suppose E is abelian. Take a section s, then the cocycle $f(x,y) = s_x s_y s_{xy}^{-1} = s_y s_x s_{yx}^{-1} = f(y,x)$ is symmetric.
Now suppose $f(x,y) = s_x s_y s_{xy}^{-1}$ is symmetric, given s. We have supposed that G (and K) are abelian. Take $ks_x, hs_y \in E$ in its unique representation. Then, by Lemma 4.3 $(ks_x)(hs_y) = k\,{}^x h f(x,y)s_{xy} = khf(x,y)s_{xy}$ and $(hs_y)(ks_x) = h\,{}^y k f(y,x)s_{yx} = hkf(y,x)s_{xy} = khf(x,y)s_{xy}$ so $(ks_x)(hs_y) = (hs_y)(ks_x)$ for all $ks_x, hs_y \in E$ but this is a representation of all the elements of E, so E is abelian. □

Remark 4.3. *If we define $Ext(G,K)$ the set of abelian extensions of G by K (up to equivalence), since $H^2(G,K)_s$ is a subgroup of $H^2(G,K)$, there is a induced group structure in $Ext(G,K) \cong H^2(G,K)_s$.*

Moreover, we can now prove a results of cohomology using group extensions:

Corollary 4.4. *Suppose $\mathbb{Z}_n$ acts trivially con K. Then*

$$H^2(\mathbb{Z}_n, K) = H^2(\mathbb{Z}_n, K)_s$$

Proof. Let $1 \longrightarrow K \xrightarrow{i} E \xrightarrow{\pi} \mathbb{Z}_n \longrightarrow 1$ be an extension of $\mathbb{Z}_n$ by K and $f \in Z^2(G,K)$ any cocycle associated to the extensions. Then, we have that E is a group with a normal abelian subgroup K with quotient group $E/K \cong \mathbb{Z}_n$ cyclic and then so E is abelian and the cocycle is symmetric. □

With this corollary and the results we had in the first section, we have also another consequence:

Corollary 4.5. *Let $G = \mathbb{Z}_{t_i} \times \cdots \times \mathbb{Z}_{t_n}$ be a finite abelian group acting trivially on K. Then*

$$H^2(G,K)_s \cong \prod_{i=1}^{n} (K/K^{t_i})$$

4.2.4 Non-trivial example: Extensions of $\mathbb{Z}_2 \times \mathbb{Z}_2$ by $\mathbb{Z}_2$

In this section we will call G to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and K to $\mathbb{Z}_2$. The only action of G on K is the trivial action. Then, we will only have only one cohomology group $H^2(G,K)$ and the extensions of G by K are in bijection to $H^2(G,K)$.

There are five groups of order 8 that have K as a normal subgroup:

1. The elementary abelian group $E_8 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

2. The direct product $H = \mathbb{Z}_4 \times \mathbb{Z}_2$.

3. The dihedral group D_8.

4. The quaternion group Q_8.

5. The cyclic group $\mathbb{Z}_8$. This group doesn't define an extension of G by K because $\mathbb{Z}_8/K$ is a cyclic group, but G isn't.

At first, we will calculate $H^2(G, K)$. Using the same notation as in Theorem 3.3 we have that $G_1 = G_2 = \mathbb{Z}_2$. Using the theorems of the second cohomology group with trivial action, we know all we want:

$$H^2(\mathbb{Z}_2, \mathbb{Z}_2) = H^2(\mathbb{Z}_2, \mathbb{Z}_2)_s \cong \mathbb{Z}_{(2,2)} = \mathbb{Z}_2$$

and with this, we can calculate $H^2(G, K)$ and $H^2(G, K)_s$:

$$H^2(G, K) \cong H^2(\mathbb{Z}_2, K) \times H^2(\mathbb{Z}_2, K) \times \mathrm{Hom}(\mathbb{Z}_2 \otimes \mathbb{Z}_2, K)$$
$$H^2(G, K)_s \cong H^2(\mathbb{Z}_2, K)_s \times H^2(\mathbb{Z}_2, K)_s$$

and using that $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2$ we have finished:

$$H^2(G, K) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$
$$H^2(G, K)_s \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

Secondly, we will compute the eight cocycles that generate $H^2(G, K)$. Using the isomorphism in the proof of Theorem 3.3, we know that:

$$f((x_1, y_1), (x_2, y_2)) = \alpha_1(x_1, x_2) + \alpha_2(y_1, y_2) + \beta(x_1, y_2)$$

where f are the cocycles that generate $H^2(G, K)$, α_1 the cocycles that generate $H^2(G_1, K)$ and α_2 the cocycles that generate $H^2(G_2, K)$. Also, $\beta \in P(\mathbb{Z}_2, K)$. With this, since G_1, G_2 are cyclic groups, the cocycle that generates $H^2(G_1, K)$ is $\alpha_1(x_1, x_2) = x_1 x_2$ and the cocycle that generates $H^2(G_2, K)$ if $\alpha_2(y_1, y_2) = y_1 y_2$. Also, the pairing that generates $P(\mathbb{Z}_2, K)$ is $\beta(x_1, y_2) = x_1 y_2$. So the eight cocycles that generate $H^2(G, K)$ are the following:

1. $f_{(0,0,0)}((x_1, y_1), (x_2, y_2)) = 0$

2. $f_{(1,0,0)}((x_1, y_1), (x_2, y_2)) = x_1 x_2$.

3. $f_{(0,1,0)}((x_1, y_1), (x_2, y_2)) = y_1 y_2$

4. $f_{(1,1,0)}((x_1, y_1), (x_2, y_2)) = x_1 x_2 + y_1 y_2$

5. $f_{(0,0,1)}((x_1, y_1), (x_2, y_2)) = x_1 y_2$

6. $f_{(1,0,1)}((x_1, y_1), (x_2, y_2)) = x_1x_2 + x_1y_2$

7. $f_{(0,1,1)}((x_1, y_1), (x_2, y_2)) = y_1y_2 + x_1y_2$

8. $f_{(1,1,1)}((x_1, y_1), (x_2, y_2)) = x_1x_2 + y_1y_2 + x_1y_2$

Where if we have $f_{(a,b,c)}$, then (a, b, c) is its representation in $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$. Since the coefficient of β is 0 in the first four cocycles, $f_{(0,0,0)}, f_{(1,0,0)}, f_{(0,1,0)}, f_{(1,1,0)}$ are the cocycles associated to abelian extensions.
Then we know there are 8 extensions of G by K; 4 abelian and 4 non-abelian:

a) E abelian.
The group E_8 is the direct product of G and K so it has the extension associated with the trivial cocycle.
The other abelian group is H, so it has 3 associated inequivalent extensions of G by K. We will define $\pi : H \longrightarrow G$ in 3 following ways and see that we have $f_{(1,0,0)}, f_{(0,1,0)}, f_{(1,1,0)}$ associated to each extension.
First, we will define π_1 to be $\pi_1(a, b) = (a, b)$. Then, we will choose $e_{(x_1,y_1)} = (x_1, y_1)$ and the cocycle associated to the extension will be

$$2f((x_1, y_1), (x_2, y_2)) = (x_1 + x_2) - ((x_1 + x_2) \mod 2) = x_1x_2$$

so we observe that the associated cocycle is $f_{(1,0,0)}$. In the same way, we define $\pi_2(a, b) = (b, a)$, and the cocycle associated to the extension will be $f_{(0,1,0)}$.
The choice of π_3 will be $\pi_3((1, 0)) = (1, 1)$ and $\pi_3(0, 1) = (1, 0)$. With this, we choose $e_{(x_1,y_1)} = (x_1 + y_1, y_1)$ and the cocycle is

$$2f(x, y) = (x_1 + x_2) - ((x_1 + x_2) \mod 2) + (y_1 + y_2) - ((y_1 + y_2) \mod 2)$$

the associated cocycle $f_{(1,1,0)}$ since it is the sum of $f_{(1,0,0)}$ and $f_{(0,1,0)}$.

b) E non-abelian.
We will see we have 3 inequivalent extensions with D_8 as the group in the middle:
We define $\pi_4 : D_8 \longrightarrow G$ with $\pi_4(r) = (0, 1)$ and $\pi_4(s) = (1, 0)$, we observe that K in this case will be $K = \langle r^2\rangle$. With the choice of $e_{(x_1,y_1)} = r^{y_1}s^{x_1}$, we compute the cocycle associated to it:

$$f(x, y) = r^{y_1}s^{x_1}r^{y_2}s^{x_2}s^{x_1+x_2}r^{-(y_1+y_2)} = r^{y_1}s^{x_1}r^{y_2}s^{x_1}r^{-(y_1+y_2)}$$
$$= \begin{cases} r^{y_1}r^{y_2}r^{-(y_1+y_2)} = 1 \;\; if\ x_1 = 0 \\ r^{y_1}s^{x_1}s^{x_1}r^{-y_1} = 1 \;\; if\ y_2 = 0 \\ r^2\ otherwise \end{cases}$$

so the associated cocycle is $f_{(0,0,1)}(x,y) = x_1 y_2$.
Secondly, we define $\pi_6(r) = (1,1)$ and $\pi_6(s) = (1,0)$. Then $e_{(x_1,x_2)} = r^{x_2} s^{x_1+x_2}$ so

$$f(x,y) = r^{y_1} s^{x_1-y_1} r^{y_2} s^{x_2+y_2} s^{x_1+y_1+x_2+y_2} r^{-(y_1+y_2)} = r^{y_1} s^{x_1-y_1} r^{y_2} s^{x_1+y_2} r^{-(y_1+y_2)}$$
$$= \begin{cases} 1 \ \ if \ x_2 = y_2 \ or \ y_1 = 0 \\ r^2 \ otherwise \end{cases}$$

and we observe that $f(x,y) = y_2(x_1+y_1)$ and then the cocycle associated is $f_{(0,1,1)}$.
Finally, in the same way, if we define $\pi_5(r) = (1,1)$ and $\pi_5(s) = (0,1)$ the cocycle associated to the extension will be $f_{(1,0,1)}$.
At last, $f_{(1,1,1)}$ will be the cocycle associated to any of the extensions with Q_8.

Putting everything together, we have eight inequivalent extensions of $\mathbb{Z}_2 \times \mathbb{Z}_2$ by $\mathbb{Z}_2$:

1. $1 \longrightarrow K \xrightarrow{i} \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{\pi} \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$ with $K = \langle (0,0,1) \rangle$, π de canonical projection in H/K and trivial cocycle.

2. $1 \longrightarrow K \xrightarrow{i} \mathbb{Z}_4 \times \mathbb{Z}_2 \xrightarrow{\pi_1} \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$ with $K = \langle (2,0) \rangle$, $\pi_1(a,b) = (a,b)$ and cocycle $f_{(1,0,0)}$.

3. $1 \longrightarrow K \xrightarrow{i} \mathbb{Z}_4 \times \mathbb{Z}_2 \xrightarrow{\pi_2} \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$ with $K = \langle (2,0) \rangle$, $\pi_2(a,b) = (b,a)$ and cocycle $f_{(0,1,0)}$.

4. $1 \longrightarrow K \xrightarrow{i} \mathbb{Z}_4 \times \mathbb{Z}_2 \xrightarrow{\pi_3} \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$ with $K = \langle (2,0) \rangle$, $\pi_3(a,b) = (a+b,a)$ and cocycle $f_{(1,1,0)}$.

5. $1 \longrightarrow K \xrightarrow{i} D_8 \xrightarrow{\pi_4} \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$ with $K = \langle r^2 \rangle$, $\pi_4(r^a s^b) = (a,b)$ and cocycle $f_{(0,0,1)}$.

6. $1 \longrightarrow K \xrightarrow{i} D_8 \xrightarrow{\pi_5} \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$ with $K = \langle r^2 \rangle$, $\pi_5(r^a s^b) = (a, a+b)$ and cocycle $f_{(1,0,1)}$.

7. $1 \longrightarrow K \xrightarrow{i} D_8 \xrightarrow{\pi_6} \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$ with $K = \langle r^2 \rangle$, $\pi_6(r^a s^b) = (a+b, a)$ and cocycle $f_{(0,1,1)}$.

8. $1 \longrightarrow K \xrightarrow{i} Q_8 \xrightarrow{\pi} \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$ with $K = \langle -1 \rangle$, π the canonical projection in Q_8/K and cocycle $f_{(1,1,1)}$.

4.3 Schur-Zassenhaus theorem

Group extensions have a lot of useful applications in group theory and Galois theory. Here we will prove an important theorem of group theory using our results of extensions with abelian kernel. During all this chapter we will suppose E to be a finite group and K a normal Hall subgroup of E, with $|E| = mn$ and $|K| = m$ (so $|E/K| = n$ and $(m, n) = 1$). We want to study if K has a complement and its properties.

Proposition 4.9. *Suppose K is abelian. Then K has a complement.*

Proof. Since K is normal in E, we can define the group $G = E/K$ and E will be an extension of G by K. But K is a Hall subgroup, so $(|K|, |G|) = 1$ and then by Corollary 4.3 all the extensions of G by K are semidirect products of K and G, so E is a semidirect product of K and G. By Proposition 4.3, K has a complement in E. □

Proposition 4.10. *If K is abelian, let G_1, G_2 be complements of K, then G_1 and G_2 are conjugates.*

Proof. We know that E is a semidirect product of G_1 by K and a semidirect product of G_2 by K. So we can construct $1 \longrightarrow K \xrightarrow{i} E \xrightarrow{\pi_1} G_1 \longrightarrow 1$ and $1 \longrightarrow K \xrightarrow{i} E \xrightarrow{\pi_2} G_2 \longrightarrow 1$ the corresponding extensions, and we can indentify G_1 and G_2 with E/K. Then, since E is a semidirect product, we can construct the corresponding sections of π_1 and π_2, and we will have both homomorphism sections $s_1, s_2 : E/K \longrightarrow E$ such that $s_i(E/K) = G_i$.

Now define $h : E/K \longrightarrow K$, with $s_1(x) = h(x)s_2(x)$, and f_1, f_2 the cocycle corresponding to each s_i. We know that f_1 and f_2 are trivial, so:

$$1 = f_1(x, y)f_2(x, y) = {}^{x}h(y)h(xy)^{-1}h(x)$$

and multiplying for all $y \in E/K$ gives us an equality

$$1 = {}^{x}a_0 a_0^{-1}({}^{n}h(x))$$

with $a_0 = \prod_{y \in E/K} h(y)$. Now, using that $(s, m) = 1$, there exists $s, t \in \mathbb{Z}$ such that $sm + tn = 1$, and define $b_0 = ta_0$. Finally, using the last equality and $|K| = m$:

$$h(x) = h(x)({}^{sm}h(x))^{-1} = {}^{-tn}h(x) = {}^{x}b_0 b_0^{-1}$$

Finally, $b_0 G_1 b_0^{-1} = G_2$ because $b_0 s_1(x) b_0^{-1} = s_2(x)$:

$$(b_0 s_1(x) b_0^{-1}) = b_0^{-1}({}^{x}b_0)s_1(x) = h(x)^{-1}s_1(x) = h(x)^{-1}h(x)s_2(x) = s_2(x)$$

□

Now we will use the two last results to prove this important theorem:

Theorem 4.2 (Schur-Zassenhaus, 1937)**.** *Let K be a normal Hall subgroup of a finite group E. Then K has a complement.*

Proof. We will prove by induction on $|K| = m$ that E contains a subgroup of order n. The case $m = 1$ is true (trivially). Now we will divide it in 2 cases:

1. K contains a proper subgroup K' which $K' \lhd E$.

   Then we have $K/K' \lhd E/K'$ and $(E/K')/(K/K') \cong E/K$ has order $(mn)/n = m$, so K/K' is a normal Hall subgroup of E/K', since $(m, n) = 1$ and $|K'|$ divides m. By induction, E/K' contains a subgroup G/K' of order n, so $|G| = n|K'|$ and $(n, |K'|) = 1$, thus K' is a normal Hall subgroup of G (K' is normal in E). Using again the induction, G has a subgroup of order n which is also a subgroup of E.

2. K does not contain a proper normal subgroup in E.

   Take p a prime which divides m, and P a p-Sylow subgroup of E. Then, Proposition 2.4 states $E = KN_E(P)$. Moreover, by the second isomorphism theorem $E/K = KN_E(P) \cong N_E(P)/(K \cap N_E(P)) = N_E(P)/N_K(P)$. In consequence, $n = |N_E(P)|/|N_K(P)| \Rightarrow |N_E(P)| = n|N_K(P)|$. If $N_E(P)$ is not the total group E, then $|N_K(P)| < m$ and it is a normal Hall subgroup of $N_E(P)$; by induction, $N_E(P)$ has a subgroup of order n and so E. On the other hand, if $N_E(P) = E$, then $P \lhd E$, but we have supposed that K has no proper normal subgroup, so $P = E$ and K is a p-group. Now using Corollary 2.1 we deduce that K is abelian, and then, by Proposition 4.9 we have finished.

□

Example 4.17. *Reclassifying groups of order $|E| = pq$, with $p < q$ distinct primes, using Schur-Zassenhaus.*
Let K be a cyclic subgroup of order q of E, then E is a semidirect product of K by $G = \mathbb{Z}_p$. If p does not divide $q - 1$, then the only action of G on K is the trivial action so $E \cong \mathbb{Z}_q \times \mathbb{Z}_p \cong \mathbb{Z}_{pq}$. Otherwise, E can be $\mathbb{Z}_{pq}$ or a semidirect product $\mathbb{Z}_q \rtimes \mathbb{Z}_p$.

Schur-Zassenhaus theorem can be complemented with this second, which give us the idea that complements of K have similar properties as p-Sylows of a group:

Theorem 4.3. *If K or G/K is solvable, then any two complements of K in E are conjugates.*

Proof. We won't prove this theorem since the proof is not related with group extensions. It can be found in Rotman's book [2], p.190-191. □

4.4 Parameter systems

The purpose of this section is to generalize the construction of Section 4.2 to K non necessarily abelian. Let c_k denote the conjugation automorphism by k.

Definition 4.8. *A parameter system of G in K is a pair (α, f) of maps $\alpha : G \to \mathrm{Aut}(K)$ with $\alpha(x) := \alpha_x$ denoting $\alpha_x(k) := {}^x k$, and $f : G \times G \to K$ with the properties:*

1. $\alpha_x \circ \alpha_y = c_{f(x,y)} \circ \alpha_{xy}$, $\forall x, y \in G$.

2. $f(x,y)f(xy,z) = {}^x f(y,z)f(x,yz)$, $\forall x, y, z \in G$.

We will denote the set of parameter systems of G in K by $\mathrm{Par}(G,K)$.

Lemma 4.5. *Given $(\alpha, f) \in \mathrm{Par}(G,K)$, we have that $\alpha_1 = c_{f(1,1)}$, $f(1,1) = f(1,x)$ and $f(x,1) = {}^x f(1,1)$ for all $x \in G$.*

Proof. The first property says that $\alpha_1 \circ \alpha_1 = c_{f(1,1)} \circ \alpha_1$ and this implies that $\alpha_1 = c_{f(1,1)}$. On the other hand, the second property applied to $x, y = 1, z = 1$ says $f(x,1)f(x,1) = {}^x f(1,1)f(x,1) \Rightarrow f(x,1) = {}^x f(1,1)$, and applied to $x = 1, y = 1, z = 1$ results in

$$f(1,1)f(1,x) = {}^1 f(1,x)f(1,x) = f(1,1)f(1,x)(f(1,1))^{-1}f(1,x) \Rightarrow f(1,x) = f(1,1)$$

□

Remark 4.4. *It is important to observe that α isn't necessarily an action (i.e. a homomorphism) and so f is not necessarily a 2-cocycle. If K is abelian, then α is an homorphism and so and action, and the second condition says that f would be a 2-cocycle, so this is a generalization of the construction in the case of K abelian, like we wanted.*

So we want to stablish a relation between this new construction and the extensions of K by G like we did in Section 4.2. First we will see that this is not an arbitrary definition. Suppose we have $1 \longrightarrow K \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1$ an extension of G by K.

Definition 4.9. *Given an extension of G by K and a choice of a section s, we can define the parameter system (α, f) associated to the extension and s as:*

$$\alpha_x(k) = {}^x k = s_x k s_x^{-1}, \;\; f(x,y) = s_x s_y s_{xy}^{-1}$$

We can suppose that we take s such that $s_1 = 1$ and, as before, this will not give us any problem.

Proposition 4.11. *A parameter system (α, f) of an extension belongs to $\mathrm{Par}(G,K)$. Moreover, $f(1,x) = f(1,x) = 1$ and $\alpha_1 = 1$ if we take $s_1 = 1$.*

Proof. We only have to prove that $\alpha_x \circ \alpha_y = c_{f(x,y)} \circ \alpha_{xy}$. The rest of the properties were proved in Section 4.3, because in that proofs we didn't need K to be abelian, so they work well here:

$$\alpha_x(\alpha_y(k)) = s_x s_y k s_y^{-1} s_x^{-1} = s_x s_y k (s_x s_y)^{-1} = f(x,y) s_{xy} k (f(x,y) s_{xy})^{-1} = c_{f(x,y)}(\alpha_{xy}(k))$$

□

Lemma 4.6. *Take two sections s, s' and construct $(\alpha, f), (\alpha', f') \in \mathrm{Par}(G, K)$ associated to s and s' respectively. Define $g : G \to K$ such that $s'_x = s_x g(x)$, then $\forall x, y \in G$*

$$\alpha'_x = c_{h(x)} \circ \alpha_x \text{ and } f'(x,y) = h(x)\alpha_x(h(y)) f(x,y) h(xy)^{-1}$$

with $h : G \longrightarrow K$ defined by $h(x) = \alpha_x(g(x))$. Moreover, setting ${}^h(\alpha, f) := (\alpha', f')$ this defines a group action of $C(G, K)$ on $\mathrm{Par}(G, K)$.

Proof. First, we prove the equalities:

$$\begin{aligned}\alpha'_x(k) = s'_x k {s'_x}^{-1} &= s_x g(x) k g(x)^{-1} s_x^{-1} = s_x g(x) s_x^{-1} s_x k s_x^{-1} s_x g(x)^{-1} s_x^{-1} \\ &= (s_x g(x) s_x^{-1}) s_x k s_x^{-1} (s_x g(x) s_x^{-1})^{-1} = c_{h(x)} \circ \alpha_x(k)\end{aligned}$$

$$\begin{aligned}f'(x,y) &= s_x g(x) s_y g(y) g(xy)^{-1} s_{xy}^{-1} = (s_x g(x) s_x^{-1})(s_x s_y s_{xy}^{-1})(s_{xy} g(y) g(xy)^{-1} s_{xy}^{-1}) \\ &= h(x) f(x,y) \alpha_{xy}(g(y) g(xy)^{-1}) = h(x) f(x,y) \alpha_{xy}(g(y)) \alpha_{xy}(g(xy))^{-1} \\ &= h(x) f(x,y) \alpha_{xy}(g(y)) f(x,y)^{-1} f(x,y) h(xy)^{-1} \\ &= h(x) \alpha_x(\alpha_y(g(y))) f(x,y) h(xy)^{-1} = h(x) \alpha_x(h(y)) f(x,y) h(xy)^{-1}\end{aligned}$$

Finally, we must prove that it is an action. If $g = 1$ then $\alpha'_x = c_1 \circ \alpha_x = \alpha_x$ and $f'(x,y) = f(x,y)$ so ${}^1(\alpha, f) = (\alpha, f)$. Now take $g, h \in C(G, K)$ and define ${}^h(\alpha, f) = (\alpha', f')$, ${}^g(\alpha', f') = (\alpha'', f'')$, we want to see that $(\alpha'', f'') = {}^{gh}(\alpha, f)$:

$$\alpha''_x = c_{g(x)} \circ \alpha'_x = c_{g(x)} \circ c_{h(x)} \circ \alpha_x = c_{gf(x)} \circ \alpha_x$$

$$\begin{aligned}f''(x,y) &= g(x) \alpha'_x(g(y)) f'(x,y) g(xy)^{-1} \\ &= g(x) h(x) \alpha_x(g(y)) h(x)^{-1} h(x) \alpha_x(h(y)) f(x,y) h(xy)^{-1} g(xy)^{-1} \\ &= (gh)(x) \alpha_x(g(y)) \alpha_x(h(y)) f(x,y) ((gh)(xy))^{-1} \\ &= (gh)(x) \alpha_x((gh)(y)) f(x,y) ((gh)(xy))^{-1}\end{aligned}$$

□

Definition 4.10. *We call two parameter systems of G in K equivalent if they belong to the same $C(G, K) -$ orbit. We denote this set of equivalent classes by* $\mathrm{par}(G, K)$.

Proposition 4.12. *Equivalent extensions have equivalent parameter systems.*

Proof. In the proof of Proposition 4.6 we didn't use K to be abelian, so it works the same proof since the definitions of $\alpha_x = c_{s_x}$ and $f = s_x s_y s_{xy}^{-1}$ are the same. $\square$

Definition 4.11. *Given $(\alpha, f) \in \mathrm{Par}(G, K)$, we define the group $E_{(\alpha,f)}$ to be, as a set, $E_{(\alpha,f)} := K \times G$ and the operation in $E_{(\alpha,f)}$ defined by*

$$(k, x)(h, y) = (k\, {}^{x}h f(x, y), xy)$$

Proposition 4.13. *Let $E_{(\alpha,f)}$ be the construction in the previous definition. Then, it has the following properties:*

1. *$E_{(\alpha,f)}$ is a group, with identity element $(1, 1)$ and inverse element*

$$(k, x)^{-1} = (f(x^{-1}, x)^{-1} ({}^{x^{-1}}k)^{-1}, x^{-1})$$

2. *We have that $1 \longrightarrow K \xrightarrow{i} E_{(\alpha,f)} \xrightarrow{\pi} G \longrightarrow 1$ is an extension of G by K with $i(k) = (k, 1)$ and $\pi(k, x) = x$.*

3. *Given $(\alpha', f') \in \mathrm{Par}(G, K)$ such that it is in the same class of (α, f) in $\mathrm{par}(G, K)$, then the extensions constructed as above from (α, f) and (α', f') are equivalent.*

Proof. It is almost the same proof as Proposition 4.7. $\square$

Now we can prove the theorem that generalises the Main theorem of this thesis when K is non-abelian, which is the objective of this final subsection:

Theorem 4.4. *There's a bijection between $\mathrm{ext}(G, K)$ and $\mathrm{par}(G, K)$.*

Proof. We will show that the constructions in Definition 4.8 and Definition 4.11 are mutually inverse bijections between $\mathrm{par}(G, K)$ and $\mathrm{ext}(G, K)$; Definition 4.8 induces a map $\phi : \mathrm{ext}(G, K) \longrightarrow \mathrm{par}(G, K)$ and Definition 4.11 a map $\theta : \mathrm{par}(G, K) \longrightarrow \mathrm{ext}(G, K)$, so we need to show that $\theta \circ \phi = id_{\mathrm{ext}(G,K)}$ and $\phi \circ \theta = id_{\mathrm{par}(G,K)}$:

1. $\theta \circ \phi = id_{\mathrm{ext}(G,K)}$

   Take an extension $1 \longrightarrow K \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1$ and $(\alpha, f) \in \mathrm{Par}(G, K)$ associated to it, so $\alpha_x = c_{s_x}$ and $f(x, y) = s_{xy} s_{xy}^{-1}$. With α and f, make the extension $1 \longrightarrow K \xrightarrow{i'} E_{(\alpha,f)} \xrightarrow{\pi'} G \longrightarrow 1$ as in Definition 4.11. We want to show that both extensions are equivalent.

   Define $\varphi : E_{(\alpha,f)} \longrightarrow E$ such that $\varphi(k, x) = ks_x$. We have to prove that φ is a homomorphism, $\varphi|K = id_K$ and $\pi \circ \varphi = \pi'$:

$$\varphi((k, x))\varphi((h, y)) = ks_x hs_y$$

$$\varphi((k,x)(h,y)) = \varphi((k\,{}^{x}hf(x,y),xy) = k\,{}^{x}hf(x,y)s_{xy}$$
$$= k\,{}^{x}hs_xs_y = ks_xhs_x^{-1}s_xs_y = ks_xhs_y = \varphi((k,x))\varphi((h,y))$$

$$\varphi((k,1)) = k1 = k = (k,1) \text{ in } E_{(\alpha,f)}$$
$$\pi(\varphi((k,x))) = \pi(ks_x) = \pi(k)\pi(s_x) = x = \pi'((k,x))$$

2. $\phi \circ \theta = id_{\mathrm{par}(G,K)}$

   Take $(\alpha, f) \in \mathrm{Par}(G,K)$, and define the extension associated to this parameter system $1 \longrightarrow K \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1$, supposing that i is the inclusion map, and the section s defined by $s_x = (1,x)$. We want to show that, given $k \in K$, then $s_xk = {}^{x}ks_x$ and $s_xs_y = f(x,y)s_{xy}$ and we have finished:

$$s_xk = (1,x)(k,1) = ({}^{x}kf(x,1),x) = ({}^{x}k,x) = ({}^{x}kf(1,x),x) = \alpha_x(k)s_x$$
$$s_xs_y = (1,x)(1,y) = ({}^{x}1f(x,y),xy) = (f(x,y),xy) = f(x,y)s_{xy}$$

□

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