NORM ESTIMATES FOR THE HARDY OPERATOR IN TERMS OF B_p WEIGHTS

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ABSTRACT. We study the explicit dependence of the B_p-constant of the weight, $[w]_{B_p}$, in the estimates of the norm of the Hardy operator acting on non-increasing functions in $L^p(w)$ or $L^{p,\infty}(w)$.

1. INTRODUCTION

The study of the sharp dependence on the class of weights characterizing the boundedness of some important operators in classical and harmonic analysis has received a lot of attention in the very recent years. In this sense we can mention the contributions of [3], [11] or [12], dealing with the Hardy-Littlewood maximal operator, and [15], [16] or [10] for the case of Hilbert, Riesz transform or general Calderón-Zygmund operators, respectively.

To establish the kind of results that we are referring to, let us consider the case of classical Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where the supremum is taken over all cubes $Q \in \mathbb{R}^n$ containing x. Let w be a weight, that is, a positive locally integrable function, and, for a given measurable set E, let $w(E) = \int_E w(x)\, dx$, and for $p > 1$, set $\sigma = w^{-1/(p-1)}$. We say that w satisfies the A_p condition if

$$[w]_{A_p} = \sup_Q \frac{w(Q)\sigma(Q)^{p-1}}{|Q|^p} < \infty.$$

For $p = 1$, the class A_1 of weights is characterized as those for which, for all cubes Q,

$$\frac{w(Q)}{|Q|} \le C \inf_{x \in Q} w(x).$$

and the best constant C in the above inequality it is denoted by the $[w]_{A_1}$ constant.

In [13], B. Muckenhoupt proved the following fundamental result: the maximal operator M is bounded on $L^p(w)$, $1 < p < \infty$, if and only if $w \in A_p$. S. Buckley (see [3]) proved the sharp dependence of $\|M\|_{L^p(w)}$ on $[w]_{A_p}$ in the following result (from now on, in all the paper, the notation $\simeq$ or $\lesssim$ preceding $[w]_{C_p}^\alpha$ will denote quantities depending linearly on $[w]_{A_p}^\alpha$ up to constants independent on the weight w belonging to some class C_p depending on p).

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Theorem 1.1. *Let $1 < p < \infty$. Then $\|M\|_{L^p(w)} \lesssim [w]_{A_p}^{1/(p-1)}$, and the exponent $1/(p-1)$ is the best possible.*

Also in [3], the sharp constant in the weak-type boundedness of M on $L^p(w)$ was studied and it was obtained that, for $1 \leq p < \infty$

$$\|M\|_{L^p(w) \longrightarrow L^{p,\infty}(w)} \simeq [w]_{A_p}^{1/p}.$$

We remark here that, as a consequence of this last inequality and in combination with Theorem 1.1, we obtain the following

$$[w]_{A_p}^{1/p} \lesssim \|M\|_{L^p(w)} \lesssim [w]_{A_p}^{1/(p-1)}, \tag{1}$$

due to the embedding $L^p(w) \hookrightarrow L^{p,\infty}(w)$.

Also in the work [3], Buckley proves that for decreasing power weights defined on $\mathbb{R}^n$, $w_\alpha(x) = |x|^\alpha$, $-n < \alpha < 0$, $[w_\alpha]_{A_p} \simeq \frac{n}{n+\alpha}$ independently of $p \geq 1$. This fact joint with the estimate $\|M\|_{L^p(w)} \lesssim [w]_{A_1}^{1/p}$, valid for any $w \in A_1$ and which can be easily obtained by the use of Marcinkiewicz interpolation theorem, implies

$$\|M\|_{L^p(w_\alpha)} \lesssim [w_\alpha]_{A_p}^{1/p}.$$

This observation combined with (1) shows that for this family of non-increasing weights $\|M\|_{L^p(w_\alpha)} \simeq [w_\alpha]_{A_p}^{1/p}$.

Since the family of weights w_α is such that $\lim_{\alpha \to -n^+} [w_\alpha]_{A_p} = \infty$, this proves that the exponents $1/p$ and $1/(p-1)$ are sharp in (1).

Let us consider $Sf(t) = t^{-1} \int_0^t f(s)\, ds$, the classical Hardy operator, since $(Mf)^* \approx S(f^*)$, where f^* denotes the classical decreasing rearrangement with respect the Lebesgue measure (see [2] for standard notation and the results involved), the action of the maximal operator M on classical Lorentz spaces with respect some weight w, $\Lambda^p(w) := \{f;\ \|f\|_{\Lambda^p(w)} := ((f^*(t))^p\, w(t)\, dt)^{1/p} < +\infty\}$, $0 < p < \infty$, can be described looking at the action of the Hardy operator S on non-increasing functions.

The same consideration is also valid in the case of weak-type Lorentz spaces $\Lambda^{p,\infty}(w)$, $0 < p < \infty$. This space is defined (see [6]) as

$$\Lambda^{p,\infty}(w) = \{f;\ \|f\|_{\Lambda^{p,\infty}(w)} = \sup_{t>0} f^*(t) \left(\int_0^t w(s)\, ds\right)^{1/p} < \infty\},$$

For $p > 0$, we recall here that a positive and measurable function $w \in B_p$ if there exists a positive constant $C > 0$ such that

$$r^p \int_r^\infty \frac{w(x)}{x^p}\, dx \leq C \int_0^r w(x)\, dx. \tag{2}$$

We observe that the best constant $C > 0$ appearing in the B_p condition (2) appears explicitly as a necessary condition to ensure the boundedness of the Hardy operator restricted to non-increasing functions in $L^p(w)$ (see [1], [14] and also [18] for results about the action of Hardy operator on monotone functions). Indeed, if we test the boundedness of S on characteristic functions, $f(x) = \chi_{(0,r)}(x)$, we obtain the following in terms of this optimal constant C

$$\int_0^\infty \left(\int_0^r \frac{\chi_{(0,x)}(t)}{x}\, dt\right)^p w(x)\, dx \leq (1 + C) \int_0^r w(x)\, dx. \tag{3}$$

For this reason it is natural to express the dependence on the B_p condition (2) of the weight in terms of the quantity

$$[w]_{B_p} := 1 + \sup_{r>0} \frac{r^p \displaystyle\int_r^\infty \frac{w(x)}{x^p}\, dx}{\displaystyle\int_0^r w(x)\, dx}.$$

Let $\|S\|$ denote the weighted norm of Hardy operator S in the following three cases:

- $S : L^p_{dec}(w) \longrightarrow L^p(w)$, $0 < p < \infty$.
- $S : L^{p,\infty}_{dec}(w) \longrightarrow L^{p,\infty}(w)$, $0 < p < \infty$.
- $S : L^p_{dec}(w) \longrightarrow L^{p,\infty}(w)$, $1 < p < \infty$.

We will use the following notation: $\|S\|_{p,w}$,$\|S\|_{(p,\infty),w}$ and $\|S\|^*_{p,w}$, respectively, for denoting the norm $\|S\|$ in each of the three cases described above.

The finiteness of $[w]_{B_p}$ in any of the three cases, is a necessary and sufficient condition to ensure that S maps one space into the other.

We remark that for $0 < p \le 1$, the necessary and sufficient condition for the boundedness of $S : L^p_{dec}(w) \longrightarrow L^{p,\infty}(w)$ is (see [6] and [4]) that the primitive of the weight $W(t) = \displaystyle\int_0^t w(x)\, dx$ is a p-quasi concave function, that is, for all $0 < s \le r < \infty$,

$$\frac{W(r)}{r^p} \le C \frac{W(s)}{s^p}.$$

We are interested in the study of sharp bounds for the exponents α and β, for which the following holds

$$[w]^\alpha_{B_p} \lesssim \|S\| \lesssim [w]^\beta_{B_p}. \tag{4}$$

The sharpness of the exponents α and β, respectively, in (4) for the three cases described, will be determined by an estimate of the following quantities

$$\alpha_p := \sup \left\{ \alpha \ge 0 : \inf_{w \in B_p} \frac{\|S\|}{[w]^\alpha_{B_p}} > 0 \right\},$$

and

$$\beta_p := \inf \left\{ \beta \ge 0 : \sup_{w \in B_p} \frac{\|S\|}{[w]^\beta_{B_p}} < \infty \right\}.$$

In the case of the strong boundedness of the operator S, E. Sawyer in [18] who solve the question in a more general context for $p > 1$, gives an expression of the explicit norm $\|S\|$ based on the so called duality principle. Moreover, his result implies ([18], Theorem 2) that, for $p > 1$ and denoting by p' its conjugate exponent,

$$\|S\|_{p,w} \simeq 1 + \sup_{t>0} \left(\int_t^\infty \frac{w(x)}{x^p}\, dx \right)^{1/p} \left(\int_0^t \left(\frac{W(x)}{x} \right)^{-p'} w(x)\, dx \right)^{1/p'}.$$

In Theorem 2.1 we study the explicit dependence on $[w]_{B_p}$ of this quantity, for all $p > 0$. The proof consists in the use of the following result appeared in [6] using the distribution function of a measurable function f defined as
$\lambda_f(y) = |\{x : |f(x)| > y\}|$, where $|A|$ denotes the Lebesgue measure of a set A.

Theorem 1.2. *Suppose w is a weight in $(0,\infty)$, and $0<p<\infty$. Then,*

$$\int_0^\infty (f^*(t))^p\ w(t)\ dt = p\int_0^\infty y^{p-1}\left(\int_0^{\lambda_f(y)} w(t)\ dt\right)\ dy.$$

In [19] necessary and sufficient conditions for the boundedness of the Hardy operator from $L^{p,\infty}_{\rm dec}(w)$ into itself were obtained. The result is the following:

Theorem 1.3. *([19] Theorem 3.1) Let $0<p<\infty$ and w be a weight in $\mathbb{R}^+$. Then, the following facts are equivalent*

i) $S: L^{p,\infty}_{dec}(w) \longrightarrow L^{p,\infty}(w)$.

ii) the weight w satisfies that, for all $t>0$,

$$\int_0^t \frac{1}{W^{1/p}(s)}\ ds \le C\frac{t}{W^{1/p}(t)}; \tag{5}$$

iii) $w\in B_p$.

Moreover, if $\|S\|_{(p,\infty),w}$ denotes the norm of the Hardy operator restricted to non-increasing functions in the weak space $L^{p,\infty}(w)$ into itself, and $\|w\|_{\widetilde{B_p}}$ is the optimal constant C in (5), that is,

$$[w]_{\widetilde{B_p}} := \sup_{t>0}\frac{1}{t}\left(\int_0^t \frac{1}{W^{1/p}(s)}\ ds\right)W^{1/p}(t), \tag{6}$$

then $\|S\|_{(p,\infty),w} = [w]_{\widetilde{B_p}}$. In Theorem 2.2, lower and upper bounds are established for the exponents in the $[w]_{B_p}$ constant in comparison with the exact norm of the operator $[w]_{\widetilde{B_p}}$.

Remark 1.4. We observe here that there is no weight w for which the operator $S: L^{p,\infty}_{dec}(w) \longrightarrow L^p(w)$ is bounded. Since, if it would be the case, the inclusion $L^{p,\infty}_{dec}(w) \hookrightarrow L^p(w)$ would be continuous, and it implies (see [5], Theorem 3.3)

$$\int_0^\infty \frac{w(t)}{W(t)}\ dt<\infty,$$

which lead us to contradiction.

Concerning the explicit expression for the norm of S as an operator from $L^p_{\rm dec}(w)$ into $L^{p,\infty}(w)$, for $p>1$, we observe, that again as a consequence of the duality principle of E. Sawyer (see [18], Theorem 1), we have the following

$$\begin{aligned}\|S\|^*_{p,w} = \sup_{f\ \text{dec}} \frac{\|Sf\|_{L^{p,\infty}(w)}}{\|f\|_{L^p(w)}} &= \sup_{t>0}\sup_{f\ \text{dec}} \frac{\int_0^\infty f(x)\chi_{(0,t)}(x)\ dx}{\int_0^\infty f^p(s)\ w(s)\ ds)^{1/p}}\frac{W^{1/p}(t)}{t}\\ &\simeq \sup_{t>0}\left(\int_0^t x^{p'-1}W(x)^{1-p'}\ dx\right)^{1/p'}\frac{W^{1/p}(t)}{t}.\end{aligned} \tag{7}$$

In Theorem 2.4, they are established lower and upper bounds for the exponents in the $[w]_{B_p}$ in comparison with this expression of the norm arising from Sawyer's duality principle.

The proofs of the estimates for the norm, in both cases $\|S\|_{(p,\infty),w}$ and $\|S\|^*_{p,w}$ are based in the following generalization of a result due to Y. Sagher [17]:

Proposition 1.5. *Let m be a positive function and ε a positive number, then:*

i) The existence of two positive constants A and B such that, for every $r > 0$

$$Am(r) \leq \int_0^r \frac{m(s)}{s}\, ds \leq Bm(r),$$

implies

$$\frac{A^{\varepsilon+1}}{\varepsilon B^{\varepsilon}} \frac{1}{m^{\varepsilon}(r)} \leq \int_r^{\infty} \frac{1}{m^{\varepsilon}(s)} \frac{ds}{s} \leq \frac{B^{\varepsilon+1}}{\varepsilon A^{\varepsilon}} \frac{1}{m^{\varepsilon}(r)}.$$

ii) Conversely, the existence of two positive constants C and D such that, for every $r > 0$

$$\frac{C}{m(r)} \leq \int_r^{\infty} \frac{1}{m(s)} \frac{ds}{s} \leq \frac{D}{m^{\varepsilon}(r)},$$

implies

$$\frac{C^{\varepsilon+1}}{\varepsilon D^{\varepsilon}} m^{\varepsilon}(r) \leq \int_0^r \frac{m^{\varepsilon}(s)}{s} ds \leq \frac{D^{\varepsilon+1}}{\varepsilon C^{\varepsilon}} m^{\varepsilon}(r).$$

Proof. We observe that ii) follows from i) applied to the positive function $\widetilde{m}(t) := \dfrac{1}{m(1/t)}$, by a change of variables. So, we will restrict to prove i).

For $r > 0$, let us define $\varphi(r) = \displaystyle\int_0^r \frac{m(s)}{s}\, ds$. We observe that, as a consequence of the hypothesis, $\lim_{r\to\infty} \varphi(r) = \infty$. The second inequality in the hypotheses can be written in the form

$$\frac{\varphi(s)}{s} \leq B\frac{m(s)}{s} = B\varphi'(s),$$

then, since $\varphi(\infty) = \infty$, it follows

$$\begin{aligned} \int_r^{\infty} \frac{1}{m^{\varepsilon}(s)} \frac{ds}{s} &\leq B^{\varepsilon} \int_r^{\infty} \frac{1}{\varphi^{\varepsilon}(s)} \frac{ds}{s} \leq B^{\varepsilon+1} \int_r^{\infty} \frac{s\varphi'(s)}{\varphi^{\varepsilon+1}(s)} \frac{ds}{s} \\ &= \frac{B^{\varepsilon+1}}{\varepsilon} \frac{1}{\varphi^{\varepsilon}(r)} \leq \frac{B^{\varepsilon+1}}{\varepsilon A^{\varepsilon}} \frac{1}{m^{\varepsilon}(r)}, \end{aligned}$$

where this last inequality is a consequence of the first inequality in the hypothesis.

To check the first inequality of the statement i), we proceed similarly by using, in this case

$$\varphi'(s) = \frac{m(s)}{s} \leq \frac{\varphi(s)}{A\ s},$$

then

$$\begin{aligned} \int_r^{\infty} \frac{1}{m^{\varepsilon}(s)} \frac{ds}{s} &\geq A^{\varepsilon} \int_r^{\infty} \frac{1}{\varphi^{\varepsilon}(s)} \frac{ds}{s} \geq A^{\varepsilon+1} \int_r^{\infty} \frac{\varphi'(s)}{\varphi^{\varepsilon+1}(s)}\, ds \\ &= \frac{A^{\varepsilon+1}}{\varepsilon} \frac{1}{\varphi^{\varepsilon}(r)} \geq \frac{A^{\varepsilon+1}}{\varepsilon B^{\varepsilon}} \frac{1}{m^{\varepsilon}(r)}, \end{aligned}$$

and the proof is complete. ∎

2. Main results

Although not explicitly written in a quantitative form, the results concerning the dependence on the B_p constant (2) of the weight w in the boundedness of the Hardy operator $S : L^p_{dec}(w) \longrightarrow L^p(w)$ are contained in the following (see [7] Proposition 2.6 for the case $0 < p \leq 1$ and also [7] Theorem 4.1 for $p > 1$).

Theorem 2.1. *Let $p > 0$ and w a weight in $\mathbb{R}^+$, the Hardy operator S is bounded from L^p_{dec} into $L^p(w)$ if and only if $w \in B_p$ and then:*

a) For $0 < p \leq 1$

$$\|S\|_{p,w} \simeq [w]^{1/p}_{B_p}.$$

That is, in this case, $\alpha_p = \beta_p = 1/p$.

b) For $p > 1$

$$[w]^{1/p}_{B_p} \leq \|S\|_{p,w} \leq [w]_{B_p}.$$

In this case, $1/p \leq \alpha_p \leq 1$ and $\beta_p = 1$.

Proof. Case $0 < p \leq 1$.

The inequality on the left hand side and the necessity condition follows by testing the boundedness of the operator on characteristic functions $\chi_{(0,r)}$.

To check the right hand side inequality, let us take f a non-increasing function the use of Theorem 1.2, the embedding $\Lambda^1(1) \hookrightarrow \Lambda^p(y^{p-1})$ for $0 < p \leq 1$ (see [5] Theorem 3.1), and formula (3)

$$\begin{aligned}
\int_0^\infty (Sf(x))^p\, w(x)\, dx &= \int_0^\infty \left(\int_0^\infty \int_0^{\lambda_f(y)} \frac{\chi_{(0,x)}(t)}{x}\, dtdy \right)^p w(x)\, dx \\
&\leq p \int_0^\infty \left(\int_0^\infty y^{p-1} \left(\int_0^{\lambda_f(y)} \frac{\chi_{(0,x)}(t)}{x}\, dt \right)^p dy \right) w(x)\, dx \\
&\lesssim [w]_{B_p} \int_0^\infty \left(\int_0^{\lambda_f(y)} w(x)\, dx \right) y^{p-1}\, dy \\
&\simeq [w]_{B_p} \int_0^\infty f^p(x)\, w(x)\, dx.
\end{aligned}$$

Case $p > 1$.

Clearly, as in the case $0 < p \leq 1$, to prove the necessity condition and the left inequality is enough to apply the hypothesis to $f = \chi_{(0,r)}$.

Conversely, let us observe that

$$\begin{aligned}
\left(\int_0^x f(t)\, dt \right)^p &= p \int_0^x \left(\int_0^t f(s)\, ds \right)^{p-1} f(t)\, dt \\
&= p \int_0^x \left(\frac{1}{t} \int_0^t f(s)\, ds \right)^{p-1} f(t)\, t^{p-1}\, dt.
\end{aligned}$$

Let us consider, then, the nonincreasing function $g(t) = Sf(t)^{p-1} f(t)$. Hence,

$$\|Sf\|_{p,w} = p^{1/p} \left(\int_0^\infty \left(\int_0^x g(t) t^{p-1}\, dt \right) x^{-p}\, w(x)\, dx \right)^{1/p}. \tag{8}$$

We use the distribution function formula included in Theorem 1.2 and obtain that the inner integral in this last expression is

$$\int_0^x g(t)t^{p-1}\,dt = \int_0^\infty \int_0^{\lambda_g(r)} \chi_{(0,x)}(t)\; t^{p-1}\,dt\,dr = \frac{1}{p}\left(\int_0^{g(x)} x^p\,dr + \int_{g(x)}^\infty \lambda_g^p(r)\,dr\right).$$

By substitution of this last expression in (8) and using Fubini's theorem, we get

$$\begin{aligned}
\|Sf\|_{p,w}^p &= \int_0^\infty \left(\int_0^{g(x)} x^p\,dr + \int_{g(x)}^\infty \lambda_g^p(r)\,dr\right)\frac{w(x)}{x^p}\,dx \\
&= \int_0^\infty \left(\int_0^{\lambda_g(r)} w(x)\,dx + \int_{\lambda_g(r)}^\infty \frac{w(x)}{x^p}\,dx\right)\lambda_g^p(r)\,dr \\
&\le [w]_{B_p}\int_0^\infty \int_0^{\lambda_g(r)} w(x)\,dx\,dr = [w]_{B_p}\int_0^\infty g(r)w(r)\,dr \\
&= [w]_{B_p}\int_0^\infty Sf(t)^{p-1}f(t)w(t)\,dt \\
&\le [w]_{B_p}\|Sf\|_{p,w}^{p/p'}\|f\|_{p,w},
\end{aligned}$$

where the last inequality is a consequence of Hölder's inequality.

The optimality of the inequality in the right hand side follows by considering the family of weights $w_\alpha(x) = x^\alpha$, $-1 < \alpha < p-1$, then (see [9])

$$\|S\|_{L^p(w_\alpha)} = \frac{p}{p-\alpha-1} = [w_\alpha]_{B_\alpha}.$$

We observe that since $\lim_{\alpha\to(p-1)^-}[w_\alpha] = \infty$, the sharpness in the upper bound of the statement holds. □

Looking at the result contained in Theorem 1.3 it is natural to ask for the exactly dependence of $\|S\|_{(p,\infty),w}$ on the constant $[w]_{B_p}$. In other words, how the constants $[w]_{B_p}$ and $[w]_{\widetilde{B_p}}$ are related. The following theorem gives an answer to this question:

Theorem 2.2. *For $0 < p < \infty$ and w weight in B_p, then*

$$[w]_{B_p}^{1/(p+1)} \le \|S\|_{(p,\infty),w} \le [w]_{B_p}^{(p+1)/p}.$$

Hence, we conclude that, in this case, $1/(p+1) \le \alpha_p \le \beta_p \le (p+1)/p$.

Proof. Let w such that $S : L^{p,\infty}_{dec}(w) \longrightarrow L^{p,\infty}(w)$ is bounded, that is, as a consequence of Theorem 1.3,

$$\int_0^t \frac{1}{W^{1/p}(s)}\,ds \le [w]_{\widetilde{B_p}}\frac{t}{W^{1/p}(t)}.$$

Then, since

$$\frac{t}{W^{1/p}(t)} \le \int_0^t \frac{1}{W^{1/p}(s)}\,ds \le [w]_{\widetilde{B_p}}\frac{t}{W^{1/p}(t)},$$

the function $m(r) = \dfrac{r}{W^{1/p}(r)}$ satisfies the hypothesis of statement i) in Proposition 1.5 with constants $A = 1$ and $B = [w]_{\widetilde{B_p}}$, respectively. Then, the use of the

Proposition for $\varepsilon = p$ implies

$$\int_t^\infty \frac{W(s)}{s^{p+1}}\,ds \le \frac{[w]_{\widetilde{B_p}}^{p+1}}{p}\frac{W(t)}{t^p}. \tag{9}$$

On the other hand, Fubini's theorem implies that, since $w \in B_p$

$$p\int_t^\infty \frac{W(s)}{s^{p+1}}\,ds = \int_t^\infty \frac{w(s)}{s^p}\,ds + \frac{W(t)}{t^p} \le [w]_{B_p}\frac{W(t)}{t^p}.$$

And, hence

$$\frac{1}{p}\frac{W(t)}{t^p} \le \int_t^\infty \frac{W(s)}{s^{p+1}}\,ds \le \frac{[w]_{B_p}}{p}\frac{W(t)}{t^p}. \tag{10}$$

This inequality combined with (9) proves that

$$[w]_{B_p} \le [w]_{\widetilde{B_p}}^{p+1},$$

which gives the left inequality in the statement.

To complete the proof, we consider the positive function $m(r) = \frac{r^p}{W(r)}$ then, equation (10) shows that the conditions of Proposition 1.5 ii) holds for this m with $C = 1/p$ and $D = [w]_{B_p}/p$. Then, taking $\varepsilon = 1/p$, Proposition 1.5 ii) implies that

$$\int_0^t \frac{1}{W^{1/p}(s)}\,ds \le [w]_{B_p}^{(p+1)/p}\frac{t}{W^{1/p}(t)},$$

and, hence, taking into account that $[w]_{\widetilde{B_p}}$ is the best constant C in the inequality (5), this implies that

$$[w]_{\widetilde{B_p}} \le [w]_{B_p}^{(p+1)/p}.$$

□

Remark 2.3. Let us observe here that for power weights in the B_p class, $w_\alpha(t) = t^\alpha$ $-1 < \alpha < p-1$, we can explicitly (6) and obtain

$$\|S\|_{(p,\infty),w_\alpha} = \frac{p}{p-\alpha-1} = [w_\alpha]_{B_p}.$$

Hence, in the case of the optimal exponents α_p and β_p in Theorem 2.2, we obtain the following $1/(p+1) \le \alpha_p \le 1 \le \beta_p \le (p+1)/p$.

On the other hand, in [8] it is proved that only for $1 < p < \infty$, the B_p condition is equivalent to the weak boundedness of the Hardy operator S restricted to decreasing functions. Concerning the explicit relation between the weak-bound of the operator S and the B_p constant of the weight, we can establish the following:

Theorem 2.4. *Let us denote by $\|S\|_{p,w}^*$ the norm of the Hardy operator S acting from $L^p_{dec}(w)$ into $L^{p,\infty}(w)$, then for $p > 1$,*

$$[w]_{B_p}^{1/(pp')} \lesssim \|S\|_{p,w}^* \le [w]_{B_p}.$$

Hence, we conclude that, in this case, $1/(pp') \le \alpha_p \le \beta_p \le 1$.

Proof. As it was pointed out in the introduction, a consequence of the so called duality principle of E. Sawyer (see [18], Theorem 1) is the following expression for weak-type norm

$$\|S\|^*_{p,w} = \sup_{f \text{ dec}} \frac{\|Sf\|_{L^{p,\infty}(w)}}{\|f\|_{L^p(w)}} = \sup_{t>0} \sup_{f \text{ dec}} \frac{\int_0^\infty f(x)\chi_{(0,t)}(x)\, dx}{\int_0^\infty f^p(s)\, w(s)\, ds)^{1/p}} \frac{W^{1/p}(t)}{t}$$

$$\simeq \sup_{t>0} \left(\int_0^t x^{p'-1} W(x)^{1-p'}\, dx \right)^{1/p'} \frac{W^{1/p}(t)}{t}. \tag{11}$$

We obtain then, the following estimates

$$\frac{t^{p'}}{p' W^{p'-1}(t)} \le \int_0^t x^{p'-1} W(x)^{1-p'}\, dx \lesssim (\|S\|^*_{p,w})^{p'} \frac{t^{p'}}{W^{p'-1}(t)}.$$

From this, applying part i) of Proposition 1.5 to the function $m(r) = r^{p'} W(r)^{1-p'}$ and $\varepsilon = 1/(p'-1)$, we obtain

$$\int_t^\infty \frac{W(x)}{x^{p+1}}\, dx \lesssim (\|S\|^*_{p,w})^{pp'} \frac{W(t)}{t^p}.$$

Using Fubini's theorem, we can express the integral in left hand side of the inequality as

$$\int_t^\infty \frac{W(x)}{x^{p+1}}\, dx = \frac{1}{p} \left(\frac{W(t)}{t^p} + \int_t^\infty \frac{w(s)}{s^p}\, ds \right), \tag{12}$$

and then we obtain

$$[w]_{B_p}^{1/(pp')} \lesssim \|S\|^*_{p,w}.$$

In order to obtain the second inequality, we observe that again using (12), we obtain, up to constants depending on p,

$$\frac{W(t)}{t^p} \le \int_t^\infty \frac{W(x)}{x^{p+1}}\, dx \le [w]_{B_p} \frac{W(t)}{t^p}.$$

Another application of part ii) of Proposition 1.5, in this case to the function $m(r) = r^p/W(r)$ and $\varepsilon = p'-1$, gives, up to constants depending on p

$$\int_0^t x^{p'-1} W(x)^{1-p'}\, dx \le [w]_{B_p}^{p'} \frac{t^{p'}}{W^{p'-1}(t)}.$$

This inequality implies, as a consequence of the expression (11),

$$\|S\|^*_{p,w} \lesssim [w]_{B_p}.$$

□

Remark 2.5. In the case of power weights in the B_p class, $w_\alpha(t) = t^\alpha$ $-1 < \alpha < p-1$, we can explicitly calculate the expression in (7) and obtain

$$\|S\|^*_{p,w_\alpha} \simeq \left(\frac{p-1}{p-\alpha-1} \right)^{1/p'} = \left(\frac{1}{p'} \right)^{1/p'} ([w]_\alpha)^{1/p'}.$$

Hence, in the case of the optimal exponents α_p and β_p in Theorem 2.4, we obtain the following $1/(pp') \le \alpha_p \le 1/p' \le \beta_p \le 1$.

Remark 2.6. We observe that, unlike what it happens in the boundedness of the Hardy operator in $L^p(w)$ and was used in Theorem 2.1, if we test the boundedness of $S : L^{p,\infty}_{dec}(w) \longrightarrow L^{p,\infty}(w)$ or $S : L^p_{dec}(w) \longrightarrow L^{p,\infty}(w)$ on characteristic funcions $f(t) = \chi_{(0,r)}(t)$, straightforward calculations show that $\|S\|^*_{p,w}$ or $\|S\|_{(p,\infty),w}$ has, as a lower bound, the best constant in the $p-$ quasi concave condition of the function W, that is

$$\sup_{0<r<t} \frac{r}{t} \left(\frac{W(t)}{W(r)} \right)^{1/p}.$$

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