Tame systems of linear and semilinear mappings

Tatiana Klimchuk, Dmitry Kovalenko
Kiev National Taras Shevchenko University, Kiev, Ukraine
klimchuk.tanya@gmail.com kovalenko.d.y@gmail.com

Tetiana Rybalkina, Vladimir V. Sergeichuk
Institute of Mathematics, Tereshchenkivska 3, Kiev, Ukraine
rybalkina_t@ukr.net sergeich@imath.kiev.ua

Abstract

We study systems of linear and semilinear mappings considering them as representations of a directed graph G with full and dashed arrows: a representation of G is given by assigning to each vertex a complex vector space, to each full arrow a linear mapping, and to each dashed arrow a semilinear mapping of the corresponding vector spaces. We extend to such representations the classical theorems by Gabriel about quivers of finite type and by Nazarova, Donovan, and Freislich about quivers of tame types.

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1 Introduction

We study systems of linear and semilinear mappings on complex vector spaces. A mapping $\mathcal{A}$ from a complex vector space U to a complex vector space V is called *semilinear* if

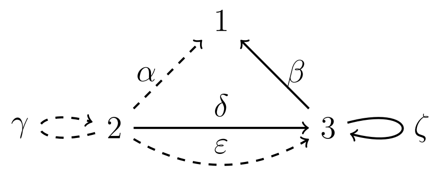
$$\mathcal{A}(u+u') = \mathcal{A}u + \mathcal{A}u', \qquad \mathcal{A}(\alpha u) = \bar{\alpha}\mathcal{A}u$$

for all $u, u' \in U$ and $\alpha \in \mathbb{C}$. We write $\mathcal{A} : U \to V$ if $\mathcal{A}$ is a linear mapping and $\mathcal{A} : U \dashrightarrow V$ (using a dashed arrow) if $\mathcal{A}$ is a semilinear mapping.

We study systems of linear and semilinear mappings considering them as representations of biquivers introduced by Sergeichuk [11, Section 5] (see also [4]); they generalize the notion of representations of quivers introduced by Gabriel [8].

Definition 1.1.

- A *biquiver* is a directed graph G with vertices $1, 2, \dots, t$ and with full and dashed arrows; for example,

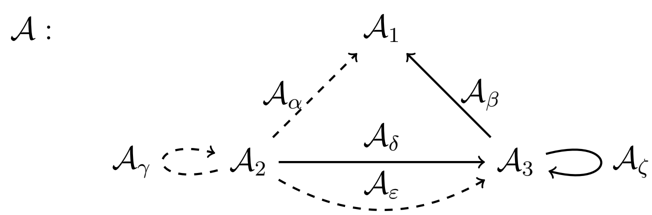


(1)

- A *representation* $\mathcal{A}$ of a biquiver G is given by assigning to each vertex v a complex vector space $\mathcal{A}_v$, to each full arrow $\alpha : u \longrightarrow v$ a linear mapping $\mathcal{A}_\alpha : \mathcal{A}_u \to \mathcal{A}_v$, and to each dashed arrow $\alpha : u \dashrightarrow v$ a semilinear mapping $\mathcal{A}_\alpha : \mathcal{A}_u \dashrightarrow \mathcal{A}_v$. The vector

$$\dim \mathcal{A} := (\dim \mathcal{A}_1, \dots, \dim \mathcal{A}_t)$$

is called the *dimension* of a representation $\mathcal{A}$. For example, a representation



of (1) is formed by complex spaces $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$, linear mappings $\mathcal{A}_\beta$, $\mathcal{A}_\delta$, $\mathcal{A}_\zeta$, and semilinear mappings $\mathcal{A}_\alpha$, $\mathcal{A}_\gamma$, $\mathcal{A}_\varepsilon$.

- A *morphism* $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ between representations $\mathcal{A}$ and $\mathcal{B}$ of a biquiver G is a family of linear mappings $\mathcal{F}_1 : \mathcal{A}_1 \to \mathcal{B}_1, \dots, \mathcal{F}_t : \mathcal{A}_t \to \mathcal{B}_t$ such that for each arrow α from u to v the diagram

$$\begin{array}{ccc} \mathcal{A}_u & \xrightarrow{\mathcal{A}_\alpha} & \mathcal{A}_v \\ {\scriptstyle \mathcal{F}_u}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{F}_v} \\ \mathcal{B}_u & \xrightarrow{\mathcal{B}_\alpha} & \mathcal{B}_v \end{array} \quad \text{if } u \xrightarrow{\alpha} v \quad \text{or} \quad \begin{array}{ccc} \mathcal{A}_u & \overset{\mathcal{A}_\alpha}{\dashrightarrow} & \mathcal{A}_v \\ {\scriptstyle \mathcal{F}_u}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{F}_v} \\ \mathcal{B}_u & \overset{\mathcal{B}_\alpha}{\dashrightarrow} & \mathcal{B}_v \end{array} \quad \text{if } u \overset{\alpha}{\dashrightarrow} v \tag{2}$$

is commutative (i.e., $\mathcal{B}_\alpha \mathcal{F}_u = \mathcal{F}_v \mathcal{A}_\alpha$). We write $\mathcal{A} \simeq \mathcal{B}$ if $\mathcal{A}$ and $\mathcal{B}$ are isomorphic; i.e., if all $\mathcal{F}_i$ are bijections.

For example, each cycle of linear and semilinear mappings

$$\mathcal{A}: \qquad \overbrace{V_1 \overset{\mathcal{A}_1}{\overline{}} V_2 \overset{\mathcal{A}_2}{\overline{}} \ \dots \ \overset{\mathcal{A}_{t-2}}{\overline{}} V_{t-1} \overset{\mathcal{A}_{t-1}}{\overline{}} V_t}^{\mathcal{A}_t}$$

(in which each edge is a full or dashed arrow $\longrightarrow$, $\longleftarrow$, $\dashrightarrow$, or $\dashleftarrow$) is a representation of the biquiver

$$C: \qquad \overbrace{1 \overset{\alpha_1}{\overline{}} 2 \overset{\alpha_2}{\overline{}} \ \dots \ \overset{\alpha_{t-2}}{\overline{}} (t-1) \overset{\alpha_{t-1}}{\overline{}} t}^{\alpha_t}$$

its representations were classified in [5].

Note that a biquiver without dashed arrows is a quiver and its representations are the quiver representations. The quivers, for which the problem of classifying their representations does not contain the problem of classifying pairs of matrices up to similarity (i.e., the problem of classifying representations of the quiver $\circlearrowright 1 \circlearrowleft$), are called *tame* (this definition is informal; formal definitions are given in [9, Section 14.10]). The list of all tame quivers and the classification of their representations were obtained independently by Donovan and Freislich [3] and Nazarova [10]. We extend their results to representations of biquivers.

2 Formulation of the main results

The *direct sum* of representations $\mathcal{A}$ and $\mathcal{B}$ of a biquiver is the representation $\mathcal{A} \oplus \mathcal{B}$ formed by the spaces $\mathcal{A}_v \oplus \mathcal{B}_v$ and the mappings $\mathcal{A}_\alpha \oplus \mathcal{B}_\alpha$. A representation of nonzero dimension is *indecomposable* if it is not isomorphic to a direct sum of representations of nonzero dimensions.

By analogy with quiver representations, we say that a biquiver is *representation-finite* if it has only finitely many nonisomorphic indecomposable representations. A biquiver is *wild* if the problem of classifying its representations contains the problem of classifying matrix pairs up to similarity transformations

$$(A, B) \mapsto (S^{-1}AS, S^{-1}AS), \qquad S \text{ is nonsingular},$$

otherwise the biquiver is *tame*. Clearly, each representation-finite biquiver is tame. The problem of classifying matrix pairs up to similarity is the problem of classifying representations of the quiver $\circlearrowleft 1 \circlearrowright$; it contains the problem of classifying representations of each quiver (see [2]) and so it is considered as hopeless. An analogous statement for representations of biquivers was proved in [4]: the problem of classifying representations of the biquiver $\circlearrowleft 1 \circlearrowright$ (dashed loops) contains the problem of classifying representations of each biquiver.

The *Tits form* of a biquiver G with vertices $1, \dots, t$ is the integral quadratic form

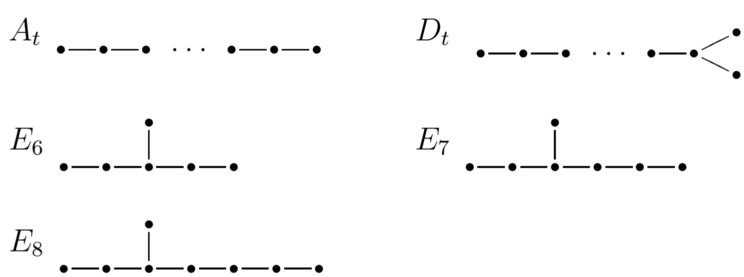
$$q_G(x_1, \dots, x_t) := x_1^2 + \dots + x_t^2 - \sum x_u x_v,$$

in which the sum $\sum$ is taken over all arrows $u \longrightarrow v$ and $u \dashrightarrow v$ of the biquiver.

The following theorem extends Gabriel's theorem [8] (see also [6, Theorem 2.6.1]) to each biquiver G and coincides with it if G is a quiver.

Theorem 2.1 (proved in Section 4)**.** *Let G be a connected biquiver with vertices $1, 2, \dots, t$.*

(a) *G is representation-finite if and only if G can be obtained from one of the Dynkin diagrams*



(3)

by replacing each edge with a full or dashed arrow, if and only if the Tits form q_G is positive definite.

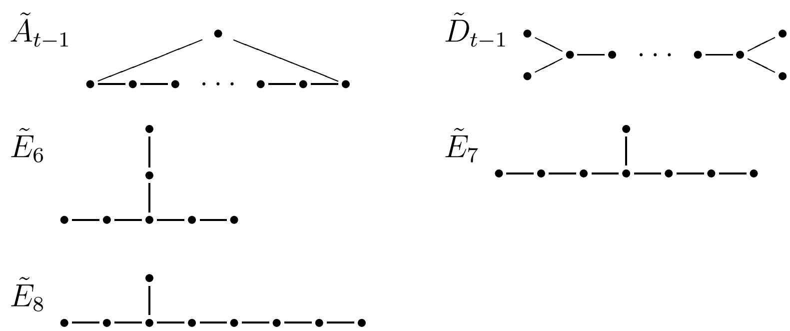
(b) *Let G be representation-finite and let $z = (z_1, \dots, z_t)$ be an integer vector with nonnegative components. There exists an indecomposable representation of dimension z if and only if $q_G(z) = 1$; this representation is determined by z uniquely up to isomorphism.*

Representations of representation-finite quivers were classified by Gabriel [8]; see also [6, Theorem 2.6.1].

The following theorem extends the Donovan–Freislich–Nazarova theorem [3, 10] (see also [6, Chapter 2]) to each biquiver G and coincides with it if G is a quiver.

Theorem 2.2 (proved in Section 5). *Let G be a connected biquiver with vertices $1, 2, \dots, t$.*

(a) *G is tame if and only if G can be obtained from one of the Dynkin diagrams (3) or from one of the extended Dynkin diagrams*



by replacing each edge with a full or dashed arrow, if and only if the Tits form q_G is positive semidefinite.

(b) *Let G be tame and let $z = (z_1, \dots, z_t)$ be an integer vector with nonnegative components. There exists an indecomposable representation of dimension z if and only if $q_G(z) = 0$ or $q_G(z) = 1$.*

Representations of tame quivers were classified by Donovan and Freislich [3] and independently by Nazarova [10].

The following theorem is a special case of the Krull–Schmidt theorem for additive categories [1, Chapter I, Theorem 3.6] (it holds for representations of a biquiver since they form an additive category in which all idempotents split).

Theorem 2.3. *Each representation of a biquiver is isomorphic to a direct sum of indecomposable representations. This direct sum is uniquely determined, up to permutations and isomorphisms of direct summands, which means that if*

$$\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_r \simeq \mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_s,$$

in which all $\mathcal{A}_i$ and $\mathcal{B}_j$ are indecomposable representations, then $r = s$ and all $\mathcal{A}_i \simeq \mathcal{B}_i$ after a suitable renumbering of $\mathcal{A}_1, \dots, \mathcal{A}_r$.

3 Matrix representations of biquivers

Let us recall some elementary facts about semilinear mappings.

We denote by $[v]_e$ the coordinate vector of v in a basis $e_1, \dots, e_n$, and by $S_{e \to e'}$ the transition matrix from a basis $e_1, \dots, e_n$ to a basis $e'_1, \dots, e'_n$. If $A = [a_{ij}]$ then $\bar{A} := [\bar{a}_{ij}]$.

Let $\mathcal{A} : U \dashrightarrow V$ be a semilinear mapping. We say that an $m \times n$ matrix $\mathcal{A}_{fe}$ is the *matrix of* $\mathcal{A}$ in bases $e_1, \dots, e_n$ of U and $f_1, \dots, f_m$ of V if

$$[\mathcal{A}u]_f = \overline{\mathcal{A}_{fe}[u]_e} \qquad \text{for all } u \in U. \tag{4}$$

Therefore, the columns of $\mathcal{A}_{fe}$ are $\overline{[\mathcal{A}e_1]_f}, \dots, \overline{[\mathcal{A}e_n]_f}$. We write $\mathcal{A}_e$ instead of $\mathcal{A}_{ee}$ if $U = V$.

If $e'_1, \dots, e'_n$ and $f'_1, \dots, f'_m$ are other bases of U and V, then

$$\mathcal{A}_{f'e'} = \bar{S}^{-1}_{f \to f'} \mathcal{A}_{fe} S_{e \to e'} \tag{5}$$

since the right hand matrix satisfies (4) with e', f' instead of e, f:

$$\overline{\bar{S}^{-1}_{f \to f'} \mathcal{A}_{fe} S_{e \to e'} [v]_{e'}} = S^{-1}_{f \to f'} \overline{\mathcal{A}_{fe}[v]_e} = S^{-1}_{f \to f'} [\mathcal{A}v]_f = [\mathcal{A}v]_{f'}$$

Lemma 3.1. *Let U, V, and W be vector spaces with bases $e_1, e_2, \dots,$ $f_1, f_2, \dots,,$ and $g_1, g_2, \dots$.*

(a) *The composition of a linear mapping $\mathcal{A} : U \to V$ and a semilinear mapping $\mathcal{B} : V \dashrightarrow W$ is the semilinear mapping with matrix*

$$(\mathcal{B}\mathcal{A})_{ge} = \mathcal{B}_{gf} \mathcal{A}_{fe} \tag{6}$$

(b) *The composition of a semilinear mapping $\mathcal{A} : U \dashrightarrow V$ and a linear mapping $\mathcal{B} : V \to W$ is the semilinear mapping with matrix*

$$(\mathcal{B}\mathcal{A})_{ge} = \overline{\mathcal{B}}_{gf} \mathcal{A}_{fe} \tag{7}$$

Proof. The identity (6) follows from observing that $\mathcal{A}B$ is a semilinear mapping and

$$[(\mathcal{B}\mathcal{A})u]_{ge} = [\mathcal{B}(\mathcal{A}u)]_{ge} = \overline{\mathcal{B}_{gf}[\mathcal{A}u]_{fe}} = \overline{(\mathcal{B}_{gf}\mathcal{A}_{fe})[u]_e}$$

for each $u \in U$. The identity (7) follows from observing that $\mathcal{A}B$ is a semilinear mapping and

$$[(\mathcal{B}\mathcal{A})u]_{ge} = [\mathcal{B}(\mathcal{A}u)]_{ge} = \mathcal{B}_{gf}[\mathcal{A}u]_{fe} = \mathcal{B}_{gf}\overline{\mathcal{A}_{fe}[u]_e} = \overline{(\overline{\mathcal{B}}_{gf}\mathcal{A}_{fe})[u]_e}$$

for each $u \in U$. ☐

Let $\mathcal{A} : V \dashrightarrow V$ be a semilinear mapping; let $\mathcal{A}_e$ and $\mathcal{A}_{e'}$ be its matrices in bases $e_1, \dots, e_n$ and $e'_1, \dots, e'_n$ of V. By (5),

$$\mathcal{A}_{e'} = \bar{S}^{-1}_{e \to e'}\mathcal{A}_e S_{e \to e'} \tag{8}$$

and so $\mathcal{A}_{e'}$ and $\mathcal{A}_e$ are consimilar: recall that two matrices A and B are *consimilar* if there exists a nonsingular matrix S such that $\bar{S}^{-1}AS = B$; a canonical form of a square complex matrix under consimilarity is given in [7, Theorem 4.6.12].

Each representation $\mathcal{A}$ of a biquiver G can be given by the set A of matrices A_α of its mappings $\mathcal{A}_\alpha$ in fixed bases of the spaces $\mathcal{A}_1, \dots, \mathcal{A}_t$. Changing the bases, we can reduce $\mathcal{A}_\alpha$ by transformations $S_v^{-1}A_\alpha S_u$ if $\alpha : u \longrightarrow v$ and $\bar{S}_v^{-1}A_\alpha S_u$ if $\alpha : u \dashrightarrow v$, in which $S_1, \dots, S_t$ are the transition matrices, which reduces the problem of classifying representations of G up to isomorphism to the problem of classifying the sets A up to these transformations. This leads to the following definition.

Definition 3.2. Let G be a biquiver with vertices $1, \dots, t$.

- A *matrix representation A of dimension* $(d_1, \dots, d_t)$ of G is given by assigning an $d_v \times d_u$ complex matrix A_α to each arrow $\alpha : u \longrightarrow v$ or $u \dashrightarrow v$.

- Two matrix representations A and B of G are *isomorphic* if there exist nonsingular matrices $S_1, \dots, S_t$ such that

$$B_\alpha = \begin{cases} S_v^{-1}A_\alpha S_u & \text{for every full arrow } \alpha : u \longrightarrow v, \\ \bar{S}_v^{-1}A_\alpha S_u & \text{for every dashed arrow } \alpha : u \dashrightarrow v. \end{cases} \tag{9}$$

Each matrix representation A of dimension $d = (d_1, \ldots, d_t)$ can be identified with the representation $\mathcal{A}$ from the Definition 1.1, whose vector spaces have the form

$$\mathcal{A}_v = \mathbb{C} \oplus \cdots \oplus \mathbb{C} \quad (d_v \text{ summands})$$

for all vertices v and the linear or semilinear mappings $\mathcal{A}_\alpha$ are defined by the matrices A_α. A *morphism* $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ of such representations of dimensions d and d' is given by a set of matrices $F_i \in \mathbb{C}^{d'_i \times d_i}$ such that

$$B_\alpha F_u = \begin{cases} F_v A_\alpha & \text{for every arrow } \alpha : u \longrightarrow v, \\ \overline{F}_v A_\alpha & \text{for every arrow } \alpha : u \dashrightarrow v \end{cases} \tag{10}$$

(these equalities are obtained from (2) in view of Lemma 3.1.) In particular, if $\mathcal{F}$ is isomorphism, then we can put $S_v := F_v^{-1}$ for all vertices v and rewrite (10) in the form (9).

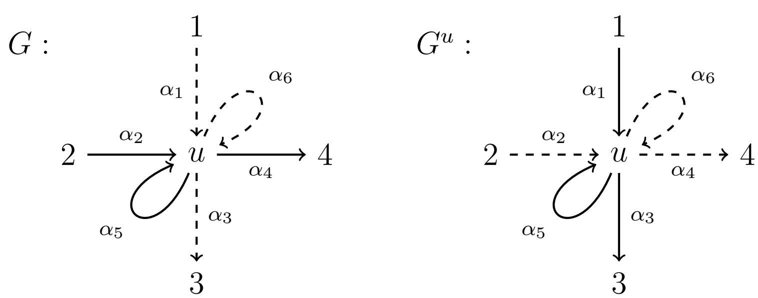
By (8) and (9),

two matrix representations are isomorphic if and only if they give the same representation $\mathcal{A}$ but in possible different bases.

Denote by $M(G)$ the set of matrix representations of a biquiver G.

4 Proof of Theorem 2.1

For each biquiver G and its vertex u, we denote by G^u the biquiver obtained from G by replacing all arrows $u \longrightarrow v$ and $v \longrightarrow u$ for each $v \neq u$ by $u \dashrightarrow v$ and $v \dashrightarrow u$, and vice versa. For example,



(11)

We say that G^u is obtained from G by *conjugation* at the vertex u. For each $A \in M(G)$, define $A^u \in M(G^u)$ as follows:

$$A^u_\alpha := \begin{cases} A_\alpha & \text{if } \alpha \text{ does not start at } u, \\ \bar{A}_\alpha & \text{if } \alpha \text{ starts at } u. \end{cases}$$

We say that A^u is obtained from A by *conjugation* at the vertex u.

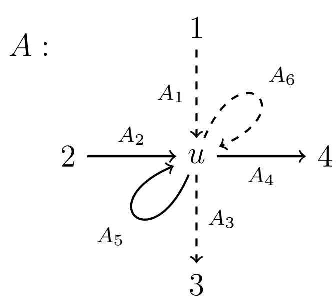
Lemma 4.1. *Let $A, B \in M(G)$ and let u be any vertex of G. Then $A \simeq B$ if and only if $A^u \simeq B^u$.*

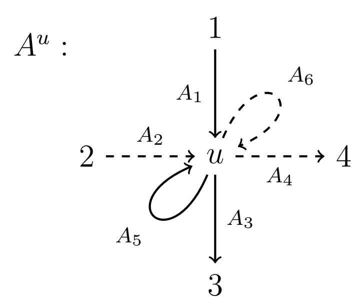
Proof. It suffices to prove that

$$\text{if } A \simeq B \text{ via } S_1, \dots, S_t \text{ (see (9)), then } A^u \simeq B^u \text{ via } R_1, \dots, R_t, \text{ in which } R_v := S_v \text{ if } v \neq u \text{ and } R_u := \bar{S}_u. \tag{12}$$

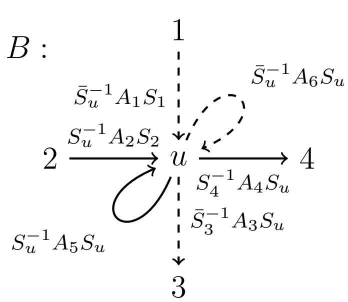
Moreover, it suffices to prove (12) for matrix representations of the biquiver G defined in (11), which contains all possible arrows from the vertex u and to the vertex u.

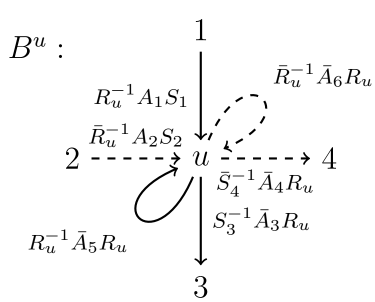
Let us consider an arbitrary matrix representation A of G and the corresponding matrix representation A^u of G^u:





Let B be any matrix representation of G that is isomorphic to A via $S_1, \dots S_4, S_u$. Then B and the corresponding matrix representation B^u of G^u have the form:





in which R_u is defined by (12).

Therefore, B^u is isomorphic to A^u via $S_1, \dots S_4, R_u$, which proves (12). □

Proof of Theorem 2.1. Let G be a connected bigraph with t vertices.

(a) Suppose first that G is a tree. Let us prove that by a sequence of conjugations we can transform G to the quiver $Q(G)$ obtained from G by replacing each dashed arrow $v \dashrightarrow w$ with the full arrow $v \longrightarrow w$.

Let w be a pendant vertex of G (i.e., a vertex of degree 1). Let α be the arrow for which w is one of its vertices. Denote by $G\backslash\alpha$ the biquiver obtained from G by deleting w and α. Reasoning by induction on the number of vertices, we assume that $G\backslash\alpha$ can be transformed to $Q(G\backslash\alpha)$ by a sequence of conjugations. The same sequence of conjugations transforms G to some biquiver G' in which only the arrow that is obtained from α can be dashed. If it is dashed, we make it full by conjugation of G' at the vertex w and obtain $Q(G)$. Theorem 2.1 holds for $Q(G)$ by Gabriel's theorem [8]. Lemma 4.1 ensures that Theorem 2.1 holds for G too.

Suppose now that G is not a tree. Then G contains a cycle C that up to renumbering of vertices of G has the form

$$C: \qquad 1 \overset{\alpha_1}{\text{———}} 2 \overset{\alpha_2}{\text{———}} \dots \overset{\alpha_{r-2}}{\text{———}} (r-1) \overset{\alpha_{r-1}}{\text{———}} r \quad (1 \overset{\alpha_r}{\text{———}} r) \tag{13}$$

If $r > 1$ and the sequence of arrows $\alpha_1, \dots, \alpha_{r-1}$ contains a dashed arrow, then we take the first dashed arrow α_ℓ and make it full by conjugation of G at the vertex $\ell + 1$. Repeating this procedure, we make all arrows $\alpha_1, \dots, \alpha_{r-1}$ full.

For each $n \times n$ matrix M, let us construct the matrix representation $P(M)$ of G by assigning I_n to each of the arrows $\alpha_1, \dots, \alpha_{r-1}$ (if $r > 1$), M to α_r, and 0_n to the other arrows. It is easy to see that $P(M) \simeq P(N)$ if and only if either α_r is full and M is similar to N, or α_r is dashed and M is consimilar to N (see (8)). The Jordan canonical form and a canonical form under consimilarity [7, Theorem 4.6.12] ensure that G is of infinite type. Since G contains a cycle, it cannot be obtained by directing edges in one of the Dynkin diagrams (3), and so q_G is not positive definite by Gabriel's theorem [8].

(b) If G is of finite type, then G is a tree. By the part (a) of the proof, G can be transformed to the quiver $Q(G)$ by a sequence of conjugations. By Lemma 4.1, this sequence of conjugations transforms all indecomposable representations of G to all indecomposable representations of $Q(G)$, and nonisomorphic representations are transformed to nonisomorphic representations. This proves (b) for G since (b) holds for $Q(G)$. $\square$

5 Proof of Theorem 2.2

Lemma 5.1. *The problem of classifying representations of each of the biquivers*

$$G_1: \quad \alpha_1 \circlearrowright 1 \xrightarrow{\ \alpha\ } 2 \qquad\qquad G_2: \quad \alpha_1 \circlearrowright 1 \xleftarrow{\ \alpha\ } 2 \tag{14}$$

contains the problem of classifying representations of the biquiver

$$G_3: \quad \alpha_1 \circlearrowright 1 \circlearrowleft \alpha_2 \tag{15}$$

Proof. The problem of classifying representations of the biquivers (14) is the problem of classifying matrix pairs up to transformations

$$(M, N) \mapsto (\bar{S}^{-1}MS, R^{-1}NS), \tag{16}$$
$$(M, N) \mapsto (\bar{S}^{-1}MS, S^{-1}NR), \tag{17}$$

respectively.

Let us consider G_1. Let

$$M := \begin{bmatrix} 0 & P \\ I & Q \end{bmatrix}, \qquad M' := \begin{bmatrix} 0 & P' \\ I & Q' \end{bmatrix}, \qquad N := \begin{bmatrix} 0 & I \end{bmatrix},$$

in which all blocks are n-by-n. Let (M, N) be reduced to (M', N) by transformations (16); i.e., there exist nonsingular S and R such that

$$MS = \bar{S}M', \qquad NS = RN. \tag{18}$$

By the second equality in (18), S has the form

$$S = \begin{bmatrix} S_1 & S_2 \\ 0 & R \end{bmatrix}.$$

Equating the 1,1 blocks in the first equality in (18) gives $S_2 = 0$; equating the 2,1 blocks gives $S_1 = \bar{R}$; equating the 1,2 and 2,2 blocks gives

$$PR = RP', \qquad QR = \bar{R}Q'.$$

Therefore, (M, N) and (M', N) define isomorphic representations of G_1 if and only if (P, Q) and (P', Q') define isomorphic representations of (15), and so the problem of classifying representations of G_1 contains the problem of classifying representations of (15).

Let us consider G_2. Taking $N := \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$ and reasoning as for G_1, we prove that if (M, N) is reduced to (M', N) by transformations (17); i.e., there exist nonsingular S and R such that $MS = \bar{S}M'$ and $NR = SN$, then S is upper block triangular and so (P, Q) and (P', Q') define isomorphic representations of (15). ☐

Lemma 5.2. *The problems of classifying representations of the biquiver* G_3 *defined in* (15) *and the biquiver*

$$G_4: \qquad \alpha_1 \circlearrowright 1 \circlearrowleft \alpha_2$$

contain the problem of classifying matrix pairs up to similarity.

Proof. The problems of classifying representations of the biquivers G_3 and G_4 are the problems of classifying matrix pairs up to transformations

$$(M, N) \mapsto (S^{-1}MS, \bar{S}^{-1}NS), \tag{19}$$
$$(M, N) \mapsto (\bar{S}^{-1}MS, \bar{S}^{-1}NS), \tag{20}$$

respectively.

Let us consider G_3. Let

$$M := \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \qquad N := \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}, \qquad N' := \begin{bmatrix} P' & 0 \\ 0 & Q' \end{bmatrix},$$

in which all blocks are n-by-n. Let (M, N) be reduced to (M, N') by transformations (19); i.e., there exists a nonsingular S such that

$$MS = SM, \qquad NS = \bar{S}N'. \tag{21}$$

By the first equality in (21), S has the form

$$S = \begin{bmatrix} S_1 & S_2 \\ 0 & S_1 \end{bmatrix}.$$

Equating the 1,1 and 2,2 blocks in the second equality in (21) gives

$$\bar{S}_1^{-1}PS_1 = P', \qquad \bar{S}_1^{-1}QS_1 = Q'.$$

Therefore, (M,N) and (M,N') define isomorphic representations of G_3 if and only if (P,Q) and (P',Q') define isomorphic representations of G_4, and so the problem of classifying representations of G_3 contains the problem of classifying representations of G_4.

Let us consider G_4. Let

$$M := \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad N := \begin{bmatrix} 0 & 0 & 0 & 0 \\ P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Q & 0 \end{bmatrix}, \quad N' := \begin{bmatrix} 0 & 0 & 0 & 0 \\ P' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Q' & 0 \end{bmatrix}$$

in which all blocks are n-by-n. Let (M,N) be reduced to (M,N') by transformations (20); i.e., there exists a nonsingular S such that

$$MS = \bar{S}M, \qquad NS = \bar{S}N'. \tag{22}$$

By the first equality in (22), S has the form

$$S = \begin{bmatrix} S_1 & S_2 & S_3 & S_4 \\ 0 & \bar{S}_1 & \bar{S}_2 & \bar{S}_3 \\ 0 & 0 & S_1 & S_2 \\ 0 & 0 & 0 & \bar{S}_1 \end{bmatrix}.$$

Equating the 2,1 and 4,3 blocks in the second equality in (22) gives

$$S_1^{-1}PS_1 = P', \qquad S_1^{-1}QS_1 = Q'.$$

Therefore, (M,N) and (M,N') define isomorphic representations of G_4 if and only if (P,Q) and (P',Q') are similar, and so the problem of classifying representations of G_4 is wild. □

Proof of Theorem 2.2. (a) Suppose first that G is a tree. Reasoning as in the proof of Theorem 2.1, we transform G to the quiver $Q(G)$ by a sequence of conjugations. Theorem 2.2 holds for $Q(G)$ by the Donovan–Freislich–Nazarova theorem [3, 10]. Lemma 4.1 ensures that Theorem 2.2 holds for G too.

Suppose now that G is not a tree. Then G contains a cycle C that up to renumbering of vertices of G has the form (13) in which $r \geqslant 1$ and each edge is a full or dashed arrow.

If $G = C$, then G is of tame type; all its representations were classified in [5].

Let us suppose that $G \neq C$ and prove that G is of wild type. The biquiver G contains a biquiver C' obtained by adjoining to C an edge $\alpha : u \to v$ or $v \to u$, in which $u \in \{1, \dots, r\}$, we suppose that $u = 1$. If C' is of wild type, then G is of wild type too: we can identify all representations of C' with those representations of G, in which the vertices outside of C' are assigned by 0; two representations of C' are isomorphic if and only if the corresponding representations of G are isomorphic. Further we suppose that $G = C'$.

Reasoning as in the proof of Theorem 2.1(a), we can transform the subbiquiver

$$2 \overset{\alpha_2}{\text{———}} 3 \overset{\alpha_3}{\text{———}} \ \dots \ \overset{\alpha_{r-1}}{\text{———}} r \overset{\alpha_r}{\text{———}} 1$$

of G to a quiver by a sequence of conjugations in some of the vertices $3, 4, \dots, 1$. Thus, we can suppose that the arrows $\alpha_2, \dots, \alpha_r$ of G are full arrows.

Suppose first that v is not a vertex of C. If $\alpha : u \text{ — } v$ is dashed, we make it full by conjugation at v. If α_1 is a full arrow, then G is a quiver of wild type by the Donovan–Freislich–Nazarova theorem. Thus, we can suppose that α_1 is a dashed arrow. Let $\ell \in \{1, 2\}$ be such that α in G has the same direction as α_1 in G_ℓ defined in (14). The biquiver G is of wild type since G_ℓ is of wild type and each matrix representation A of G_ℓ can be identified with the matrix representation of G obtained from A by assigning the identity matrix to $\alpha_2, \dots, \alpha_r$; two representations of G_ℓ are isomorphic if and only if the corresponding representations of G are isomorphic.

Suppose now that v is a vertex of C. If α and α_1 are full arrows, then G is a quiver of wild type by the Donovan–Freislich–Nazarova theorem. Let α or α_1 be a dashed arrow. Denote by G' the biquiver obtained from G by deleting its arrows $\alpha_2, \dots, \alpha_r$ and its vertices $2, 3, \dots, r-1$, and by identifying the vertices 1 and r. By Lemma 5.2, G' is of wild type. Hence, G is of wild type too since each matrix representation A of G' can be identified with the matrix representation of G obtained from A by assigning the identity matrix to $\alpha_2, \dots, \alpha_r$; two representations of G' are isomorphic if and only if the corresponding representations of G are isomorphic.

(b) Let G be tame. Then G is a tree or cycle. If G is a tree, then it can be transformed to the quiver $Q(G)$ by a sequence of conjugations. By Lemma 4.1, this sequence of conjugations transforms all indecomposable representations of G to all indecomposable representations of $Q(G)$, and nonisomorphic representations are transformed to nonisomorphic representations. This proves (b) for G since (b) holds for quivers by the Donovan–Freislich–

Nazarova theorem. If G is a cycle then (b) follows from the classification of its representations given in [5]. $\square$

References

[1] H. Bass, *Algebraic K-theory*, Benjamin, New York, 1968.

[2] G.R. Belitskii and V.V. Sergeichuk, *Complexity of matrix problems*, Linear Algebra Appl. **361** (2003), 203–222.

[3] P. Donovan and M.R. Freislich, *The representation theory of finite graphs and associated algebras*, Carleton Math. Lecture Notes, 5, Carleton University, Ottawa, 1973.

[4] D. Duarte de Oliveira, R.A. Horn, T. Klimchuk, and V.V. Sergeichuk, *Remarks on the classification of a pair of commuting semilinear operators*, Linear Algebra Appl. **436** (2012), 3362–3372.

[5] D. Duarte de Oliveira, V. Futorny, T. Klimchuk, D. Kovalenko, and V.V. Sergeichuk, *Cycles of linear and semilinear mappings*, Linear Algebra Appl. **438** (2013), 3442–3453.

[6] M. Hazewinkel, N. Gubareni, and V.V. Kirichenko. *Algebras, Rings and Modules*. Vol. 2. Springer, 2007.

[7] R.A. Horn and C.R. Johnson, *Matrix Analysis*, 2nd ed., Cambridge University Press, Cambridge, 2013.

[8] P. Gabriel, *Unzerlegbare Darstellungen I*, Manuscripta Math. **6** (1972), 71–103.

[9] P. Gabriel and A.V. Roiter, *Representations of finite-dimensional algebras*, Encyclopaedia of Math. Sci. Vol. 73 (Algebra VIII), Springer-Verlag, 1992.

[10] L.A. Nazarova, *Representations of quivers of infinite type*, Math. USSR Izv. **7** (1973), 749–792.

[11] V.V. Sergeichuk, *Linearization method in classification problems of linear algebra*, São Paulo J. Math. Sci. **1** (no. 2) (2007), 219–240.