# On the vanishing and non-rigidity of the André-Quillen (co)homology

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**Abstract**

Let $I$ be an ideal of a commutative ring A, $B = A/I$. Given $n \geq 2$, we characterize the vanishing of the André-Quillen homology modules $H_p(A, B, W)$ for all $B$-module $W$ and for all $p$, $2 \leq p \leq n$, in terms of some canonical morphisms. As a corollary, we obtain a new proof of a theorem of André. Finally, we construct an example of an ideal $I$ of a commutative ring $A$ such that $H_2(A, B, W) = 0$ and $H_3(A, B, W) = W$ for all $B$-module $W$.

## 1 Introduction

Let $I$ be an ideal of a commutative ring $A$ and $B = A/I$. Let $\alpha : \mathbf{S}(I) \to \mathbf{R}(I)$ denote the canonical morphism from the symmetric algebra of $I$ onto its Rees algebra. Let $\beta : \mathbf{S}^B(I/I^2) \to \mathbf{G}(I)$ denote the canonical morphism from the symmetric algebra of the conormal module of $I$ onto its associated graded ring. Let $\gamma : \mathbf{\Lambda}^B(I/I^2) \to \mathrm{Tor}_*^A(B, B)$ denote the canonical morphism from the exterior algebra of $I/I^2$ to the anticommutative graded $B$-algebra $\mathrm{Tor}_*^A(B, B)$. Moreover, we stand $\tau_{p,q} : \mathrm{Tor}_p^A(B, A/I^q) \to \mathrm{Tor}_p^A(B, A/I^{q-1})$ for the canonical morphism for any two given integers $p, q \geq 1$.

Let $H_p(A, B, W)$ denote the $p$-th André-Quillen homology module of the $A$-algebra $B$ with coefficients in the $B$-module $W$. The first purpose of this paper is to show the following theorem:

**Theorem 1.1** *Given $n \geq 2$, the following conditions are equivalent:*

*(i)* $H_p(A, B, W) = 0$ *for all $B$-module $W$ and for all $p$, $2 \leq p \leq n$.*

*(ii)* $I/I^2$ *is a flat $B$-module, $\alpha$ is an isomorphism and $\tau_{p,q} = 0$ for all $p$, $3 \leq p \leq n$, for all $q \geq 2$.*

*(iii)* $I/I^2$ *is a flat $B$-module, $\beta$ is an isomorphism and $\tau_{p,q} = 0$ for all $p$, $2 \leq p \leq n$, for all $q \geq 2$.*

*(iv)* $I/I^2$ *is a flat $B$-module and $\gamma_p$ is an isomorphism for all $p$, $2 \leq p \leq n$.*

The equivalence between $(i)$ and $(iv)$, for $n = \infty$, is proved by Quillen in 10.3 of [8] (see also 6.13 of [9]). The proof of this equivalence for a given $n \geq 2$ follows carefully that one of [8].

The equivalence between $(i)$ and $(iii)$, for $n = \infty$, is due to André (see Théorème A of [2]). The proof of this equivalence for a given $n \geq 2$ consists in proving firstly that one of $(iii)$ with $(iv)$. To do this, we shall recover a diagram build by Quillen in [8] and then apply Theorem 4.2 of [7] (see also [6]). Since we will use this theorem several times we recall it here:

**Theorem** (see 4.2 of [7]) *The following conditions are equivalent:*

$(i)$ $H_2(A, B, W) = 0$ *for all* $B$*-module* $W$.

$(ii)$ $I/I^2$ *is a flat* $B$*-module and* $\alpha$ *is an isomorphism.*

$(iii)$ $I/I^2$ *is a flat* $B$*-module,* $\beta$ *is an isomorphism and* $\tau_{2,q} = 0$ *for all* $q \geq 2$.

$(iv)$ $I/I^2$ *is a flat* $B$*-module,* $\beta_2$ *is an isomorphism and* $\tau_{2,2} = 0$.

In this way, Théorème A of André in [2] is obtained as a consequence of Theorem 4.2 of [7] and the methods used by Quillen in [8].

Finally, the equivalence between $(ii)$ and $(iii)$ in Theorem 1.1 is clearly a corollary of the same Theorem 4.2 of [7].

When $A$ is a noetherian ring, it is well-known that the vanishing of the second homology functor already implies the vanishing of all higher homology functors. In fact, $H_2(A, B, W) = 0$, for all $B$-module $W$, is equivalent to $I$ being locally generated by a regular sequence (see 6.25 of [1] or 10.7 of [8]).

The second purpose of this paper is to give an example of the non-rigidity of the André-Quillen homology when $A$ fails to be noetherian. Concretely, we construct a commutative local ring $A$ of Krull dimension 2, with maximal ideal $I$ generated by two elements, and such that, if we denote by $B = A/I$ the residual field, then $H_2(A, B, W) = 0$ and $H_3(A, B, W) = W$, for all $B$-module $W$. In particular, $\gamma_2 : \mathbf{\Lambda}_2^B(I/I^2) \to \mathrm{Tor}_2^A(B, B)$ is an isomorphism, but $\gamma_3$ it is not. Moreover, $\tau_{2,q} = 0$ for all $q \geq 2$, but $\tau_{3,2} \neq 0$ (see Proposition 2.2).

The author wishes to thank José M. Giral for several discussions concerning this paper. He would also thank Michel André and Antonio G. Rodicio for their comments and interest.

## 2 Proof of the Theorem

Let $I$ be an ideal of $A$, $B = A/I$. For every $q \geq 1$, the short exact sequence

$$0 \to I^q/I^{q+1} \to A/I^{q+1} \to A/I^q \to 0$$

leads to the correspondent long exact sequence of $\mathrm{Tor}_*^A(B,\cdot)$:

$$\ldots \xrightarrow{c_{p+1,q}} \mathrm{Tor}_p^A(B, I^q/I^{q+1}) \xrightarrow{i_{p,q}} \mathrm{Tor}_p^A(B, A/I^{q+1}) \xrightarrow{\tau_{p,q+1}} \mathrm{Tor}_p^A(B, A/I^q) \xrightarrow{c_{p,q}} \ldots \tag{1}$$

Let $\mathrm{d}_{p,q} : \mathrm{Tor}_p^A(B, I^q/I^{q+1}) \to \mathrm{Tor}_{p-1}^A(B, I^{q+1}/I^{q+2})$ be defined as the composition

$$\mathrm{d}_{p,q} : \mathrm{Tor}_p^A(B, I^q/I^{q+1}) \xrightarrow{i_{p,q}} \mathrm{Tor}_p^A(B, A/I^{q+1}) \xrightarrow{c_{p,q+1}} \mathrm{Tor}_{p-1}^A(B, I^{q+1}/I^{q+2})\,.$$

It is shown (see 8.2 of [8]) that $\mathrm{d}_{p,q}$ defines in the bigraded $B$-algebra $\mathrm{Tor}_*^A(B, \mathbf{G}_*(I))$ a differential. Moreover, the isomorphism $I/I^2 \simeq \mathrm{Tor}_1^A(B,B)$ extends naturally to a canonical morphism of differential bigraded $B$-algebras:

$$\psi_{p,q} : \mathbf{\Lambda}_p^B(I/I^2) \underset{B}{\otimes} \mathbf{S}_q^B(I/I^2) \longrightarrow \mathrm{Tor}_p^A(B, I^q/I^{q+1})\,,$$

where the left side is endowed with the Koszul differential. In other words, for every $p, q \geq 1$, one has the following commutative diagram:

$$\begin{array}{ccccccc}
 & \mathrm{Tor}_{p+1}^A(B, A/I^{q-2}) & & \mathrm{Tor}_p^A(B, A/I^{q-1}) & & & \\
 & {\scriptstyle i_{p+1,q-3}}\nearrow \quad \searrow{\scriptstyle c_{p+1,q-2}} & & {\scriptstyle i_{p,q-2}}\nearrow \quad \searrow{\scriptstyle c_{p,q-1}} & & & \\
\cdots \to \mathrm{Tor}_{p+1}^A(B, I^{q-3}/I^{q-2}) & \xrightarrow{\mathrm{d}_{p+1,q-3}} & \mathrm{Tor}_p^A(B, I^{q-2}/I^{q-1}) & \xrightarrow{\mathrm{d}_{p,q-2}} & \mathrm{Tor}_{p-1}^A(B, I^{q-1}/I^q) \to \cdots \\
\uparrow{\scriptstyle \psi_{p+1,q-3}} & & \uparrow{\scriptstyle \psi_{p,q-2}} & & \uparrow{\scriptstyle \psi_{p-1,q-1}} \\
\cdots \to \mathbf{\Lambda}_{p+1}^B(I/I^2) \underset{B}{\otimes} \mathbf{S}_{q-3}^B(I/I^2) & \xrightarrow[\partial_{p+1,q-3}]{} & \mathbf{\Lambda}_p^B(I/I^2) \underset{B}{\otimes} \mathbf{S}_{q-2}^B(I/I^2) & \xrightarrow[\partial_{p,q-2}]{} & \mathbf{\Lambda}_{p-1}^B(I/I^2) \underset{B}{\otimes} \mathbf{S}_{q-1}^B(I/I^2) \to \cdots
\end{array}$$

Quillen's diagram: $\mathrm{QD}_{p+q-2}$

The bottom row of the diagram $\mathrm{QD}_{p+q-2}$ is the homogeneous part of degree $p+q-2$ of the Koszul complex $\mathbf{\Lambda}^B(I/I^2) \otimes \mathbf{S}^B(I/I^2)$. It is known to be acyclic whenever $I/I^2$ is a flat $B$-module or $A$ contains the field of rational numbers (see, for instance, 9.3 of [4]).

Remark also that for each $p, q \geq 0$, the morphism $\psi_{p,q}$ factorizes through

$$\psi_{p,q} : \mathbf{\Lambda}_p^B(I/I^2) \underset{B}{\otimes} \mathbf{S}_q^B(I/I^2) \xrightarrow{\gamma_p \otimes \beta_q} \mathrm{Tor}_p^A(B,B) \underset{B}{\otimes} I^q/I^{q+1} \longrightarrow \mathrm{Tor}_p^A(B, I^q/I^{q+1})\,.$$

Notice that the second morphism is bijective if $I^q/I^{q+1}$ is a flat $B$-module.

**Lemma 2.1** *If $I/I^2$ is a flat $B$-module, then $\gamma$ is a monomorphism.*

*Proof.* Let us prove, by induction on $p \geq 1$, that $\gamma_p : \mathbf{\Lambda}_p^B(I/I^2) \to \mathrm{Tor}_p^A(B,B)$ is a monomorphism. For $p = 1$, it is clear. Suppose $p \geq 2$ and $\gamma_{p-1}$ is a monomorphism. Consider the diagram $\mathrm{QD}_p$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Tor}_p^A(B,B) & \xrightarrow{\mathrm{d}_{p,0}} & \mathrm{Tor}_{p-1}^A(B,I/I^2) & \xrightarrow{\mathrm{d}_{p-1,1}} & \mathrm{Tor}_{p-2}^A(B,I^2/I^3) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \gamma_p} & & \uparrow{\scriptstyle \psi_{p-1,1}} & & \uparrow{\scriptstyle \psi_{p-2,2}} & & \\
0 & \longrightarrow & \mathbf{\Lambda}_p^B(I/I^2) & \xrightarrow[\partial_{p,0}]{} & \mathbf{\Lambda}_{p-1}^B(I/I^2)\underset{B}{\otimes} I/I^2 & \xrightarrow[\partial_{p-1,1}]{} & \mathbf{\Lambda}_{p-2}^B(I/I^2)\underset{B}{\otimes} \mathbf{S}_2^B(I/I^2) & \longrightarrow & \cdots
\end{array}
$$

Since the bottom row is exact, then $\partial_{p,0}$ is injective. Since $\gamma_{p-1}$ is injective, then $\psi_{p-1,1}$ is also injective. Therefore, by the commutativity of $\mathrm{QD}_p$, $\gamma_p$ is injective too. ∎

**Proposition 2.2** *Given $n \geq 2$ and if $I/I^2$ is a flat $B$-module, then the following conditions are equivalent:*

*(i) $\beta$ is an isomorphism and $\tau_{p,q} = 0$ for all $p$, $2 \leq p \leq n$, for all $q \geq 2$.*

*(ii) $\beta_2$ is an isomorphism and $\tau_{p,2} = 0$ for all $p$, $2 \leq p \leq n$.*

*(iii) $\gamma_p$ is an isomorphism for all $p$, $2 \leq p \leq n$.*

*Proof.* It is clear that $(i)$ implies $(ii)$. Let us prove $(ii)$ implies $(iii)$ by induction on $p \geq 2$. If $p = 2$, we have that $\psi_{1,1}$ and $\psi_{0,2} = \beta_2$ are two isomorphisms. Since $\tau_{2,2} = 0$ and $\mathrm{d}_{2,0} = c_{2,1}$, then $\mathrm{Kerd}_{2,0} = \mathrm{Im}\tau_{2,2} = 0$. Therefore, using $\mathrm{QD}_2$, one deduces that $\gamma_2$ is an epimorphism. Suppose $p \geq 3$. Since $I/I^2$ is a flat $B$-module and $\gamma_{p-1}$, $\gamma_{p-2}$ and $\beta_2$ are all three isomorphisms, then $\psi_{p-1,1}$ and $\psi_{p-2,2}$ are two isomorphisms. Since $\tau_{p,2} = 0$ and $\mathrm{d}_{p,0} = c_{p,1}$, then $\mathrm{Kerd}_{p,0} = \mathrm{Im}\tau_{p,2} = 0$. Thus, using $\mathrm{QD}_p$ and the same argument used in the case $p = 2$, one deduces that $\gamma_p$ is an epimorphism. Remark that by Lemma 2.1, $\gamma$ is a monomorphism since $I/I^2$ is a flat $B$-module.

Let us prove now $(iii)$ implies $(i)$. Since $\gamma_p$ and $\psi_{p-1,1}$ are isomorphisms and $\partial_{p,0}$ is injective, then, by the commutativity of $\mathrm{QD}_p$, $\mathrm{d}_{p,0}$ is injective. In particular, $\tau_{p,2} = 0$. Moreover, for $p = 2$, $\psi_{0,2} = \beta_2$ and by similar arguments to the lemma of five applied to $\mathrm{QD}_2$, we deduce that $\beta_2$ is an isomorphism. In particular, using Theorem 4.2 of [7], we deduce that $\beta$ is an isomorphism. To finish it suffices to prove, by induction on $q \geq 2$ and for every given $p$, $2 \leq p \leq n$, the following

*CLAIM*: If $\beta$, $\gamma_{p-1}$ and $\gamma_p$ are isomorphisms, then $\tau_{p,q} = 0$.

For $q = 2$, we have already seen $\tau_{p,2} = 0$. Suppose $q \geq 3$. Since $I/I^2$ is a flat $B$-module, the bottom row of the diagram $\mathrm{QD}_{p+q-2}$ is exact, and, as $\beta$, $\gamma_{p-1}$ and $\gamma_p$ are isomorphisms, the morphisms $\psi_{p,q-2}$ and $\psi_{p-1,q-1}$ are isomorphisms. In particular, a piece of the top row of the diagram $\mathrm{QD}_{p+q-2}$ is exact. Concretely, the following short complex is an exact sequence:

$$\mathrm{Tor}_{p+1}^A(B, I^{q-3}/I^{q-2}) \overset{\mathrm{d}_{p+1,q-3}}{\longrightarrow} \mathrm{Tor}_p^A(B, I^{q-2}/I^{q-1}) \overset{\mathrm{d}_{p,q-2}}{\longrightarrow} \mathrm{Tor}_{p-1}^A(B, I^{q-1}/I^q)\,. \qquad (2)$$

Let us see $\tau_{p,q} = 0$. Since (1) is an exact sequence, then $\mathrm{Im}\tau_{p,q} = \mathrm{Ker}c_{p,q-1}$ and, therefore, it suffices to prove that $c_{p,q-1}$ is a monomorphism. Take $x \in \mathrm{Ker}c_{p,q-1}$. The induction hypothesis on $q \geq 3$, assures that $\tau_{p,q-1} = 0$ and the exactness of (1) says that $\mathrm{Im}i_{p,q-2} = \mathrm{Ker}\tau_{p,q-1}$. Therefore, $i_{p,q-2}$ is an epimorphism. So, there exists $y \in \mathrm{Tor}_p^A(B, I^{q-2}/I^{q-1})$ such that $i_{p,q-2}(y) = x$. Thus, $\mathrm{d}_{p,q-2}(y) = 0$. As (2) is an exact sequence, there exists $z \in \mathrm{Tor}_{p+1}^A(B, I^{q-3}/I^{q-2})$ with $\mathrm{d}_{p+1,q-3}(z) = y$. Since (1) is exact, $i_{p,q-2} \circ c_{p+1,q-2} = 0$, and, therefore $x = i_{p,q-2}(\mathrm{d}_{p+1,q-3}(z)) = i_{p,q-2} \circ c_{p+1,q-2} \circ i_{p+1,q-3}(z) = 0$. ∎

*Proof of Theorem 1.1.* The equivalence of $(ii)$ with $(iii)$ follows from Theorem 4.2 of [7]. The equivalence of $(iii)$ with $(iv)$ is Proposition 2.2. The proof of the equivalence between $(i)$ and $(iv)$ consists in following the proof of 10.3 in [8]. We sketch it here briefly.

Taking a free presentation of each $B$-module $W$ and applying the homology functors $H_p(A, B, \cdot)$ to the chosen presentation, it is easy to see that condition $(i)$ is equivalent to:

$(i')$ $I/I^2$ is a flat $B$-module and $H_p(A, B, B) = 0$ for all $p$, $2 \leq p \leq n$.

But, this condition is shown to be equivalent to:

$(i'')$ $I/I^2$ is a flat $B$-module and the canonical morphism $\xi : \mathbb{L}_{B|A} \to \mathrm{K}(I/I^2, 1)$ is an $n$-equivalence.

where $\mathbb{L}_{B|A}$ stands for the cotangent complex of the $A$-algebra $B$ and $\mathrm{K}(I/I^2, 1)$ stands for the simplicial $B$-module whose normalisation is the chain complex with $I/I^2$ in dimension 1 and zero elsewhere.

Let $P$ stand for a free simplicial $A$-algebra resolution of $B$. Consider the simplicial augmented $B$-algebra $Q = P \otimes_A B$ and denote by $J$ its augmentation ideal. By filtering $Q$ with the powers $J^q$ of $J$ one obtains the spectral sequence

$$\mathrm{E}^2_{p,q} = H_{p+q}(J^q/J^{q+1}) = H_{p+q}(\mathbf{S}_q^B(\mathbb{L}_{B|A}))\ ,\ \mathrm{d}^r : \mathrm{E}^r_{p,q} \to \mathrm{E}^r_{p-r,q+r-1}\,, \tag{3}$$

which converges to $\mathrm{Tor}_{p+q}^A(B, B) = H_{p+q}(Q)$ (see 6.8 of [8]).

In 10.3 of [8], Quillen shows that if $I/I^2$ is a flat $B$-module and $\xi$ is an $n$-equivalence, then $\mathrm{E}^2_{p,q} = 0$ for all $p, q$ with $p + q \leq n$ and $p > 0$. Moreover, in this case, one can deduce the following exact sequence:

$$\mathrm{Tor}_{n+1}^A(B, B) \longrightarrow H_{n+1}(A, B, B) \xrightarrow{\mathrm{d}^2_{n,1}} \mathrm{E}^2_{n-2,2} = 0\,, \tag{4}$$

where the first morphism is the edge morphism.

To finish, it suffices to prove, under the flatness assumption on $I/I^2$, that $\xi$ is an $n$-equivalence if and only if $\gamma_q : \mathbf{\Lambda}_q^B(I/I^2) \to \mathrm{Tor}_q^A(B, B)$ is surjective for all $q$, $2 \leq q \leq n$. Suppose $\xi_n$ is an $n$-equivalence, then $\mathrm{E}^\infty_{p,q} = \mathrm{E}^2_{p,q} = 0$ for all $p + q \leq n$, $p > 0$. Therefore, the edge morphism

$$\gamma_q : \mathrm{E}^2_{0,q} \to \mathrm{E}^3_{0,q} \to \ldots \to \mathrm{E}^r_{0,q} = \mathrm{E}^\infty_{0,q} = H_q(Q) = \mathrm{Tor}_q^A(B, B)$$

is an epimorphism. Reciprocally, if $\xi$ is a $(q-1)$-equivalence and $\gamma_q$ is an epimorphism, then using the sequence (4) and the fact that the edge morphism $\mathrm{Tor}_q^A(B,B) \to H_q(A,B,B)$ vanishes the decomposable elements of the $B$-algebra $\mathrm{Tor}_*^A(B,B)$, we deduce $H_q(A,B,B) = 0$. Hence, the conclusion follows by induction on $q \geq 2$. ∎

**Remark 2.3** By changing the flatness condition on $I/I^2$ for the projectiveness one, we can replace homology for cohomology in Theorem 1.1.

# 3 An example of the non-rigidity

Let $I$ be an ideal of $A$, $B = A/I$. Let $f : F \to A$ be the free presentation of $I$ associated to an arbitrary set of generators $\mathbf{x}$ of $I$. Denote by $\mathcal{K}(f) = \mathcal{K}(\mathbf{x}) = \mathcal{K}(I)$ the Koszul complex of $f$. Then, for each $B$-module $W$, there exists an exact sequence of $B$-modules (15.12 [1]):

$$0 \longrightarrow H_2(A,B,W) \longrightarrow H_1(\mathcal{K}(I)) \underset{B}{\otimes} W \longrightarrow F \underset{A}{\otimes} W \longrightarrow I \underset{A}{\otimes} W \longrightarrow 0\,. \tag{5}$$

On the other hand, $H_3(A,B,W) = 0$ for all $B$-module $W$ is equivalent to $H_1(\mathcal{K}(I))$ being a flat $B$-module and $\mathbf{\Lambda}_2^B(H_1(\mathcal{K}(I))) \to H_2(\mathcal{K}(I))$ being surjective ([3] or [10]).

Thus, if we find a ring $A$ with an ideal $I$, $B = A/I$, such that $H_1(\mathcal{K}(I)) = 0$ and $H_2(\mathcal{K}(I)) \neq 0$, then $H_2(A,B,W) = 0$ for all $B$-module $W$ and $H_3(A,B,W_0) \neq 0$ for some $B$-module $W_0$. Actually, taking a free presentation of $W_0$, one would deduce $H_3(A,B,B) \neq 0$.

If $I = < x >$ is a principal ideal, then the second Koszul homology group of $x$ is always zero. We thus have to look for an ideal $I = < x, y >$ generated by at least two elements $x, y$. Recall that the second Koszul homology group of $x, y$ is $H_2(\mathcal{K}(x,y)) = (0 : I)$. Next two lemmas characterize the vanishing of $H_1(\mathcal{K}(x,y))$ and how the elements of $H_2(\mathcal{K}(x,y)) = (0 : I)$ look like when $H_1(\mathcal{K}(x,y)) = 0$.

**Lemma 3.1** *The following two conditions are equivalent:*

*(i)* $H_1(\mathcal{K}(x,y)) = 0$.

*(ii)* $(x : y) = < x >$, $(0 : x) \subset < y >$ *and* $(0 : xy) = (0 : x) + (0 : y)$.

*Proof.* Let us denote by $Z_1$ and $B_1$ the modules of 1-cycles and 1-boundaries of $\mathcal{K}(x,y)$. Let $\pi_2 : Z_1 \to A$ be the morphism of $A$-modules defined by $\pi_2(a,b) = b$. It is clear that $\pi_2(Z_1) = (x : y)$. Consider $g : Z_1 \to (x : y)/ < x >$ the composition of $\pi_2$ with the projection onto the quotient of $(x : y)$ by $< x >$. One has $B_1 \subseteq \mathrm{Ker} g = \pi_2^{-1}(< x >)$, from where we deduce the following exact sequence:

$$0 \longrightarrow \frac{\pi_2^{-1}(< x >)}{B_1} \longrightarrow H_1(\mathcal{K}(x,y)) \longrightarrow \frac{(x : y)}{< x >} \longrightarrow 0\,.$$

Finally, it is not difficult to prove that $\pi_2^{-1}(< x >) = B_1$ is equivalent to $(0 : x) \subset < y >$ and $(0 : xy) = (0 : x) + (0 : y)$. ∎

**Lemma 3.2** *If $H_1(\mathcal{K}(x,y)) = 0$ and $t_0 \in (0 :< x,y >)$, then there exists a sequence $t_0, t_1, t_2, \ldots, t_n, \ldots$ such that, for each $n \geq 1$, $t_n \in (0 :< x^{n+1}, y^{n+1} >)$ and $t_{n-1} = t_n xy$.*

*Proof.* For each pair $p, q \geq 1$, let us denote by $Z_1^{(p,q)} = \{(a,b) \in A^2 \mid ax^p + by^q = 0\}$ and by $B_1^{(p,q)} = \{c(-y^q, x^p) \in A^2 \mid c \in A\}$ the modules of 1-cycles and 1-boundaries of the Koszul complex $\mathcal{K}(x^p, y^q)$ on the two elements $x^p, y^q \in A$. Since $H_1(\mathcal{K}(x,y)) = 0$, then $H_1(\mathcal{K}(x^p, y^q)) = 0$ (exercise 9.9 [4]). Suppose $t_{n-1} \in (0 :< x^n, y^n >)$ for a given $n \geq 1$ (for $n = 1$, take $t_0 \in (0 :< x,y >)$ given by the hypothesis). Then, $(t_{n-1}, 0) \in Z_1^{(n,n)} = B_1^{(n,n)}$. So, there exists $u_n \in A$ such that $t_{n-1} = u_n y^n$ and $u_n x^n = 0$. Analogously, since $(0, t_{n-1}) \in Z_1^{(n,n)} = B_1^{(n,n)}$, there exists $v_n \in A$ such that $t_{n-1} = v_n x^n$ and $v_n y^n = 0$. Therefore, $t_{n-1} = u_n y^n = v_n x^n$ and $(v_n, -u_n) \in Z_1^{(n,n)} = B_1^{(n,n)}$. So, there exists $w_n \in A$ such that $v_n = w_n y^n$ and $u_n = w_n x^n$. Hence, $t_{n-1} = u_n y^n = w_n x^n y^n$. Take $t_n = w_n x^{n-1} y^{n-1}$. Then, $t_{n-1} = t_n xy$ with $t_n x^{n+1} = w_n x^n x^n y^{n-1} = u_n x^n y^{n-1} = 0$ and, analogously, $t_n y^{n+1} = 0$. ∎

**Example 3.3** Let $k$ be a field and $R = k[X, Y, T_0, T_1, T_2, \ldots]$ the polynomial ring in the variables $X, Y, T_0, T_1, T_2, \ldots$. Let $J$ be the ideal of $R$ defined by

$$J =< T_n X^{n+1}, T_n Y^{n+1}, T_n - T_{n+1} XY \mid n \geq 0 > \ .$$

Take $A = R/J = k[x, y, t_0, t_1, t_2, \ldots]$, where $x, y, t_0, t_1, t_2, \ldots,$ denote the classes in $A$ of the variables $X, Y, T_0, T_1, T_2, \ldots$. Let $I =< x, y >$ be the ideal of $A$ generated by $x, y$ and $B = A/I$. Then, $H_1(\mathcal{K}(x,y)) = 0$ and $H_2(\mathcal{K}(x,y)) \neq 0$. In particular, $H_2(A,B,W) = 0$ for all $B$-module $W$ and $H_3(A,B,B) \neq 0$. Moreover, $H_3(A,B,W) = W$ for all $B$-module $W$, and if $k$ is of characteristic zero, then $H_6(A,B,W) = W$ and $H_p(A,B,W) = 0$ for all $p \geq 4$, $p \neq 6$.

*Proof.* Let us begin by proving $H_2(\mathcal{K}(x,y)) \neq 0$. By construction $t_0 \in (0 : I)$. Let us see $t_0 \neq 0$. Consider $J_n$ the ideal of the polynomial ring $R_n = k[X, Y, T_0, T_1, \ldots, T_n]$ defined by

$$J_n =< T_0 X, T_0 Y, T_0 - T_1 XY, T_1 X^2, T_1 Y^2, \ldots, T_{n-1} - T_n XY, T_n X^{n+1}, T_n Y^{n+1} > \ .$$

Note that we have:

$$J_n =< T_0 - T_n X^n Y^n, \ldots, T_i - T_n X^{n-i} Y^{n-i}, \ldots, T_{n-1} - T_n XY, T_n X^{n+1}, T_n Y^{n+1} > \ .$$

Suppose $T_0 \in J$. Then, there exists $n \geq 0$ such that $T_0 \in J_n$ in $R_n$. For such $n \geq 0$, consider the morphism of $k$-algebras $\varphi : R_n \longrightarrow k[X, Y, T_n]$ defined by $\varphi(X) = X$, $\varphi(Y) = Y$, $\varphi(T_0) = T_n X^n Y^n$, $\varphi(T_1) = T_n X^{n-1} Y^{n-1}$, ..., $\varphi(T_{n-1}) = T_n XY$ and $\varphi(T_n) = T_n$. Then, applying $\varphi$ to the expression of $T_0$ as an element of $J_n$ one gets an equality in $k[X, Y, T_n]$ of the form: $T_n X^n Y^n = a T_n X^{n+1} + b T_n Y^{n+1}$, where $a, b \in k[X, Y, T_n]$, which would imply the contradiction $1 = a' X + b' Y$.

Now and using Lemma 3.1, let us prove $H_1(\mathcal{K}(x,y)) = 0$. It is not difficult to see $(x : y) =< x >$ and $(0 : x) \subset< y >$. On the other hand, we have

*CLAIM*: $(0:t_n) =< x^{n+1}, y^{n+1} >$ for all $n \geq 0$.

To see this, write any $a \in A$, for a given $n \geq 0$, as $a = cx^{n+1} + dy^{n+1} + f_n(x,y)$, where $c, d \in A$ and with each monomial of $f_n(x,y) \in k[x,y]$ being of the form $\lambda x^i y^j$, $\lambda \in k$ and $i, j \leq n$. Let us prove by induction on $n \geq 0$, that if $t_n f_n(x,y) = 0$, then $f_n(x,y) = 0$. For $n = 0$, it is just to say that $t_0 \neq 0$. For $n \geq 1$, write $f_n(x,y) = f_{n-1}(x,y) + g_n(x,y)$, where $g_n(x,y) = \lambda_{n,0}x^n + \lambda_{n,1}x^n y + \ldots + \lambda_{n,n}x^n y^n + \ldots + \lambda_{1,n}xy^n + \lambda_{0,n}y^n$ and with each monomial of $f_{n-1}(x,y) \in k[x,y]$ being of the form $\mu x^i y^j$, $\mu \in k$ and $i, j \leq n-1$.

As $t_n f_n(x,y) = 0$, then $0 = t_n f_n(x,y)xy = t_{n-1}f_{n-1}(x,y) + t_n xy g_n(x,y)$. But, since $t_n x^{n+1} = t_n y^{n+1} = 0$, then $t_n xy g_n(x,y) = 0$. So, $t_{n-1}f_{n-1}(x,y) = 0$ and, by the induction hypothesis, $f_{n-1}(x,y) = 0$. Multiplying $t_n g(x,y) = 0$ by $x$ and using the induction hypothesis, we deduce $\lambda_{n-1,n} = \ldots = \lambda_{1,n} = \lambda_{0,n} = 0$. Multiplying $t_n g_n(x,y) = 0$ by $y$ and using again the induction hypothesis, we deduce $\lambda_{n,0} = \lambda_{n,1} = \ldots = \lambda_{n,n-1} = 0$. So $f_n(x,y) = \lambda_{n,n}x^n y^n$. Since $0 = t_n f_n(x,y) = \lambda_{n,n}t_n x^n y^n = \lambda_{n,n}t_0$ and by the case $n = 0$, we deduce $\lambda_{n,n} = 0$ and, therefore, $f_n(x,y) = 0$.

Thus, $(0:xy) = (0:x) + (0:y)$. Indeed, if $a = P + J \in (0:xy)$, then $PXY \in J$, and since $J \subset < T_0, T_1, \ldots >$, $P \in < T_0, T_1, \ldots >$. Therefore, $a = bt_{n+1}$ for some $n \geq 1$. We have $0 = axy = bt_{n+1}xy = bt_n$. Thus, $b \in (0:t_n) =< x^{n+1}, y^{n+1} >$. So $b = cx^{n+1} + dy^{n+1}$ and $a = ct_{n+1}x^{n+1} + dt_{n+1}y^{n+1}$, where $t_{n+1}x^{n+1} \in (0:x)$ and $t_{n+1}y^{n+1} \in (0:y)$.

Therefore, $H_1(\mathcal{K}(x,y)) = 0$ and $H_2(\mathcal{K}(x,y)) \neq 0$. So $H_2(A,B,W) = 0$ for all $B$-module $W$ and $H_3(A,B,B) \neq 0$. Finally, let us prove $H_3(A,B,W) = W$ for all $B$-module $W$. Since $I$ is a maximal ideal of residual field $B = A/I = k$, it is enough to prove that $H_3(A,B,B) = B$.

The five-term exact sequence associated to the spectral sequence (3) is (see 6.12 [8]):

$$\mathrm{Tor}_3^A(B,B) \overset{h}{\to} H_3(A,B,B) \to \mathbf{\Lambda}_2^B(I/I^2) \overset{\gamma_2}{\to} \mathrm{Tor}_2^A(B,B) \to H_2(A,B,B) \to 0\,.$$

Since $\gamma_2$ is an isomorphism, then $h : \mathrm{Tor}_3^A(B,B) \longrightarrow H_3(A,B,B)$ is surjective. Therefore, it suffices to show that $\mathrm{Tor}_3^A(B,B) = B$. Since for all $n \geq 1$, $< t_n > \cap (0:x) =< t_n x^n >$ and $< t_n > \cap (0:y) =< t_n y^n >$, then it is not difficult to see that $(0:I) =< t_0 >$. It follows that the following complex is a free resolution of the $A$-module $B$:

$$\ldots \longrightarrow A \xrightarrow{\partial_5} A^2 \xrightarrow{\partial_4} A \xrightarrow{\partial_3} A \xrightarrow{\partial_2} A^2 \xrightarrow{\partial_1} A \longrightarrow B \longrightarrow 0\,, \tag{6}$$

where, for each $n \geq 0$, $\partial_{1+3n} : A^2 \to A$ is defined by sending $(1,0)$ to $x$ and $(0,1)$ to $y$; $\partial_{2+3n} : A \to A^2$ is defined by sending 1 to $(y,-x)$ and, finally, $\partial_{3+3n} : A \to A$ is defined by sending 1 to $t_0$. Therefore, $\mathrm{Tor}_{1+3n}^A(B,B) = B^2$ and $\mathrm{Tor}_{2+3n}^A(B,B) = \mathrm{Tor}_{3n}^A(B,B) = B$ for all $n \geq 0$.

If $k$ is of characteristic zero and $p \geq 4$, then $H_6(A,B,W) = W$ and $H_p(A,B,W) = 0$ otherwise. Indeed, the free resolution (6) of $B$ has a multiplicative structure, since it can be obtained from the Koszul complex $\mathcal{K}(x,y)$ by adjoining the necessary variables in order to kill the cycle $t_0$ in degree 3 and 6. Using this DG-algebra, free resolution of $B$, one can compute the modules $H_p(A,B,W)$ (see [5]). ∎

**Remark 3.4** In Example 3.3, it can be proven that $A$ has Krull dimension 2. So, localizing at the maximal ideal $I$, we get a local commutative ring of Krull dimension 2.

To finish, we remark that for principal ideals, the André-Quillen homology is rigid.

**Proposition 3.5** *Let $I =< x >$ be a principal ideal of $A$, $B = A/I$. If $H_2(A, B, W) = 0$ for all $B$-module $W$, then $H_p(A, B, W) = 0$ for all $p \geq 3$ and for all $B$-module $W$.*

*Proof.* Consider $0 \to (0 : x) \to A \to I =< x >\to 0$. Thus, $H_1(\mathcal{K}(x)) = (0 : x)$. By the exactness of (5), $H_2(A, B, W) = \mathrm{Tor}_1^A(I, W)$. Therefore, the vanishing of $H_2(A, B, \cdot)$ is equivalent to the flatness, as an $A$-module, of the ideal $I$. On the other hand, if $J$ is an ideal of $A$, flat as an $A$-module, then $J/J^2$ is a flat $A/J$-module and hence, by Lemma 2.1, $\gamma : \mathbf{\Lambda}^{A/J}(J/J^2) \to \mathrm{Tor}_*^A(A/J, A/J)$ is a monomorphism. Moreover, as $J$ is flat, $\mathrm{Tor}_p^A(A/J, A/J) = \mathrm{Tor}_{p-1}^A(J, A/J) = 0$ for all $p \geq 2$. In particular, $\gamma$ is an epimorphism and, by Theorem 1.1, $H_p(A, A/J, \cdot) = 0$ for all $p \geq 2$. ∎

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