The spectral excess theorem for graphs with few eigenvalues whose distance-2 or distance-1-or-2 graph is strongly regular *

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Abstract

We study regular graphs whose distance-2 graph or distance-1-or-2 graph is strongly regular. We provide a characterization of such graphs Γ (among regular graphs with few distinct eigenvalues) in terms of the spectrum and the mean number of vertices at maximal distance d from every vertex, where $d+1$ is the number of different eigenvalues of Γ. This can be seen as another version of the so-called spectral excess theorem, which characterizes in a similar way those regular graphs that are distance-regular.

Keywords: Distance-regular graph; distance-2 graph; spectrum; predistance polynomials.

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1 Preliminaries

Let Γ be a distance-regular graph with adjacency matrix $\boldsymbol{A}$ and $d+1$ distinct eigenvalues. The distance-i graph (associated with Γ) is the graph Γ_i having the same vertices as Γ and in which two vertices are adjacent if and only if they are at distance i in Γ. Similarly, the distance-i-or-j graph is the graph $\Gamma_{i,j}$ with the same vertices as Γ and in which two vertices are adjacent if and only if they are at distance i or j in Γ. In the recent works of Brouwer and Fiol [4, 15], it was studied the situation in which the distance-d graph Γ_d of Γ (or the Kneser graph K of Γ) with adjacency matrix $\boldsymbol{A}_d(=p_d(\boldsymbol{A}))$, where p_d is the distance-d polynomial, has fewer distinct eigenvalues than Γ. Examples are the so-called half antipodal (K with only one negative eigenvalue, up to multiplicity), and antipodal distance-regular graphs (where K consists of disjoint copies of a complete graph).

Here we study the cases in which Γ has few eigenvalues and its distance-2 graph Γ_2 or its distance-1-or-2 graph $\Gamma_{1,2}$ are strongly regular. The main result of this paper is a characterization of such (partially) distance-regular graphs, among regular graphs with $d\in\{3,4\}$ distinct nontrivial eigenvalues, in terms of the spectrum and the mean number of vertices at maximal distance d from every vertex. This can be seen as another version of the so-called spectral excess theorem. Other related characterizations of some of these cases were given by Fiol in [11, 12, 13]. For background on distance-regular graphs and strongly regular graphs, we refer the reader to Brouwer, Cohen, and Neumaier [3], Brouwer and Haemers [5], and Van Dam, Koolen and Tanaka [9].

Let Γ be a regular (connected) graph with degree k, n vertices, and spectrum $\operatorname{sp}\Gamma=\{\theta_0^{m_0},\theta_1^{m_1},\dots,\theta_d^{m_d}\}$, where $\theta_0(=k)>\theta_1>\cdots>\theta_d$, and $m_0=1$. In this work, we use the following scalar product on the $(d+1)$-dimensional vector space of real polynomials modulo $m(x)=\prod_{i=0}^d(x-\theta_i)$, that is, the minimal polynomial of $\boldsymbol{A}$.

$$\langle p,q\rangle_\Gamma=\frac{1}{n}\operatorname{tr}(p(\boldsymbol{A})q(\boldsymbol{A}))=\frac{1}{n}\sum_{i=0}^{d}m_i p(\theta_i)q(\theta_i),\qquad p,q\in\mathbb{R}_d[x]/(m(x)). \tag{1}$$

This is a special case of the inner product of symmetric $n\times n$ real matrices $\boldsymbol{M}$ and $\boldsymbol{N}$, defined by $\langle\boldsymbol{M},\boldsymbol{N}\rangle=\frac{1}{n}\operatorname{tr}(\boldsymbol{MN})$. The *predistance polynomials* $p_0,p_1,\dots,p_d$, introduced by Fiol and Garriga [17], are a sequence of orthogonal polynomials with respect to the inner product (1), normalized in such a way that $\|p_i\|_\Gamma^2=p_i(k)$ (this makes sense since it is known that $p_i(k)>0$ for any $i=0,\dots,d$, see for instance Szegö [20]). As every sequence of orthogonal polynomials, the predistance polynomials satisfy a three-term recurrence of the form

$$xp_i=\beta_{i-1}p_{i-1}+\alpha_i p_i+\gamma_{i+1}p_{i+1}\qquad i=0,1,\dots,d,$$

where the constants β_{i-1}, α_i, and γ_{i+1} are called *preintersection numbers* and are the Fourier coefficients of xp_i in terms of p_{i-1}, p_i, and p_{i+1}, respectively (and $\beta_{-1}=\gamma_{d+1}=0$), beginning with $p_0=1$ and $p_1=x$.

Some basic properties of the predistance polynomials and preintersection numbers are included in the following result (see Cámara, Fàbrega, Fiol, and Garriga [6], and Diego, Fàbrega, and Fiol [10]).

Lemma 1. *Let G be a k-regular graph with $d+1$ distinct eigenvalues and predistance polynomials $p_0,\dots,p_d$. Given an integer $\ell \geq 0$, let $\overline{\mathcal{C}}_\ell$ be the average number of circuits of length ℓ rooted at every vertex, that is, $\overline{\mathcal{C}}_\ell = \frac{1}{n}\sum_{i=0}^{d} m_i\theta_i^\ell$. Then,*

- *(i)* $p_0(x)=1$, $p_1(x)=x$, $p_2(x)=\frac{1}{\gamma_2}(x^2-\alpha_1 x-k)$.
- *(ii) For $i=0,\dots,d$, the two highest terms of the predistance polynomial p_i are given by*

$$p_i(x)=\frac{1}{\gamma_1\cdots\gamma_i}[x^i-(\alpha_1+\cdots+\alpha_{i-1})x^{i-1}+\cdots].$$

- *(iii)* $\alpha_i+\beta_i+\gamma_i=k$, *for* $i=0,\dots,d$.
- *(iv)* $\alpha_0=0$, $\beta_0=k$, $\gamma_1=1$, $\alpha_1=\overline{\mathcal{C}}_3/\overline{\mathcal{C}}_2$, *and*

$$\gamma_2=\frac{\overline{\mathcal{C}}_3^2-\overline{\mathcal{C}}_4 k+k^3}{k(\overline{\mathcal{C}}_3+k-k^2)}. \tag{2}$$

- *(v)* $p_0+p_1+\cdots+p_d=H$, *where H is the Hoffman polynomial [19].*
- *(vi) For every $i=0,\dots,d$, (any multiple of) the sum polynomial $q_i=p_0+\cdots+p_i$ maximizes the quotient $r(\theta_0)/\|r\|_\Gamma$ among the polynomials $r\in\mathbb{R}_i[x]$ (notice that $q_i(\theta_0)^2/\|q_i\|_\Gamma^2=q_i(\theta_0)$), and*

$$(1=)q_0(\theta_0)<q_1(\theta_0)<\cdots<q_d(\theta_0)(=H(\theta_0)=n).$$

A graph G with diameter D is called *m-partially distance-regular*, for some $m=0,\dots,D$, if its predistance polynomials satisfy $p_i(\boldsymbol{A})=\boldsymbol{A}_i$ for every $i\leq m$. In particular, every m-partially distance-regular with $m\geq 1$ must be regular (see Abiad, Van Dam, and Fiol [1]). As an alternative characterization, a graph G is m-partially distance-regular when the intersection numbers c_i $(i\leq m)$, a_i $(i\leq m-1)$, b_i $(i\leq m-1)$ are well-defined. In this case, these intersection numbers are equal to the corresponding preintersection numbers γ_i $(i\leq m)$, α_i $(i\leq m-1)$, β_i $(i\leq m-1)$, and also k_i is well-defined and equal to $p_i(\theta_0)$ for $i\leq m$. We refer to Dalfó, Van Dam, Fiol, Garriga, and Gorissen [7] for more background.

Then, with this definition, a graph Γ with diameter $D=d$ is distance-regular if and only it is d-partially distance-regular. In fact, in this case we have the following strongest proposition, which is a combination of results in Fiol, Garriga and Yebra [18], and Dalfó, Van Dam, Fiol, Garriga and Gorissen [7].

Proposition 2. *A regular graph Γ with $d+1$ different eigenvalues (and, hence, with diameter $D\leq d$) is distance-regular if and only if there exists a polynomial p of degree d such that $p(\boldsymbol{A})=\boldsymbol{A}_d$, in which case $p=p_d$.* □

Lemma 3. *Let Γ be a regular graph with diameter D, and let $m \leq D$ be a positive integer. Let $n_i(u)$ be the number of vertices at distance at most $i \leq D$ from vertex u in Γ, and let $\overline{n_i} = \frac{1}{n}\sum_{u\in V} n_i(u)$ be the average of these numbers of vertices for all $u \in V$. Then, for any nonzero polynomial $r \in \mathbb{R}_i[x]$ we have*

$$\frac{r(\theta_0)^2}{\|r\|_\Gamma^2} \leq \overline{n_i}, \tag{3}$$

with equality if and only if r is a multiple of $q_i = p_0 + \cdots + p_i$, and $q_i(\boldsymbol{A}) = \boldsymbol{A}_0 + \cdots + \boldsymbol{A}_i$.

Proof. Let $\boldsymbol{S}_i = \boldsymbol{A}_0 + \cdots + \boldsymbol{A}_i$. As $\deg r \leq i$, we have $\langle r(\boldsymbol{A}), \boldsymbol{J}\rangle = \langle r(\boldsymbol{A}), \boldsymbol{S}_i\rangle$, where $\boldsymbol{J}$ is the all-one matrix. But $\langle r(\boldsymbol{A}), \boldsymbol{J}\rangle = \langle r, H\rangle_\Gamma = r(\theta_0)$. Then, the Cauchy-Schwarz inequality gives

$$r^2(\theta_0) \leq \|r(\boldsymbol{A})\|^2\|\boldsymbol{S}_i\|^2 = \|r\|_\Gamma^2\overline{n_i},$$

whence (3) follows. Besides, in case of equality we have that r is multiple of q_i, by Lemma 1(vi), with $q_i(\theta_0) = \overline{n_i}$. Therefore, $q_i(\boldsymbol{A}) = \alpha\boldsymbol{S}_i$ for some nonzero constant α and taking norms we conclude that $\alpha = 1$. □

In fact, as it was shown in Fiol [14], the above result still holds if we change the arithmetic mean of the numbers $n_i(u)$, $u \in V$, by its harmonic mean.

As a consequence of Lemma 3 and Proposition 2, we have the following generalization of the spectral excess theorem, due to Fiol and Garriga [17] (for short proofs, see Van Dam [8], and Fiol, Gago and Garriga [16]).

Theorem 4. *Let Γ be a regular graph with $d+1$ distinct eigenvalues $\theta_0 > \cdots > \theta_d$, and diameter $D = d$. Let $m \leq D$ be a positive integer.*

(i) *If Γ is $(m-1)$-partially distance-regular for some $m < d$, and $q_m(\theta_0) = \overline{n_m}$, then Γ is m-partially distance-regular.*

(ii) *If $q_{d-1}(\theta_0) = \overline{n_{d-1}}$, then Γ is distance-regular.*

2 The case of distance-regular graphs

Here we study the case when Γ is a distance-regular graph with diameter three or four. In fact, in the first case everything is basically known (see Brouwer [2]), although only a combinatorial characterization was provided, whereas we think that the spectral characterization is also important. Indeed, Brouwer [2] proved the following (see also Proposition 4.2.17(i) in Brouwer, Cohen, and Neumaier [3]):

Proposition 5. *[2] Let Γ be a distance-regular graph with degree k and diameter $d = 3$. Then,* (*i*) Γ_2 *is strongly regular* $\iff c_3(a_3 + a_2 - a_1) = b_1 a_2$.

(*ii*) $\Gamma_{1,2}$ *is strongly regular* $\iff$ Γ *has eigenvalue* $-1 \iff k = b_2 + c_3 - 1$.

Notice that, in this case, $\Gamma_{1,2}$ is strongly regular if and only if its complement Γ_3 is. As commented in the Introduction, the last case was studied for general diameter by Brouwer and Fiol [4] and Fiol [15].

Proposition 6. *Let* Γ *be a distance-regular graph with diameter* $D = d = 3$*, and eigenvalues* $\theta_0(= k) > \theta_1 > \theta_2 > \theta_3$*.*

(*i*) *The distance-*2 *graph* Γ_2 *is strongly regular if and only if* $a_2 - c_3$ *is an eigenvalue of* Γ*.*

(*ii*) *The distance-*1*-or-*2 *graph* $\Gamma_{1,2}$ *is strongly regular if and only if* $a_3 - b_2$ *is an eigenvalue of* Γ*.*

Proof. We only prove (*i*), as the proof of (*ii*) is similar. As Γ_2 has adjacency matrix $\boldsymbol{A}_2 = p_2(\boldsymbol{A})$, where $p_2(x) = \frac{1}{c_2}(x^2 - a_1 x - k)$, it has eigenvalues $\frac{1}{c_2}(\theta^2 - a_1\theta - k)$, where θ is an eigenvalue of Γ. For two non-trivial eigenvalues η, θ of Γ, assume that $\eta^2 - a_1\eta - k = \theta^2 - a_1\theta - k$. This implies $\theta = \eta$ or $\theta + \eta = a_1$. Let τ be the third non-trivial eigenvalue of Γ. Then $k + \theta + \eta + \tau = a_1 + a_2 + a_3$ and the result follows. The other direction is trivial to see. □

For diameter $D = d = 4$ only the case of the distance-1-or-2 graph is known (see Proposition 4.2.18 in Brouwer, Cohen, and Neumaier [3]). In the following result, we give an equivalent characterization of this case and, moreover, we study the case of the distance-2 graph which, as far as we know, it is new. The proof is as in Proposition 6. For instance, notice that, in case (*i*), for Γ_2 to have only two nontrivial distinct eigenvalues, the only possibility is that $p_2(\theta_1) = p_2(\theta_4)$ and $p_2(\theta_2) = p_2(\theta_3)$.

Proposition 7. *Let* Γ *be a distance-regular graph with diameter four and eigenvalues* $\theta_0(= k) > \theta_1 > \theta_2 > \theta_3 > \theta_4$*.*

(*i*) *The distance-*2 *graph* Γ_2 *is strongly regular if and only if* $\theta_1 + \theta_4 = a_1 = \theta_2 + \theta_3$*.*

(*ii*) *The distance-*1*-or-*2 *graph* Γ_2 *is strongly regular if and only if* $\theta_1 + \theta_4 = a_1 - c_2 = \theta_2 + \theta_3$*.*

An example of distance-regular graph satisfying the conditions of Proposition 7 is the Hamming graph $H(4, 3)$ (see Example 2 in the next section). Another example would be the (possible) graph corresponding to the feasible array $\{39, 32, 20, 2; 1, 4, 16, 30\}$ (see Brouwer, Cohen, and Neumaier [3, p. 420]). If it exists, this would be a graph with $n = 768$ vertices and spectrum $39^1, 15^{52}, 7^{117}, -1^{468}, -9^{130}$. In this case, its distance-2 graph would have spectrum 312^1, 24^{182}, -8^{585}.

3 The case of regular graphs

Now we want to conclude the same result as above but only requiring that the graph Γ is regular. In this case, we use the predistance polynomials and preintersection numbers. Notice that now $p_i(\boldsymbol{A})$ is not necessarily the distance-i matrix $\boldsymbol{A}_i$ (usually not even a 0-1 matrix). However, as above, we consider that $p_2(\boldsymbol{A})$ has only three distinct eigenvalues.

3.1 The case of diameter three

We begin with the case of $d = 3$ (that is, assuming that Γ has four distinct eigenvalues).

Theorem 8. *Let Γ be a regular graph with degree k, n vertices, spectrum $\operatorname{sp}\Gamma = \{\theta_0, \theta_1^{m_1}, \theta_2^{m_2}, \theta_3^{m_3}\}$, where $\theta_0 (= k) > \theta_1 > \theta_2 > \theta_3$, and preintersection number γ_2 given by (2). Let $\overline{k_3} = \frac{1}{n}\sum_{u\in V} k_3(u)$ be the average number of vertices at distance 3 from every vertex in Γ. Consider the polynomials*

$$s_1(x) = x^2 - (\theta_1 + \theta_3 - \gamma_2)x + \gamma_2 + \theta_2(\theta_1 - \theta_2 + \theta_3), \quad (4)$$
$$s_2(x) = x^2 - (\theta_1 + \theta_2 - \gamma_2)x + \gamma_2 + \theta_3(\theta_1 - \theta_3 + \theta_2). \quad (5)$$

Then,

$$\overline{k_3} \leq \frac{n\sum_{i=1}^{3} m_i (s_j(\theta_i) - \tau_j)^2}{\sum_{i=0}^{3} m_i (s_j(\theta_i) - \tau_j)^2}, \quad j = 1, 2, \quad (6)$$

where

$$\tau_j = \frac{s_j(\theta_0)\sum_{i=1}^{3} m_i s_j(\theta_i) - \sum_{i=1}^{3} m_i s_j(\theta_i)^2}{s_j(\theta_0)(n-1) - \sum_{i=1}^{3} m_i s_j(\theta_i)}, \quad j = 1, 2. \quad (7)$$

Equality in (6) holds for some $j \in \{1, 2\}$ if and only if Γ is a distance-regular graph and its distance-2 graph Γ_2 is strongly regular, with eigenvalues

$$\lambda_0 = n - \overline{k_3} - \theta_0 - 1, \quad \lambda_1 = ((\theta_1 - \theta_2)(\theta_2 - \theta_3) - \tau_1)/\gamma_2, \text{ and } \lambda_2 = -\tau_1/\gamma_2,$$

or

$$\lambda_0 = n - \overline{k_3} - \theta_0 - 1, \quad \lambda_1 = -\tau_2/\gamma_2, \text{ and } \lambda_2 = ((\theta_1 - \theta_3)(\theta_3 - \theta_2) - \tau_2)/\gamma_2,$$

where $k_3 = \overline{k_3}$ is the constant value of the number of vertices at distance 3 from any vertex in Γ.

Proof. Taking into account that the eigenvalues of $p_2(\boldsymbol{A})$ interlace those of Γ (because of the orthogonality of the predistance polynomials with respect to the scalar product in (1), see for instance Cámara, Fàbrega, Fiol, and Garriga [6], or Szegö [20]), the only possible cases are:

1. $p_2(\theta_1) = p_2(\theta_3) = \sigma_1/\gamma_2$ and $p_2(\theta_2) = -\tau_1/\gamma_2$,

2. $p_2(\theta_1) = p_2(\theta_2) = \sigma_2/\gamma_2$ and $p_2(\theta_3) = -\tau_2/\gamma_2$,

where σ_j and τ_j, for $j = 1, 2$, are constants. We only prove the first case, as the other is similar. The main idea is to apply Lemma 3 with a polynomial $r \in \mathbb{R}_2[x]$ having the desired properties of (any multiple of) q_2. To this end, let us assume that $p_2(\theta_1) = p_2(\theta_3) = \sigma_1/\gamma_2$, and $p_2(\theta_2) = -\tau_1/\gamma_2$ where σ_1 and τ_1 are constants. Thus, if we consider a generic monic polynomial $r(x) = x^2 + \alpha x + \beta = \gamma_2 q_2(x)$, where $q_2(x) = p_2(x) + x + 1$, we must have

$$
\begin{aligned}
r(\theta_1) &= \theta_1^2 + \alpha\theta_1 + \beta = \sigma_1 + \gamma_2\theta_1 + \gamma_2,\\
r(\theta_2) &= \theta_2^2 + \alpha\theta_2 + \beta = -\tau_1 + \gamma_2\theta_2 + \gamma_2,\\
r(\theta_3) &= \theta_3^2 + \alpha\theta_3 + \beta = \sigma_1 + \gamma_2\theta_3 + \gamma_2.
\end{aligned}
$$

From the first and last equation we get $\alpha = \gamma_2 - \theta_1 - \theta_3$ and, hence, the second equation yields $\beta = \gamma_2 + \theta_2(\theta_1 - \theta_2 + \theta_3) - \tau_1$. Then, we must take $r(x) = s_1(x) - \tau_1$, where $s_1(x)$ is as in (4), and (3) yields

$$
\Phi(\tau_1) = \frac{r(\theta_0)^2}{\|r\|_\Gamma^2} = \frac{n(s(\theta_0) - \tau_1)^2}{\sum_{i=0}^3 m_i(s(\theta_i) - \tau_1)^2} \le \overline{s_2} = n - \overline{k_3}. \tag{8}
$$

Now, to have the best result in (8), and, since we are mostly interested in the case of equality, we find the maximum of the function Φ, which is attained at τ_1 given by (7). Then, as $\overline{s_2} = n - \overline{k_3}$, the claimed inequality follows. Moreover, in case of equality, we know, by Theorem 4, that Γ is distance-regular with $r(x) = \gamma q_{d-1}(x)$ for some constant γ, which it is $\gamma = \gamma_2$. Then, we get (with standard notation $P_{ij} = p_j(\theta_i)$)

$$
\begin{aligned}
P_{22} &= p_2(\theta_2) = -\frac{\tau_1}{\gamma_2},\\
P_{i2} &= p_2(\theta_i) = \frac{\sigma_1}{\gamma_2} = \frac{1}{\gamma 2}((\theta_1 - \theta_2)(\theta_2 - \theta_3) - \tau_1), \qquad i = 1, 3.
\end{aligned}
$$

To prove the converse, we only need to carry out a simple computation. Indeed, assume that Γ is a distance-regular graph, with k_i being the vertices at distance $i = 1, 2, 3$ from any vertex ($k_1 = k$), and $p_2(\theta_1) = p_d(\theta_3)$. Then, the same reasoning as in Proposition 6 gives $a_1 = \theta_1 + \theta_3$. Then, from $kb_1 = c_2k_2 = c_2(n - k_3 - k - 1)$ and $a_1 + b_1 + 1 = k$, we get that $c_2 = \frac{k(k-1-\theta_1-\theta_3)}{n-k_3-k-1}$. Thus, by putting $\gamma_2 = c_2$ in $s_1(x)$ of (4) to compute τ_1 in (7), the inequality (6) becomes an equality (since $\overline{k_3} = k_3$). □

The following result gives similar conditions for Γ to be distance-regular with the distance-1-or-2 graph $\Gamma_{1,2}$ being strongly regular.

Theorem 9. *Let Γ be a regular graph with degree k, n vertices, spectrum* $\operatorname{sp}\Gamma = \{\theta_0, \theta_1^{m_1}, \theta_2^{m_2}, \theta_3^{m_3}\}$, *where $\theta_0(= k) > \theta_1 > \theta_2 > \theta_3$, and preintersection number γ_2. Let* $\overline{k_3} =$

$\frac{1}{n}\sum_{u\in V} k_3(u)$ *be the average number of vertices at distance* 3 *from every vertex in* Γ. *Consider the polynomials*

$$s_1(x) = x^2 - (\theta_1 + \theta_3)x + \gamma_2 + \theta_2(\theta_1 - \theta_2 + \theta_3), \quad (9)$$
$$s_2(x) = x^2 - (\theta_1 + \theta_2)x + \gamma_2 + \theta_3(\theta_1 - \theta_3 + \theta_2). \quad (10)$$

Then,

$$\overline{k_3} \le \frac{n\sum_{i=1}^{3} m_i(s_j(\theta_i) - \tau_j)^2}{\sum_{i=0}^{3} m_i(s_j(\theta_i) - \tau_j)^2}, \quad j = 1, 2, \quad (11)$$

where

$$\tau_j = \frac{s_j(\theta_0)\sum_{i=1}^{3} m_i s_j(\theta_i) - \sum_{i=1}^{3} m_i s_j(\theta_i)^2}{s_j(\theta_0)(n-1) - \sum_{i=1}^{3} m_i s_j(\theta_i)}, \quad j = 1, 2. \quad (12)$$

Equality in (11) *holds with some* $j \in \{1, 2\}$ *if and only if* Γ *is a distance-regular graph and its distance-*1*-or-*2 *graph* $\Gamma_{1,2}$ *is strongly regular,*

Example 1. The Odd graph O(4) with 7 points, has $n = \binom{7}{3} = 35 = 1+4+12+18$ vertices, diameter $d = 3$, intersection array $\{4, 3, 3; 1, 1, 2\}$, and spectrum $4^1, 2^{14}, -1^{14}, -3^6$. Then, the functions $\Phi(\tau_j)$ in (6) with $j = 1, 2$ have maximum values at $\tau_1 = 18/5$ and $\tau_2 = -8$, respectively, and their values are $\Phi(18/5) = 138/7$ and $\Phi(-8) = 22$. Then, since both numbers are greater than $k_3 = 18$, its distance-2 graph Γ_2 is not strongly regular.

On the other hand the function $\Phi(\tau_1)$ in (11) has maximum value at $\tau_1 = 4$, and $\Phi(4) = 18 = k_3$. Hence, its distance-1-or-2 graph $\Gamma_{1,2}$ (and, hence, also Γ_3) is strongly regular with $p_1(x) + p_2(x) = x^2 + x - 4$, and spectrum $16^1, 2^{20}, -4^{14}$.

3.2 The case of diameter four

The following result deals with the case of $d = 4$. As in the case of Theorem 9, we omit this proof as goes along the same lines of reasoning as in Theorem 8.

Theorem 10. *Let* Γ *be a regular graph with degree* k, n *vertices, spectrum* $\mathrm{sp}\,\Gamma = \{\theta_0, \theta_1^{m_1}, \theta_2^{m_2}, \theta_3^{m_3}, \theta_4^{m_4}\}$, *where* $\theta_0(= k) > \theta_1 > \theta_2 > \theta_3 > \theta_4$, *such that* $\theta_1 + \theta_4 = \theta_2 + \theta_3$, *and preintersection number* γ_2. *Let* $\overline{n_2} = \frac{1}{n}\sum_{u\in V} n_2(u)$ *be the average number* $n_2(u) = |N_2(u)|$ *of vertices at distance at most* 2 *from every vertex* u *in* Γ. *Consider the polynomials*

$$s_1(x) = x^2 - (\theta_2 + \theta_3 - \gamma_2)x + \theta_2\theta_3,$$
$$s_2(x) = x^2 - (\theta_2 + \theta_3)x + \gamma_2 - \theta_2\theta_3.$$

Then,

$$\overline{n_2} \ge \Phi(\tau_j) = \frac{n(s_j(\theta_0) - \tau_j)^2}{\sum_{i=0}^{4} m_i(s_j(\theta_i) - \tau_j)^2}, \quad j = 1, 2, \quad (13)$$

where

$$\tau_j = \frac{s_j(\theta_0)\sum_{i=1}^{4} m_i s_j(\theta_i) - \sum_{i=1}^{4} m_i s_j(\theta_i)^2}{s_j(\theta_0)(n-1) - \sum_{i=1}^{4} m_i s_j(\theta_i)}, \quad j = 1, 2. \quad (14)$$

Equality in (13) *holds with* $j = 1$ *or* $j = 2$ *if and only if* Γ *is a 2-partially distance-regular graph and its distance-*2 *or distance-*1*-or-*2 *graph, respectively, is strongly regular.*

Example 2. The Hamming graph $H(4,3)$, with $n = 3^4 = 81$ vertices and diameter $d = 4$, has intersection array $\{8,6,4,2;1,2,3,4\}$, so that $k_4 = 16$, and spectrum $8^1, 5^8, 2^{24}, -1^{32}, -4^{16}$. Then, the function $\Phi(\tau_j)$ in (13) with $j = 1$ has a maximum at $\tau_1 = 4$, and its value is $\Phi(4) = 33 = s_2$. Then, $P_{14} = P_{44}$ and $P_{24} = P_{34}$. Indeed, its distance-2 polynomial is $p_2(x) = \frac{1}{2}(x^2 - x - 8)$ with values $p_4(8) = 24$, $p_4(5) = 6$, $p_4(2) = -3$, $p_4(-1) = -3$, and $p_4(-4) = 6$. Hence, the distance-2 graph Γ_2 is strongly regular with spectrum $24^1, 6^{24}, -3^{56}$.

Example 3. The Odd graph O(5) with 9 points, has $n = \binom{9}{4} = 126 = 1 + 5 + 20 + 40 + 60$ vertices, diameter $d = 4$, intersection array $\{5,4,4,3;1,1,2,2\}$, and spectrum $5^1, 3^{27}, 1^{42}, -2^{48}, -4^8$. Then, the function $\Phi(\tau_j)$ in (13) with $j = 2$ has a maximum at $\tau_2 = 3$, and its value is $\Phi(4) = 26 = s_2$. Then, its distance-1-or-2 polynomial is $p_{1,2}(x) = p_1(x) + p_2(x) = x^2 + x - 5$ with values $p_{1,2}(5) = 25$, $p_{1,2}(3) = 7$, $p_{1,2}(1) = -3$, $p_{1,2}(-2) = -3$, and $p_{1,2}(-4) = 7$. Hence, the distance-1-or-2 graph $\Gamma_{1,2}$ is strongly regular with spectrum $25^1, 7^{35}, -3^{90}$.

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