Exponentially small splitting for whiskered tori in Hamiltonian systems: Flow-box coordinates and upper bounds

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Abstract

We consider a singular or weakly hyperbolic Hamiltonian, with $n+1$ degrees of freedom, as a model for the behaviour of a nearly-integrable Hamiltonian near a simple resonance. The model consists of an integrable Hamiltonian possessing an n-dimensional hyperbolic invariant torus with fast frequencies $\omega/\sqrt{\varepsilon}$ and coincident whiskers, plus a perturbation of order $\mu = \varepsilon^p$. The vector ω is assumed to satisfy a Diophantine condition.

We provide a tool to study, in this singular case, the splitting of the perturbed whiskers for ε small enough, as well as their homoclinic intersections, using the Poincaré–Melnikov method. Due to the exponential smallness of the Melnikov function, the size of the error term has to be carefully controlled. So we introduce flow-box coordinates in order to take advantage of the quasiperiodicity properties of the splitting. As a direct application of this approach, we obtain quite general upper bounds for the splitting.

Keywords. Hyperbolic KAM theory, flow-box coordinates, Poincaré–Melnikov method.

1 Introduction and main results

The existence of transverse intersection of the unstable and stable manifolds of a (partially) hyperbolic transitive invariant object has been pointed out as one of the main causes of chaotic behavior in a dynamical system. In nearly-integrable Hamiltonian systems with $n+1$ degrees of freedom, $n+1 \geq 2$, the rôle of the invariant objects is played by the whiskered tori, of dimension at most n, which are associated to the resonances of the frequencies of the unperturbed integrable Hamiltonian. An expansion in the perturbation parameter, say ε, reveals that both the unstable and stable manifolds (the whiskers) of a whiskered torus coincide up to any finite order in ε, giving rise to separatrices. However, in general, the whiskers do not coincide, but their distance turns out to be exponentially small with respect to ε. This phenomenon, called *exponentially small splitting* of separatrices, takes place in nearly-integrable Hamiltonian systems, and was first detected by Poincaré [Poi90, §19] by means of what is nowadays known as the *Poincaré–Melnikov method*.

Moreover, for a nearly-integrable Hamiltonian with $n+1$ degrees of freedom, the computation of the splitting of separatrices is a very important problem. Indeed, the existence of splitting with *transverse homoclinic orbits* implies the existence of heteroclinic orbits connecting close enough whiskered tori (at least for whiskered tori whose distance is similar to the splitting distance), and gives rise to the transition chains mechanism, due to Arnold [Arn64], designed to detect, for $n+1 \geq 3$, the phenomenon of instability called *Arnold diffusion*.

In the exponentially small case, the validation of the Poincaré–Melnikov method for detecting the splitting becomes a difficult problem due to its *singular* character. This validation was first carried out for hyperbolic periodic orbits of Hamiltonian systems with 2 degrees of freedom (the case $n=1$ in our setting) and for saddle fixed points of area preserving maps (see, for instance, [DS97, DR98]), at least when the perturbation was small enough near the *complex* *singularities* of natural parameterizations of the separatrices. In fact, Lazutkin was the first to introduce complex parameterizations of the separatrices, in the case of the standard map (see [Laz84, Gel99]), although in this case one needs to consider a different unperturbed model in order to validate the method.

In more dimensions, $n \geq 2$, the arithmetic properties of the frequencies of the n-dimensional whiskered torus, together with the quasiperiodicity properties of the splitting, also have to be taken into account in the validation of the Poincaré–Melnikov approximation to the size of the splitting. This was detected in [Sim94], and rigurously established in [DGJS97] for the quasiperiodically forced pendulum.

On the other hand, for n-dimensional whiskered tori of a Hamiltonian with $n+1$ degrees of freedom, it turns out [Eli94, DG00] that the splitting vector distance and the Melnikov vector function are the gradient of scalar functions, called respectively splitting potential and Melnikov potential. This property, closely related to the Lagrangian character of the whiskers, implies directly the existence of homoclinic intersections and translates the problem of searching for such intersections to the problem of searching for critical points of a scalar function. In the same way, the transverse intersections can be translated to nondegenerate critical points.

Such study has been carried out in [Sau01, RW00, LMS99], by looking at two different solutions of a Hamilton–Jacobi equation, which correspond to both whiskers of the torus. Their difference is shown to be exponentially small in the parameter perturbation ε by expressing it in some flow-box variables for the $n+1$ variables of the whiskers, where it is a quasiperiodic function.

In the present paper, *flow-box coordinates* are introduced in a small neighbourhood containing a piece of both whiskers (but excluding the whiskered torus), straightening in this way all the $2n+2$ coordinates and not only half of them (see also [PV01]). These coordinates are global in the angular variables of the whiskers, and we construct them with the help of an iterative process which allows us to control their complex domain.

The flow-box coordinates provide a tool to study the splitting of the perturbed whiskers in the singular case, as well as their homoclinic intersections, using the Poincaré–Melnikov method. Indeed, these coordinates allow us to take advantage of the quasiperiodicity properties of the splitting, closely related to its exponential smallness.

As a direct application of the results obtained, sharp *upper bounds* follow for the exponentially small splitting of separatrices. It is worth noting that the straightening of all the $2n+2$ coordinates seems to indicate that such result is also valid in other non-Hamiltonian settings. Therefore, we believe that such flow-box coordinates can be useful in other situations where a total description of the dynamics close to a whisker of a Hamiltonian system is needed, as well as in other settings, like reversible systems.

Besides, we provide an accurate upper bound for the size of the error term of the Melnikov function. In the paper [DG02], for some more concrete perturbations in the case of 3 degrees of freedom ($n=2$), it is seen as a consequence of this bound that the Melnikov function dominates the error term. This implies the validity of the Poincaré–Melnikov method to give asymptotic estimates for the splitting, showing the existence of transverse homoclinic orbits. This can be done thanks to the fact that all the results obtained in the present paper are quantitative enough.

A more precise description of the setting and of the results of the present paper follows.

1.1 Setup: A singular or weakly hyperbolic Hamiltonian

We consider a Hamiltonian system, with $n+1 \geq 3$ degrees of freedom, depending on two perturbation parameters ε and μ. In canonical coordinates $(x, y, \varphi, I) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}^n \times \mathbb{R}^n$, with the symplectic form $\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}\varphi \wedge \mathrm{d}I$, our Hamiltonian is defined by

$$H(x, y, \varphi, I) = H_0(x, y, I) + \mu H_1(x, \varphi), \tag{1}$$

$$H_0(x, y, I) = \langle \omega_\varepsilon, I \rangle + \frac{1}{2} \langle \Lambda I, I \rangle + \frac{y^2}{2} + \cos x - 1, \tag{2}$$

$$H_1(x, \varphi) = h(x) f(\varphi), \tag{3}$$

where Λ is a symmetric $(n \times n)$-matrix, and $h(x)$ and $f(\varphi)$ are analytic periodic functions. We work with *fast frequencies* of the form

$$\omega_\varepsilon = \frac{\omega}{\sqrt{\varepsilon}}, \tag{4}$$

where $\omega \in \mathbb{R}^n$ is fixed, and $\varepsilon > 0$. (We also assume $\mu > 0$ with no loss of generality.)

Notice that H_0 consists of a pendulum, given by $P(x,y) = y^2/2 + \cos x - 1$, and n rotators with fast frequencies, $\dot{\varphi} = \omega_\varepsilon + \Lambda I$. Then, the Hamiltonian H_0 has an n-parameter family of n-dimensional whiskered tori (or hyperbolic tori) given by the equations $I = \text{const}$, $x = y = 0$. The stable and unstable whiskers of each torus coincide, forming in this way a unique homoclinic whisker. We shall focus our attention on a concrete *whiskered torus*, located at $I = 0$, whose frequencies are assumed to satisfy a Diophantine condition (see hypothesis (H2) below). We denote $\mathcal{T}_0$ this torus and $\mathcal{W}_0$ its homoclinic whisker.

The two parameters of the Hamiltonian will not be independent. On the contrary, they will be linked by a power-like relation of the type $\mu = \varepsilon^p$ with a suitable $p > 0$ (the smaller p the better). Then, one usually says that the considered problem is *singular* for $\varepsilon \to 0$ (one can also say that the Hamiltonian is *weakly hyperbolic*, or *a-priori stable*). A motivation for the singular problem is given in Section 1.2. In fact, our approach will be to work with (1–3) first as a *regular* problem: with $\varepsilon > 0$ fixed and $\mu \to 0$ as the perturbation parameter (i.e. starting with a hyperbolic situation for $\mu = 0$). By proving our results under smallness conditions of the type $\mu \leq \varepsilon^p$, they will also be valid for the singular case.

The following hypotheses will be assumed:

(H1) The Hamiltonian H_0 is *isoenergetically nondegenerate*:

$$\det \begin{pmatrix} \Lambda & \omega \\ \omega^\top & 0 \end{pmatrix} \neq 0.$$

(H2) The frequencies ω satisfy a *Diophantine condition*: for some $\tau \geq n-1$ and $\gamma > 0$,

$$|\langle k, \omega \rangle| \geq \frac{\gamma}{|k|^\tau} \qquad \forall k \in \mathbb{Z}^n \setminus \{0\}.$$

(H3) The function $h(x)$ is a trigonometric polynomial of degree $l \geq 1$.

(H4) The function $f(\varphi)$ is analytic in a complex strip $|\operatorname{Im}\varphi| < \rho$ and there exists $\alpha \geq 0$ and a constant $c > 0$ such that, for any $0 < \delta < \rho$,

$$\|f\|_{\rho-\delta} \leq \frac{c}{\delta^\alpha}.$$

(We denote $\|f\|_{\rho-\delta}$ the norm of $f(\varphi)$ on the complex strip $\{\varphi : |\operatorname{Im}\varphi| \leq \rho - \delta\}$; see Section 1.5.) Notice that this hypothesis provides a control on the size of the perturbation near a "pole-like singularity of order α".

1.2 Context and motivation: Simple resonances of nearly-integrable Hamiltonians

We will see from KAM theory that, under hypotheses (H1–H2), for μ small enough the whiskered torus persists (we denote $\mathcal{T}$ the perturbed torus), as well as its local whiskers. However, if the local whiskers are extended to global ones, one can also expect in general the existence of *splitting* between the perturbed stable and unstable whiskers (denoted $\mathcal{W}^+$ and $\mathcal{W}^-$), since they will no longer coincide. The study of this splitting, which is important in relation with chaotic behaviour and Arnold diffusion, is simpler in the case of transverse intersections between the whiskers (which give rise to *transverse homoclinic orbits*, contained in both whiskers). If one considers ε fixed and μ small enough (i.e. the regular case), such transverse intersections can be detected by applying the Poincaré–Melnikov method, which provides a first order approximation in μ for the splitting. However, a difficulty that goes back to [Arn64] is that the Melnikov function is *exponentially small* with respect to ε. Then, a direct application of the method also requires μ to be exponentially small in ε.

Such an approach does not work in the problem considered in the present paper, since we work with a singular situation: the two parameters are linked by a relation of the type $\mu = \varepsilon^p$. To motivate such a situation, we stress that the singular Hamiltonian (1–3) can be considered as a model for the behaviour of a nearly-integrable Hamiltonian near a *simple resonance*. Indeed, after rescaling the coordinates y, I and time, we can rewrite (1–3) in the form

$$\begin{aligned} &N_0(y,I) + \varepsilon N_1(x) + \varepsilon\mu H_1(x,\varphi), \\ &N_0(y,I) = \langle \omega, I\rangle + \frac{1}{2}\langle \Lambda I, I\rangle + \frac{y^2}{2}, \\ &N_1(x) = \cos x - 1. \end{aligned}$$

Note that N_0 is a completely integrable Hamiltonian, whose trajectories all lie in $(n+1)$-dimensional invariant tori with frequencies $(y, \omega + \Lambda I)$. Then, we could consider $N_0 + \varepsilon N_1$ as the truncated resonant normal form of some perturbed Hamiltonian, near the simple resonance given by $y = 0$. This normal form is integrable, and for $\varepsilon > 0$ it has a family of n-dimensional whiskered tori with coincident whiskers, along the resonance. Finally, the term $\varepsilon\mu H_1$ could be the remainder of the normal form, containing higher-order terms in ε. This is the reason to consider $\mu = \varepsilon^p$ with $p > 0$. (For a more general case and details, see for instance [DG01].)

The key point to overcome the difficulties of the singular case and obtain exponentially small estimates is to carry out the bounds on *complex domains*. Under hypotheses (H3–H4), we will obtain bounds for the splitting function on a complex domain. Then, the quasiperiodicity properties of the splitting function, together with an analysis of its Fourier coefficients that can be done under hypothesis (H2), lead to an exponentially small upper bound for the splitting.

In this way, in a general setting like the one described in (1–3), it is possible to obtain exponentially small upper bounds for the splitting distance. However, one cannot establish the existence of transverse homoclinic orbits, unless some more information on ω, $h(x)$ and $f(\varphi)$ is provided. This is carried out in the paper [DG02], for a much more concrete example with 3 degrees of freedom ($n = 2$), giving asymptotic estimates for the splitting and showing the existence of transverse homoclinic orbits, under the condition $\mu \le \varepsilon^p$ (with a suitable p). This requires to establish the validity of the Poincaré–Melnikov method in this singular case, despite the fact that the Melnikov function is exponentially small.

1.3 The unperturbed torus and its whisker

For the unperturbed Hamiltonian H_0 defined in (2), recall that we denote $\mathcal{T}_0$ the whiskered torus located at $I = 0$, and $\mathcal{W}_0$ its associated homoclinic whisker. The torus can obviously be parameterized by

$$\mathcal{T}_0 : \qquad Z_0^*(\theta) = (0,0,\theta,0), \quad \theta \in \mathbb{T}^n.$$

The whisker $\mathcal{W}_0$ (or more precisely its positive part, $y > 0$) can be easily parameterized from the well-known homoclinic trajectory of the pendulum $P(x,y)$,

$$\mathcal{W}_0 : \qquad Z_0(s,\theta) = (x_0(s), y_0(s), \theta, 0), \quad s \in \mathbb{R},\ \theta \in \mathbb{T}^n, \tag{5}$$

$$x_0(s) = 4\arctan e^s, \quad y_0(s) = \frac{2}{\cosh s}. \tag{6}$$

Note that the inner Hamiltonian flow on $\mathcal{W}_0$, associated to H_0, is given by $\dot{s} = 1$, $\dot{\theta} = \omega_\varepsilon$, i.e. the trajectories are $Z_0(s+t, \theta + \omega_\varepsilon t)$, $t \in \mathbb{R}$, for any initial s, θ. In fact, we shall consider complex values for the parameters s, θ, and the singularity at $s = \pm i\pi/2$ is going to play an important rôle.

1.4 Description of the results

The main contributions of this paper can be summarized as follows. We express the Hamiltonian (1–3) in flow-box coordinates, in a neighbourhood containing a piece of both whiskers. This allows us to exploit the quasiperiodicity properties of the splitting, leading to exponentially small upper bounds for the splitting function. On the other hand, we provide an accurate upper bound for the error term in the Poincaré–Melnikov method, to be used in the paper [DG02].

In Section 2, we establish the persistence of the hyperbolic torus and its local whiskers under the perturbation. To such end, we put our Hamiltonian in local *hyperbolic coordinates* (u, v, φ, I), in which the unperturbed whiskers become coordinate planes. Then, we apply the *hyperbolic KAM theorem* (in the version of [Nie00]), which provides, after a symplectic change, a normal form for the perturbed Hamiltonian, ensuring the persistence of the torus and its local whiskers. It is important to control the loss δ of complex domain in the angles φ (this will be useful in order to estimate the exponentially small splitting, by choosing $\delta = \varepsilon^a$ for some $a > 0$). We also give an improvement of the estimates valid if $h(x) = \mathcal{O}_2(x)$ in (3), i.e. when the torus $\mathcal{T}_0$ remains fixed under the perturbation (for instance $h(x) = \cos x - 1$, see also [DG02]). The improvement consists in smaller exponents of δ in the estimates.

In Section 3, we prove the *flow-box theorem* (Theorem 4): near a piece of the local stable whisker (excluding the torus), we introduce symplectic *flow-box coordinates* (S, E, φ, I) in such a way that the expression of the Hamiltonian becomes very simple. As before, in this theorem we keep a control on the loss δ in the angles. We stress that the flow-box coordinates cannot be defined in a direct way, because the normal form provided by the hyperbolic KAM theorem is not explicitly known. To overcome this difficulty, we first put the integrable part of this normal form in flow-box coordinates and, afterwards, we carry out an iterative process that removes the remainder. In fact, the part coming from the non-small term $\frac{1}{2}\langle \Lambda I, I\rangle$ in (2) also has to be removed (in order to obtain a quasiperiodic splitting function in Section 4). Although this makes us accept a strong reduction of domain in the actions I, the domain of the flow-box coordinates will contain a piece of both the local stable whisker and the global unstable whisker, and this will be suitable enough for our purposes.

In Section 4, we mainly show that a piece of both perturbed whiskers actually enter in the domain of the flow-box coordinates, define a function measuring the distance between the whiskers, and compare this function with the Melnikov function. To such end, we first define parameterizations $Z^{\pm}(s, \theta)$ for the local perturbed whiskers (Section 4.1), which can be extended in a natural way to global whiskers with the help of the *extension theorem* (Section 4.2). Next we define the *Melnikov function* $M(s, \theta)$, and show that it is the gradient of a scalar function $L(s, \theta)$, called the *Melnikov potential* (Section 4.3). We show that the Melnikov function provides a first order approximation in μ for the splitting distance (Section 4.4). However, we have to translate the parameterizations of the whiskers to the flow-box coordinates in order to have a *quasiperiodic* function giving the splitting distance (Section 4.5). After a further reparameterization, we define the *splitting function* $\mathcal{M}(s, \theta)$, which becomes also a gradient of a scalar function $\mathcal{L}(s, \theta)$, called the *splitting potential* (Section 4.6). In Theorem 10, we provide bounds for both the splitting function and the *error term* $\mathcal{R}(s, \theta)$ (defined as the difference between the splitting function and the Melnikov function), in a complex domain (always with a control on the loss δ in the complex domain). We also establish the quasiperiodicity of $\mathcal{M}$ and $\mathcal{R}$.

Finally, in Section 5 we use the quasiperiodicity of the splitting and the bounds of Section 4, and obtain *exponentially small upper bounds* for the splitting function $\mathcal{M}$ restricted to the real domain (see Theorem 12): under a condition $\mu \le c_1 \varepsilon^{p^*}$, we provide a bound of the type

$$|\mathcal{M}| \le \frac{c_2 \mu}{\varepsilon^{p^{**}}} \exp\left\{ -\frac{C\left(\frac{\pi}{2}, \rho\right)}{\varepsilon^{1/(2\tau+2)}} \right\}, \tag{7}$$

with a constant which depends on the complex widths in the parameters s, θ:

$$C\left(\frac{\pi}{2}, \rho\right) = \left(1 + \frac{1}{\tau}\right) \left(\frac{\pi \tau \rho^\tau \gamma}{2}\right)^{1/(\tau+1)}, \tag{8}$$

and some exponents $p^*, p^{**} > 0$ depending on n, τ, l, α. However, in the general case considered in this paper one cannot ensure the existence of transverse intersections (i.e. simple zeros with respect to θ of the splitting function, or nondegenerate critical points of the splitting potential), because lower bounds for the Melnikov function are necessary in order to see that it actually dominates the error term $\mathcal{R}$. This is carried out in the paper [DG02], where a much more particular case is considered, and asymptotic estimates for the splitting are given. For such a case, the asymptotic estimates obtained in [DG02] show the *optimality* of the upper bound (7), since the constant $C(\frac{\pi}{2}, \rho)$ is replaced in that paper by a function of ε (periodic in $\ln \varepsilon$) whose minimum value is $C(\frac{\pi}{2}, \rho)$.

1.5 Some notations

To express the bounds of functions in a given norm $|\cdot|$, we write $|f| \preceq |g|$ if we can bound $|f| \le c\,|g|$, with some constant c not depending on any of the parameters that will be relevant to us: ε, μ, δ. In this way, we do not describe the (usually complicated) dependence on amounts like $n, \tau, r, \rho, \gamma, \ldots$ and include this dependence in the 'constants'. We also write $f \sim g$ if we can bound $c_1 |g| \le |f| \le c_2 |g|$ with $c_1 > 0$. In particular, the expression $|f| \preceq 1$ means that $|f|$ is smaller than a suitable constant, and $|f| \sim 1$ means that $|f|$ can be bounded from above and from below by positive constants.

In Section 2, we introduce "hyperbolic coordinates" (u, v, φ, I). Later, in Section 3, we introduce "flow-box coordinates" (S, E, φ, I). For a vector-valued function f with images in the (u, v, φ, I)-space, the notation

$$|f| \preceq (|g_1|, |g_2|)$$

will be used to express separate bounds for the (u, v)-components and the (φ, I)-components of the function f. We use the same notation for a vector-valued function with images in the (S, E, φ, I)-space. In this way, we achieve some improvement of the exponents involved in the bounds.

We also introduce here the complex domains of functions in the different kinds of variables or coordinates. In the hyperbolic coordinates, we define the "cross-like" domain

$$\begin{aligned} \mathcal{S}_{r,\gamma,\rho} = \{(u, v, \varphi, I) : (|u| \le r, |v| \le \gamma \text{ or } |u| \le \gamma, |v| \le r), \\ \operatorname{Re}\varphi \in \mathbb{T}^n, \ |\operatorname{Im}\varphi| \le \rho, \ |I| \le \gamma\}, \end{aligned} \tag{9}$$

which contains the whiskered torus and its local whiskers. In the flow-box coordinates, we define

$$\begin{aligned} \mathcal{B}_{\kappa,\sigma,\eta,\rho,\zeta} = \{(S, E, \varphi, I) : |\operatorname{Re} S| \le \kappa, \ |\operatorname{Im} S| \le \sigma, \ |E| \le \eta, \\ \operatorname{Re}\varphi \in \mathbb{T}^n, \ |\operatorname{Im}\varphi| \le \rho, \ |I| \le \zeta\}, \end{aligned} \tag{10}$$

a domain which will contain a piece of the local stable whisker and a piece of the global unstable whisker, but not the whiskered torus.

For an analytic function $f(u, v, \varphi, I)$ in (a neighbourhood of) the domain $\mathcal{S}_{r,\gamma,\rho}$, the supremum norm in this domain will be denoted $|f|_{r,\gamma,\rho}$. However, we mainly will use the following norm, which takes into account the Fourier expansion in the angular variables:

$$\|f\|_{r,\gamma,\rho} = \sum_{k\in\mathbb{Z}^n} |f_k|_{r,\gamma} e^{|k|\rho}, \qquad \text{where } f(x, y, \varphi, I) = \sum_{k\in\mathbb{Z}^n} f_k(x, y, I) e^{i\langle k,\varphi\rangle} \tag{11}$$

($|f_k|_{r,\gamma}$ denotes the supremum of each coefficient). In the same way, we denote $|f|_{\kappa,\sigma,\eta,\rho,\zeta}$ and $\|f\|_{\kappa,\sigma,\eta,\rho,\zeta}$ the analogous norms for a function $f(S, E, \varphi, I)$ in the domain $\mathcal{B}_{\kappa,\sigma,\eta,\rho,\zeta}$.

The width of the complex domains has to be reduced along (a finite number of) successive normalizing transformations: KAM, flow-box, ... Of course, there is a lot of freedom in the choice of the reductions from the initial widths $r_0 = r$, $\gamma_0 = \gamma$ and $\rho_0 = \rho$. Thus, we denote r_j, γ_j and ρ_j, $j \ge 1$, the successively reduced widths, but we do not have to worry about the concrete values of r_j and γ_j. Indeed, it suffices that $r_{j-1} - r_j$ and $\gamma_{j-1} - \gamma_j$ are ~ 1, because these reductions only influence the unwritten constants in the bounds. On the other hand, the reduction in the angles has to be more carefully controlled, and we need $\rho_{j-1} - \rho_j \sim \delta$, where $\delta \ll \rho$ is a free parameter to be chosen later as a power of ε (see Section 5.2 and also the paper [DG02]). For instance, we can take $\rho_j = \rho - j\delta$.

We will also bound functions of the parameters s, θ of the whiskers (these parameters have been introduced in (5–6)). We define the following domain of complex parameters

$$\mathcal{P}_{\kappa,\nu,\rho} = \{(s, \theta) : |\operatorname{Re} s| \le \kappa, \ |\operatorname{Im} s| \le \nu, \ \operatorname{Re}\theta \in \mathbb{T}^n, \ |\operatorname{Im}\theta| \le \rho\} \tag{12}$$

and, for a function $g(s, \theta)$, we denote $|g|_{\kappa,\nu,\rho}$ its supremum norm on this domain (note that $|g|_{\kappa,0,0}$ is then the supremum norm on the real domain). In the same way, we can denote $|\cdot|_{\kappa,\nu}$ or $|\cdot|_{\rho}$ the norms of functions depending only on s or only on θ, respectively.

On the other hand, recall that ω_ε denotes the (fast) frequency vector of the unperturbed torus, introduced in (4). We shall denote $\tilde{\omega}_\varepsilon$ the frequency vector of the perturbed torus, and $\hat{\omega}_\varepsilon$ the vector that indicates the quasiperiodicity of the splitting function. Those vectors, defined in (28) and (51) respectively, are close and proportional to the initial one.

We say that a function $g(s, \theta)$ is $\hat{\omega}_\varepsilon$*-quasiperiodic* if it satisfies the equality

$$g(s, \theta) = g(0, \theta - \hat{\omega}_\varepsilon s) \tag{13}$$

for any s, θ. In other words, the function g only depends on $\theta - \hat{\omega}_\varepsilon s$. This property can also be expressed by the following partial differential equation:

$$\partial_s g + \langle \hat{\omega}_\varepsilon, \partial_\theta g \rangle = 0.$$

We also stress that the θ-average of a quasiperiodic function does not depend on s, i.e. $\overline{g(s, \cdot)} = \text{const}$. If a function g is analytic on $\mathcal{P}_{\kappa,\nu,\rho}$ and $\hat{\omega}_\varepsilon$-quasiperiodic, this implies that the function $g(0, \cdot)$, that in principle is defined only for $|\mathrm{Im}\,\theta| \le \rho$, is actually analytic on a much wider domain:

$$\{\theta - \hat{\omega}_\varepsilon s \ : \ |\mathrm{Im}\,\theta| \le \rho, \ |\mathrm{Im}\, s| \le \nu\}.$$

To end this section, we introduce several exponents of δ or ε that will appear along the paper. These exponents depend on τ, l, α and n, and can explicitly be computed in concrete cases (see an example in [DG02]). In fact, there will be some improvement of the general estimates if $h(x) = \mathcal{O}_2(x)$ in (3), i.e. when the torus $\mathcal{T}_0$ remains fixed under the perturbation. The first three exponents, in the "*general case*", are

$$p_1 = 3\tau + \alpha + 3, \qquad p_2 = \tau + \alpha + 1, \qquad p_3 = 2\tau + \alpha + 1, \tag{14}$$

and, in the "*case of a fixed torus*",

$$p_1 = \tau + \alpha + 2, \qquad p_2 = \alpha, \qquad p_3 = \tau + \alpha. \tag{15}$$

Then, the remaining three exponents are computed as follows from the three initial ones:

$$p_4 = \max(p_3 + 1, 2l + \alpha), \tag{16}$$
$$p_5 = \max(p_3 + 2, 2l + \alpha), \tag{17}$$
$$p_6 = \max(p_3 + 2l + \alpha + 3, 4l + 2\alpha + 1, 2p_3 + 4), \tag{18}$$
$$p_7 = \max(p_2 + 2l + \alpha + 1, p_2 + p_3 + 3), \tag{19}$$
$$p_8 = \frac{\max(p_1, p_3 + 3, 2l + \alpha + 2)}{2\tau + 2}, \tag{20}$$
$$p_9 = \frac{\max(p_3 + n + 1, 2l + \alpha + n)}{2\tau + 2}. \tag{21}$$

2 Hyperbolic coordinates and local normal form

2.1 Moser's transformation to hyperbolic coordinates

We put the unperturbed Hamiltonian H_0 in some local coordinates (u, v, φ, I), that we call *hyperbolic coordinates*, in which the local whiskers (in a neighbourhood around the torus $\mathcal{T}_0$) become coordinate planes. This comes from a well-known Moser's result [Mos56] on the convergence of the Birkhoff normal form for a 1-degree-of-freedom Hamiltonian near a hyperbolic equilibrium point.

More precisely, there exists a symplectic transformation $(x, y) = \Gamma_0(u, v)$, defined in a neighbourhood around $(0, 0)$ of some radius $r > 0$, taking the pendulum P to a function of uv:

$$P \circ \Gamma_0(u, v) = g(uv) = uv + \mathcal{O}_2(uv).$$

The linear part of Γ_0 can be chosen as

$$\mathrm{D}\Gamma_0(0, 0) = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}. \tag{22}$$

It is then clear that the transformation $\Gamma(u, v, \varphi, I) = (\Gamma_0(u, v), \varphi, I)$ takes our Hamiltonian H into the form

$$G = H \circ \Gamma = G_0 + \mu G_1,$$
$$G_0(u, v, I) = \langle \omega_\varepsilon, I \rangle + \frac{1}{2} \langle \Lambda I, I \rangle + g(uv), \tag{23}$$
$$G_1(u, v, \varphi) = H_1 \circ \Gamma = (h \circ \Gamma_0(u, v)) \cdot f(\varphi). \tag{24}$$

In order to apply the hyperbolic KAM theorem of [Nie00], we restrict G to the domain $\mathcal{S}_{r,\gamma,\rho_1}$ defined in (9). Recall that γ is the constant in hypothesis (H2) of Section 1.1 and also that, according to the notations introduced in Section 1.5, we write $\rho_1 = \rho - \delta$. Note that hypothesis (H4) provides a bound for H_1 and hence for G_1:

$$\|G_1\|_{r,\gamma,\rho_1} \preceq \frac{1}{\delta^\alpha}. \tag{25}$$

2.2 The hyperbolic KAM theorem

The persistence, for ε small enough, of the Diophantine whiskered torus and its local stable and unstable whiskers concerns KAM theory. In fact, we need a normalizing transformation convergent in a whole neighbourhood of the torus, in which the existence of the perturbed torus and its local whiskers becomes transparent, and making possible the study of the dynamics near them. We give in this section the hyperbolic KAM theorem, in a version established in [Nie00], which provides the refined estimates that we need for our purposes. Indeed, in that paper there is a control on the loss δ of complex domain in the angles φ; this is important because it leads to exponentially small estimates for the splitting (taking δ as some power of ε; see Section 5.2).

Another fact to be pointed out is that the paper [Nie00] deals with exact symplectic transformations. Using this, one can establish the existence of homoclinic intersections. This idea, that goes back to [Eli94], was used later in order to introduce the splitting potential as a function whose gradient gives a measure of the splitting distance [DG00] (see also [Sau01, LMS99, RW00] as related papers). Denoting $\eta = -(y\mathrm{d}x + I\mathrm{d}\varphi)$, we recall that a symplectic map Ψ is exact if the 1-form $\Psi^*\eta - \eta$ is exact (i.e. it has a global scalar primitive). The symplectic transformation to normal form that we provide in Theorem 1 is not exact, but it can be made exact by composing it with a translation in the actions, $I \mapsto I + a$.

Another useful point (that we follow) is that [Nie00] establishes the theorem under the isoenergetic nondegeneracy. It is a well-known fact (see for instance [DG96]) that under this condition one obtains a perturbed torus lying exactly on the same energy level, though its frequencies $\tilde\omega_\varepsilon$ are *proportional* to the unperturbed frequencies ω_ε.

The following statement comes from [Nie00, Th. 2.2]. The main point is the form of the remainder, $R = \mathcal{O}_2(uv, I)$, which gives directly the expression (in the normal form coordinates) of the perturbed torus and its local whiskers (see Section 4.1).

As said in Section 1.5, there is an improvement of the estimates valid in the case of a fixed torus, i.e. $h(x) = \mathcal{O}_2(x)$ in (3) and hence $G_1 = \mathcal{O}_2(u,v)$ in (24). Indeed, since in this case the torus $\mathcal{T}_0$ with frequencies ω_ε remains fixed under the perturbation, the proof of the theorem becomes simpler because one only has to worry about the whiskers. (Recall that, as defined in Section 1.5, we take $r_1 < r$, $\gamma_1 < \gamma$, $\rho_2 = \rho_1 - \delta$.)

Theorem 1 (hyperbolic KAM theorem) *Let the Hamiltonian $G(u,v,\varphi,I) = G_0 + \mu G_1$ as given in (23–24), real analytic on $\mathcal{S}_{r,\gamma,\rho_1}$, satisfying hypotheses (H1–H2), $\tau > n-1$, and with G_1 satisfying (25). Assume the conditions*

$$\varepsilon \preceq 1, \qquad \mu \preceq \delta^{p_1}, \qquad \mu \preceq \delta^{p_2}\sqrt{\varepsilon}. \tag{26}$$

Then, there exists an analytic symplectic map $\Phi : \mathcal{S}_{r_1,\gamma_1,\rho_2} \longrightarrow \mathcal{S}_{r,\gamma,\rho_1}$, and there exist a, b, b', such that $\widetilde{G} = G \circ \Phi$ takes the form

$$\widetilde{G} = \langle \tilde\omega_\varepsilon, I\rangle + \frac{1}{2}\langle \Lambda I, I\rangle + bg(uv) + R(u,v,\varphi,I), \qquad R = \mathcal{O}_2(uv, I), \tag{27}$$

where we write

$$\tilde\omega_\varepsilon = b'\omega_\varepsilon = \frac{b'\omega}{\sqrt{\varepsilon}}, \tag{28}$$

and Φ is such that $\Phi \circ T_a^{-1}$ is exact symplectic, with $T_a : (u,v,\varphi,I) \mapsto (u,v,\varphi,I+a)$. Besides, the following bounds hold:

$$|\Phi - \mathrm{id}|_{r_1,\gamma_1,\rho_2} \preceq \frac{\mu}{\delta^{p_3}} \cdot \left(1, \frac{1}{\delta}\right), \tag{29}$$

$$|a|, |b-1|, |b'-1| \preceq \frac{\mu}{\delta^{p_2}}, \tag{30}$$

$$\|R\|_{r_1,\gamma_1,\rho_2} \preceq \frac{\mu}{\delta^{p_3+1}}. \tag{31}$$

In the case of a fixed torus one has $\Phi - \mathrm{id} = \mathcal{O}(u,v,I)$, $a=0$, $b'=1$. The exponents p_1, p_2, p_3, in both the general case and the case of the fixed torus, have been defined in (14–15).

Proof. We write the integrable part (23) in the form

$$G_0 = \langle \omega_\varepsilon, I\rangle + uv + f(uv,I),$$

where

$$f(uv,I) = \frac{1}{2}\langle \Lambda I, I\rangle + g(uv) - uv = \mathcal{O}_2(uv,I).$$

So we are in the framework of [Nie00], with $\sqrt{\varepsilon}$ instead of ε, and a change of time scale that does not affect the results. According to bound (25), we also have μ/δ^α instead of μ. We have to check that, for some $m>0$,

$$\left| \begin{pmatrix} \Lambda & \omega_\varepsilon \\ \omega_\varepsilon^\top & 0 \end{pmatrix} v \right| \geq m\,|v| \qquad \forall v \in \mathbb{R}^{n+1}.$$

The isoenergetic condition (H1) implies a bound of the type

$$\left| \begin{pmatrix} \Lambda & \omega \\ \omega^\top & 0 \end{pmatrix} v \right| \geq m\,|v| \qquad \forall v \in \mathbb{R}^{n+1},$$

and we easily obtain

$$\begin{aligned} \left| \begin{pmatrix} \Lambda & \omega_\varepsilon \\ \omega_\varepsilon^\top & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right| &= \left| \begin{pmatrix} \Lambda v_1 + \omega_\varepsilon v_2 \\ \omega_\varepsilon^\top v_1 \end{pmatrix} \right| \geq \left| \begin{pmatrix} \Lambda v_1 + \omega_\varepsilon v_2 \\ \omega^\top v_1 \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} \Lambda & \omega \\ \omega^\top & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ \frac{v_2}{\sqrt{\varepsilon}} \end{pmatrix} \right| \geq m \left| \begin{pmatrix} v_1 \\ \frac{v_2}{\sqrt{\varepsilon}} \end{pmatrix} \right| \geq m \left| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right|. \end{aligned}$$

Then, we can apply the results of [Nie00, Th. 2.2] and obtain a symplectic map $\Phi : \mathcal{S}_{r_1,\gamma_1,\rho_2} \longrightarrow \mathcal{S}_{r,\gamma,\rho_1}$, with $\Phi \circ T_a^{-1}$ exact, such that

$$G\circ\Phi = \langle \tilde\omega_\varepsilon, I\rangle + buv + F(u,v,\varphi,I), \qquad F = \mathcal{O}_2(uv,I),$$

and we get bounds (29–30) for $|\Phi - \mathrm{id}|_{r_1,\gamma_1,\rho_2}$, $|a|$ and $|b'-1|$ (with the use of the notation '$\preceq$', we avoid the complicated constants of these bounds). Although a bound for $|b-1|$ (the variation in the Lyapunov exponent) is missing in [Nie00], it is not hard to deduce such a bound from the proof.

We get the expression (27) by writing

$$R = F - \frac{1}{2}\langle \Lambda I, I\rangle - b(g(uv) - uv) = F - f - (b-1)(g(uv)-uv).$$

A bound for $\|F-f\|_{r_1,\gamma_1,\rho_2}$ is also given in [Nie00]. Then, using also the bound for $|b-1|$, we obtain (31).

The bounds provided in [Nie00] concern the case of a general perturbation. The case of a fixed torus, $G_1 = \mathcal{O}_2(u,v)$, is carried out by reviewing carefully the proof given in [Nie00] and taking into account that, for this special case, several terms vanish in the normalizing procedure. □

Remarks.

1. In the paper [DG00] (that follows an approach introduced in [Eli94]), the translation T_a^{-1} was included in the normalizing transformation and then the normal form in (27) was expressed in terms of $I-a$ instead of I. However, this is not strictly necessary in the current paper.

2. Another feature of the paper [DG00] (also following [Eli94]) is that the normal form was expressed in the original coordinates (x,y,φ,I) instead of the hyperbolic coordinates (u,v,φ,I) defined in Section 2.1. To get this, the normal form can be composed again with Γ^{-1} in order put it back (locally) in the (x,y,φ,I)-coordinates. Also, we do not need this here.

3. As a technical point, the bound for $|\Phi - \mathrm{id}|$ is given in [Nie00, Th. 2.2] with more detail than in (29), separating it in four components: $\frac{\mu}{\delta^{p_3}} \cdot \left(1,1,1,\frac{1}{\delta}\right)$. However, this feature does not allow us to obtain better estimates for the splitting.

4. Another technical remark is that the result of [Nie00, Th. 2.2] is stated using always the supremum norm. However, a close look at the proof shows that the bounds given in that paper do not change with the norm (11).

5. It can also be seen from the proof given in [Nie00] that the third condition of (26) can be removed in the case of fixed tori. This is due to the fact that $b' = 1$ in this case, i.e. the frequencies remain unchanged: $\tilde{\omega}_\varepsilon = \omega_\varepsilon$. Nevertheless, the estimates for the splitting that we obtain if this condition is kept in (26) are not worse.

3 Flow-box coordinates

We are going to introduce, near a piece of the local stable whisker, symplectic flow-box coordinates in which the expression of the Hamiltonian will become very simple. In fact, the local stable whisker will be a coordinate plane, and then the global unstable whisker will be considered a manifold close to this plane. In the flow-box coordinates, we will easily deduce the quasiperiodicity of the splitting function (that will measure the splitting distance) and, with a good control of the complex domain in the angles, we can get that the splitting is exponentially small with respect to ε.

Our aim is to express the normal form $\widetilde{G}$ in (27), in terms of flow-box coordinates. But, unlike [DS97, DGJS97], the normal form is not integrable in general and the change to flow-box coordinates cannot be defined explicitly. On the other hand, flow-box coordinates for a system that is not integrable are defined in the paper [DG00] but, due to the use of Poincaré sections and implicit functions, it becomes more difficult to carry out an accurate control of the loss δ of complex domain in the angles.

To overcome those difficulties, we carry out two steps. We write the normal form $\widetilde{G}$ in (27) as some integrable Hamiltonian $\widetilde{G}_0$ plus a perturbation:

$$\widetilde{G} = \widetilde{G}_0 + \frac{1}{2}\langle \Lambda I, I\rangle + R(u,v,\varphi,I), \qquad \widetilde{G}_0 = \langle \tilde{\omega}_\varepsilon, I\rangle + bg(uv)$$

(compare also with (23–24)). First, in Section 3.1 we define (in a simple way) flow-box coordinates for the integrable part $\widetilde{G}_0$. In Section 3.2, we obtain the flow-box coordinates for the whole Hamiltonian $\widetilde{G}$ by means of a convergent iterative process; this makes it easier to control the loss of complex domain.

We stress that a different approach, but related to ours, appears in [Sau01] (see also [LMS99, RW00]). In these papers, the whiskers are given by two solutions of the Hamilton–Jacobi equation. Their difference is shown to be exponentially small by putting it in flow-box coordinates, which only involve the variables (s,θ) of the whiskers instead of all coordinates of the phase space. The iterative process we carry out in order to construct the flow-box coordinates is reminiscent of the one used in [Sau01].

On the other hand, in [PV01] the flow-box coordinates are constructed without using an iterative process. Indeed, this paper considers flows defined for complex times, in order to construct a suitable analytic conjugation of vector fields giving the flow-box coordinates.

3.1 Unperturbed flow-box coordinates

We define here flow-box coordinates for the integrable part $\widetilde{G}_0$. We choose some fixed v_0 with $0 < v_0 < r_1$, for instance $v_0 = r_1/2$, and consider first a symplectic change $(u,v) = \Delta_0(S,E)$, in a neighbourhood of $(0,v_0)$, defined in the following standard way:

$$S = \frac{\log(v_0/v)}{g'(uv)}, \qquad E = g(uv). \tag{32}$$

Note that $(0,v_0) = \Delta_0(0,0)$. The new coordinates (S,E) are the usual time–energy coordinates for the Hamiltonian $g(uv)$.

Defining $\Delta(S,E,\varphi,I) = (\Delta_0(S,E),\varphi,I)$, the Hamiltonian $K = \widetilde{G}\circ\Delta$ has the expression

$$K = K_0 + \frac{1}{2}\langle \Lambda I, I\rangle + Q, \tag{33}$$

where

$$K_0 = \widetilde{G}_0 \circ \Delta = \langle \tilde{\omega}_\varepsilon, I \rangle + bE, \qquad Q = R \circ \Delta(S, E, \varphi, I) = \mathcal{O}_2(E, I). \tag{34}$$

According to (31), the remainder Q will be small with respect to μ. In fact, this remainder Q and the term $\frac{1}{2}\langle \Lambda I, I \rangle$ will be removed with additional symplectic transformations in Section 3.2. After these transformations, the whole Hamiltonian will be reduced to K_0, whose associated Hamiltonian equations are very simple:

$$\dot{S} = b, \quad \dot{E} = 0, \quad \dot{\varphi} = \tilde{\omega}_\varepsilon, \quad \dot{I} = 0. \tag{35}$$

Recall that the normal form $\widetilde{G}$ is valid in the domain $\mathcal{S}_{r_1,\gamma_1,\rho_2}$. The following lemma provides a domain where the new expression K of the Hamiltonian holds (recall the definition (10) for the new domain).

Lemma 2 *For suitable $\kappa, \sigma, \eta \sim 1$, with $\sigma > \pi/2$, one has $\Delta : \mathcal{B}_{\kappa,\sigma,\eta,\rho_2,\gamma_1} \longrightarrow \mathcal{S}_{r_1,\gamma_1,\rho_2}$.*

Proof. Note that the change (32) can be inverted as

$$u = \frac{g^{-1}(E)}{v_0} \cdot e^{g'(g^{-1}(E))S}, \qquad v = v_0 e^{-g'(g^{-1}(E))S}.$$

Let $|\operatorname{Re} S| \le \kappa$, $|\operatorname{Im} S| \le \sigma$, $|E| \le \eta$, and we want to see that $|u| \le \gamma_1$, $|v| \le r_1$. Since $g(0) = 0$ and $g'(0) = 1$, we can assume (taking η small enough) that, for $|E| \le \eta$,

$$\left| g^{-1}(E) \right| \le 2\,|E|\,, \qquad \left| g'\left(g^{-1}(E)\right) - 1 \right| \le 2c\,|E|\,,$$

with $c = g''(0)$. We have

$$\begin{aligned} \left|\operatorname{Re}\left[g'\left(g^{-1}(E)\right) S\right]\right| &\le \left|\operatorname{Re}\left[g'\left(g^{-1}(E)\right)\right]\right| \cdot |\operatorname{Re} S| + \left|\operatorname{Im}\left[g'\left(g^{-1}(E)\right)\right]\right| \cdot |\operatorname{Im} S| \\ &\le (1 + 2c\eta)\,\kappa + 2c\eta\sigma \le 2\kappa, \end{aligned}$$

provided $2c\eta \le \kappa/(\kappa + \sigma)$. Then,

$$\left| e^{\pm g'(g^{-1}(E))S} \right| \le e^{2\kappa} \le 2$$

if $\kappa \le (\ln 2)/2$. Finally, recalling our choice of v_0,

$$|u| \le \frac{4\eta}{v_0} \le \gamma_1, \qquad |v| \le 2v_0 = r_1,$$

if $\eta \le r_1 \gamma_1 / 8$. □

Remark. It is a technical point that we need $\sigma > \pi/2$ (for instance $\sigma = 2$); this can be reached by taking η small enough, but still ~ 1 (independent of ε, μ, δ). The reason for this requirement on σ is that we need the splitting function defined near the singularity at $\pm i\pi/2$ in order to get exponentially small lower bounds on the real domain (see Section 5).

3.2 Perturbed flow-box coordinates

Now our aim is to put the whole Hamiltonian in flow-box coordinates. In this section, we are going to carry out further symplectic transformations in order to remove *completely* the "perturbation" from the expression described in (33–34). The expression of the Hamiltonian in the new flow-box coordinates will be just K_0.

Nevertheless, a difficulty comes up when constructing this transformation. In order to get a quasiperiodic splitting function, we should also remove the term $\frac{1}{2}\langle \Lambda I, I \rangle$, appearing in (33), and coming from the unperturbed Hamiltonian H_0. This term is not "small" with respect to the perturbation parameter μ, but we overcome this difficulty by restricting the domain to a small neighbourhood of the stable whisker (given by $E = 0$, $I = 0$). The width of this neighbourhood will be $\sim \delta$ in the actions I, and we will show in Section 4.5 that the new domain contains (a piece of) both invariant manifolds if μ is small enough.

To be more precise, we carry out a previous step consisting of a symplectic transformation $\Upsilon^{(0)}$ that removes the term $\frac{1}{2}\langle \Lambda I, I\rangle$. This transformation is very simple and can be given explicitly, and the new Hamiltonian $K^{(0)} = K\circ\Upsilon^{(0)}$ can be written as $K_0 + Q^{(0)}$, where $Q^{(0)}$ is now small (with respect to μ), but the domain in the actions I shrinks to a width $\sim \delta$.

In a further step, we are going to obtain a symplectic transformation (near to the identity) removing the small perturbation $Q^{(0)}$. This transformation is constructed using a standard iterative process, based in Lie series, quite simple because there do not appear small divisors. In fact, we do not use at all in this section that the frequency vector ω satisfies the Diophantine condition (H2) of Section 1.1.

It has to be stressed that the flow-box coordinates we obtain are global in the angles φ. The following two results describe the two normalizing steps.

Proposition 3 *Let $K = K_0 + \frac{1}{2}\langle \Lambda I, I\rangle + Q$ as in (33–34), analytic on $\mathcal{B}_{\kappa,\sigma,\eta,\rho_2,\gamma_1}$. Then, there exists a symplectic map $\Upsilon^{(0)} : \mathcal{B}_{\kappa,\sigma,\eta_1,\rho_3,\beta\delta} \longrightarrow \mathcal{B}_{\kappa,\sigma,\eta,\rho_2,\gamma_1}$, with $\beta \sim 1$, such that*

$$K^{(0)} = K\circ\Upsilon^{(0)} = K_0 + Q^{(0)}, \qquad Q^{(0)} = \mathcal{O}_2(E,I), \tag{36}$$

and one has

$$\left\| Q^{(0)} \right\|_{\kappa,\sigma,\eta_1,\rho_3,\beta\delta} \le \|Q\|_{\kappa,\sigma,\eta,\rho_2,\gamma_1}. \tag{37}$$

Proof. We define the map

$$\Upsilon^{(0)}(S,E,\varphi,I) = \left(S, E - \frac{1}{2b}\langle \Lambda I, I\rangle, \varphi + \frac{S}{b}\Lambda I, I \right), \tag{38}$$

clearly symplectic because it is the time-one flow of the Hamiltonian $U_0 = \frac{S}{2b}\langle \Lambda I, I\rangle$. We obtain the expression (36), with $Q^{(0)} = Q\circ\Upsilon^{(0)} = \mathcal{O}_2(E,I)$.

Concerning the domains, note that we can assume $b \ge 1/2$ and then

$$\left| \frac{1}{2b}\langle \Lambda I, I\rangle \right| \le |\Lambda|(\beta\delta)^2 \le \eta - \eta_1, \qquad \left| \frac{S}{b}\Lambda I \right| \le 2(\kappa+\sigma)|\Lambda|\beta\delta \le \delta, \tag{39}$$

provided we choose β small enough, but ~ 1 (independent of ε, μ, δ).

Finally, the norm of $Q^{(0)}$ is easily bounded. Taking the Fourier expansion $Q = \sum_{k\in\mathbb{Z}^n} Q_k(S,E,I) e^{i\langle k,\varphi\rangle}$ and analogously for $Q^{(0)}$, we have

$$Q_k^{(0)}(S,E,I) = Q_k\left(S, E - \frac{1}{2b}\langle \Lambda I, I\rangle, I \right) e^{i\frac{S}{b}\langle k,\Lambda I\rangle},$$

and we deduce from (39) that

$$\left| Q_k^{(0)} \right|_{\kappa,\sigma,\eta_1,\beta\delta} \le |Q_k|_{\kappa,\sigma,\eta,\gamma_1} \cdot e^{|k|\delta},$$

and then

$$\left\| Q^{(0)} \right\|_{\kappa,\sigma,\eta_1,\rho_3,\beta\delta} \le \sum_{k\in\mathbb{Z}^n} |Q_k|_{\kappa,\sigma,\eta,\gamma_1} \cdot e^{|k|(\delta+\rho_3)} = \|Q\|_{\kappa,\sigma,\eta,\rho_2,\gamma_1}.$$

□

Remarks.

1. To have a reduction δ of the complex domain in the angles φ, we have accepted a drastic reduction of domain in the actions I. However, this is not important because for μ small enough the reduced domain still contains a neighbourhood of both whiskers, as shown in Proposition 9.
2. According to this proposition, an "anisochronous" system with equations $\dot S = b$, $\dot\varphi = \tilde\omega_\varepsilon + \Lambda I$ can be taken into an "isochronous" system $\dot S = b$, $\dot\varphi = \tilde\omega_\varepsilon$. At a first view, this fact can seem strange, but it is due to that the flow-box coordinates are defined for (S,φ) in (a subset of) the cylinder $\mathbb{R}\times\mathbb{T}^n$, and the invariant torus (at $S=\pm\infty$) has been excluded from the domain of these coordinates.

Theorem 4 (flow-box theorem) *Let $K^{(0)}$ as in (36), analytic on $\mathcal{B}_{\kappa,\sigma,\eta_1,\rho_3,\beta\delta}$. Assume that*

$$\left\| Q^{(0)} \right\|_{\kappa,\sigma,\eta_1,\rho_3,\beta\delta} \preceq \delta^2. \tag{40}$$

Then, there exists a symplectic map $\Upsilon : \mathcal{B}_{\kappa_1,\sigma_1,\eta_2,\rho_4,\beta_1\delta} \longrightarrow \mathcal{B}_{\kappa,\sigma,\eta_1,\rho_3,\beta\delta}$, with $\Upsilon - \mathrm{id} = \mathcal{O}(E, I)$, such that

$$K^{(0)} \circ \Upsilon = K_0,$$

and the following bound holds:

$$|\Upsilon - \mathrm{id}|_{\kappa_1,\sigma_1,\eta_2,\rho_4,\beta_1\delta} \preceq \left\| Q^{(0)} \right\|_{\kappa,\sigma,\eta_1,\rho_3,\beta\delta} \cdot \left(1, \frac{1}{\delta}\right). \tag{41}$$

The proof of this result involves a standard convergent iterative process, based in Lie series, and we defer it to Appendix B.

Remark. We see from (37), (34) and (31) that

$$\left\| Q^{(0)} \right\|_{\kappa,\sigma,\eta_1,\rho_3,\beta\delta} \le \|Q\|_{\kappa,\sigma,\eta,\rho_2,\gamma_1} \le \|R\|_{r_1,\gamma_1,\rho_2} \preceq \frac{\mu}{\delta^{p_3+1}}, \tag{42}$$

and hence condition (40) can be written as

$$\mu \preceq \delta^{p_3+3}. \tag{43}$$

4 The perturbed whiskers and the splitting function

4.1 Local whiskers

First, we translate the initial torus and its local unperturbed whiskers to the hyperbolic coordinates (u, v, φ, I), introduced in Section 2.1. Recall that, in these coordinates, G_0 is the unperturbed Hamiltonian and $G = G_0 + \mu G_1$ is the perturbed one. The unperturbed torus is given by the parameterization

$$\mathcal{T}_0 : \qquad Y_0^*(\theta) = (0, 0, \theta, 0), \quad |\mathrm{Im}\,\theta| \le \rho_1. \tag{44}$$

Its stable whisker can be locally parameterized by

$$\mathcal{W}_0^+ : \qquad Y_0^+(s,\theta) = (0, Ae^{-s}, \theta, 0), \quad \mathrm{Re}\,s \ge q_0,\ |\mathrm{Im}\,\theta| \le \rho_1, \tag{45}$$

where the constant $A > 0$ is such that $(x_0(s), y_0(s)) = \Gamma_0(0, Ae^{-s})$, and hence $Z_0(s,\theta) = \Gamma\left(Y_0^+(s,\theta)\right)$. In fact, we see from (22) that $A = 4\sqrt{2}$. We can also take $q_0 = \ln(A/r)$ (and then one has $Y_0^+(s,\theta) \in \mathcal{S}_{r,\gamma,\rho_1}$, the domain of G_0). Analogously, we parameterize the unperturbed unstable whisker by

$$\mathcal{W}_0^- : \qquad Y_0^-(s,\theta) = (Ae^{s}, 0, \theta, 0), \quad \mathrm{Re}\,s \le -q_0,\ |\mathrm{Im}\,\theta| \le \rho_1 \tag{46}$$

(with the same A and q_0). Note that the inner flow of G_0 on these whiskers is given by $\dot{s} = 1$, $\dot{\theta} = \omega_\varepsilon$.

According to Theorem 1, the perturbed Hamiltonian in the normal form coordinates is $\widetilde{G} = G \circ \Phi$ (see (27)). It has exactly the same torus, and the local whiskers as in the parameterizations (45–46). The only difference is that the inner flow associated to $\widetilde{G}$ becomes $\dot{s} = b$, $\dot{\theta} = \tilde{\omega}_\varepsilon$, and the parameterizations have to be slightly restricted:

$$\pm\,\mathrm{Re}\,s \ge q_1, \quad |\mathrm{Im}\,\theta| \le \rho_2,$$

where $q_1 = \ln(A/r_1)$ (in order to have $Y_0^\pm(s,\theta) \in \mathcal{S}_{r_1,\gamma_1,\rho_2}$, the domain of $\widetilde{G}$).

Now, we can translate the parameterizations of the perturbed torus and the local perturbed whiskers, from the normal form coordinates to the hyperbolic coordinates (using $\widetilde{G} = G \circ \Phi$) and the original coordinates (using $G = H \circ \Gamma$):

$$\begin{aligned}
\mathcal{T} : \qquad & Y^*(\theta) = \Phi\left(Y_0^*(\theta)\right), \\
& Z^*(\theta) = \Gamma\left(Y^*(\theta)\right), \quad |\mathrm{Im}\,\theta| \le \rho_2, \\
\mathcal{W}^\pm_{\mathrm{loc}} : \qquad & Y^\pm(s,\theta) = \Phi\left(Y_0^\pm(s,\theta)\right), \\
& Z^\pm(s,\theta) = \Gamma\left(Y^\pm(s,\theta)\right), \quad \pm\,\mathrm{Re}\,s \ge q_1,\ |\mathrm{Im}\,\theta| \le \rho_2.
\end{aligned} \tag{47}$$

In components, we shall write $Z^\pm = \left(Z_x^\pm, Z_y^\pm, Z_\varphi^\pm, Z_I^\pm\right)$ and similarly for the other parameterizations.

The following lemma provides an asymptotic formula for the local whiskers $\mathcal{W}_{\text{loc}}^\pm$ near the torus $\mathcal{T}$. It includes also bounds for the distance from the perturbed objects $\mathcal{W}_{\text{loc}}^\pm$ and $\mathcal{T}$ to the unperturbed ones $\mathcal{W}_0$ and $\mathcal{T}_0$ respectively. A much simpler statement is also given for the case of a fixed torus.

We state this lemma in terms of the hyperbolic coordinates, because it gives local information around the whiskered torus. It only concerns the stable whisker, but it is completely analogous for the unstable one. We write $q_2 = \ln(A/r_2)$.

Lemma 5 *For any* $\operatorname{Re} s \geq q_2$, $|\operatorname{Im}\theta| \leq \rho_2$, *the parameterizations defined above satisfy the bounds:*

(a) $\left|Y^+(s,\theta) - Y_0^+(s,\theta) - Y^*(\theta) + Y_0^*(\theta)\right| \preceq \dfrac{e^{-\operatorname{Re} s}\mu}{\delta^{p_3}} \cdot \left(1, \dfrac{1}{\delta}\right).$

(b) $\left|Y^+(s,\theta) - Y_0^+(s,\theta)\right| \preceq \dfrac{\mu}{\delta^{p_3}} \cdot \left(1, \dfrac{1}{\delta}\right).$

(c) $\left|Y^*(\theta) - Y_0^*(\theta)\right| \preceq \dfrac{\mu}{\delta^{p_3}} \cdot \left(1, \dfrac{1}{\delta}\right).$

In the case of a fixed torus, one has $Y^*(\theta) = Y_0^*(\theta)$ *and hence bounds (a) and (b) can be written together as*

(a') $\left|Y^+(s,\theta) - Y_0^+(s,\theta)\right| \preceq \dfrac{e^{-\operatorname{Re} s}\mu}{\delta^{p_3}} \cdot \left(1, \dfrac{1}{\delta}\right).$

Proof. First, we write

$$\begin{aligned} Y^+(s,\theta) - Y_0^+(s,\theta) - Y^*(\theta) + Y_0^*(\theta) &= \mu\left[\Phi_1\left(Y_0^+(s,\theta)\right) - \Phi_1\left(Y_0^*(\theta)\right)\right] \\ &= \mu\left[\Phi_1(0, Ae^{-s}, \theta, 0) - \Phi_1(0,0,\theta,0)\right], \end{aligned}$$

where we have written $\Phi = \mathrm{id} + \mu\Phi_1$. Now, taking a partial derivative we get

$$|\mu\partial_v\Phi_1|_{r_2,\gamma_2,\rho_2} \preceq |\Phi - \mathrm{id}|_{r_1,\gamma_1,\rho_2}$$

(due to that the reduction of domain is ~ 1; see Section 1.5), and applying (29) we obtain part (a), with the slight additional restriction that $\operatorname{Re} s \geq q_2$.

To prove (b), we write

$$Y^+(s,\theta) - Y_0^+(s,\theta) = \mu\Phi_1\left(Y_0^+(s,\theta)\right),$$

and we use (29) again. We obtain (c) similarly.

Finally, in the case of a fixed torus we have $Y^*(\theta) = Y_0^*(\theta)$, and bound (a') is then an obvious consequence of (a). □

Remark. These bounds are of the same order if we express them in the original coordinates. Using that $|D\Gamma| \preceq 1$ (because Γ is defined independently of ε, μ, δ) we can obtain, for instance,

$$\left|Z^+(s,\theta) - Z_0(s,\theta)\right| \preceq \frac{\mu}{\delta^{p_3}} \cdot \left(1, \frac{1}{\delta}\right), \qquad \operatorname{Re} s \geq q_2,\ |\operatorname{Im}\theta| \leq \rho_2. \tag{48}$$

4.2 Global whiskers and the extension theorem

Now, we extend the parameterizations of the whiskers $\mathcal{W}_{\text{loc}}^\pm$, valid for $\pm\operatorname{Re} s \geq q_2$, to further values of s using the fact that the whiskers are formed by trajectories of our Hamiltonian H. We denote $\mathcal{W}^\pm$ the *global whiskers* obtained in this way.

The extension theorem given below provides a bound for the distance between the global perturbed whiskers and the unperturbed homoclinic one. As one can expect, this bound is worse than (48) because of the closeness of trajectories to the complex singularity at $s = \pm i\pi/2$. Previous results for somewhat simpler but similar situations are given in [DS97, DGJS97, RW98].

The initial parameterization (5–6) of the global unperturbed homoclinic whisker $\mathcal{W}_0$ can be defined in a complex strip of the parameters s, θ:

$$\operatorname{Re} s \in \mathbb{R}, \qquad |\operatorname{Im} s| < \frac{\pi}{2}, \qquad |\operatorname{Im} \theta| < \rho.$$

The reduction of the widths of this complex strip has to be carefully controlled, in order to get exponentially small estimates for the splitting. Recall that we write $\rho_j = \rho - j\delta$, $j \geq 1$ (see Section 1.5). Here, we shall also use the notation $\nu_j = \pi/2 - j\delta$ for the successively reduced width of $|\operatorname{Im} s|$. We consider δ as a free parameter ($\delta \ll \rho$ and $\delta \ll \pi/2$) to be chosen later.

Again, this theorem is established here for the stable whisker but, of course, a symmetric result can also be given for the unstable whisker. In the stable case, we extend the parameterization from $\operatorname{Re} s \geq q_2$ to $\operatorname{Re} s \geq -q_*$, where q_* is a fixed value such that

$$q_2 < q_* \sim 1.$$

Theorem 6 (extension theorem) *Assume condition (26) and also that*

$$\mu \preceq \delta^{p_4+2}, \quad \mu \preceq \delta^{p_2+1}\sqrt{\varepsilon}. \tag{49}$$

Then, the parameterization $Z^+(s,\theta)$ of $\mathcal{W}^+$ can be extended to

$$\operatorname{Re} s \geq -q_*, \qquad |\operatorname{Im} s| \leq \nu_1, \qquad |\operatorname{Im} \theta| \leq \rho_3,$$

and with the exponents p_i defined in (14–17), one has the bounds:

(a) $\left|Z^+(s,\theta) - Z_0(s,\theta)\right| \preceq \left(\frac{\mu}{\delta^{p_5}}, \frac{\mu}{\delta^{p_4}} + \frac{\mu}{\delta^{p_2}\sqrt{\varepsilon}}\right).$

(b) $$\left|\int_0^T \left[\partial_\varphi H_1\left(Z^+\left(s+bt, \theta+\tilde{\omega}_\varepsilon t\right)\right) - \partial_\varphi H_1\left(Z_0\left(s+bt,\theta+\tilde{\omega}_\varepsilon t\right)\right)\right] \mathrm{d}t\right|$$
$$\preceq \frac{1}{\delta^{2l+\alpha+1}}\left(\frac{\mu}{\delta^{p_4}} + \frac{\mu}{\delta^{p_2}\sqrt{\varepsilon}}\right), \text{ for any } 0 < T \preceq 1.$$

The proof relies essentially on Gronwall estimates and is straightforward. However, it involves some technicalities like a good choice of the solutions of the variational equations close to the singularities of the homoclinic whisker $Z_0(s,\theta)$, so it is deferred to Appendix A.

4.3 The Melnikov potential and the Melnikov function

It is well-known that the Poincaré–Melnikov method provides a first order approximation in μ for the splitting between the global whiskers. In our case, the unperturbed Hamiltonian H_0 decouples in a pendulum and n fast rotators (a more general situation is considered in [DG00]). Then, the *Melnikov function* can be defined as the following absolutely convergent integral:

$$M(s,\theta) = -\int_{-\infty}^{\infty} \left[\partial_\varphi H_1\left(Z_0\left(s+bt,\theta+\tilde{\omega}_\varepsilon t\right)\right) - \partial_\varphi H_1\left(Z_0^*\left(\theta+\tilde{\omega}_\varepsilon t\right)\right)\right] \mathrm{d}t, \tag{50}$$

which is analytic for $|\operatorname{Im} s| < \pi/2$, $|\operatorname{Im} \theta| < \rho$. As pointed out in [DG00], the (vector) Melnikov function is the gradient with respect to θ of a scalar function, called the *Melnikov potential*: $M(s,\theta) = \partial_\theta L(s,\theta)$, where

$$L(s,\theta) = -\int_{-\infty}^{\infty} \left[H_1\left(Z_0\left(s+bt,\theta+\tilde{\omega}_\varepsilon t\right)\right) - H_1\left(Z_0^*\left(\theta+\tilde{\omega}_\varepsilon t\right)\right)\right] \mathrm{d}t + \text{const}$$

(the constant can be chosen in such a way that L has zero average with respect to θ; this average does not depend on s). The Melnikov potential L is very useful because its nondegenerate critical points (with respect to θ) give rise to simple zeroes of M and, for μ small enough, to transverse homoclinic orbits.

It is an important point the quasiperiodicity in s of both the functions M and L. More precisely, defining

$$\hat{\omega}_\varepsilon = \frac{\tilde{\omega}_\varepsilon}{b} = \frac{b'\omega_\varepsilon}{b} = \frac{b'\omega}{b\sqrt{\varepsilon}}, \tag{51}$$

these functions are $\hat{\omega}_\varepsilon$-quasiperiodic (this notion has been defined in (13)). Indeed, one easily checks the equality $L(s,\theta) = L(0, \theta - \hat{\omega}_\varepsilon s)$ for any real s, and therefore also for complex s (using analytic prolongation).

As a slight difference with respect to [DG00], we stress that the perturbed frequencies and Lyapunov exponent ($\tilde{\omega}_\varepsilon$ and b) have been introduced in the formulas, instead of the unperturbed ones (ω_ε and 1). In this way, the Melnikov function and the splitting function will have the same quasiperiodicity vector $\hat{\omega}_\varepsilon$. Then, the "error term" that we define in (70) will also be quasiperiodic and it is possible to obtain exponentially small estimates for it (see Lemma 11).

Note that, because of the particular form (3) of the perturbation, the Melnikov potential and function can be written as

$$\begin{aligned} L(s,\theta) &= -\int_{-\infty}^{\infty} [h(x_0(s+bt)) - h(0)] \cdot f(\theta + \tilde{\omega}_\varepsilon t)\, \mathrm{d}t + \text{const}, \\ M(s,\theta) &= -\int_{-\infty}^{\infty} [h(x_0(s+bt)) - h(0)] \cdot \partial_\varphi f(\theta + \tilde{\omega}_\varepsilon t)\, \mathrm{d}t. \end{aligned} \tag{52}$$

Next we provide an upper bound for the Melnikov function, to be used later.

Lemma 7 *For any (s,θ) with $|\mathrm{Im}\, s| \le \nu_1$, $|\mathrm{Im}\, \theta| \le \rho_2$, one has in (52) the bound $|M(s,\theta)| \preceq 1/\delta^{2l+\alpha}$.*

Proof. We first use hypothesis (H3), which says that $h(x)$ is a trigonometric polynomial of degree l and hence $h(x_0(s))$ has poles of order $2l$ at $s = \pm i\pi/2$ (this comes from the equality $e^{ix_0(s)} = (i - \sinh s)^2/\cosh^2 s$). This implies, for $|\mathrm{Im}\, s| \le \nu_1 = \pi/2 - \delta$, the bound

$$\int_{-\infty}^{\infty} |h(x_0(s+bt)) - h(0)|\, \mathrm{d}t \preceq \frac{1}{\delta^{2l-1}},$$

which can be deduced as a consequence of Lemma 15 in Appendix A. On the other hand, we deduce from hypothesis (H4) that $|\partial_\varphi f|_{\rho_2} \preceq 1/\delta^{\alpha+1}$, and we then obtain the expected bound for $|M(s,\theta)|$. □

4.4 A first order approximation for the splitting

In this section we show that, for μ small enough, the Melnikov function M gives a first order approximation for the splitting between the global whiskers $\mathcal{W}^\pm$ (in the original coordinates) providing, in addition, an estimate for the remainder (often called the "error term") in terms of the loss δ of complex domain.

Recall that, in Theorem 6, the parameterizations $Z^+(s,\theta)$ and $Z^-(s,\theta)$ have been extended to $\mathrm{Re}\, s \ge -q_*$ and $\mathrm{Re}\, s \le q_*$ respectively. Hence their splitting can be measured for $|\mathrm{Re}\, s| \le q_*$. In fact, we measure the splitting in the (n-dimensional) I-direction, considering the I-component of the parameterizations: $Z_I^-(s,\theta) - Z_I^+(s,\theta)$.

Proposition 8 *Consider the parameterizations $Z^\pm(s,\theta)$ as in Theorem 6, and assume conditions (26) and (49). Then, for*

$$|\mathrm{Re}\, s| \le q_*, \qquad |\mathrm{Im}\, s| \le \nu_1, \qquad |\mathrm{Im}\, \theta| \le \rho_3,$$

and with the exponents p_i defined in (14–16), one has the bound

$$\left| Z_I^-(s,\theta) - Z_I^+(s,\theta) - \mu M(s,\theta) \right| \preceq \frac{1}{\delta^{2l+\alpha+1}} \left(\frac{\mu^2}{\delta^{p_4}} + \frac{\mu^2}{\delta^{p_2}\sqrt{\varepsilon}} \right).$$

Proof. To begin, we write

$$Z_I^-(s,\theta) - Z_I^+(s,\theta) = \left[Z_I^-(s,\theta) - Z_I^*(\theta)\right] - \left[Z_I^+(s,\theta) - Z_I^*(\theta)\right].$$

Using that $\dot I = -\mu\partial_\varphi H_1$ is one of the Hamiltonian equations associated to $H = H_0 + \mu H_1$, we get

$$\begin{aligned} Z_I^+(s,\theta) - Z_I^*(\theta) &= -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\left[Z_I^+\left(s+bt, \theta+\tilde\omega_\varepsilon t\right) - Z_I^*\left(\theta+\tilde\omega_\varepsilon t\right)\right]\mathrm{d}t \\ &= \mu\int_0^\infty \left[\partial_\varphi H_1\left(Z^+\left(s+bt,\theta+\tilde\omega_\varepsilon t\right)\right) - \partial_\varphi H_1\left(Z^*\left(\theta+\tilde\omega_\varepsilon t\right)\right)\right]\mathrm{d}t. \end{aligned}$$

Proceeding similarly with $Z_I^-(s,\theta) - Z_I^*(\theta)$, and taking into account the expression (50) of the Melnikov function, we obtain

$$\begin{aligned} & Z_I^-(s,\theta) - Z_I^+(s,\theta) - \mu M(s,\theta) \\ = & -\mu\left[\int_{-\infty}^0 \left[\partial_\varphi H_1\left(Z^-\right) - \partial_\varphi H_1\left(Z_0\right) - \partial_\varphi H_1\left(Z^*\right) + \partial_\varphi H_1\left(Z_0^*\right)\right]\mathrm{d}t\right. \\ & \left. + \int_0^\infty \left[\partial_\varphi H_1\left(Z^+\right) - \partial_\varphi H_1\left(Z_0\right) - \partial_\varphi H_1\left(Z^*\right) + \partial_\varphi H_1\left(Z_0^*\right)\right]\mathrm{d}t\right] \end{aligned} \tag{53}$$

where, for shortness, we denote

$$Z^\pm = Z^\pm\left(s+bt,\theta+\tilde\omega_\varepsilon t\right), \quad Z^* = Z^*\left(\theta+\tilde\omega_\varepsilon t\right), \tag{54}$$

$$Z_0 = Z_0\left(s+bt,\theta+\tilde\omega_\varepsilon t\right), \quad Z_0^* = Z_0^*\left(\theta+\tilde\omega_\varepsilon t\right). \tag{55}$$

Since the two integrals in (53) are analogous, we are going to bound only one of them, say the second one. We break the integral $\int_0^\infty$ in three parts, $\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3$, defined as follows, with a suitable $t_0(s)$:

$$\begin{aligned} \mathcal{I}_1 &= -\int_0^{t_0(s)} \left[\partial_\varphi H_1\left(Z^*\right) - \partial_\varphi H_1\left(Z_0^*\right)\right]\mathrm{d}t, \\ \mathcal{I}_2 &= \int_0^{t_0(s)} \left[\partial_\varphi H_1\left(Z^+\right) - \partial_\varphi H_1\left(Z_0\right)\right]\mathrm{d}t, \\ \mathcal{I}_3 &= \int_{t_0(s)}^\infty \left[\partial_\varphi H_1\left(Z^+\right) - \partial_\varphi H_1\left(Z_0\right) - \partial_\varphi H_1\left(Z^*\right) + \partial_\varphi H_1\left(Z_0^*\right)\right]\mathrm{d}t, \end{aligned}$$

and our aim is to apply the bounds of Lemma 5 to the integrals $\mathcal{I}_1$ and $\mathcal{I}_3$, and Theorem 6 to the integral $\mathcal{I}_2$. Recalling that $|\operatorname{Re} s| \le q_*$ and that we consider $q_* > q_2$, we choose

$$t_0(s) = \begin{cases} (q_2 - \operatorname{Re} s)/b & \text{if } \operatorname{Re} s < q_2, \\ 0 & \text{if } \operatorname{Re} s \ge q_2, \end{cases}$$

so that, in the integral $\mathcal{I}_3$, we have $\operatorname{Re} s + bt \le q_2$. Note that $t_0(s) \le (q_2 + q_*)/b \preceq 1$.

Among the three integrals, the second one will have the largest bound, which comes from a direct application of part (b) of Theorem 6:

$$|\mathcal{I}_2| \preceq \frac{1}{\delta^{2l+\alpha+1}}\left(\frac{\mu}{\delta^{p_4}} + \frac{\mu}{\delta^{p_2}\sqrt{\varepsilon}}\right).$$

To bound the integrals $\mathcal{I}_1$ and $\mathcal{I}_3$, we first write them in terms of the parameterizations 'Y' introduced in (44–45) and (47):

$$\begin{aligned} \mathcal{I}_1 &= -\int_0^{t_0(s)} \left[F\left(Y^*\right) - F\left(Y_0^*\right)\right]\mathrm{d}t, \\ \mathcal{I}_3 &= \int_{t_0(s)}^\infty \left[F\left(Y^+\right) - F\left(Y_0^+\right) - F\left(Y^*\right) + F\left(Y_0^*\right)\right]\mathrm{d}t, \end{aligned}$$

where we denote $F = \partial_\varphi H_1 \circ \Gamma = \partial_\varphi G_1$ (see Section 2.1), and Y^+, Y_0^+, Y^*, Y_0^* are analogous to (54–55). We have $Y_0^+, Y_0^* \in \mathcal{S}_{r_2,\gamma_2,\rho_3}$ and, since bounds (b) and (c) of Lemma 5 are $\preceq \delta$, we also have $Y^+, Y^* \in \mathcal{S}_{r_1,\gamma_1,\rho_2}$. The results of

Lemma 5 can be applied replacing s by $s+bt$, and provide the following three bounds:

$$|Y^* - Y_0^*| \preceq \frac{\mu}{\delta^{p_3}} \cdot \left(1, \frac{1}{\delta}\right), \qquad \left|Y^+ - Y_0^+\right| \preceq \frac{\mu}{\delta^{p_3}} \cdot \left(1, \frac{1}{\delta}\right),$$
$$\left|Y^+ - Y_0^+ - Y^* + Y_0^*\right| \preceq \frac{e^{-(\operatorname{Re} s+bt)}\mu}{\delta^{p_3}} \cdot \left(1, \frac{1}{\delta}\right).$$

Note also from (44–45) that

$$\left|Y_0^+ - Y_0^*\right| \preceq e^{-(\operatorname{Re} s+bt)} \cdot (1,0).$$

We are also going to use the next bounds, easily deduced from (25):

$$|\mathrm{D}F|_{r_1,\gamma_1,\rho_2} \preceq \frac{1}{\delta^{\alpha+1}} \cdot \left(1, \frac{1}{\delta}\right), \qquad \left|\mathrm{D}^2 F\right|_{r_1,\gamma_1,\rho_2} \preceq \frac{1}{\delta^{\alpha+1}} \begin{pmatrix} 1 & 1/\delta \\ 1/\delta & 1/\delta^2 \end{pmatrix}, \tag{56}$$

where the elements of the matrix give bounds for the second partial derivatives (analogously to the notation of Section 1.5).

Using these ingredients, the integral $\mathcal{I}_1$ is easily bounded:

$$|\mathcal{I}_1| \preceq \left\langle |\mathrm{D}F|_{r_1,\gamma_1,\rho_2}, |Y^* - Y_0^*| \right\rangle \preceq \frac{\mu}{\delta^{p_3+\alpha+3}}.$$

The integral $\mathcal{I}_3$ requires some extra work. First, we write

$$\begin{aligned}
&F\left(Y^+\right) - F\left(Y_0^+\right) - F\left(Y^*\right) + F\left(Y_0^*\right) \\
&= \int_0^1 \left\langle \mathrm{D}F\left(Y_0^+ + u\left(Y^+ - Y_0^+\right)\right), Y^+ - Y_0^+\right\rangle \mathrm{d}u \\
&\quad - \int_0^1 \left\langle \mathrm{D}F\left(Y_0^* + u\left(Y^* - Y_0^*\right)\right), Y^* - Y_0^*\right\rangle \mathrm{d}u \\
&= \int_0^1 \left\langle \mathrm{D}F\left(Y_0^+ + u\left(Y^+ - Y_0^+\right)\right) - \mathrm{D}F\left(Y_0^* + u\left(Y^* - Y_0^*\right)\right), Y^+ - Y_0^+\right\rangle \mathrm{d}u \\
&\quad + \int_0^1 \left\langle \mathrm{D}F\left(Y_0^* + u\left(Y^* - Y_0^*\right)\right), Y^+ - Y_0^+ - Y^* + Y_0^*\right\rangle \mathrm{d}u.
\end{aligned}$$

Then, using the previous bounds we obtain:

$$\begin{aligned}
&\left|F\left(Y^+\right) - F\left(Y_0^+\right) - F\left(Y^*\right) + F\left(Y_0^*\right)\right| \\
&\preceq \left\langle \left|\mathrm{D}^2 F\right|_{r_1,\gamma_1,\rho_2} \cdot \left(\left|Y_0^+ - Y_0^*\right| + \left|Y^+ - Y_0^+ - Y^* + Y_0^*\right|\right), \left|Y^+ - Y_0^+\right| \right\rangle \\
&\quad + \left\langle |\mathrm{D}F|_{r_1,\gamma_1,\rho_2}, \left|Y^+ - Y_0^+ - Y^* + Y_0^*\right| \right\rangle \\
&\preceq \frac{e^{-(\operatorname{Re} s+bt)}\mu}{\delta^{p_3+\alpha+3}},
\end{aligned} \tag{57}$$

and then

$$|\mathcal{I}_3| \preceq \frac{\mu}{\delta^{p_3+\alpha+3}}.$$

Note that it is essential to have the exponential factor in (57), because this bound has to be integrated over an infinite interval.

Now we compare the bounds obtained for $|\mathcal{I}_1|$, $|\mathcal{I}_2|$ and $|\mathcal{I}_3|$. Using that $p_4 + 2l \geq p_3 + 2$, we see that $|\mathcal{I}_2|$ has the largest bound (coming from the extension theorem), which gives the expected bound for (53). □

Remark. The case of fixed tori is a bit simpler, because in this case it is enough to write the integral $\int_0^\infty$ in (53) as $\mathcal{I}_1' + \mathcal{I}_2'$, with

$$\mathcal{I}_1' = \int_0^{t_0(s)} \left[\partial_\varphi H_1\left(Z^+\right) - \partial_\varphi H_1\left(Z_0\right)\right] \mathrm{d}t,$$
$$\mathcal{I}_2' = \int_{t_0(s)}^\infty \left[\partial_\varphi H_1\left(Z^+\right) - \partial_\varphi H_1\left(Z_0\right)\right] \mathrm{d}t.$$

However, no extra improvement in the bounds is obtained.

4.5 The splitting distance in the flow-box coordinates

Next, we translate the parameterizations of the whiskers to the flow-box coordinates (S, E, φ, I) introduced in Section 3. Recall that our Hamiltonian, in the flow-box coordinates, becomes

$$K_0 = H \circ \Psi = \widetilde{G} \circ \Delta \circ \Upsilon^{(0)} \circ \Upsilon = \langle \tilde{\omega}_\varepsilon, I \rangle + bE,$$

on the domain $\mathcal{B}_{\kappa_1,\sigma_1,\eta_2,\rho_4,\beta_1\delta}$. We have denoted

$$\Psi = \Gamma \circ \Phi \circ \Delta \circ \Upsilon^{(0)} \circ \Upsilon \tag{58}$$

the whole transformation from the original coordinates to the flow-box coordinates.

The domain of K_0 contains a piece of the *local stable whisker* $\mathcal{W}^+_{\mathrm{loc}}$, which can be parameterized as

$$W^+_0(s,\theta) = (s, 0, \theta, 0), \qquad (s,\theta) \in \mathcal{P}_{\kappa_1,\sigma_1,\rho_4} \tag{59}$$

(recall the notation (12) for the domain of the parameters (s,θ)). The inner flow on this whisker is given by $\dot{s} = b$, $\dot{\theta} = \tilde{\omega}_\varepsilon$. It is easy to establish the relation between this parameterization and the one introduced in (47):

$$\Psi\left(W^+_0(s,\theta)\right) = Z^+(s+s_0,\theta),$$

This comes from the following steps:

$$\begin{aligned} W^+_0(s,\theta) &\overset{\Upsilon}{\longmapsto} W^+_0(s,\theta) \overset{\Upsilon^{(0)}}{\longmapsto} W^+_0(s,\theta) \overset{\Delta}{\longmapsto} Y^+_0(s+s_0,\theta) \overset{\Phi}{\longmapsto} Y^+(s+s_0,\theta) \\ &\overset{\Gamma}{\longmapsto} Z^+(s+s_0,\theta), \end{aligned} \tag{60}$$

where we have used that $\Delta\left(W^+_0(s,\theta)\right) = (0, v_0 e^{-s}, \theta, 0) = Y^+_0(s+s_0,\theta)$; in this way we define

$$s_0 = \ln(A/v_0) \tag{61}$$

(recall that in (45) we introduced the constant $A = 4\sqrt{2}$) as a translation between both parameterizations, in order to have $W^+_0(s,\theta)$ defined around $s = 0$. Recall also that $\Upsilon^{(0)}$ and Υ leave this whisker unchanged, since these maps are $\mathrm{id} + \mathcal{O}(E, I)$.

According to the extension theorem (Section 4.2), a large piece of the *global unstable whisker* $\mathcal{W}^-$ remains close to the stable one for μ small enough. Then, it is natural to define the following parameterization for the part of this whisker that enters in the domain of the flow-box coordinates:

$$W^-(s,\theta) := \Psi^{-1}\left(Z^-(s+s_0,\theta)\right). \tag{62}$$

In components, we shall write $W^- = \left(W^-_S, W^-_E, W^-_\varphi, W^-_I\right)$. We are going to establish in Proposition 9 that this parameterization can be defined essentially in the domain given in Theorem 6 and Proposition 8 for the parameters s and θ, with a reduction of only $\sim \delta$ in their imaginary parts. Note that this requires to take into account the drastic reduction of domain for the flow-box coordinates, in the I-direction, that we had to carry out in Proposition 3.

Comparing the I-components in (59) and (62), we can consider the function $W^-_I(s,\theta)$ as giving the splitting distance. Our use of the flow-box coordinates implies that this function is $\hat{\omega}_\varepsilon$-quasiperiodic (recall that $\hat{\omega}_\varepsilon$ is defined in (51)). The next theorem provides a bound for this function, and also says that the approximation given in Proposition 8, in terms of the Melnikov function M, remains true after changing from the original coordinates to the flow-box coordinates. In fact, since the translation (61) in the variable s has to be taken into account, we use the notation $M_{s_0}(s,\theta) = M(s+s_0,\theta)$.

Proposition 9 *Under conditions (26) and (49), the parameterization $W^-(s,\theta)$ can be defined for $(s,\theta) \in \mathcal{P}_{\kappa_2,\nu_2,\rho_5}$, and with the exponents p_i defined in (14–19), one has:*

(a) $\left|W^- - W^+_0\right|_{\kappa_2,\nu_2,\rho_5} \preceq \left(\dfrac{\mu}{\delta^{p_5}} + \dfrac{\mu}{\delta^{p_2}\sqrt{\varepsilon}}, \dfrac{\mu}{\delta^{p_4}} + \dfrac{\mu}{\delta^{p_2}\sqrt{\varepsilon}}\right).$

(b) $\left|W^-_I - \mu M_{s_0}\right|_{\kappa_2,\nu_2,\rho_5} \preceq \dfrac{\mu^2}{\delta^{p_6}} + \dfrac{\mu^2}{\delta^{p_7}\sqrt{\varepsilon}}.$

(c) $W^-(s,\theta) - W_0^+(s,\theta) = \left(W_S^- - s, W_E^-, W_\varphi^- - \theta, W_I^-\right)$ *is* $\hat\omega_\varepsilon$*-quasiperiodic.*

Proof. First, we point out that the flow-box coordinates can be defined if conditions (26) and (43) are fulfilled. Notice that condition (43), needed in Theorem 4, is included in condition (49), required here.

To prove part (a), we have to consider the effect of the successive transformations (58) on the bounds provided by Theorem 6 and Proposition 8. In order to write the successive bounds in a simple way, we introduce the following notations for the intermediate steps in (62), to be compared with (60):

$$\begin{aligned} & Z^-(s+s_0,\theta) \stackrel{\Gamma^{-1}}{\longmapsto} Y^-_{(1)}(s+s_0,\theta) \stackrel{\Phi^{-1}}{\longmapsto} Y^-_{(2)}(s+s_0,\theta) \stackrel{\Delta^{-1}}{\longmapsto} W^-_{(1)}(s,\theta) \\ & \stackrel{\left(\Upsilon^{(0)}\right)^{-1}}{\longmapsto} W^-_{(2)}(s,\theta) \stackrel{\Upsilon^{-1}}{\longmapsto} W^-(s,\theta). \end{aligned}$$

We begin with the following bound, given by Theorem 6:

$$\begin{aligned} & \left|Z^-(s+s_0,\theta) - Z^+(s+s_0,\theta)\right| \preceq \left|Z^- - Z_0\right| + \left|Z^+ - Z_0\right| \\ & \preceq \left(\frac{\mu}{\delta^{p_5}}, \frac{\mu}{\delta^{p_4}} + \frac{\mu}{\delta^{p_2}\sqrt{\varepsilon}}\right), \\ & |\mathrm{Re}(s+s_0)| \le q_*, \ |\mathrm{Im}\, s| \le \nu_1, \ |\mathrm{Im}\,\theta| \le \rho_3. \end{aligned} \tag{63}$$

Let us study the effect of the successive transformations on this bound. The key points, in order to show that the final bound does not get essentially worse than (63), are that Γ^{-1} and Δ^{-1} do not touch the I-component, and that Φ^{-1} and Υ^{-1} are close to the identity. On the other hand, the transformation $\left(\Upsilon^{(0)}\right)^{-1}$ will easily be controlled.

First, recall the form of the change to the hyperbolic coordinates: $\Gamma(u,v,\varphi,I) = (\Gamma_0(u,v),\varphi,I)$. We have $\left|\mathrm{D}\left(\Gamma_0^{-1}\right)\right| \preceq 1$ on its domain, and then from (63) we directly obtain

$$\left|Y^-_{(1)}(s+s_0,\theta) - Y^+(s+s_0,\theta)\right| \preceq \left(\frac{\mu}{\delta^{p_5}}, \frac{\mu}{\delta^{p_4}} + \frac{\mu}{\delta^{p_2}\sqrt{\varepsilon}}\right). \tag{64}$$

Next, we consider the transformation to the normal form coordinates, provided by Theorem 1. For this transformation, we easily see from (29) that $|\Phi - \mathrm{id}| \preceq (1,\delta)$ and we have $\Phi^{-1}: \mathcal{S}_{r_2,\gamma_2,\rho_3} \longrightarrow \mathcal{S}_{r_1,\gamma_1,\rho_2}$. Writing $\Phi^{-1} = \mathrm{id} + \mu\hat\Phi$ and applying now (29) to $\left|\Phi^{-1} - \mathrm{id}\right|$, we obtain

$$\left|\mu \mathrm{D}\hat\Phi\right|_{r_3,\gamma_3,\rho_4} \preceq \frac{\mu}{\delta^{p_3}} \begin{pmatrix} 1 & 1/\delta \\ 1/\delta & 1/\delta^2 \end{pmatrix}, \tag{65}$$

where we use the notation of (56). Then, in the same domain we have $\left|\mathrm{D}\left(\Phi^{-1}\right)\right| \le \mathrm{Id} + \left|\mu\mathrm{D}\hat\Phi\right|$. Applying this bound, we obtain again

$$\left|Y^-_{(2)}(s+s_0,\theta) - Y_0^+(s+s_0,\theta)\right| \preceq \left(\frac{\mu}{\delta^{p_5}}, \frac{\mu}{\delta^{p_4}} + \frac{\mu}{\delta^{p_2}\sqrt{\varepsilon}}\right).$$

The change Δ^{-1} to the unperturbed flow-box coordinates works like Γ^{-1}, and then the bound remains also unchanged:

$$\left|W^-_{(1)}(s,\theta) - W_0^+(s,\theta)\right| \preceq \left(\frac{\mu}{\delta^{p_5}}, \frac{\mu}{\delta^{p_4}} + \frac{\mu}{\delta^{p_2}\sqrt{\varepsilon}}\right).$$

Next we consider the transformation $\left(\Upsilon^{(0)}\right)^{-1}$, which can explicitly be written from (38), and the $\mathcal{O}_2(I)$-term in this transformation provides a new term in our bound. Indeed, denoting $(\cdot)_S$, $(\cdot)_I$ the S,I-components, we have

$$W^-_{(2)} - W^-_{(1)} = \left(0, \frac{1}{2b}\left\langle \Lambda\left(W^-_{(1)}\right)_I, \left(W^-_{(1)}\right)_I\right\rangle, -\frac{1}{b}\left(W^-_{(1)}\right)_S \Lambda\left(W^-_{(1)}\right)_I, 0\right)$$

and we deduce the bound

$$\left|W^-_{(2)}(s,\theta) - W_0^+(s,\theta)\right| \preceq \left(\frac{\mu}{\delta^{p_5}} + \frac{\mu}{\delta^{p_2}\sqrt{\varepsilon}}, \frac{\mu}{\delta^{p_4}} + \frac{\mu}{\delta^{p_2}\sqrt{\varepsilon}}\right). \tag{66}$$

Finally, for the transformation Υ^{-1} provided by Theorem 4, we proceed as with Φ^{-1}. So we write $\Upsilon^{-1} = \mathrm{id} + \mu\hat{\Upsilon}$ and, applying (41) and (42) to $\left|\Upsilon^{-1} - \mathrm{id}\right|$, we obtain

$$\left|\mu \mathrm{D}\hat{\Upsilon}\right| \preceq \frac{\mu}{\delta^{p_3+1}} \begin{pmatrix} 1 & 1/\delta \\ 1/\delta & 1/\delta^2 \end{pmatrix}, \tag{67}$$

and hence $\left|\mathrm{D}\left(\Upsilon^{-1}\right)\right| \leq \mathrm{Id} + \left|\mu \mathrm{D}\hat{\Upsilon}\right|$. Applying this to (66), we obtain the bound of part (a).

We also have to establish the condition on μ, for which $W^-(s,\theta)$ can actually be defined in $\mathcal{P}_{\kappa_2,\nu_2,\rho_5}$, without leaving the domain of K_0. The only important restriction comes from Proposition 3, where the domain in the actions I shrinks to a width $\sim \delta$. In order to keep $W^-_{(2)}(s,\theta) = \left(\Upsilon^{(0)}\right)^{-1}\left(W^-_{(1)}(s,\theta)\right)$ in the domain, according to (66) we clearly need

$$\frac{\mu}{\delta^{p_4}} + \frac{\mu}{\delta^{p_2}\sqrt{\varepsilon}} \preceq \delta,$$

and this condition is included in (49).

To obtain part (b), we have to relate the function W^-_I with the difference $Z^-_I - Z^+_I$ that has been estimated in Proposition 8. This is quite simple because the only two transformations that touch the I-component are Φ^{-1} and Υ^{-1}. We write $\left(\Phi^{-1}\right)_I = I + \mu\hat{\Phi}_I$ and $\left(\Upsilon^{-1}\right)_I = I + \mu\hat{\Upsilon}_I$ (the subscript $(\cdot)_I$ always denotes the I-component). Recalling that $\left(W^+_0\right)_I = 0$ and $\left(Y^+_0\right)_I = 0$, we have

$$\begin{aligned}
W^-_I(s,\theta) &= \left(W^-_{(2)}\right)_I(s,\theta) + \mu\left[\hat{\Upsilon}_I\left(W^-_{(2)}(s,\theta)\right) - \hat{\Upsilon}_I\left(W^+_0(s,\theta)\right)\right] \\
&= \left(W^-_{(1)}\right)_I(s,\theta) + \mu\left[\hat{\Upsilon}_I\left(W^-_{(2)}(s,\theta)\right) - \hat{\Upsilon}_I\left(W^+_0(s,\theta)\right)\right] \\
&= \left(Y^-_{(2)}\right)_I(s,\theta) + \mu\left[\hat{\Upsilon}_I\left(W^-_{(2)}(s,\theta)\right) - \hat{\Upsilon}_I\left(W^+_0(s,\theta)\right)\right] \\
&= \left[\left(Y^-_{(1)}\right)_I(s+s_0,\theta) - Y^+_I(s+s_0,\theta)\right] \\
&\quad + \mu\left[\hat{\Phi}_I\left(Y^-_{(1)}(s+s_0,\theta)\right) - \hat{\Phi}_I\left(Y^+(s+s_0,\theta)\right)\right] \\
&\quad + \mu\left[\hat{\Upsilon}_I\left(W^-_{(2)}(s,\theta)\right) - \hat{\Upsilon}_I\left(W^+_0(s,\theta)\right)\right] \\
&= \left[Z^-_I(s+s_0,\theta) - Z^+_I(s+s_0,\theta)\right] \\
&\quad + \mu\left[\hat{\Phi}_I\left(Y^-_{(1)}(s+s_0,\theta)\right) - \hat{\Phi}_I\left(Y^+(s+s_0,\theta)\right)\right] \\
&\quad + \mu\left[\hat{\Upsilon}_I\left(W^-_{(2)}(s,\theta)\right) - \hat{\Upsilon}_I\left(W^+_0(s,\theta)\right)\right].
\end{aligned}$$

Now we substract $\mu M(s+s_0,\theta)$ and apply Proposition 8, and bounds (64) and (66) together with the second row of (65) and (67). We obtain statement (b):

$$\begin{aligned}
&\left|W^-_I(s,\theta) - \mu M(s+s_0,\theta)\right| \\
&\quad \preceq \frac{1}{\delta^{2l+\alpha+1}}\left(\frac{\mu^2}{\delta^{p_4}} + \frac{\mu^2}{\delta^{p_2}\sqrt{\varepsilon}}\right) + \frac{\mu^2}{\delta^{p_3+p_4+3}} + \frac{\mu^2}{\delta^{p_2+p_3+3}\sqrt{\varepsilon}} \\
&\quad \preceq \frac{\mu^2}{\delta^{p_6}} + \frac{\mu^2}{\delta^{p_7}\sqrt{\varepsilon}},
\end{aligned}$$

since $p_6 = \max(p_4 + 2l + \alpha + 1, p_3 + p_4 + 3)$ and $p_7 = \max(p_2 + 2l + \alpha + 1, p_2 + p_3 + 3)$.

Finally, part (c) is a direct consequence of the fact that the inner flow on $\mathcal{W}^-$ is given by $\dot{s} = b$, $\dot{\theta} = \tilde{\omega}_\varepsilon$, and the Hamiltonian equations (35) in the flow-box coordinates. For instance, from the equation $\dot{I} = 0$ we get that the I-component $W^-_I(s,\theta)$ is $\hat{\omega}_\varepsilon$-quasiperiodic:

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} W^-_I\left(s+bt, \theta + \tilde{\omega}_\varepsilon t\right) = b\partial_s W^-_I(s,\theta) + \left\langle \tilde{\omega}_\varepsilon, \partial_\theta W^-_I(s,\theta)\right\rangle.$$

□

4.6 The splitting potential and the splitting function

It has to be pointed out that the quasiperiodicity of the function $W_I^-(s,\theta)$, established in Proposition 9, does not lead directly to exponentially small estimates for the splitting, because an essential ingredient still missing is that the function considered has to have zero average with respect to the angles. This is closely related to the main result of [DG00], which says that, after a suitable change of parameters $(s,\theta)=\mathcal{A}\left(\tilde{s},\tilde{\theta}\right)$, the function $W_I^-(s,\theta)$ becomes the gradient of a scalar function $\mathcal{L}\left(\tilde{s},\tilde{\theta}\right)$, called *splitting potential.* The change $\mathcal{A}$ is defined as the inverse of

$$\tilde{s}=W_S^-(s,\theta), \qquad \tilde{\theta}=W_\varphi^-(s,\theta). \tag{68}$$

Then, the global unstable whisker $\mathcal{W}^-$ can be seen as a graphic over the local stable whisker $\mathcal{W}_{\rm loc}^+$ given in (59). Indeed, in the new parameters the whisker $\mathcal{W}^-$ becomes

$$\begin{aligned} W^-(s,\theta) &= \left(W_S^-(s,\theta),W_E^-(s,\theta),W_\varphi^-(s,\theta),W_I^-(s,\theta)\right) \\ &= \left(\tilde{s},\mathcal{E}\left(\tilde{s},\tilde{\theta}\right),\tilde{\theta},\mathcal{M}\left(\tilde{s},\tilde{\theta}\right)\right). \end{aligned}$$

We stress that part (c) of Proposition 9 implies that, in the new parameters, the inner flow on $\mathcal{W}^-$ is still given by $\dot{\tilde{s}}=b$, $\dot{\tilde{\theta}}=\tilde{\omega}_\varepsilon$. To study the splitting, we only need to consider the function $\mathcal{M}$, that we call the *splitting function.* The function $\mathcal{E}$ is directly related to $\mathcal{M}$ by the energy conservation, as will be seen in (72). Besides, as in [DG00] we shall have:

$$\mathcal{E}=\partial_{\tilde{s}}\mathcal{L}, \qquad \mathcal{M}=\partial_{\tilde{\theta}}\mathcal{L}. \tag{69}$$

So the function $\mathcal{M}\left(\tilde{s},\tilde{\theta}\right)$ is a reparameterization of the function $W^-(s,\theta)$, becoming now the gradient of the splitting potential. We then define the *error term* for the Poincaré–Melnikov method as the function

$$\mathcal{R}\left(\tilde{s},\tilde{\theta}\right)=\mathcal{M}\left(\tilde{s},\tilde{\theta}\right)-\mu M\left(\tilde{s}+s_0,\tilde{\theta}\right). \tag{70}$$

Theorem 10 *Under conditions (26) and (49), the splitting function $\mathcal{M}\left(\tilde{s},\tilde{\theta}\right)$ can be defined for $\left(\tilde{s},\tilde{\theta}\right)\in\mathcal{P}_{\kappa_3,\nu_3,\rho_6}$, and with the exponents p_i defined in (14–19), one has:*

(a) $|\mathcal{M}|_{\kappa_3,\nu_3,\rho_6}\preceq\dfrac{\mu}{\delta^{p_4}}+\dfrac{\mu}{\delta^{p_2}\sqrt{\varepsilon}}$.

(b) $|\mathcal{R}|_{\kappa_3,\nu_3,\rho_6}\preceq\dfrac{\mu^2}{\delta^{p_6}}+\dfrac{\mu^2}{\delta^{p_7}\sqrt{\varepsilon}}$.

(c) $\mathcal{M}=\partial_\theta\mathcal{L}$ *for a scalar and $\hat{\omega}_\varepsilon$-quasiperiodic function* $\mathcal{L}\left(\tilde{s},\tilde{\theta}\right)$.

Proof. First, using part (a) of Proposition 9, we see that the change (68) is close to the identity. More precisely, for $(s,\theta)\in\mathcal{P}_{\kappa_2,\nu_2,\rho_5}$,

$$|\tilde{s}-s|\preceq\frac{\mu}{\delta^{p_5}}+\frac{\mu}{\delta^{p_2}\sqrt{\varepsilon}}, \qquad \left|\tilde{\theta}-\theta\right|\preceq\frac{\mu}{\delta^{p_4}}+\frac{\mu}{\delta^{p_2}\sqrt{\varepsilon}}. \tag{71}$$

We see from condition (49) that these bounds are $\preceq\delta$ and we deduce that $(s,\theta)=\mathcal{A}\left(\tilde{s},\tilde{\theta}\right)$, i.e. the inverse of (68), can be defined for $\left(\tilde{s},\tilde{\theta}\right)\in\mathcal{P}_{\kappa_3,\nu_3,\rho_6}$. Then, part (a) is obvious from part (a) of Proposition 9 (taking only the I-component).

To establish part (b), we write

$$\mathcal{R}\left(\tilde{s},\tilde{\theta}\right)=\left[W_I^-(s,\theta)-\mu M(s+s_0,\theta)\right]+\mu\left[M(s+s_0,\theta)-M\left(\tilde{s}+s_0,\tilde{\theta}\right)\right].$$

The most significant term in this sum is the first one, and has been bounded in part (b) of Proposition 9. For the second term, we deduce from Lemma 7 the bounds

$$|\partial_s M_{s_0}|_{\kappa_1,\nu_2,\rho_2}\preceq\frac{1}{\delta^{2l+\alpha+1}}, \qquad |\partial_\theta M_{s_0}|_{\kappa,\nu_1,\rho_3}\preceq\frac{1}{\delta^{2l+\alpha+1}}$$

and, using also (71), we obtain

$$\begin{aligned}
&\left|M(s+s_0,\theta) - M\left(\tilde{s}+s_0,\tilde{\theta}\right)\right| \\
&\quad \preceq \frac{1}{\delta^{2l+\alpha+1}}\left(\frac{\mu}{\delta^{p_5}}+\frac{\mu}{\delta^{p_2}\sqrt{\varepsilon}}\right)+\frac{1}{\delta^{2l+\alpha+1}}\left(\frac{\mu}{\delta^{p_4}}+\frac{\mu}{\delta^{p_2}\sqrt{\varepsilon}}\right) \\
&\quad \preceq \frac{1}{\delta^{2l+\alpha+1}}\left(\frac{\mu}{\delta^{p_5}}+\frac{\mu}{\delta^{p_2}\sqrt{\varepsilon}}\right).
\end{aligned}$$

Putting this bound and the one of part (b) of Proposition 9 together, from the inequality $p_5+2l+\alpha+1\le p_6$ we can keep p_6 as in (18) and obtain the expected bound for $|\mathcal{R}|$.

Concerning part (c), the existence of the splitting potential $\mathcal{L}\left(\tilde{s},\tilde{\theta}\right)$ was established in [DG00], from the fact that there exists an exact symplectic map Θ (close to the identity), defined in a neighbourhood of a piece of the local stable whisker $\mathcal{W}^+_{\mathrm{loc}}$, taking this whisker into the global unstable one, $\mathcal{W}^-$. In fact, the parameterizations of these whiskers are linked by this map: $\Theta\left(Z^+(s+s_0,\theta)\right) = Z^-(s+s_0,\theta)$. This map can be moved to the flow-box coordinates: $\widehat{\Theta} = \Psi^{-1}\circ\Theta\circ\Psi$, and hence $\widehat{\Theta}\left(W_0^+(s,\theta)\right) = W^-(s,\theta)$. Nevertheless, there is a small difference between [DG00] and this paper. Indeed, in this paper one of the ingredients of Ψ is not exact in general, since the transformation Φ to normal form (provided by Theorem 1) is not exact, but rather a translation of it, $\Phi\circ T_a^{-1}$, is exact. But it is not hard to see that, even with this remark, the map $\widehat{\Theta}$ is exact. Once this is established, the existence of a splitting potential $\mathcal{L}\left(\tilde{s},\tilde{\theta}\right)$, satisfying (69), follows as in [DG00]. Its $\hat{\omega}_\varepsilon$-quasiperiodicity is a consequence of the Hamilton–Jacobi equation and the fact that both whiskers belong to the same energy level:

$$\begin{aligned}
0 &= K_0\left(\tilde{s},0,\tilde{\theta},0\right) = K_0\left(\tilde{s},\mathcal{E}\left(\tilde{s},\tilde{\theta}\right),\tilde{\theta},\mathcal{M}\left(\tilde{s},\tilde{\theta}\right)\right) \\
&= b\mathcal{E}\left(\tilde{s},\tilde{\theta}\right)+\left\langle\tilde{\omega}_\varepsilon,\mathcal{M}\left(\tilde{s},\tilde{\theta}\right)\right\rangle = b\partial_{\tilde{s}}\mathcal{L}\left(\tilde{s},\tilde{\theta}\right)+\left\langle\tilde{\omega}_\varepsilon,\partial_{\tilde{\theta}}\mathcal{L}\left(\tilde{s},\tilde{\theta}\right)\right\rangle. \qquad (72)
\end{aligned}$$

□

Remark. It is an obvious consequence of part (c) that the error term $\mathcal{R}$ is also $\hat{\omega}_\varepsilon$-quasiperiodic, since both $\mathcal{M}$ and M have the same quasiperiodicity parameter. This has motivated definition (50). It is also clear that $\mathcal{R}$ is a gradient, because so are $\mathcal{M}$ and M. These facts are used in the paper [DG02] (for a concrete example) to show that the Melnikov function M dominates the error term $\mathcal{R}$ and, consequently, gives asymptotic estimates for the splitting function $\mathcal{M}$.

5 Exponentially small estimates

5.1 Quasiperiodic functions and exponentially small estimates

The notion of a quasiperiodic function of the parameters (s,θ) has been introduced in Section 1.5. The following standard lemma allows us to deduce that the splitting function (and some related functions), which are quasiperiodic, with zero average, and analytic on a complex neighbourhood of the type (12), become exponentially small (with respect to ε) when restricted to a real domain. For simplicity, we assume in this lemma that the quasiperiodicity vector is ω_ε instead of $\hat{\omega}_\varepsilon$.

For a function $g(s,\theta)$, we consider its Fourier expansion $\sum_{k\in\mathbb{Z}^n} g_k(s)e^{i\langle k,\theta\rangle}$. Note that, if g is ω_ε-quasiperiodic, then

$$g_k(s) = g_k(0)e^{-is\langle k,\omega_\varepsilon\rangle}. \qquad (73)$$

In particular, we see that the average value g_0 does not depend on s.

We next provide a result concerning the exponential smallness in ε of an ω_ε-quasiperiodic function. This is nowadays standard (similar results are given in [DGJS97, Sau01]), but we include here the proof for the sake of completeness. Recall that the vector ω_ε is defined in (4) in terms of the initial frequency vector ω, assumed to satisfy the Diophantine condition (H2) with $\tau\ge n-1$ and $\gamma>0$. In fact, the vector ω_ε will have to be replaced by the true quasiperiodicity vector $\hat{\omega}_\varepsilon$, defined in (51).

We use the following notation for the constant appearing in the exponentials:

$$C(\nu,\rho)=\left(1+\frac{1}{\tau}\right)(\tau\nu\rho^\tau\gamma)^{1/(\tau+1)}. \tag{74}$$

Recall from Section 1.5 that $|\cdot|_{\kappa,0,0}$ is the notation for the supremum norm on the real domain: $s\in\mathbb{R}$, $|s|\le\kappa$, $\theta\in\mathbb{T}^n$.

Lemma 11 *Let $g(s,\theta)$ analytic on $\mathcal{P}_{\kappa,\nu,\rho}$ and ω_ε-quasiperiodic. One has:*

(a) $|g_k|_{\kappa,0}\le|g|_{\kappa,\nu,\rho}\,e^{-(\rho|k|+\nu|\langle k,\omega_\varepsilon\rangle|)}\quad\forall k\in\mathbb{Z}^n.$

(b) $|g-g_0|_{\kappa,0,0}\preceq\dfrac{|g|_{\kappa,\nu,\rho}}{\varepsilon^{n/(2\tau+2)}}\exp\left\{-\dfrac{C(\nu,\rho)}{\varepsilon^{1/(2\tau+2)}}\right\}.$

Proof. We easily deduce part (a) from the inequalities $|g_k|_{\kappa,\nu}\le|g|_{\kappa,\nu,\rho}\,e^{-\rho|k|}$, using also the relations (73). To prove part (b), we consider some δ to be chosen, with $0<\delta<\rho$. For given $s\in\mathbb{R}$, $|s|\le\kappa$, $\theta\in\mathbb{T}^n$, we write

$$\begin{aligned}|g(s,\theta)-g_0|&\le\sum_{k\ne0}|g_k(s)|\le|g|_{\kappa,\nu,\rho}\sum_{k\ne0}\left(e^{-(\rho_1|k|+\nu|\langle k,\omega_\varepsilon\rangle|)}\cdot e^{-\delta|k|}\right)\\&\preceq\frac{|g|_{\kappa,\nu,\rho}}{\delta^n}\exp\left\{-\frac{C(\nu,\rho_1)}{\varepsilon^{1/(2\tau+2)}}\right\},\end{aligned}$$

with $\rho_1=\rho-\delta$. We have used the bound $\sum_k e^{-\delta|k|}\preceq1/\delta^n$ and the following standard inequality, in which the Diophantine condition (H2) plays an essential rôle:

$$\rho_1|k|+\nu|\langle k,\omega_\varepsilon\rangle|\ge\rho_1|k|+\frac{\nu\gamma}{|k|^\tau\sqrt{\varepsilon}}\ge\frac{C(\nu,\rho_1)}{\varepsilon^{1/(2\tau+2)}},\qquad\forall k\ne0$$

(this inequality comes from finding the minimum of a function like $ax+bx^{-\tau}$ with respect to $x=|k|$). Finally, the choice

$$\delta=\varepsilon^{1/(2\tau+2)} \tag{75}$$

allows us to replace ρ_1 by ρ. Indeed, we have

$$\left|\frac{C(\nu,\rho)}{\varepsilon^{1/(2\tau+2)}}-\frac{C(\nu,\rho_1)}{\varepsilon^{1/(2\tau+2)}}\right|\preceq1,$$

and we deduce that

$$\exp\left\{-\frac{C(\nu,\rho_1)}{\varepsilon^{1/(2\tau+2)}}\right\}\sim\exp\left\{-\frac{C(\nu,\rho)}{\varepsilon^{1/(2\tau+2)}}\right\},$$

which implies estimate (b), because the replacement of ρ_1 by ρ only affects some constants that we do not write down. □

5.2 Exponentially small upper bounds for the splitting function

As a simple consequence of Lemma 11, we can deduce an exponentially small upper bound, on the real domain, for the splitting function $\mathcal{M}\left(\tilde{s},\tilde{\theta}\right)$. Notice that the constant $C\left(\frac{\pi}{2},\rho\right)$ in the exponential is the one introduced in (8).

Theorem 12 *Assuming*

$$\varepsilon\preceq1,\qquad\mu\preceq\varepsilon^{p_8}, \tag{76}$$

with the exponents p_8 and p_9 defined in (20–21), one has the upper bound

$$|\mathcal{M}|_{\kappa_3,0,0}\preceq\frac{\mu}{\varepsilon^{p_9}}\exp\left\{-\frac{C\left(\frac{\pi}{2},\rho\right)}{\varepsilon^{1/(2\tau+2)}}\right\}.$$

Proof. Recall from Theorem 10 that $\mathcal{M}$ is defined in $\mathcal{P}_{\kappa_3,\nu_3,\rho_6}$ and $\hat{\omega}_\varepsilon$-quasiperiodic, with $\hat{\omega}_\varepsilon$ as in (51). Note also that $\mathcal{M}_0 = 0$, since $\mathcal{M}$ is the gradient of the splitting potential $\mathcal{L}$. Now, we apply Lemma 11 to the bound given in part (a) of Theorem 10. Taking δ as in (75), we see that conditions (26) and (49), the ones required in Theorem 10, give rise to the smallness condition (76), since $p_8 = \max(p_1, p_4 + 2)/(2\tau + 2)$. Note that, from our choice of δ, the conditions containing the exponent p_2 in (26) and (49) have been ignored, since $p_4 \geq p_3 + 1 \geq p_2 + \tau + 1$. Then, we deduce from Theorem 10 the bound

$$|\mathcal{M}|_{\kappa_3,0,0} \preceq \frac{\mu}{\varepsilon^{p_9}} \exp\left\{ -\frac{\widetilde{C}(\nu_3,\rho_6)}{\varepsilon^{1/(2\tau+2)}} \right\}, \tag{77}$$

since $p_9 = (p_4 + n)/(2\tau + 2)$ (as before, the bound containing the exponent p_2 has been ignored). We have written, instead of (74),

$$\widetilde{C}(\nu_3,\rho_6) = \left(1 + \frac{1}{\tau}\right)\left(\tau\nu_3\rho_6^\tau \cdot \frac{b'\gamma}{b}\right)^{1/(\tau+1)}$$

(note that, since we have replaced ω_ε by $\hat{\omega}_\varepsilon$, the Diophantine constant is now $b'\gamma/b$). However, recalling that $\nu_3 = \pi/2 - 3\delta$, $\rho_6 = \rho - 6\delta$, $b'/b = 1 + \mathcal{O}(\mu\delta^{-p_2})$, and proceeding as in the proof of Lemma 11, we can replace $\widetilde{C}(\nu_3,\rho_6)$ by $C\left(\frac{\pi}{2},\rho\right)$ in (77), obtaining the expected bound. □

Remarks.

1. The upper bound obtained for the splitting function $\mathcal{M}$ has been obtained without using at all the Melnikov function M introduced in Section 4.3. Nevertheless, using it one could obtain a somewhat better upper bound for $\mathcal{M}$ under a somewhat stronger smallness condition on μ, i.e. a smaller exponent p_9 but a larger exponent p_8. To get this, one could deduce from Lemma 7 an exponentially small upper bound for the Melnikov function M, and see that this upper bound dominates the upper bound for the error term that one can deduce from part (b) of Theorem 10. This is done better in the paper [DG02] with a concrete example, for which accurate upper and lower bounds for the dominant coefficients of the Melnikov function can be obtained.
2. In a concrete example of Hamiltonian (1–3), the exponents p_8 and p_9 can easily be computed applying (20–21), from the values of n, τ, l, α.

A Proof of the extension theorem

In this section we are going to prove Theorem 6. In fact, we first prove the following result, which bounds the distance between actual trajectories on the perturbed stable whisker $\mathcal{W}^+$, and the unperturbed one $\mathcal{W}_0$. Afterwards, we will deduce statements (a) and (b) of Theorem 6.

We recall that the exponents p_i appearing in this appendix have been defined in (14–17).

Theorem 13 *Let be $Z^+(s,\theta)$ the stable whisker defined for complex (s,θ) as in (47) and verifying bounds (48). Then, if the condition*

$$\mu \preceq \delta^{p_4+2}$$

holds, for any (s,θ) such that $|\operatorname{Im} s| \leq \nu_1$, $q_2 \leq \operatorname{Re} s \leq 2q_2$ and $|\operatorname{Im}\theta| \leq \rho_3$, this whisker can be extended by the flow for real times $-T \leq t \leq 0$, where $T \preceq 1$, verifying:

$$|Z^+(s + bt, \theta + \tilde{\omega}_\varepsilon t) - Z_0(s + t, \theta + \omega_\varepsilon t)| \preceq \left(\frac{\mu}{\delta^{p_5}}, \frac{\mu}{\delta^{p_4}}\right).$$

Before proceeding to the proof of this theorem, we will introduce some notations as well as some auxiliary results.

In the sequel, s, θ are complex parameters in the strip $|\operatorname{Im} s| \leq \nu_1 = \pi/2 - \delta$, $|\operatorname{Im}\theta| \leq \rho_3 = \rho - 3\delta$, and t will be the *real* time. The complex singularities of the unperturbed whisker at $s = \pm i\pi/2$ are going to play an important rôle in our bounds. For the case $0 \leq \operatorname{Im} s \leq \nu_1$, which will be first considered, we define

$$\tau := |t + s - i\pi/2|$$

as a variable which controls the distance to the singularity $i\pi/2$. The case $-\nu_1 \leq \operatorname{Im} s \leq 0$ will be analogous.

We introduce the vector-valued function:

$$u(t) = Z^+(s + bt, \theta + \tilde{\omega}_\varepsilon t) - Z_0(s + t, \theta + \omega_\varepsilon t) = (\xi(t), \eta(t), \zeta(t), J(t)),$$

which satisfies the system of differential equations with respect to the variable t:

$$\begin{aligned} \dot{\xi} &= \eta, \\ \dot{\eta} &= \sin(x_0(t+s) + \xi) - \sin(x_0(t+s)) - \mu h'(x_0(t+s) + \xi) f(\theta + \omega_\varepsilon t + \zeta) \\ \dot{\zeta} &= \Lambda J \\ \dot{J} &= -\mu h(x_0(t+s) + \xi) \partial_\varphi f(\theta + \omega_\varepsilon t + \zeta). \end{aligned} \tag{78}$$

One can easily see [DGJS97] that the variational equations around the homoclinic solution (6) of the pendulum equation can be solved, and a fundamental system of solutions is given by the functions

$$\begin{aligned} \Psi(u) &= y_0(u) = \dot{x}_0(u), \\ \Phi(u) &= y_0(u) W(u) = \Psi(u) W(u), \end{aligned}$$

where

$$W(u) = \int_b^u \frac{d\sigma}{y_0(\sigma)^2}, \tag{79}$$

b being an arbitrary complex number. It is very important to choose adequately this parameter b to get a function $W(u)$ as regular as possible, near the singularities of y_0. For the case $0 \leq \operatorname{Im} s \leq \nu_1$, we choose $b = \pi i/2$. In this way, at the point $u = \pi i/2$, since $y_0(u)$ has a simple pole, $W(u)$ has a triple zero and $y_0(u)W(u)$ has a double zero.

Using these functions one can obtain, as in [DGJS97], integral expressions for the solutions:

$$\begin{aligned} \xi(t) = \Psi(t+s) &\left[\Phi'(s)\xi(0) - \Phi(s)\eta(0) - \int_0^t \Phi(\sigma+s) g(\xi(\sigma), \zeta(\sigma), \sigma)\, d\sigma \right] \\ &+ \Phi(t+s) \left[-\Psi'(s)\xi(0) + \Psi(s)\eta(0) + \int_0^t \Psi(\sigma+s) g(\xi(\sigma), \zeta(\sigma), \sigma)\, d\sigma \right] \end{aligned} \tag{80}$$

$$\begin{aligned} \eta(t) = \Psi'(t+s) &\left[\Phi'(s)\xi(0) - \Phi(s)\eta(0) - \int_0^t \Phi(\sigma+s) g(\xi(\sigma), \zeta(\sigma), \sigma)\, d\sigma \right] \\ &+ \Phi'(t+s) \left[-\Psi'(s)\xi(0) + \Psi(s)\eta(0) + \int_0^t \Psi(\sigma+s) g(\xi(\sigma), \zeta(\sigma), \sigma)\, d\sigma \right] \end{aligned} \tag{81}$$

$$\zeta(t) = \zeta(0) + \Lambda J(0) t - \int_0^t \Lambda (t-\sigma) N(\xi(\sigma), \zeta(\sigma), \sigma) d\sigma \tag{82}$$

$$J(t) = J(0) - \int_0^t N(\xi(\sigma), \zeta(\sigma), \sigma) d\sigma \tag{83}$$

where we define

$$\begin{aligned} g(\xi, \zeta, t) &= \sin(x_0(t+s) + \xi) - \sin(x_0(t+s)) - \cos(x_0(t+s))\xi \\ &\quad -\mu h'(x_0(t+s) + \xi) f(\theta + \omega_\varepsilon t + \zeta), \end{aligned} \tag{84}$$

$$N(\xi, \zeta, t) = \mu h(x_0(t+s) + \xi) \partial_\varphi f(\theta + \omega_\varepsilon t + \zeta). \tag{85}$$

Using these integral equations we have to solve a system of implicit equations for (ξ, ζ) given by (80) and (82), and later, we can obtain explicitely the actions (η, J) using (81) and (83). In order to proceed we need the following technical lemma, whose proof is straighforward, using hypotheses (H3–H4) of Section 1.1, Taylor formula and Cauchy estimates (for more details, see [DGJS97]):

Lemma 14 *For $0 \leq \operatorname{Im} s \leq \nu_1 = \pi/2 - \delta$, $|\operatorname{Im} \theta| \leq \rho_3 = \rho - 3\delta$ and $\tau = |t + s - \pi i/2| \preceq T$, the following bounds hold, where l and α are defined in hypotheses (H3–H4):*

(a) $|\Psi(t+s)| \preceq \frac{1}{\tau}, \quad |\Phi(t+s)| \preceq \tau^2, \quad |\Psi'(t+s)| \preceq \frac{1}{\tau^2}, \quad |\Phi'(t+s)| \preceq \tau.$

(b) $|h^{(j)}(x_0(t+s))| \preceq \frac{1}{\tau^{2l}}$, *where* $h^{(j)}$ *means the* j*-derivative of* $h(x)$.

(c) $|f(\theta+\omega_\varepsilon t)| \preceq \frac{1}{\delta^\alpha}, \quad |\partial_\varphi f(\theta+\omega_\varepsilon t)| \preceq \frac{1}{\delta^{\alpha+1}}, \quad |\partial_\varphi^2 f(\theta+\omega_\varepsilon t)| \preceq \frac{1}{\delta^{\alpha+2}}.$

(d) *If* $|\xi_j(t)| \preceq \lambda/\tau^\beta \leq 1$ *and* $|\zeta_j(t)| \preceq \Omega \preceq \delta$, $j=1,2$, *then the function* g *defined in (84) verifies*

$$\begin{aligned} |g(\xi_1(t),\zeta_1(t),t) - g(\xi_2(t),\zeta_2(t),t)| &\preceq \left(\frac{\lambda}{\tau^{\beta+2}} + \frac{\mu\delta^{-\alpha}}{\tau^{2l}}\right)|\xi_1(t)-\xi_2(t)| \\ &\quad + \frac{\mu\delta^{-(\alpha+1)}}{\tau^{2l}}|\zeta_1(t)-\zeta_2(t)|, \end{aligned}$$

and the function N *defined in (85) verifies:*

$$\begin{aligned} |N(\xi_1(t),\zeta_1(t),t) - N(\xi_2(t),\zeta_2(t),t)| &\preceq \frac{\mu\delta^{-(\alpha+1)}}{\tau^{2l}}|\xi_1(t)-\xi_2(t)| \\ &\quad + \frac{\mu\delta^{-(\alpha+2)}}{\tau^{2l}}|\zeta_1(t)-\zeta_2(t)|. \end{aligned}$$

The proof of Theorem 13, for the moment for $s \in D^+ = \{s \in \mathbb{C} : 0 \leq \operatorname{Im} s \leq \pi/2 - \delta\}$, is based on the next Propositions 16 and 17. In the first one, the solutions of system (78) with initial conditions verifying bounds (48), will be extended up to some time $t = -t_1(s)$ defined bellow. In the second proposition, we take $t = -t_1(s)$ as the initial time.

We divide the complex strip D^+ in two parts

$$\begin{aligned} D^{\text{up}} &= \{s \in \mathbb{C} : \pi/2 - \delta^{2/3} \leq \operatorname{Im} s \leq \pi/2 - \delta\}, \\ D^{\text{down}} &= \{s \in \mathbb{C} : 0 \leq \operatorname{Im} s \leq \pi/2 - \delta^{2/3}\} \end{aligned}$$

and define the separation point $-t_1(s)$ by

$$-t_1(s) + \operatorname{Re} s = \begin{cases} -\delta^{2/3}, & \text{for} \quad s \in D^{\text{up}}, \\ 0, & \text{for} \quad s \in D^{\text{down}}. \end{cases}$$

During this proof we will use the following technical result (an analogous one is proved in [DS92]).

Lemma 15 *Let* t, t_0 *real,* s *complex, such that*

$$|\operatorname{Im} s| < \pi/2, \quad q_2 \leq \operatorname{Re} s \leq 2q_2, \quad -T \leq t \leq t_0 \leq 0.$$

Then, given $\beta \in \mathbb{R}$, *the following inequality holds:*

$$\int_t^{t_0} \frac{d\sigma}{|\sigma + s - \pi i/2|^\beta} \leq K \cdot \rho_{[t,t_0]}^{-(\beta-1)}(s),$$

with $K = K(q_2, T, \beta) > 0$, *and*

$$\rho_{[t,t_0]}^{-\beta}(s) := \begin{cases} \sup \dfrac{1}{|\sigma + s - \pi i/2|^\beta}, & \textit{if } \beta \neq 0, \\ \sup |\ln(|\sigma + s - \pi i/2|)|, & \textit{if } \beta = 0, \end{cases}$$

where the supremum is taken for $\sigma \in [t, t_0]$.

Proposition 16 *Let* $u(t) = (\xi(t), \eta(t), \zeta(t), J(t))$ *a solution of system (78) with initial conditions satisfying*

$$|u(0)| \preceq \frac{\mu}{\delta^{p_3}} \cdot \left(1, \frac{1}{\delta}\right). \tag{86}$$

Then, under the hypotheses of Theorem 13, $u(t)$ can be extended for $t \in [-t_1(s), 0]$ and satisfies there the following bound:

$$\left|\left(\tau\xi(t), \tau^2\eta(t), \zeta(t), J(t)\right)\right| \preceq \left(\frac{\mu}{\delta^{p_3}} + \frac{\mu}{\delta^{2l+\alpha-2}}, \frac{\mu}{\delta^{p_3+1}} + \frac{\mu}{\delta^{2l+\alpha}}\right). \tag{87}$$

Proof. We shall use the method of successive approximations. We begin the iteration process with $\xi_0(t) = 0$, $\zeta_0(t) = 0$ and consider, for $n \geq 0$, the recurrence suggested by the system of equations (80), (82):

$$\begin{aligned}
\xi_{n+1}(t) &= \Psi(t+s)\left[\Phi'(s)\xi(0) - \Phi(s)\eta(0) - \int_0^t \Phi(\sigma+s) g(\xi_n(\sigma), \zeta_n(\sigma), \sigma)\, d\sigma\right] \\
&\quad + \Phi(t+s)\left[-\Psi'(s)\xi(0) + \Psi(s)\eta(0) + \int_0^t \Psi(\sigma+s) g(\xi_n(\sigma), \zeta_n(\sigma), \sigma)\, d\sigma\right] \\
\zeta_{n+1}(t) &= \zeta(0) + \Lambda J(0) t - \int_0^t \Lambda(t-\sigma) N(\xi_n(\sigma), \zeta_n(\sigma), \sigma) d\sigma.
\end{aligned}$$

For the first iteration we obtain (ξ_1, ζ_1) as:

$$\begin{aligned}
\xi_1(t) &= \Psi(t+s)\left[\Phi'(s)\xi(0) - \Phi(s)\eta(0) - \int_0^t \Phi(\sigma+s) g(0,0,\sigma)\, d\sigma\right] \\
&\quad + \Phi(t+s)\left[-\Psi'(s)\xi(0) + \Psi(s)\eta(0) + \int_0^t \Psi(\sigma+s) g(0,0,\sigma)\, d\sigma\right] \qquad (88) \\
\zeta_1(t) &= \zeta(0) + \Lambda J(0) t - \int_0^t \Lambda(t-\sigma) N(0,0,\sigma) d\sigma \qquad (89)
\end{aligned}$$

This first iterate can easily be bounded, using the initial conditions (86), and Lemmas 14 and 15:

$$|\xi_1(t)| \preceq \frac{\mu}{\tau}\left[\frac{1}{\delta^{p_3}} + \left(\rho_{[t,0]}^{-(2l-3)}(s) + 1\right)\frac{1}{\delta^\alpha}\right] + \mu\tau^2\left[\frac{1}{\delta^{p_3}} + \rho_{[t,0]}^{-2l}(s)\frac{1}{\delta^\alpha}\right]. \tag{90}$$

An analogous bound for $\tau\xi_1(t)$ follows immediately. Now, using that for $s \in D^{\mathrm{down}}$ we have

$$\rho_{[t,0]}^{-2l}(s) \preceq \tau^{-2l}$$

and for $s \in D^{\mathrm{up}}$ we have

$$\begin{aligned}
\rho_{[t,0]}^{-2l}(s) \preceq \tau^{-2l}, &\quad \text{for} \quad 0 \leq t + \mathrm{Re}\, s \leq \mathrm{Re}\, s, \\
\rho_{[t,0]}^{-2l}(s) \preceq \delta^{-2l}, \quad \tau^3 \preceq \delta^2, &\quad \text{for} \quad -t_1(s) + \mathrm{Re}\, s \leq t + \mathrm{Re}\, s \leq 0,
\end{aligned}$$

and, consequently, in all the strip D^+, i.e. for $s \in D^+$ and $t \in [-t_1(s), 0]$, it follows that

$$\tau^3 \rho_{[t,0]}^{-2l}(s) \preceq \delta^{-(2l-2)}. \tag{91}$$

We remark that the value of $t_1(s)$ has been chosen just in order that bound (91) holds. In this way, we can bound $|\tau\xi_1(t)|$ in a uniform way:

$$|\tau\xi_1(t)| \preceq \mu\left(\frac{1}{\delta^{p_3}} + \frac{1}{\delta^{2l-2+\alpha}}\right).$$

On the other hand, the bound for ζ_1 is immediate:

$$|\zeta_1(t)| \preceq \frac{\mu}{\delta^{p_3+1}} + \mu\rho_{[t,0]}^{-(2l-1)}(s)\frac{1}{\delta^{\alpha+1}} \preceq \frac{\mu}{\delta^{p_3+1}} + \frac{\mu}{\delta^{2l+\alpha}}.$$

In order to prove the convergence we proceed by induction. To begin the iteration process, we introduce the norm

$$\|(\xi, \zeta)\| := \sup\left(|\tau\xi(t)| + \left|\delta^2\zeta(t)\right|\right),$$

where the supremum is taken for $s \in D^+$ and $t \in [-t_1(s), 0]$. The above bound on $\tau\xi_1(t)$ and $\zeta_1(t)$ reads now as

$$\|(\xi_1, \zeta_1)\| \preceq \mu\left(\frac{1}{\delta^{p_3}} + \frac{1}{\delta^{2l+\alpha-2}}\right).$$

Assuming that $\|(\xi_{n-1},\zeta_{n-1})\|, \|(\xi_n,\zeta_n)\| \preceq \mu\left(\frac{1}{\delta^{p_3}}+\frac{1}{\delta^{2l+\alpha-2}}\right)$, we now consider

$$\begin{aligned}\xi_{n+1}(t)-\xi_n(t) &= -\Psi(t+s)\int_0^t \Phi(\sigma+s)\left[g_n-g_{n-1}\right]d\sigma \\ &\quad +\Phi(t+s)\int_0^t \Psi(\sigma+s)\left[g_n-g_{n-1}\right]d\sigma,\end{aligned}$$

where g_n denotes $g(\xi_n(\sigma),\zeta_n(\sigma),\sigma)$. Now, we apply part (d) of Lemma 14, with $\lambda=\mu(\frac{1}{\delta^{p_3}}+\frac{1}{\delta^{2l+\alpha-2}})$, $\beta=1$ and $\Omega=\frac{\mu}{\delta^{p_3+2}}+\frac{\mu}{\delta^{2l+\alpha}}$. By the hypotheses of Theorem 13, these amounts verify that $\lambda\preceq\delta^3\preceq 1$, and $\Omega\preceq\delta$, which are the conditions needed in this lemma. We will use the notation $\tau(\sigma)=|\sigma+T-\pi i/2|$, and we will use again Lemma 15, as well as inequality (91). We obtain:

$$\begin{aligned}&|\tau(\xi_{n+1}(t)-\xi_n(t))| \\ &\preceq \int_0^t \tau(\sigma)^2\left[\left(\frac{\lambda}{\tau(\sigma)^3}+\frac{\mu\delta^{-\alpha}}{\tau(\sigma)^{2l}}\right)|\xi_n(\sigma)-\xi_{n-1}(\sigma)| \right.\\ &\qquad\left. +\frac{\mu\delta^{-(\alpha+1)}}{\tau(\sigma)^{2l}}|\zeta_n(\sigma)-\zeta_{n-1}(\sigma)|\right]d\sigma \\ &\quad +\tau^3\int_0^t \frac{1}{\tau(\sigma)}\left[\left(\frac{\lambda}{\tau(\sigma)^3}+\frac{\mu\delta^{-\alpha}}{\tau(\sigma)^{2l}}\right)|\xi_n(\sigma)-\xi_{n-1}(\sigma)|\right. \\ &\qquad\left. +\frac{\mu\delta^{-(\alpha+1)}}{\tau(\sigma)^{2l}}|\zeta_n(\sigma)-\zeta_{n-1}(\sigma)|\right]d\sigma \\ &\preceq \left[\left(\lambda\rho_{[t,0]}^{-1}(s)+\mu\delta^{-\alpha}\rho_{[t,0]}^{-(2l-2)}(s)+\mu\delta^{-(\alpha+3)}\left(1+\rho_{[t,0]}^{-(2l-3)}(s)\right)\right)\right. \\ &\quad\left. +\tau^3\left(\lambda\rho_{[t,0]}^{-4}(s)+\mu\delta^{-\alpha}\rho_{[t,0]}^{-(2l+1)}(s)+\mu\delta^{-(\alpha+3)}\rho_{[t,0]}^{-2l}(s)\right)\right] \\ &\quad\cdot\|(\xi_n-\xi_{n-1},\zeta_n-\zeta_{n-1})\| \\ &\preceq \left(\frac{\mu}{\delta^{p_3+2}}+\frac{\mu}{\delta^{2l+\alpha+1}}\right)\|(\xi_n-\xi_{n-1},\zeta_n-\zeta_{n-1})\|.\end{aligned}$$

We can also obtain an analogous bound for $\left|\delta^2\left(\zeta_{n+1}(t)-\zeta_n(t)\right)\right|$.

Since by the hypotheses of Theorem 13 we can take $\frac{\mu}{\delta^{p_3+2}}+\frac{\mu}{\delta^{2l+\alpha+1}}$ small enough, it follows, by induction, that the following inequalities

$$\begin{aligned}\|(\xi_n,\zeta_n)\| &\leq 2\|(\xi_1,\zeta_1)\|\preceq\mu\left(\frac{1}{\delta^{p_3}}+\frac{1}{\delta^{2l+\alpha-2}}\right), \\ \|(\xi_{n+1}-\xi_n,\zeta_{n+1}-\zeta_n)\| &\leq \frac{1}{2}\|(\xi_n-\xi_{n-1},\zeta_n-\zeta_{n-1})\|,\end{aligned}$$

are valid for $n\geq 1$, and consequently $(\xi_n,\zeta_n)_{n\geq 0}$ converges uniformly for $s\in D^+$ and $t\in[-t_1(s),0]$ to the components $\xi(t),\zeta(t)$ of a solution of system (80), (82) satisfying

$$|\tau\xi(t)|+\left|\delta^2\zeta(t)\right|\preceq\mu\left(\frac{1}{\delta^{p_3}}+\frac{1}{\delta^{2l+\alpha-2}}\right).$$

For the component $\eta(t)$, we simply use its integral equation (81), and it is straightforward to check that

$$\left|\tau^2\eta(t)\right|\preceq\mu\left(\frac{1}{\delta^{p_3}}+\frac{1}{\delta^{2l+\alpha-2}}\right).$$

One can bound $J(t)$ from its integral equation (83) by applying Lemma 14 with $\xi_1=\xi$, $\xi_2=0$, $\zeta_1=\zeta$ and $\zeta_2=0$. Then we have $\lambda=\mu\left(\frac{1}{\delta^{p_3}}+\frac{1}{\delta^{2l+\alpha-2}}\right)$, $\beta=1$ and $\Omega=\mu(\frac{1}{\delta^{p_3+2}}+\frac{1}{\delta^{2l+\alpha}})$, and we obtain

$$|J(t)|\preceq\frac{\mu}{\delta^{p_3+1}}+\frac{\mu}{\delta^{\alpha+1}}\rho_{[t,0]}^{-(2l-1)}(s)\preceq\mu\left(\frac{1}{\delta^{p_3+1}}+\frac{1}{\delta^{2l+\alpha}}\right).$$

Now, putting this bound for $J(t)$ in the integral equation for $\zeta(t)$, we obtain the desired bound (87). □

We get from bound (87) the following global estimates for the components (ξ,η):

$$|\xi(t)|\preceq\frac{\mu}{\delta^{p_3+1}}+\frac{\mu}{\delta^{2l+\alpha-1}},\quad |\eta(t)|\preceq\frac{\mu}{\delta^{p_3+2}}+\frac{\mu}{\delta^{2l+\alpha}},$$

for $t \in [-t_1(s), 0]$. On the final point $t = -t_1(s)$, bound (87) gives a better estimate:

$$|\xi(t)| \preceq \left(\frac{\mu}{\delta^{p_3}} + \frac{\mu}{\delta^{2l+\alpha-2}}\right)\frac{1}{\tau_1}, \quad |\eta(t)| \preceq \left(\frac{\mu}{\delta^{p_3}} + \frac{\mu}{\delta^{2l+\alpha-2}}\right)\frac{1}{\tau_1^2},$$

which gives, using that $\tau_1 = |t_1(s) + s - \pi i/2| \geq \delta^{2/3}$:

$$|\xi(t)| \preceq \frac{\mu}{\delta^{p_3+2/3}} + \frac{\mu}{\delta^{2l+\alpha-4/3}}, \quad |\eta(t)| \preceq \frac{\mu}{\delta^{p_3+4/3}} + \frac{\mu}{\delta^{2l+\alpha-2/3}}.$$

These and the corresponding bounds for (ζ, J) given by (87), are the initial conditions for the next proposition.

Proposition 17 *Let $u(t)$ a solution of system (78) with initial conditions satisfying (87) for $t = t_1(s)$. Then, under the hypotheses of Theorem 13, $u(t)$ can be extended for $t \in [-T, -t_1(s)]$ and satisfies there the following bound:*

$$\left|\left(\frac{\xi(t)}{\tau^2}, \frac{\eta(t)}{\tau}, \zeta(t), J(t)\right)\right| \preceq \left(\frac{\mu}{\delta^{p_3+2}} + \frac{\mu}{\delta^{2l+\alpha}}, \frac{\mu}{\delta^{p_3+1}} + \frac{\mu}{\delta^{2l+\alpha}}\right). \tag{92}$$

Proof. We shall use exactly the same method of successive approximations as in Proposition 16, but replacing the initial condition 0 by $-t_1(s)$, in the integral equations (80–83).

The first iteration gives $(\xi_1(t), \zeta_1(t))$ as provided by equations (88–89) but with $-t_1(s)$ instead of 0. Proceeding like in Proposition 16, but using now the initial conditions (87), we can bound the first iterate $\xi_1(t)$ as in (90):

$$\begin{aligned}|\xi_1(t)| \;\preceq\; & \frac{\mu}{\tau}\left[\frac{1}{\delta^{p_3}} + \frac{1}{\delta^{2l+\alpha-2}} + \left(\rho_{[t,-t_1(s)]}^{-(2l-3)}(s) + 1\right)\frac{1}{\delta^\alpha}\right] \\ & + \mu\tau^2\left[\frac{1}{\delta^{p_3}\tau_1^3} + \frac{1}{\delta^{2l+\alpha-2}\tau_1^3} + \rho_{[t,-t_1(s)]}^{-2l}(s)\frac{1}{\delta^\alpha}\right],\end{aligned}$$

where $\tau_1 = |-t_1(s) + s - i\pi/2|$. Now the following inequalities hold, for positive values of β:

$$\rho_{[t,-t_1(s)]}^{-\beta}(s) \preceq \tau_1^{-\beta} \preceq \delta^{-2\beta/3} \preceq \delta^{-\beta}, \quad \rho_{[t,-t_1(s)]}^{\beta}(s) \preceq \tau^\beta, \tag{93}$$

and consequently we can bound $\frac{\xi_1(t)}{\tau^2}$ as

$$\left|\frac{\xi_1(t)}{\tau^2}\right| \preceq \frac{1}{\tau^3}\mu\left(\frac{1}{\delta^{p_3}} + \frac{1}{\delta^{2l+\alpha-2}}\right) + \frac{1}{\tau_1^3}\mu\left(\frac{1}{\delta^{p_3}} + \frac{1}{\delta^{2l+\alpha-2}}\right) \preceq \mu\left(\frac{1}{\delta^{p_3+2}} + \frac{1}{\delta^{2l+\alpha}}\right)$$

where we have used that $\tau \geq \tau_1 \geq \delta^{2/3}$. For the component ζ_1, we have:

$$|\zeta_1(t)| \preceq \mu\left(\frac{1}{\delta^{p_3+1}} + \frac{1}{\delta^{2l+\alpha}} + \rho_{[t,-t_1(s)]}^{-(2l-1)}(s)\frac{1}{\delta^{\alpha+1}}\right) \preceq \mu\left(\frac{1}{\delta^{p_3+1}} + \frac{1}{\delta^{2l+\alpha}}\right).$$

In view of these bounds, we define now the norm

$$\|(\xi, \zeta)\| := \sup\left(\left|\frac{\xi(t)}{\tau^2}\right| + |\zeta(t)|\right),$$

with the supremum taken for $s \in D^+$ and $t \in [-T, -t_1(s)]$. With this new terminology we have proved that

$$\|(\xi_1, \zeta_1)\| \preceq \mu\left(\frac{1}{\delta^{p_3+2}} + \frac{1}{\delta^{2l+\alpha}}\right),$$

and therefore, for the component ξ, we have:

$$|\xi_1(t)| \preceq \mu\left(\frac{1}{\delta^{p_3+2}} + \frac{1}{\delta^{2l+\alpha}}\right)\tau^2.$$

For the successive iterates we apply Lemma 14, with $\lambda = \Omega = \mu\left(\frac{1}{\delta^{p_3+2}} + \frac{1}{\delta^{2l+\alpha}}\right) \preceq \delta$, $\beta = -2$, and Lemma 15, as well as inequalities (93), obtaining

$$\begin{aligned}
&\left|\frac{1}{\tau^2}(\xi_{n+1}(t) - \xi_n(t))\right| \\
&\preceq \frac{1}{\tau^3}\int_{t_1(s)}^{t} \tau(\sigma)^2 \left[\left(\lambda + \frac{\mu\delta^{-\alpha}}{\tau(\sigma)^{2l}}\right)|\xi_n(\sigma) - \xi_{n-1}(\sigma)| \right. \\
&\qquad \left. + \frac{\mu\delta^{-(\alpha+1)}}{\tau(\sigma)^{2l}}|\zeta_n(\sigma) - \zeta_{n-1}(\sigma)|\right] d\sigma \\
&\quad + \int_{t_1(s)}^{t} \frac{1}{\tau(\sigma)} \left[\left(\lambda + \frac{\mu\delta^{-\alpha}}{\tau(\sigma)^{2l}}\right)|\xi_n(\sigma) - \xi_{n-1}(\sigma)| \right. \\
&\qquad \left. + \frac{\mu\delta^{-(\alpha+1)}}{\tau(\sigma)^{2l}}|\zeta_n(\sigma) - \zeta_{n-1}(\sigma)|\right] d\sigma \\
&\preceq \left(\frac{1}{\tau^3}\left[\lambda\rho^5 + \mu\delta^{-\alpha}\rho^{-(2l-5)} + \mu\delta^{-(\alpha+1)}\rho^{-(2l-3)}\right]\right. \\
&\qquad \left. + \lambda\rho^2 + \mu\delta^{-\alpha}\rho^{-(2l-2)} + \mu\delta^{-(\alpha+1)}\rho^{-2l}\right) \|(\xi_n, \zeta_n) - (\xi_{n-1}, \zeta_{n-1})\| \\
&\preceq \left(\frac{\mu}{\delta^{p_3+2}} + \frac{\mu}{\delta^{2l+\alpha+1}}\right)\|(\xi_n, \zeta_n) - (\xi_{n-1}, \zeta_{n-1})\|,
\end{aligned}$$

and similarly for $|\zeta_{n+1}(t) - \zeta_n(t)|$. We have denoted $\tau(\sigma) = |\sigma + s - \pi i/2|$, and ρ^β instead of $\rho^\beta_{[t,-t_1(s)]}(s)$. Like in Proposition 16, it follows by induction that for $n \geq 1$

$$\|(\xi_n, \zeta_n)\| \preceq \mu\left(\frac{1}{\delta^{p_3+2}} + \frac{1}{\delta^{2l+\alpha}}\right)$$

$$\|(\xi_{n+1}, \zeta_{n+1}) - (\xi_n, \zeta_n)\| \leq \frac{1}{2}\|(\xi_n, \zeta_n) - (\xi_{n-1}, \zeta_{n-1})\|,$$

and consequently $(\xi_n, \zeta_n)_{n\geq 0}$ converges uniformly for $s \in D^+$ and $t \in [-T, -t_1(s)]$ to the components $(\xi(t), \zeta(t))$ of a solution of system (78), satisfying the required bound (92).

As in Proposition 17, we can now bound $\eta(t)$ and $J(t)$ from their integral equations (81) and (83), and we finally obtain that they also verify bound (92). □

Proof of Theorem 13. First consider $0 \leq \operatorname{Im} s \leq \nu_1$ and fix $q_2 \leq \operatorname{Re} s \leq 2q_2$. Putting Propositions 16 and 17 together, as well as bound (92), we immediately obtain Theorem 13 for the case $0 \leq \operatorname{Im} s \leq \nu_1$, with the required estimates.

For $-\nu_1 \leq \operatorname{Im} s \leq 0$ we only have to choose $b = -\pi i/2$ in definition (79) of $W(u)$, in order to get a second solution $\Phi(u)$ of the variational equations of the pendulum with a double zero at $u = -\pi i/2$. An analogous Lemma 14, as well as Propositions 16 and 17 are also valid in this case, and consequently Theorem 13 follows for any s with $|\operatorname{Im} s| \leq \nu_1$. □

Proof of Theorem 6 (extension theorem). The proof of the extension theorem is almost direct from the results of Theorem 13. In fact, there is only one difference between Theorem 13 and statement (a) of Theorem 6. Indeed, in Theorem 13 we compare trajectories on the whiskers $\mathcal{W}^+$ and $\mathcal{W}_0$, whereas in Theorem 6 we have to compare points on the two whiskers given by the same values of the parameters. Let us take some values $\tilde{s}$, $\tilde{\theta}$ with

$$q_2 \geq \operatorname{Re}\tilde{s} \geq -q_*, \qquad |\operatorname{Im}\tilde{s}| \leq \nu_1, \qquad \left|\operatorname{Im}\tilde{\theta}\right| \leq \rho_3.$$

To apply Theorem 13, consider the initial parameter $s = 2q_2 + i\operatorname{Im}\tilde{s}$ and time $t = \frac{1}{b}(\operatorname{Re}\tilde{s} - 2q_2) \leq 0$. Note that $s + bt = \tilde{s}$, and consider also θ such that $\theta + \tilde{\omega}_\varepsilon t = \tilde{\theta}$. Now we can use Theorem 13, as well as Taylor theorem for the unperturbed homoclinic solution Z_0, and bounds (30) of Theorem 1. We obtain the desired bound (a) of Theorem 6:

$$\begin{aligned}
\left|Z^+(\tilde{s}, \tilde{\theta}) - Z_0(\tilde{s}, \tilde{\theta})\right| &= \left|Z^+(s + bt, \theta + \tilde{\omega}_\varepsilon t) - Z_0(s + bt, \theta + \tilde{\omega}_\varepsilon t)\right| \\
&\leq \left|Z^+(s + bt, \theta + \tilde{\omega}_\varepsilon t) - Z_0(s + t, \theta + \omega_\varepsilon t)\right| \\
&\quad + |Z_0(s + t, \theta + \omega_\varepsilon t) - Z_0(s + bt, \theta + \tilde{\omega}_\varepsilon t)|
\end{aligned}$$

$$
\begin{aligned}
&\preceq \left(\frac{\mu}{\delta^{p_5}}, \frac{\mu}{\delta^{p_4}}\right) + \left(\frac{|b-1|}{\delta^2}, |\omega_\varepsilon - \tilde{\omega}_\varepsilon|\right) \\
&\preceq \left(\frac{\mu}{\delta^{p_5}} + \frac{\mu}{\delta^{p_2+2}}, \frac{\mu}{\delta^{p_4}} + \frac{\mu}{\delta^{p_2}\sqrt{\varepsilon}}\right) \preceq \left(\frac{\mu}{\delta^{p_5}}, \frac{\mu}{\delta^{p_4}} + \frac{\mu}{\delta^{p_2}\sqrt{\varepsilon}}\right). \qquad (94)
\end{aligned}
$$

In order to bound the integral in statement (b) of Theorem 6, we use the previous results and part (d) of Lemma 14. To this end we define

$$
\begin{aligned}
\xi_1 &= Z_x^+(s+bt, \theta + \tilde{\omega}_\varepsilon t) - x_0(s+t), \quad \zeta_1 = Z_\varphi^+(s+bt, \theta+\tilde{\omega}_\varepsilon t) - \theta - \omega_\varepsilon t \\
\xi_2 &= x_0(s+bt) - x_0(s+t), \quad \zeta_2 = \theta + \tilde{\omega}_\varepsilon t - \theta - \omega_\varepsilon t.
\end{aligned}
$$

By the results of Theorem 13 we have

$$
|\xi_1| \preceq \frac{\mu}{\delta^{p_5}} \preceq 1, \quad |\zeta_1| \preceq \frac{\mu}{\delta^{p_4}} \preceq \delta
$$

and, by Theorem 1 we have

$$
|\xi_2| \preceq \frac{|b-1|}{\delta} \preceq \frac{\mu}{\delta^{p_2+1}} \preceq 1, \quad |\zeta_2| \preceq |\tilde{\omega}_\varepsilon - \omega_\varepsilon| \preceq \frac{\mu}{\delta^{p_2}\sqrt{\varepsilon}}
$$

and the hypotheses of Theorem 6 imply $|\zeta_2| \preceq \delta$, and hence we can apply part (d) of Lemma 14. Note also that we have given bounds for $|\xi_1 - \xi_2|$ and $|\zeta_1 - \zeta_2|$ in (94). Applying also Lemma 15, we obtain the desired bound (b) of Theorem 6:

$$
\begin{aligned}
&\left| \int_0^T \left[\partial_\varphi H_1(Z^+(s+bt, \theta+\tilde{\omega}_\varepsilon t)) - \partial_\varphi H_1(Z_0(s+bt, \theta+\tilde{\omega}_\varepsilon t))\right] \mathrm{d}t \right| \\
&= \left| \frac{1}{\mu} \int_0^T \left[N(\xi_1, \zeta_1, t) - N(\xi_2, \zeta_2, t)\right] \mathrm{d}t \right| \\
&\preceq \int_0^T \frac{\delta^{-(\alpha+1)}}{\tau^{2l}} |\xi_1 - \xi_2| \,\mathrm{d}t + \int_0^T \frac{\delta^{-(\alpha+2)}}{\tau^{2l}} |\zeta_1 - \zeta_2| \,\mathrm{d}t \\
&\preceq \frac{1}{\delta^{2l+\alpha}} \cdot \frac{\mu}{\delta^{p_5}} + \frac{1}{\delta^{2l+\alpha+1}} \left(\frac{\mu}{\delta^{p_4}} + \frac{\mu}{\delta^{p_2}\sqrt{\varepsilon}}\right) \preceq \frac{1}{\delta^{2l+\alpha+1}} \left(\frac{\mu}{\delta^{p_4}} + \frac{\mu}{\delta^{p_2}\sqrt{\varepsilon}}\right).
\end{aligned}
$$

□

B Proof of the flow-box theorem

In this appendix, we rename the widths involved in Theorem 4, writing κ, σ, η, ρ, ζ, instead of κ, σ, η_1, ρ_3, $\beta\delta$, and also $\kappa - \hat{\kappa}$, $\sigma - \hat{\sigma}$, $\eta - \hat{\eta}$, $\rho - \hat{\rho}$, $\zeta - \hat{\zeta}$, instead of κ_1, σ_1, η_2, ρ_4, $\beta_1\delta$. Besides, we deviate slightly in this appendix from the notation introduced in Section 1.5: for a vector-valued function f with images in the (S, E, φ, I)-space, we now write $|f| \leq (|g_1|, |g_2|, |g_3|, |g_4|)$ to mean separate bounds for the S-component, the E-component, the φ-component and the I-component of the function f.

For the norms on $\mathcal{B}_{\kappa,\sigma,\eta,\rho,\zeta}$ defined in Section 1.5, we shall use the following standard properties, which can easily be proved (for instance, see in [Pös93] analogous properties in a somewhat different context):

(P1) $|f|_{\kappa,\sigma,\eta,\rho,\zeta} \leq \|f\|_{\kappa,\sigma,\eta,\rho,\zeta}$, $\quad \|f\|_{\kappa,\sigma,\eta,\rho-\hat{\rho},\zeta} \preceq \dfrac{|f|_{\kappa,\sigma,\eta,\rho,\zeta}}{\hat{\rho}^n}$ (we shall not use the second one).

(P2) $|\mathrm{D}f|_{\kappa-\hat{\kappa},\sigma-\hat{\sigma},\eta-\hat{\eta},\rho-\hat{\rho},\zeta-\hat{\zeta}} \leq |f|_{\kappa,\sigma,\eta,\rho,\zeta} \cdot \left(\dfrac{1}{\min(\hat{\kappa},\hat{\sigma})}, \dfrac{1}{\hat{\eta}}, \dfrac{1}{\hat{\rho}}, \dfrac{1}{\hat{\zeta}}\right)$,

$\|\mathrm{D}f\|_{\kappa-\hat{\kappa},\sigma-\hat{\sigma},\eta-\hat{\eta},\rho-\hat{\rho},\zeta-\hat{\zeta}} \leq \|f\|_{\kappa,\sigma,\eta,\rho,\zeta} \cdot \left(\dfrac{1}{\min(\hat{\kappa},\hat{\sigma})}, \dfrac{1}{\hat{\eta}}, \dfrac{1}{e\hat{\rho}}, \dfrac{1}{\hat{\zeta}}\right)$.

(P3) $\|\{f,g\}\|_{\kappa-\hat{\kappa},\sigma-\hat{\sigma},\eta-\hat{\eta},\rho-\hat{\rho},\zeta-\hat{\zeta}} \leq \dfrac{4}{\min\left(\hat{\kappa}\hat{\eta}, \hat{\sigma}\hat{\eta}, e\hat{\rho}\hat{\zeta}\right)} \|f\|_{\kappa,\sigma,\eta,\rho,\zeta} \cdot \|g\|_{\kappa,\sigma,\eta,\rho,\zeta}$.

(P4) If $|U|_{\kappa,\sigma,\eta,\rho,\zeta} \le \frac{1}{4}\min\left(\hat\kappa\hat\eta, \hat\sigma\hat\eta, \hat\rho\hat\zeta\right)$, then the Hamiltonian time-one flow of U maps $\Upsilon : \mathcal{B}_{\kappa-\hat\kappa,\sigma-\hat\sigma,\eta-\hat\eta,\rho-\hat\rho,\zeta-\hat\zeta} \longrightarrow \mathcal{B}_{\kappa-\hat\kappa/2,\sigma-\hat\sigma/2,\eta-\hat\eta/2,\rho-\hat\rho/2,\zeta-\hat\zeta/2}$, and one has

$$\begin{aligned} |\Upsilon - \mathrm{id}|_{\kappa-\hat\kappa,\sigma-\hat\sigma,\eta-\hat\eta,\rho-\hat\rho,\zeta-\hat\zeta} &\le 2\,|U|_{\kappa,\sigma,\eta,\rho,\zeta} \cdot \left(\frac{1}{\hat\eta}, \frac{1}{\min(\hat\kappa,\hat\sigma)}, \frac{1}{\hat\zeta}, \frac{1}{\hat\rho}\right) \\ &\le \frac{1}{2}\left(\min(\hat\kappa,\hat\sigma), \hat\eta, \hat\rho, \hat\zeta\right). \end{aligned}$$

(P5) If $\|U\|_{\kappa,\sigma,\eta,\rho,\zeta} \le \frac{1}{8e^2}\min\left(\hat\kappa\hat\eta, \hat\sigma\hat\eta, e\hat\rho\hat\zeta\right)$, then its Hamiltonian time-one flow Υ satisfies that, for any function f given,

$$\|f\circ\Upsilon\|_{\kappa-\hat\kappa,\sigma-\hat\sigma,\eta-\hat\eta,\rho-\hat\rho,\zeta-\hat\zeta} \le 2\,\|f\|_{\kappa,\sigma,\eta,\rho,\zeta}.$$

The proof of Theorem 4 relies on applying iteratively the following lemma.

Lemma 18 (iterative lemma) *Let $K = K_0 + R$, analytic on $\mathcal{B}_{\kappa,\sigma,\eta,\rho,\zeta}$, with $K_0 = \langle\tilde\omega_\varepsilon, I\rangle + bE = \langle b\hat\omega_\varepsilon, I\rangle + bE$ and $R = \mathcal{O}_2(E,I)$. For $\hat\kappa<\kappa$, $\hat\sigma<\sigma$, $\hat\eta<\eta$, $\hat\rho<\rho$, $\hat\zeta<\zeta$ given, assume that*

$$\|R\|_{\kappa,\sigma,\eta,\rho,\zeta} \le \frac{\min\left(\hat\kappa\hat\eta, \hat\sigma\hat\eta, e\hat\rho\hat\zeta\right)}{8e^2c}, \tag{95}$$

where $c = 2(\kappa+2\sigma)$. Then, there exists a symplectic map

$$\Upsilon^{(1)} : \mathcal{B}_{\kappa-\hat\kappa,\sigma-\hat\sigma,\eta-\hat\eta,\rho-\hat\rho,\zeta-\hat\zeta} \longrightarrow \mathcal{B}_{\kappa,\sigma,\eta,\rho,\zeta}$$

such that

$$K\circ\Upsilon^{(1)} = K_0 + R^{(1)},$$

with $\Upsilon^{(1)} - \mathrm{id} = \mathcal{O}(E,I)$ and $R^{(1)} = \mathcal{O}_3(E,I)$. Besides, one has the bounds

$$\left\|R^{(1)}\right\|_{\kappa-\hat\kappa,\sigma-\hat\sigma,\eta-\hat\eta,\rho-\hat\rho,\zeta-\hat\zeta} \le \frac{16c\,\|R\|^2_{\kappa,\sigma,\eta,\rho,\zeta}}{\min\left(\hat\kappa\hat\eta, \hat\sigma\hat\eta, e\hat\rho\hat\zeta\right)}, \tag{96}$$

$$\left|\Upsilon^{(1)} - \mathrm{id}\right|_{\kappa-\hat\kappa,\sigma-\hat\sigma,\eta-\hat\eta,\rho-\hat\rho,\zeta-\hat\zeta} \le 2c\,\|R\|_{\kappa,\sigma,\eta,\rho,\zeta} \cdot \left(\frac{1}{\hat\eta}, \frac{1}{\min(\hat\kappa,\hat\sigma)}, \frac{1}{\hat\zeta}, \frac{1}{\hat\rho}\right). \tag{97}$$

We first establish the following lemma, which provides an estimate for a solution of the linearized equation appearing in the proof of Lemma 18. We point out that such an estimate does not involve small divisors. Its only difficulty lies in the choice of the suitable solution to be estimated. Our approach is analogous to the one introduced in [Sau01] (see also [RW00]), and we give here the proof for completeness.

Lemma 19 *Let R analytic on $\mathcal{B}_{\kappa,\sigma,\eta,\rho,\zeta}$. Then, the linear equation*

$$\partial_S U + \langle\hat\omega_\varepsilon, \partial_\varphi U\rangle = \frac{1}{b}R \tag{98}$$

has a solution U satisfying the bound

$$\|U\|_{\kappa,\sigma,\eta,\rho,\zeta} \le c\,\|R\|_{\kappa,\sigma,\eta,\rho,\zeta} \tag{99}$$

(with c as in Lemma 18).

Proof of Lemma 19. We use the Fourier expansions: $U = \sum_{k\in\mathbb{Z}^n} U_k(S,E,I)e^{i\langle k,\varphi\rangle}$, and analogously for R. Then, we get from equation (98) that each coefficient U_k has to be a solution of the following simple linear ordinary differential equation (with respect to the variable S):

$$\partial_S U_k + i\,\langle k, \hat\omega_\varepsilon\rangle\, U_k = \frac{1}{b}R_k.$$

Following [Sau01] we choose, among the solutions of this equation, a very concrete one:

$$U_k(S,E,I) = \frac{1}{b}\int_{ia_k}^{S} R_k\left(S',E,I\right) e^{-i\langle k,\hat{\omega}_\varepsilon\rangle(S-S')}\mathrm{d}S',$$

where

$$a_k = \begin{cases} \sigma & \text{if } \langle k,\hat{\omega}_\varepsilon\rangle > 0, \\ 0 & \text{if } \langle k,\hat{\omega}_\varepsilon\rangle = 0, \\ -\sigma & \text{if } \langle k,\hat{\omega}_\varepsilon\rangle < 0. \end{cases}$$

It is then clear that we have $\left|e^{-i\langle k,\hat{\omega}_\varepsilon\rangle(S-S')}\right| \le 1$ along the path of integration. Then, using that $b \ge 1/2$ and $|S - ia_k| \le \kappa + 2\sigma$, we obtain

$$|U_k|_{\kappa,\sigma,\eta,\zeta} \le 2(\kappa+2\sigma)\,|R_k|_{\kappa,\sigma,\eta,\zeta}\,,$$

and we easily deduce (99). □

Remarks.

1. The bounds would be worse if, instead of the norm $\|\cdot\|$ (that takes into account the Fourier coefficients), the supremum norm $|\cdot|$ was used. In fact, a denominator $\hat{\rho}^n$, as in property (P1), would appear.
2. In this lemma (and hence in this appendix), we do not need the frequency vector to satisfy a Diophantine condition like (H2).

Proof of Lemma 18. In this proof we consider, for $j = 0,1,2$, the widths $\kappa_j = \kappa - j\hat{\kappa}/2$, $\sigma_j = \sigma - j\hat{\sigma}/2$, $\eta_j = \eta - j\hat{\eta}/2$, $\rho_j = \rho - j\hat{\rho}/2$, $\zeta_j = \zeta - j\hat{\zeta}/2$, the domains $\mathcal{B}_j = \mathcal{B}_{\kappa_j,\sigma_j,\eta_j,\rho_j,\zeta_j}$, and the associated norms $\|\cdot\|_j$ and $|\cdot|_j$ on these domains (as introduced in Section 1.5).

Using the Lie method, we are going to construct $\Upsilon^{(1)}$ as the time-one flow of a generating Hamiltonian U. We write

$$K\circ\Upsilon^{(1)} = K_0 + R + \{K_0, U\} + R^{(1)},$$

and we choose U such that $\{U, K_0\} = R$, that is, the function U has to be a solution of equation (98). According to Lemma 19, we can choose U satisfying bound (99): $\|U\|_0 \le c\,\|R\|_0$. Then, denoting Υ_t the time-t Hamiltonian flow associated to U, and $\Upsilon^{(1)} = \Upsilon_1$,

$$R^{(1)} = K\circ\Upsilon^{(1)} - K_0 = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\left[(K_0 + tR)\circ\Upsilon_t\right]\mathrm{d}t = \int_0^1 t\left[\{R,U\}\circ\Upsilon_t\right]\mathrm{d}t, \tag{100}$$

which is quadratic with respect to the size of R. Note also that if $R = \mathcal{O}_2(E,I)$, we get $U = \mathcal{O}_2(E,I)$, $\Upsilon_t - \mathrm{id} = \mathcal{O}(E,I)$, and $R^{(1)} = \mathcal{O}_3(E,I)$.

Concerning the bounds, from (95) we deduce that

$$\|U\|_0 \le \frac{1}{8e^2}\min\left(\hat{\kappa}\hat{\eta}, \hat{\sigma}\hat{\eta}, e\hat{\rho}\hat{\zeta}\right),$$

and we then have $\Upsilon_t : \mathcal{B}_2 \longrightarrow \mathcal{B}_1$ with the bound

$$|\Upsilon_t - \mathrm{id}|_2 \le 2\,\|U\|_0 \cdot \left(\frac{1}{\hat{\eta}}, \frac{1}{\min(\hat{\kappa},\hat{\sigma})}, \frac{1}{\hat{\zeta}}, \frac{1}{\hat{\rho}}\right),$$

which clearly implies (97). On the other hand, we have

$$\|\{R,U\}\circ\Upsilon_t\|_2 \le 2\,\|\{R,U\}\|_1 \le \frac{32\,\|R\|_0\cdot\|U\|_0}{\min\left(\hat{\kappa}\hat{\eta}, \hat{\sigma}\hat{\eta}, e\hat{\rho}\hat{\zeta}\right)},$$

and using (100) we obtain bound (96) for $\left\|R^{(1)}\right\|_2$. □

Proof of Theorem 4 (flow-box theorem). Recall that, in this appendix, the initial widths in the statement of Theorem 4, κ, σ, η_1, ρ_3, $\beta\delta$, are written as κ, σ, η, ρ, ζ, and that the final widths κ_1, σ_1, η_2, ρ_4, $\beta_1\delta$ are written as $\kappa-\hat{\kappa}$, $\sigma-\hat{\sigma}$, $\eta-\hat{\eta}$, $\rho-\hat{\rho}$, $\zeta-\hat{\zeta}$. We also define:

$$q_1 = \frac{\min\left(\hat{\kappa}\hat{\eta}, \hat{\sigma}\hat{\eta}, e\hat{\rho}\hat{\zeta}\right)}{8e^2c}, \qquad q_2 = \frac{e^2}{2}q_1, \qquad \widetilde{q} = 2c\cdot\left(\frac{1}{\hat{\eta}}, \frac{1}{\min(\hat{\kappa},\hat{\sigma})}, \frac{1}{\hat{\zeta}}, \frac{1}{\hat{\rho}}\right).$$

Note that, since we take $\hat{\rho}, \hat{\zeta} \sim \delta$, and $\hat{\kappa}, \hat{\sigma}, \hat{\eta} \sim 1$, we have

$$q_1, q_2 \sim \delta^2, \qquad \widetilde{q} \sim \left(1, 1, \frac{1}{\delta}, \frac{1}{\delta}\right).$$

We are going to carry out an iterative use of Lemma 18. So we begin with $K^{(0)} = K_0 + Q^{(0)}$ on the domain $\mathcal{B}_0 = \mathcal{B}_{\kappa,\sigma,\eta,\rho,\zeta}$. For $p \geq 1$, we choose the successive reductions $\hat{\kappa}_p = \hat{\kappa}/2^p$, $\hat{\sigma}_p = \hat{\sigma}/2^p$, $\hat{\eta}_p = \hat{\eta}/2^p$, $\hat{\rho}_p = \hat{\rho}/2^p$, $\hat{\zeta}_p = \hat{\zeta}/2^p$. We write $\kappa_p = \kappa - (\hat{\kappa}_1 + \cdots + \hat{\kappa}_p)$, which tends to $\kappa - \hat{\kappa}$, and analogously σ_p, η_p, ρ_p, ζ_p. We consider the domains $\mathcal{B}_p = \mathcal{B}_{\kappa_p,\sigma_p,\eta_p,\rho_p,\zeta_p}$, and denote $\|\cdot\|_p$ and $|\cdot|_p$ the norms on $\mathcal{B}_p$ (see Section 1.5). Besides, we consider the final domain $\mathcal{B}_\infty = \mathcal{B}_{\kappa-\hat{\kappa},\sigma-\hat{\sigma},\eta-\hat{\eta},\rho-\hat{\rho},\zeta-\hat{\zeta}}$ and the norms $\|\cdot\|_\infty$ and $|\cdot|_\infty$.

In order to describe the p-th iteration, we start with $K^{(p-1)} = K_0 + Q^{(p-1)}$ on $\mathcal{B}_{p-1}$, and write $\mu_{p-1} = \left\|Q^{(p-1)}\right\|_{p-1}$. The iterative lemma can be applied if

$$\mu_{p-1} \leq \frac{\min\left(\hat{\kappa}_p\hat{\eta}_p, \hat{\sigma}_p\hat{\eta}_p, e\hat{\rho}_p\hat{\zeta}_p\right)}{8e^2c} = \frac{q_1}{4^p}, \tag{101}$$

and we then get $\Upsilon^{(p)} : \mathcal{B}_p \longrightarrow \mathcal{B}_{p-1}$, with $\Upsilon^{(p)} - \mathrm{id} = \mathcal{O}(E, I)$, such that $K^{(p)} = K^{(p-1)} \circ \Upsilon^{(p)} = K_0 + Q^{(p)}$, with the bounds

$$\mu_p \leq \frac{16c\mu_{p-1}^2}{\min\left(\hat{\kappa}_p\hat{\eta}_p, \hat{\sigma}_p\hat{\eta}_p, e\hat{\rho}_p\hat{\zeta}_p\right)} = \frac{4^p\mu_{p-1}^2}{q_2}, \tag{102}$$

$$\left|\Upsilon^{(p)} - \mathrm{id}\right|_p \leq 2c\mu_{p-1}\left(\frac{1}{\hat{\eta}_p}, \frac{1}{\min(\hat{\kappa}_p,\hat{\sigma}_p)}, \frac{1}{\hat{\zeta}_p}, \frac{1}{\hat{\rho}_p}\right) = 2^p\mu_{p-1}\widetilde{q}.$$

Let us check that (101) holds for every $p \geq 1$, and this implies that all the iterations can be carried out, under the condition of Theorem 4. First, for the first iteration ($p = 1$) we need

$$\mu_0 \leq \frac{q_1}{4} \sim \delta^2.$$

Assume now that $p \geq 1$ iterations of the procedure have already been done. Using induction, we see from (102) that

$$\mu_p \leq \frac{q_2}{4^{p+2}}\left(\frac{16\mu_0}{q_2}\right)^{2^p}.$$

Then, if

$$\mu_0 \leq \frac{q_2}{16} \sim \delta^2,$$

we deduce that

$$\mu_p \leq \frac{q_2}{4^{p+2}} \cdot \frac{16\mu_0}{q_2} = \frac{\mu_0}{4^p} \leq \frac{q_1}{4^{p+1}},$$

and the $(p+1)$-th iteration can also be carried out. The condition $\mu_0 \preceq \delta^2$ is the one required in Theorem 4.

Since μ_p tends to 0 as $p \to \infty$, it becomes clear that the Hamiltonians $K^{(p)}$ tend to K_0. On the other hand, from the bound

$$\sum_{p=1}^{\infty}\left|\Upsilon^{(p)} - \mathrm{id}\right|_p \leq \sum_{p=1}^{\infty} 2^p\mu_{p-1}\widetilde{q} \leq 4\mu_0\widetilde{q} \sim \mu_0\left(1, 1, \frac{1}{\delta}, \frac{1}{\delta}\right), \tag{103}$$

we see that the maps $\Upsilon^{(1)} \circ \cdots \circ \Upsilon^{(p)}$ tend to a symplectic map $\Upsilon : \mathcal{B}_\infty \longrightarrow \mathcal{B}_0$, with $\Upsilon - \mathrm{id} = \mathcal{O}(E, I)$, and (103) also gives a bound for $|\Upsilon - \mathrm{id}|_\infty$ (notice that the two first and the two last components in this bound are put together in Theorem 4). □

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