CONTACT STRUCTURES WITH SINGULARITIES

EVA MIRANDA AND CÉDRIC OMS

ABSTRACT. We study singular contact structures, which are tangent to a given smooth hypersurface Z and satisfy certain transversality conditions. These singular contact structures are determined by the kernel of non-smooth differential forms, called b^m-contact forms having an associated critical hypersurface Z. We provide several constructions, prove local normal forms and study the induced structure on the critical hypersurface. In the last section of this paper we tackle the problem of existence of b^m-contact structures on a given manifold. We prove that convex hypersurfaces can be realized as critical set of b^{2k}-contact structures. In particular, in the 3-dimensional case, this construction yields the existence of a generic set of surfaces Z such that the pair (M, Z) is a b^{2k}-contact manifold and Z is its critical hypersurface.

1. INTRODUCTION

Contact structures appear naturally associated to regular level-sets $H = cst$ of symplectic manifolds whenever a Liouville vector field is transverse to it. This construction is connected to the study of Hamiltonian systems as the function H defining the hypersurface can be taken to be the Hamiltonian of the system. Whenever the orbits of the Hamiltonian system present singularities (for instance heteroclinic or homoclinic orbits), this associated contact structure will show a singularity and the Reeb vector field will be singular. This gives us a first motivation to analyze the singular counterpart to contact structures in order to take these situations into account.

We are also interested in these structures as the odd-dimensional cousins to singular symplectic structures largely explored in several papers by several authors recently [GMP, GMPS, KM].

From the contact perspective, there is an extra reason to consider these structures: Let M be an $(2n + 1)$-dimensional manifold with a hyperplane distribution denoted by ξ. If ξ is cooriented it can be written as the kernel of a one-form α. The distribution is contact if α satisfies the non-integrability condition $\alpha \wedge (d\alpha)^n \neq 0$ and geometrically this condition is on the antipodes of integrability. Under this light another motivation for this article is to export the notion of non-integrability to the setting of manifolds with boundary.

To the authors knowledge, the only work that has been done in this direction is the study of convex hypersurfaces initiated by Giroux [Gi1]. The approach carried out in this paper is different in the sense that we ask the hyperplane distribution ξ to be everywhere non-integrable except on the boundary, where we ask ξ to be tangent to the boundary. Hence, the hyperplane distribution flattens out when approaching the boundary. In other words, the manifold admits a hyperplane distribution that is nowhere integrable except on a codimension one submanifold that is integrable. This is in line with the programme of symplectic fillability [El] (see also [FMM] for its b-symplectic analog) since a geometrical structure is prescribed on the boundary.

Let us take an elementary, but important example in what follows. Locally, an odd-dimensional manifold with boundary is diffeomorphic to the half space $\mathbb{R}^{2n+1}_+ = \{(x_i, y_i, z) | z \geq 0, i = 1, \ldots, n\}$. Let us consider the set of vector fields tangent to the boundary and denote it by $\mathcal{S}$. Locally, $\mathcal{S}$ is spanned by $\langle z \frac{\partial}{\partial z}, \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, i = 1, \ldots, n \rangle$. One can prove that those vector fields are the sections of a

vector bundle that we call b-tangent bundle. By replacing the tangent bundle by the b-tangent bundle, we can construct differential forms which are non-smooth on the boundary, but behave well as they are evaluated on $\mathcal{S}$. The non-smooth differential forms dual to the vector fields locally generating $\mathcal{S}$ are given by $dx_i, dy_i, \frac{dz}{z}$. The kernel of the non-smooth differential form $\frac{dz}{z} + \sum_{i=1}^{n} x_i dy_i$ meets the desired conditions: away from the boundary, the form is smooth and the usual de Rham exterior derivative applies to show that it satisfies the non-integrability condition. On the boundary, the vector field $z\frac{\partial}{\partial z}$ is zero, so the hyperplane distribution becomes tangent to the boundary.

The language of those non-smooth forms in the case of manifolds with boundary is not new. The notions of b-tangent bundle were already introduced by Richard Melrose in [Me] as a framework to study differential calculus on manifolds with boundary. Recently, it regained a lot of attention in the Poisson and symplectic setting. Indeed, in the foundational work of Radko [R], she classifies a certain type of Poisson structures on closed surfaces, called *topologically stable Poisson surfaces*. Later, in [GMP], it is shown that those Poisson structures can be treated using symplectic techniques by using the Melrose language of b-tangent bundle and extending the de Rham derivative to this setting. Since then a lot has been done to understand the local and global behavior of this extension of symplectic manifolds, see for example [BDMOP, FMM, MO, GL] and references therein. This paper can be considered to be the first direction to an odd-dimensional counterpart of the aforementioned papers.

The investigation of existence of contact structures in all dimensions has a particularly rich history and led to many important developments in the field. We provide a partial answer in our setting by narrowly linking the existence problem of singular contact structures to convex hypersurfaces in Contact Geometry, thereby shedding new light on the theory of convex surfaces initiated by [Gi1].

Organization of this paper. After the introduction, we start reviewing the basics of b-symplectic geometry in Section 2 by explaining in greater details the construction of the b-tangent bundle and the extension of the de Rham exterior derivative. We also include a selection of theorems in b-symplectic geometry that we use in this paper. In Section 3, we then give the main definitions of this paper, namely the one of b-contact manifolds. We prove local normal forms for b-contact forms in Section 4. We will see in Section 5 that the right framework to study those geometric structures is the one of Jacobi manifolds. We continue by explaining the relation with b-symplectic geometry in Section 6. The induced structure by the b-contact structure on the boundary is explained in Section 7. We end up this article exploring the relation of b-contact manifolds to smooth contact structures following the techniques of [GMW] and proving existence theorems for b^m-contact structures on a given manifold. The constructions in Section 9 and 10 rely strongly on the existence of convex hypersurfaces on contact manifolds but also on the desingularization constructions in [GMW]. We include an appendix on computational aspects of the Jacobi structures associated to a given contact structure and recall the local normal theorem for Jacobi manifold proved in [DLM].

2. b-SYMPLECTIC SURVIVAL KIT

Let (M^n, Z) be a smooth manifold of dimension n with a hypersurface Z. In what follows, the hypersurface Z will be called *critical set*. Assume that there exist a global defining function for Z, that is $f : M \to \mathbb{R}$ such that $Z = f^{-1}(0)$. A vector field is said to be a b-vector field if it is everywhere tangent to the hypersurface Z. The space of b-vector fields is a Lie sub-algebra of the Lie algebra of vector fields on M. A natural question to ask is whether or not there exist a vector bundle such that its sections are given by the b-vector fields. A coordinate chart of a neighbourhood around a point $p \in Z$ is given by $\{(x_1, \ldots, x_{n-1}, f)\}$ and the b-vector fields restricted to this

neighbourhood form a locally free C^∞-module with basis

$$(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}, f\frac{\partial}{\partial f}).$$

By the Serre–Swan theorem [Sw], there exists an n-dimensional vector bundle which sections are given by the b-vector fields. We denote this vector bundle by ${}^b TM$, the b-*tangent bundle*. We now adopt the classical construction to obtain differential forms for this vector bundle. We denote the dual of this vector bundle by ${}^b T^*M := ({}^b TM)^*$ and call it the b-*cotangent bundle*. A b-form of degree k is the section of the kth exterior wedge product of the b-cotangent bundle: $\omega \in \Gamma(\Lambda^k({}^b T^*M)) := {}^b\Omega^k(M)$. To extend the de Rham differential to an exterior derivative for b-forms, we need a decomposition lemma.

Lemma 2.1. [GMP] *Let $\omega \in {}^b\Omega^k(M)$ be a b-form of degree k. Then ω decomposes as follows:*

$$\omega = \frac{df}{f} \wedge \alpha + \beta, \quad \alpha \in \Omega^{k-1}(M), \ \beta \in \Omega^k(M).$$

Equipped with this decomposition lemma, we extend the exterior derivative by putting

$$d\omega := \frac{df}{f} \wedge d\alpha + d\beta.$$

It is clear that this is indeed an extension of the usual exterior derivative and that $d^2 = 0$.

Definition 2.2. An even-dimensional b-manifold M^{2n} with a b-form $\omega \in {}^b\Omega^2(M)$ is b-symplectic if $d\omega = 0$ and $\omega^n \neq 0$ as element of $\Lambda^{2n}({}^b T^*M)$.

Outside of the critical set Z, we are dealing with symplectic manifolds. On the critical set, the local normal form of the b-symplectic form is given by the following theorem.

Theorem 1 (b-Darboux theorem). [GMP] *Let ω be a b-symplectic form on (M^{2n}, Z). Let $p \in Z$. Then we can find a local coordinate chart $(x_1, y_1, \ldots, x_n, y_n)$ centered at p such that hypersurface Z is locally defined by $y_1 = 0$ and*

$$\omega = dx_1 \wedge \frac{dy_1}{y_1} + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

The b-Darboux theorem for b-symplectic forms has been proved using two different approaches. The first proof follows Moser path method, that can be adapted in the b-setting. Another way of proving it is to show that a b-form of degree 2 on a $2n$-dimensional b-manifold is b-symplectic if and only if its dual bi-vector field is a Poisson vector field Π whose maximal wedge product is transverse to the zero section of the vector bundle $\Lambda^{2n}({}^b TM)$, that is $\Pi^n \pitchfork 0$. A Poisson manifold satisfying this condition is called a b-*Poisson manifold*. Using the transversality condition in Weinstein's splitting theorem, one sees that the Poisson structure is of the form

(2.3) $$\Pi = y_1\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}.$$

Furthermore, Weinstein splitting theorem implies that the critical set of a b-symplectic manifold is a regular codimension one foliation of symplectic leaves. Even better, it is proved in [GMP] that the critical set is a cosymplectic manifold[1].

The relation of b^m-symplectic manifolds to symplectic manifolds and the less well-known folded symplectic manifolds was investigated in [GMW].

---

[1]A cosymplectic manifold is manifold M^{2n+1} together with a closed one-form η and a closed two-form ω such that $\eta \wedge \omega^n$ is a volume form.

Theorem 2 ([GMW]). *Let ω be a b^m-symplectic structure on a manifold M and let Z be its critical hypersurface.*

- *If m is even, there exists a family of symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighborhood of Z and for which the family of bivector fields $(\omega_\epsilon)^{-1}$ converges in the C^{2k-1}-topology to the Poisson structure ω^{-1} as $\epsilon \to 0$.*
- *If m is odd, there exists a family of folded symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighborhood of Z.*

A direct consequence of this theorem is that any orientable manifold admitting a b^{2k}-symplectic structure admits a symplectic structure.

3. b-CONTACT MANIFOLDS

In this section we introduce the main objects of this paper. Inspired by the definition of b-symplectic manifolds, we define the contact case as follows:

Definition 3.1. Let (M, Z) be a (2n+1)-dimensional b-manifold. A b-contact structure is the distribution given by the kernel of a one b-form $\xi = \ker \alpha \subset {}^bTM$, $\alpha \in {}^b\Omega^1(M)$, that satisfies $\alpha \wedge (d\alpha)^n \neq 0$ as a section of $\Lambda^{2n+1}({}^bT^*M)$. We say that α is a b-contact form and the pair (M, ξ) a b-contact manifold.

The hypersurface Z is called *critical hypersurface*. In what follows, we always assume that Z is non-empty. Away from the critical set Z the b-contact structure is a smooth contact structure. The former definition fits well with what is standard in contact geometry where coorientable contact manifolds are considered (i.e. there exists a defining contact form with kernel the given contact structure).

Example 3.2. Let (M, Z) be a b-manifold of dimension n. Let $z, y_i, i = 2, \ldots, n$ be the local coordinates for the manifold M on a neighbourhood of a point in Z, with Z defined locally by $z = 0$ and $x_i, i = 1, \ldots, n$ be the fiber coordinates on ${}^bT^*M$, then the canonical one-form is given in these coordinates by

$$x_1 \frac{dz}{z} + \sum_{i=2}^n x_i dy_i.$$

The bundle $\mathbb{R} \times {}^bT^*M$ is a b-contact manifold with b-contact structure defined as the kernel of the one-form

$$dt + x_1 \frac{dz}{z} + \sum_{i=2}^n x_i dy_i,$$

where t is the coordinate on $\mathbb{R}$. The critical set is given by $\tilde{Z} = Z \times \mathbb{R}$. Using the definition of the extended de Rham derivative, one checks that $\alpha \wedge (d\alpha)^n \neq 0$. Away from $\tilde{Z}$, $\xi = \ker \alpha$ is a non-integrable hyperplane field distribution, as in usual contact geometry. On the critical set however, ξ is tangent to $\tilde{Z}$. This comes from the definition of b-vector fields. Since the rank of ξ can drop by 1 on $\tilde{Z}$, we cannot say that ξ is a hyperplane field.

As we will see in the next example, the rank does not necessarily drop.

Example 3.3. Let us take $\mathbb{R}^{2n+1}$ with coordinates $(z, x_1, \ldots, x_n, y_1, \ldots, y_n)$. We consider the distribution of the kernel of $\alpha = \frac{dz}{z} + \sum_{i=1}^n x_i dy_i$. The rank does not drop on the critical set: on the critical set, the distribution is spanned by $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, i = 1, \ldots n\}$.

Using the two last examples and a generalization of Moebius transformations, we can construct b-contact structures on the unit ball with critical set given by the unit sphere.

Example 3.4. Let us denote the unit ball in dimension n by D^n and the half-space, that is R^n where the first coordinate is positive, by $\mathbb{R}^n_+$. The Moebius transformation maps the open half plane diffeomorphically two open 2-ball by the following map:

$$\Phi : \{z \in \mathbb{C} | \Re(z) > 0\} \to D^2$$

$$z \mapsto \frac{z-1}{z+1}.$$

This map can easily be generalized to all dimension and the inverse is given by

$$\Psi : D^n \to \mathbb{R}^n_+$$

$$(x_1, \ldots, x_n) \mapsto \frac{1}{(x_1-1)^2 + \sum_{i=2}^n x_i^2}\Big(\sum_{i=1}^n x_i^2 - 1, 2x_2, \ldots, 2x_n\Big).$$

We now provide $\mathbb{R}^{2n+1}_+$ with the b-contact structures described in Example 3.2 (respectively 3.3) and pull-back the b-contact form. We obtain hence two different b-contact structures on the unit ball and the critical set is given by unit sphere S^{2n-2}.

Example 3.5. A compact example admitting a b-contact structure is given by $S^2 \times S^1$. Let us consider the 2-sphere S^2, with coordinates (θ, h) where $\theta \in [0, 2\pi]$ is the angle and $h \in [0, 1]$ is the height, and the 1-sphere S^1 with coordinate $\varphi \in [0, 2\pi]$. Then $(S^2 \times S^1, \alpha = \sin \varphi d\theta + \cos \varphi \frac{dh}{h})$ is a b-contact manifold. Once more, the rank changes when $\cos \varphi = 0$, where instead of a plane-distribution, we are dealing with a line distribution.

Example 3.6. (Product examples) Let (N^{2n+1}, α) be a b-contact manifold and let $(M^{2m}, d\lambda)$ be an exact symplectic manifold, then $(N \times M, \alpha + \lambda)$ is a b-contact manifold. It is easy to check that $\tilde{\alpha} = \alpha + \lambda$ satisfies $\tilde{\alpha} \wedge (d\tilde{\alpha})^{n+m} \neq 0$.

In the same way if (N^{2n+1}, α) is a contact manifold and $(M^{2m}, d\lambda)$ be an exact b-symplectic manifold (where exactness is understood in the b-complex), then $(N \times M, \alpha + \lambda)$ is a b-contact manifold. These product examples can even be endowed with additional structures such as group actions or integrable systems. For instance we can produce examples of toric b-contact manifolds combining the product of toric contact manifolds in [Le] with (exact) toric b-symplectic manifolds (see [GMPS]). We can also combine the techniques in [KM] for b-symplectic manifolds and [B] (among others) for contact manifolds to produce examples of integrable systems on these manifolds.

4. THE b-CONTACT DARBOUX THEOREM

In usual contact geometry, the Reeb vector field R_α of a contact form α is given by the equations

$$\begin{cases} i_{R_\alpha} d\alpha = 0 \\ \alpha(R_\alpha) = 1. \end{cases}$$

In the case where we change the tangent bundle by ${}^b TM$, the existence is given by the same reasoning: $d\alpha$ is a bilinear, skewsymmetric 2-form on the space of b-vector fields ${}^b TM$, hence the rank is an even number. As $\alpha \wedge (d\alpha)^n$ is non-vanishing and of maximum degree, the rank of $d\alpha$ must be $2n$, its kernel is 1-dimensional and α is non-trivial on that line field. So a global vector field is defined by the normalization condition.

By the same reasoning, we can define the b-contact vector fields: for every function $H \in C^\infty(M)$, there exist a unique b-vector field X_H defined by the equations

$$\begin{cases} i_{X_H} \alpha = H \\ i_{X_H} d\alpha = -dH + R_\alpha(H)\alpha. \end{cases}$$

A direct computation yields that in Example 3.2, the Reeb vector field is given by $\frac{\partial}{\partial t}$. In Example 3.3, the Reeb vector field is given by $z\frac{\partial}{\partial z}$ and hence singular. We will see that, roughly speaking, the Reeb vector field locally classifies b-contact structures.

We now prove a Darboux theorem for b-contact manifolds. The proof follows the one of usual contact geometry as in [Ge]. More precisely, it makes use of Moser's path method. There are two differences from the standard Darboux theorem: the first one is that there exist two local models, depending on whether or not the Reeb vector field is vanishing on the critical set Z. The second one is that in the case where the Reeb vector field is singular, the local expression of the contact form only holds pointwise, see for instance Example 4.5. Furthermore, in the case where the Reeb vector field is singular, this linearization is done up to multiplication of a non-vanishing function. The proof is not following Moser's path method in this case as the flow of the Reeb vector field is stationary.

Theorem 3. *Let α be a b-contact form inducing a b-contact structure ξ on a b-manifold (M, Z) of dimension $(2n + 1)$ and $p \in Z$. We can find a local chart $(\mathcal{U}, z, x_1, y_1, \ldots, x_n, y_n)$ centered at p such that on $\mathcal{U}$ the hypersurface Z is locally defined by $z = 0$ and*

  (1) *if $R_p \neq 0$*
     (a) *ξ_p is singular, then*

$$\alpha|_{\mathcal{U}} = dx_1 + y_1\frac{dz}{z} + \sum_{i=2}^{n} x_i dy_i,$$

     (b) *ξ_p is regular, then*

$$\alpha|_{\mathcal{U}} = dx_1 + y_1\frac{dz}{z} + \frac{dz}{z} + \sum_{i=2}^{n} x_i dy_i,$$

  (2) *if $R_p = 0$, then $\tilde{\alpha} = f\alpha$ for $f(p) \neq 0$, where*

$$\tilde{\alpha}_p = \frac{dz}{z} + \sum_{i=1}^{n} x_i dy_i.$$

To distinguish both local models, we call the first one *regular* and the second one *singular* model, depending whether or not the Reeb vector field is singular or not.

Proof. We may assume without loss of generality that $M = \mathbb{R}^{2n+1}$ and that p is the origin of $\mathbb{R}^{2n+1}$. Let us choose linear coordinates on $T_p\mathbb{R}^{2n+1}$. By the non-integrability condition, $d\alpha$ has rank $2n$ and α is non-trivial on the kernel of $d\alpha$. We first choose the vector belonging to the kernel of $d\alpha$ and then complete a symplectic basis of $d\alpha$.

Let us first treat the case where $\ker d\alpha \subset T_pZ$: We choose x_1 such that $\frac{\partial}{\partial x_1} \in \ker d\alpha$ and $\alpha(\frac{\partial}{\partial x_1}) = 1$. Now let us take $V \in \ker \alpha$, but $V \notin T_pZ$ such that $i_V d\alpha \neq 0$. As $V \notin T_pZ$, V belongs to the kernel of the a vector bundle morphism

$$^bTM|_Z \to TZ$$

as explained in [GMP]. We take the coordinate z such that $V = z\frac{\partial}{\partial z}$. We then choose a coordinate y_1 such that $\frac{\partial}{\partial y_1} \in \ker \alpha$ and $d\alpha(z\frac{\partial}{\partial z}, \frac{\partial}{\partial y_1}) = 1$.

We complete a symplectic basis of $d\alpha$ and we can choose the remaining $2n - 2$ coordinates x_i and y_i in both cases so that for all $i = 2, \ldots, n$ that $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \in T_pZ$.

We now set

$$(4.1) \qquad \alpha_0 = dx_1 + y_1\frac{dz}{z} + \sum_{i=2}^{n} x_i dy_i \text{ when } \ker d\alpha \subset Z$$

when ξ_p is singular and when ξ_p is regular we set

$$\tilde{\alpha}_0 = dx_1 + y_1 \frac{dz}{z} + \frac{dz}{z} + \sum_{i=2}^{n} x_i dy_i \text{ when } \ker d\alpha \subset Z.$$

By the choice of the basis, it is clear that at the origin,

$$\begin{cases} \alpha = \alpha_0 \\ d\alpha = d\alpha_0 \end{cases}$$

when ξ_p is singular. We only work out the details in this case, as the case ξ_p regular works analogously.

Note that, until this stage, we only used linear algebra arguments, which are more involved due to the structure of the vector bundle ${}^b TM$. Let us now apply Moser's path method. In a neighbourhood of p, we consider the family of b-forms of degree 1

$$\alpha_t = (1-t)\alpha_0 + t\alpha \text{ for } t \in [0,1].$$

By the choice of basis, it is clear that at the origin,

$$\begin{cases} \alpha_t = \alpha \\ d\alpha_t = d\alpha \end{cases}$$

and so α_t is a path of b-contact forms in a neighbourhood of the origin. We want to show that there exist an isotopy $\psi_t : \mathcal{U} \mapsto \mathbb{R}^{2n+1}$ satisfying

(4.2)
$$\begin{cases} \psi_t^* \alpha_t = \alpha_0 \\ \psi_t(p) = p \\ \psi_t|_Z \subset Z. \end{cases}$$

Differentiating the first equation, we obtain $\mathcal{L}_{X_t}\alpha_t + \dot{\alpha}_t = 0$, where $X_t(p) = \frac{d\psi_s}{ds}(\psi_t^{-1}(p))\big|_{s=t}$. Inserting the splitting $X_t = H_t R_{\alpha_t} + Y_t$, where $H_t \in C^\infty(M)$ and $Y_t \in \ker \alpha_t$ and applying Cartan's formula, we obtain

(4.3)
$$i_{Y_t} d\alpha_t + dH_t + \dot{\alpha}_t = 0.$$

Evaluating this differential equation in the Reeb vector field R_{α_t}, we obtain

(4.4)
$$dH_t(R_{\alpha_t}) + \dot{\alpha}_t(R_{\alpha_t}) = 0.$$

This equation can be solved locally around the point p, as we can assume without loss of generality that R_{α_t} does not have closed orbits around that point. This is due to the fact that $R_{\alpha_t} \neq 0$. In fact, by the construction of the coordinate system $R_\alpha = \frac{\partial}{\partial x_1}$. Furthermore, as $\dot{\alpha}_t(p) = 0$, $dH_t(p) = 0$, and we can choose the constant of integration such that $H_t(p) = 0$. Once H_t is chosen, let us take a look at Equation (4.3), given by

$$i_{Y_t} d\alpha_t = -(dH_t + \dot{\alpha}_t).$$

We want to solve this equation for Y_t. By the previous observation and the fact that $d\alpha_t$ is a b-symplectic form, we obtain that $Y_t(p) = 0$, so $X_t(p) = 0$. Furthermore, it is clear that Y_t is a b-vector field because $d\alpha$ is a b-form. Integrating the vector field X_t gives us the isotopy ψ_t, satisfying the conditions of (4.2). This proves the first part of the theorem.

Let us now consider the case where $\ker d\alpha \not\subseteq T_p Z$, which corresponds to the case where $R_p = 0$ and $d\alpha$ is a smooth de Rham form. A b-form decomposes as $f\frac{dz}{z} + \beta$, where z is a defining function. As $d\alpha$ is smooth, the function f can only depend on z on Z and hence, $f(p) \neq 0$ as we would be in the smooth case otherwise. We choose a neighbourhood $\mathcal{U}$ around the origin such that f is non-vanishing on that neighbourhood. By dividing by f, the b-form $\tilde{\alpha} = \frac{dz}{z} + \tilde{\beta}$ defines the same distribution. Now take a contractible $2n$-dimensional disk $D^{2n} \ni p$ in $\mathcal{U}$. As $(D, d\alpha)$ is

symplectic, we know by applying Darboux theorem for symplectic forms (we assume the disk D small enough), that there exist $2n$ functions x_i, y_i such that locally $d\alpha = \sum_{i=1}^n dx_i \wedge dy_i$. Now consider the b-form $\alpha - \sum_{i=1}^n x_i dy_i - \frac{dz}{z}$. This form is closed and smooth. Hence by Poincaré lemma for smooth forms, there exists a smooth function g such that

$$\tilde{\alpha} = \frac{dz}{z} + dg + \sum_{i=1}^n x_i dy_i.$$

We can change the defining function by $\tilde{z} = e^{-g}z$, so that $\frac{d\tilde{z}}{\tilde{z}} = \frac{dz}{z} + dg$. Now

$$\tilde{\alpha} = \frac{d\tilde{z}}{\tilde{z}} + \sum_{i=1}^n x_i dy_i.$$

As $\tilde{\alpha} \wedge (d\tilde{\alpha})^n = n\frac{d\tilde{z}}{\tilde{z}} \wedge \sum_{i=1}^n dx_i \wedge dy_i \neq 0$, the functions $\tilde{z}, x_i, y_i$ form a basis.   □

The following example shows that it is possible to have both local models appearing on one connected component of the critical set. Furthermore, it shows in the case where the Reeb vector field is singular, we can only prove the normal form pointwise and does not hold in a local neighbourhood as when the Reeb vector field is regular.

Example 4.5. $(S^2 \times S^1, \alpha = \sin\varphi d\theta + \cos\varphi\frac{dh}{h})$ where (θ, h) are the polar coordinates on S^2 and φ the coordinate on S^1. The Reeb vector field is given by $R = \sin\varphi\frac{\partial}{\partial\theta} + \cos\varphi h\frac{\partial}{\partial h}$.

Remark 4.6. It follows from the b-Darboux theorem that if $(M, \ker\alpha)$ be a b-contact manifold and $\ker\alpha_p$ is regular for $p \in Z$, then there is an open neighbourhood around p where $\ker\alpha$ is regular.

A well known result in contact geometry is Gray's stability theorem, asserting that on a closed manifold, smooth families of contact structures are isotopic. The proof uses Moser's path method that works well in b-geometry. One proves in the same line the following stability result for b-contact manifolds.

Theorem 4. *Let* (M, Z) *compact b-manifold and let* (ξ_t), $t \in [0, 1]$ *be a smooth path of b-contact structures. Then there exists an isotopy* ϕ_t *preserving the critical set Z such that* $(\phi_t)_*\xi_0 = \xi_t$, *or equivalently,* $\phi_t^*\alpha_t = \lambda_t\alpha_0$ *for a non-vanishing function* λ_t.

Proof. Assume that ϕ_t is the flow of a time dependent vector field X_t. Deriving the equation, we obtain

$$di_{X_t}\alpha_t + i_{X_t}d\alpha_t + \dot{\alpha} = \mu_t\alpha_t$$

where $\mu_t = \frac{\dot{\lambda}_t}{\lambda_t} \circ \phi_t^{-1}$. If X_t belongs to ξ_t, the first term of the last equation vanishes and applying then the Reeb vector field yields

$$\dot{\alpha}_t(R_{\alpha_t}) = \mu_t.$$

The equation given by

$$i_{X_t}d\alpha_t = \mu_t\alpha_t - \dot{\alpha}_t$$

then defines X_t because $(\mu_t\alpha_t - \dot{\alpha}_t)(R_{\alpha_t})$. We integrate the vector field X_t to find ϕ_t and as X_t is a vector field, tangent to the critical set, the flow preserves it.   □

The compactness condition is necessary as is shown in the next example.

Example 4.7. Consider the path of b-contact structures on $\mathbb{R}^3$ given by $\ker\alpha_t$ where $\alpha_t = (\cos\frac{\pi}{2}t - y\sin\frac{\pi}{2}t)\frac{dz}{z} + (\sin\frac{\pi}{2}t + y\cos\frac{\pi}{2}t)dx$. As $\alpha_0 = \frac{dz}{z} + ydx$ and $\alpha_1 = dx - y\frac{dz}{z}$, the two b-contact structures cannot be isotopic.

In the same lines, we prove the following semi-local result.

Theorem 5. *Let (M, Z) be a b-manifold and assume Z compact. Let $\xi_0 = \ker \alpha_0$ and $\xi_1 = \ker \alpha_1$ be two b-contact structures such that $\alpha_0|_Z = \alpha_1|_Z$. Then there exists a local isotopy ψ_t, $t \in [0, 1]$ in an open neighbourhood $\mathcal{U}$ around Z such that $\psi_t^* \alpha_t = \lambda_t \alpha_0$ and $\psi_t|_Z = \mathrm{Id}$ where λ_t is a family of non-vanishing smooth functions.*

Proof. The proof is done following Moser's path method. Put $\xi_t = (1-t)\xi_0 + t\xi_1$, $t \in [0, 1]$. Because the non-integrability condition is an open condition and $\xi_t|_Z = \xi_0|_Z = \xi_1|_Z$, there exists an open neighbourhood $\mathcal{U}$ containing Z such that ξ_t is a family of b-contact structures. We will prove that there exists an isotopy $\psi_t : \mathcal{U} \mapsto M$ such that $\psi_t^* \alpha_t = \lambda_t \alpha_0$, where λ_t is a non-vanishing smooth function and $\lambda_t|_Z = \mathrm{Id}$. Assume that ψ_t is the flow of a vector field X_t and differentiating, we obtain the following equation:

$$di_{X_t}\alpha_t + i_{X_t}d\alpha_t + \dot{\alpha}_t = \mu_t \alpha_t,$$

where $\mu_t = \frac{d}{dt}(\log |\lambda_t|) \circ \psi_t^{-1}$. Taking $X_t \in \xi_t$, this equation writes down

(4.8)
$$\dot{\alpha}_t + i_{X_t}d\alpha_t = \mu_t \alpha_t.$$

Applying the Reeb vector field to both sides, we obtain the equation that defines μ_t:

$$\mu_t = \dot{\alpha}_t(R_{\alpha_t}).$$

As $\dot{\alpha}_t|_Z = 0$, $\mu_t|_Z = 0$ and hence X_t is zero on Z. By non-degeneracy of $d\alpha_t$ on ξ_t there exists a unique $X_t \in \xi_t$ solving Equation 4.8. Integrating X_t yields the desired result. □

Note that this proof fails if one wants to prove stability of b-contact forms, that is we cannot assume that $\lambda_t = \mathrm{Id}$ in a neighbourhood of Z.

5. b-JACOBI MANIFOLDS

In the symplectic case, it is often helpful to look at b-symplectic manifolds as being the dual of a particular case of Poisson manifold. In contact geometry, Jacobi manifolds play this role.

Recall that a Jacobi structure on a manifold M is a triplet (M, Λ, R) where Λ is a smooth bi-vector field and R a vector field satisfying the following compatibility conditions

(5.1)
$$[\Lambda, \Lambda] = 2R \wedge \Lambda, \qquad\qquad [\Lambda, R] = 0,$$

where the bracket is the Schouten–Nijenhuis bracket. We refer the reader [V] and references therein for further information on Jacobi manifolds.

Definition 5.2. Let (M, Λ, R) be a Jacobi manifold of dimension $2n + 1$. We say that M is a b-Jacobi manifold if $\Lambda^n \wedge R$ cuts the zero section of $\Lambda^{2n+1}(TM)$ transversally.

Note that this definition is similar to the one of b-Poisson manifolds, in the sense that it also asks the top wedge power to be tranverse to the zero section. We denote the hypersurface given by the zero section of $\Lambda^{2n+1}(TM)$ by Z and we call it the *critical set*.

It is well-known that contact manifolds are a particular case of odd-dimensional Jacobi manifolds. A particular case of even-dimensional Jacobi manifolds are given by *locally conformally symplectic manifolds*.

Definition 5.3. A locally conformally symplectic manifold is a manifold M of dimension $2n$ equipped with a non-degenerate two-form $\omega \in \Omega^2(M)$ that is locally closed, which is equivalent to the existence of a closed 1-form $\alpha \in \Omega^1(M)$ such that $d\omega = \alpha \wedge \omega$.

Locally conformally symplectic manifold regained recent attention, notably in the work [CM].

We will prove that b-contact manifolds and b-Jacobi manifolds are dual in some sense, as will be explained in the next two propositions. Before doing so, let us note that in the case where the dimension of the Jacobi manifold is $\dim M = 2n$, we can given an similar definition to the one of Definition 5.2 by asking that Λ^{2n} cuts the zero-section of $\Lambda^{2n}(TM)$ transversally. It should be

possible to prove in the same lines that this case corresponds to locally conformally b-symplectic manifold.

Proposition 5.4. *Let $(M, \ker \alpha)$ be a b-contact manifold. Let Λ be the bi-vector field computed as in Equation A.1 in Appendix A and let R be the Reeb vector field. Then (M, Λ, R) is a b-Jacobi manifold.*

Proof. As being b-Jacobi is a local condition, we can work in a local coordinate chart. Outside of the critical set, α is a contact form. Hence we can compute Λ as in Equation A.1 in both local models of the Darboux theorem and Λ can smoothly be extended to the critical set Z. A straightforward computation now yields that for both local models $\Lambda^n \wedge R \pitchfork 0$. $\square$

Recall that to every Jacobi manifold (M, Λ, R), one can associate a homogenous Poisson manifold. Indeed, $(M \times \mathbb{R}, \Pi := e^{-\tau}(\Lambda + \frac{\partial}{\partial \tau} \wedge R))$ is a Poisson manifold because

$$
\begin{aligned}
[\Pi, \Pi] &= [e^{-\tau}\Lambda, e^{-\tau}\Lambda] + 2[e^{-\tau}\Lambda, e^{-\tau}\frac{\partial}{\partial \tau} \wedge R] + [e^{-\tau}\frac{\partial}{\partial \tau} \wedge R, e^{-\tau}\frac{\partial}{\partial \tau} \wedge R] \\
&= 2e^{-2\tau}[\Lambda, \Lambda] + 2(-e^{-\tau}\Lambda \wedge R) = 0.
\end{aligned}
$$

Furthermore, the later is said to be *homogenous* because the vector field $T = \frac{\partial}{\partial \tau}$ satisfies

$$ L_T P = -P. $$

This construction is called Poissonization. The same stays true in the b-scenario, although we need to assume that the b-Jacobi manifold is of odd dimension, as b-Poisson manifold are defined only for even dimensions.

Lemma 5.5. *The Poissonization of a b-Jacobi manifold of odd dimension is a homogenous b-Poisson manifold.*

Proof. The proof is a straightforward computation:

$$ \Pi^{n+1} = -e^{-(n+1)\tau}\frac{\partial}{\partial \tau} \wedge \Lambda^n \wedge R. $$

It follows from the definition of b-Jacobi that Π is transverse to the zero-section. $\square$

Proposition 5.6. *Let (M^{2n+1}, Λ, R) be a b-Jacobi manifold. Then M is a b-contact manifold.*

Proof. The proposition is based on the local normal form of Jacobi structures, which are proved in [DLM]. The main result is recalled in Appendix B. Let (M, Λ, R) be the b-Jacobi structure, so that $\Lambda^n \wedge R \pitchfork 0$. As usual, denote the critical hypersurface by $Z = (\Lambda^n \wedge R)^{-1}(0)$. First note that outside of Z, the leaf of the characteristic foliation is maximal dimensional. This is saying that outside of Z, the Jacobi structure is equivalent to a contact structure.

Consider a point $p \in Z$ and denote the leaf of the characteristic foliation by L. By the transversality condition, the dimension of the leaf needs to be of dimension $2n$ or $2n - 1$. Indeed, as $(M \times \mathbb{R}, e^{-\tau}(\frac{\partial}{\partial \tau} \wedge R + \Lambda))$ is b-Poisson, the critical set of $M \times \mathbb{R}$ is foliated by symplectic manifolds of codimension 2, that is of dimension $2n$. Hence the critical set restricted to the hypersurface $\{\tau = 0\}$, which is identified to be the critical set Z of the initial manifold M, is foliated by codimension 1 and codimension 2 leaves.

Let us first consider the case where at the point $x \in Z$, the leaf is of dimension $2n$. We will prove that this case corresponds to the case where the R is singular, vanishing linearly. Let us apply Theorem 5.9 of [DLM]. Hence the Jacobi manifold (N, Λ_N, E_N) (see Theorem 5.9) is of dimension 1, hence Λ_N is zero. Hence Λ is given by

$$ \Lambda = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_{i+n}} - \sum_{i=1}^{n} x^{i+n}\frac{\partial}{\partial x_{i+n}} \wedge E_N. $$

We now use the transversality condition on $\Lambda^n \wedge E_N$ to conclude that $E_N = z\frac{\partial}{\partial z}$. which is the same expression for the b-Jacobi structure associated to the b-contact form $\alpha = \frac{dz}{z} + \sum_{i=1}^n x_i dx_{i+n}$.

Let us consider the case where the leaf is of dimension $2n - 1$. We will see that this corresponds to the case where the Reeb vector field is regular. According to Theorem 5.11 in [DLM], the bi-vector field is given by

$$\Lambda = \Lambda_{2n-1} + \Lambda_N + E \wedge Z_N$$

where (N, Λ_N, Z_N) is a homogenous 2-dimensional Poisson manifold and $\Lambda_{2n-1} = \sum_{i=1}^{n-1}(x^{i+n-1}\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_i}) \wedge \frac{\partial}{\partial x_{i+q}}$. The transversality condition implies that $\Lambda_{2n-1}^{n-1} \wedge \Lambda_N \wedge \frac{\partial}{\partial x_0} \pitchfork 0$, hence Λ_N is a b-Poisson manifold. By [GMP], $\Lambda_N = z\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial y}$. The homogenous vector field Z_N is determined by equation $\mathcal{L}_{Z_N}\Lambda_N = -\Lambda_N$. Hence $Z_N = y\frac{\partial}{\partial y}$. Hence the Jacobi structure is given by $E = \frac{\partial}{\partial x_0}$ and

$$\Lambda = \sum_{i=1}^{n-1}(x^{i+n-1}\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_i}) \wedge \frac{\partial}{\partial x_{i+q}} + z\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial x_0} \wedge y\frac{\partial}{\partial y},$$

which is the Jacobi structure associated to the contact form $\alpha = dx_0 + y\frac{dz}{z} + \sum_{i=1}^{n-1} x_i dx_{i+q}$. $\qquad\square$

6. SYMPLECTIZATION AND CONTACTIZATION

Symplectic and contact manifolds are related to each other as follows. It is well-known that a contact manifold can be transformed into a symplectic one by *symplectization*: if (M, α) is a contact manifold, then $(M \times \mathbb{R}, d(e^t \alpha))$ (where t is the coordinate on $\mathbb{R}$) is a symplectic manifold. On the other hand, hypersurfaces of a symplectic manifold (M, ω) are contact, provided that there exist a vector field satisfying $\mathcal{L}_X\omega = \omega$ that is transverse to the hypersurface. Such a vector field is called Liouville vector field. The contact form on the hypersurface is given by the contraction of the symplectic form with the Liouville vector field, i.e. $\alpha = i_X\omega$.

We will show that the same holds in the b-category.

Example 6.1. Let $(W = \mathbb{R}^4, \omega = \frac{1}{z}dz \wedge dt + dx \wedge dy)$ be a b-symplectic manifold. A Liouville vector field is given by $X = \frac{1}{2}(z\log|z|\frac{\partial}{\partial z} + t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$. Note that Liouville vector fields are defined up to addition of symplectic vector fields, that is a vector field Y satisfying $\mathcal{L}_Y\omega = 0$[2]. Another Liouville vector field is for example given by $t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$.

Let us take a b-symplectic manifold (W, ω) of dimension $(2n + 2)$ and a Liouville vector field X on W that is transverse to a hypersurface H of W. Then $(H, i_X\omega)$ is a b-contact manifold of dimension $(2n+1)$ as $i_X\omega \wedge (di_X\omega)^n = \frac{1}{n+1}i_X(\omega^{n+1})$ is a volume form provided that X is tranverse to H. If H does not intersect the critical set, one obtains of course a smooth contact form. Due to the b-Darboux theorem, there are two local models for b-contact manifolds and we will see that we can obtain both structures, depending on the relative position of the hypersurface with the Reeb vector field on it.

Example 6.2. Let us take $(W = \mathbb{R}^4, \omega = \frac{1}{z}dz \wedge dt + dx \wedge dy)$ and the Liouville vector field $X = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$. The contraction of X with the b-symplectic form yields $i_X\omega = -\frac{t}{z}dz + xdy$. Let us take different hypersurfaces transverse to X and compute the induced b-contact form.

- If we take as hypersurface the hyperplane $M_1 = \{(1, y, -t, z), y, t, z \in \mathbb{R}\}$, which is transverse to X, we obtain $\alpha = dy + t\frac{dz}{z}$, which is the regular local model.
- If we take as hypersurface the hyperplane $M_2 = \{(x, y, -1, z), x, y, z \in \mathbb{R}\}$, which is transverse to X, we obtain $\alpha = \frac{dz}{z} + xdy$, which is the singular local model.

---

[2] We define the Lie derivative of a b-form using the Cartan formula and the extended definition of exterior derivative for b-forms.

Example 6.3. The three dimensional sphere admits b-contact structures. Consider the $\mathbb{R}^4$ with the standard b-symplectic structure $\frac{dz}{z} \wedge dt + dx \wedge dy$. The vector field $X = \frac{1}{2}(z \log |z| \frac{\partial}{\partial z} + t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y})$ is a Liouville vector field, which is transverse to the sphere centered at the origin of radius greater then 1. The critical set is given by the intersection of the sphere with the hyperplane $z = 0$. To define b-contact structures on the unit sphere, we use radial rescaling. We hence obtain a family of b-contact structures $\ker \alpha_r$ for $r > 1$. Note that by Gray stability theorem for b-contact structures, they are all isotopic.

Example 6.4. The unit cotangent bundle of a b-manifold have a natural b-contact structure. Let (M, Z) be a b-manifold of dimension n with coordinates $z, y_i, i = 2, \ldots, n$ as in Example 3.2. It is shown in [GMP] that the cotangent bundle has a natural b-symplectic structure defined by the b-form given by the exterior derivative $d\lambda = d\left(x_1 \frac{dz}{z} + \sum_{i=2}^{n} x_i dy_i\right)$. The unit b-cotangent bundle is given by ${}^bT_1^*M = \{(z_1, y_2, \ldots, y_n, x_1, \ldots, x_n) \in {}^bT^*M|\ \|x\| = 1\}$, where the norm is the usual Euclidean norm. The vector field $\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$ defined on the b-cotangent bundle ${}^bT^*M$ is a Liouville vector field, and is tranverse to the unit b-cotangent bundle, and hence induces a b-contact structure on it.

We saw that hypersurfaces of b-symplectic manifolds that are transverse to a Liouville vector field have an induced b-contact structure. The next lemma describes which model describes locally the b-contact structure.

Lemma 6.5. *Let (W, ω) be a b-symplectic manifold and X a Liouville vector field transverse to a hypersurface H. Let R be the Reeb vector field defined on H for the b-contact form $\alpha = i_X \omega$. Then $R \in H^\perp$, where $H^\perp$ is the symplectic orthogonal of H.*

Proof. The Reeb vector field defined on H satisfies $i_R(d\alpha)|_H = i_R(di_X\omega)|_H = i_R\omega|_H = 0$. $\qquad \square$

Hence if $H^\perp$ is generated by a singular vector field, the contact manifold (H, α) is locally of the second type as in the b-Darboux theorem. In the other case, the local model is given by the first type.

We now come back to the symplectization of a b-symplectic manifold.

Theorem 6. *Let (M, α) be a b-contact manifold. Then $(M \times \mathbb{R}, \omega = d(e^t\alpha))$ is a b-symplectic manifold.*

Proof. It is clear that ω is a closed b-form. Furthermore, direct computation yield

$$\left((e^t d\alpha)\right)^{n+1} = e^{t(n+1)} dt \wedge \alpha \wedge (d\alpha)^n,$$

which is non-zero as a b-form by the non-integrability condition. $\qquad \square$

It is easy to see that $\frac{\partial}{\partial t}$ is a Liouville vector field of the symplectization $(M \times \mathbb{R}, d(e^t\alpha))$, which is clearly transverse to the submanifold $M \times \{0\}$. Hence, we obtain the initial contact manifold (M, α). This gives us the following theorem.

Theorem 7. *Every b-contact manifold can be obtained as a hypersurface of a b-symplectic manifold.*

Remark 6.6. Another close relation between the symplectic and the contact world is the contactization: take an exact symplectic manifold, i.e. $(M, d\beta)$, then $(M \times \mathbb{R}, \beta + dt)$, where t is the coordinate on $\mathbb{R}$, is contact. This remains true in the b-case. Furthermore, it is clear that by this construction, we obtain b-contact forms of the first type, as the Reeb vector field is given by $\frac{\partial}{\partial t}$.

7. GEOMETRIC STRUCTURE ON THE CRITICAL SET

To determine the induced structure of the b-contact structure on the critical set, we compute the associated Jacobi structure. Let us briefly review some results on Jacobi manifolds, which can all be found in [V]. The Hamiltonian vector fields of a Jacobi manifold (M, Λ, R) are defined by $X_f = \Lambda^\sharp(df) + fR$. It can be shown that the distribution $\mathcal{C}(M) = \{X_f | f \in C^\infty(M)\}$

is involutive and invariant under the Hamiltonian flow. Stefan–Suessmann theorem asserts that $\mathcal{C}(M)$ integrates to a singular foliation. As $\mathcal{C}(M) = \mathrm{Im}\Lambda^\sharp + \langle R \rangle$, the leaves of this distribution are even-dimensional when $R \in \mathrm{Im}\Lambda^\sharp$ and odd-dimensional in the other case. The induced structure on odd-dimensional leaves turns out to be a contact structure. For odd-dimensional leaves, one obtains locally conformally symplectic leaves. The definition of locally symplectic manifolds is recalled in Definition 5.3.

The computation of a Jacobi structure associated to a contact structure is explained in Appendix A. As we have proved a local norm theorem, we can use the two local models to compute the associate Jacobi structure and check in both cases if $R \in \Lambda^\sharp$. We will prove

Theorem 8. *Let $(M^{2n+1}, \xi = \ker \alpha)$ be a b-contact manifold and $p \in Z$. Then*

(1) *if ξ_p is regular, that is of dimension $2n$, then the induced structure on the critical set is locally conformally symplectic;*
(2) *if ξ_p is singular, that is of dimension $2n-1$, then the induced structure is contact.*

Proof. By Theorem 3, if ξ is singular, the Reeb vector field is not singular and the contact form can be written locally as $\alpha = dx_1 + y_1 \frac{dz}{z} + \sum_{i=2}^n x_i dy_i)$. The Reeb vector field is given by $R = \frac{\partial}{\partial x_1}$, the dual of $d\alpha$ by $\Pi = z \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial z} + \sum_{i=2}^n \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$. As Liouville vector field with respect to $d\alpha$, we take $X = \sum_{i=1}^n y_i \frac{\partial}{\partial y_i}$. The Jacobi structure associated to this b-contact structure is given by $\Lambda = \Pi + R \wedge X$.

On the critical set, we have

$$\Lambda\big|_Z = \sum_{i=2}^n \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{i=1}^n y_i \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_i}.$$

Let us check if we can find a one form α such that $\Lambda\big|_Z^\#(\alpha) = \frac{\partial}{\partial x_1}$. For $y_1 = 0$, this cannot be solved, hence the set $\{z = 0, y_1 = 0\}$ is a leaf with an induced contact structure.

If ξ is not singular and the Reeb vector is regular, the contact form can be written locally as $\alpha = dx_1 + y_1 \frac{dz}{z} + \frac{dz}{z} + \sum_{i=2}^n x_i dy_i)$. A direct computation implies that the Reeb vector field lies in the distribution spanned by the bi-vector field Λ, hence the b-contact structure induces a locally conformally symplectic structure on the set $\{z = 0, y_1 \neq 0\}$.

Last, if ξ is not singular and the Reeb vector is singular, Theorem 3 yields that the Reeb vector field can be written as $z \frac{\partial}{\partial z}$. As the Reeb vector field is vanishing, the critical set equals the $2n$-dimensional leaf spanned by $\mathrm{Im}^\sharp$. The induced structure is locally conformally symplectic. $\qquad\square$

Remark 7.1. Let us consider the case where $\dim M = 3$ and the distribution ξ is singular. Then the induced structure on the critical set is given by $\Lambda|_Z = y_1 \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial x_1}$. As the critical set is a surface, it is clear that this is a Poisson structure and furthermore, that it is transverse to the zero section. Hence we obtain an induced b-symplectic structure on the critical set. Note that this is not true for higher dimensions.

8. HIGHER ORDER SINGULARITIES

In what follows, we consider contact structures with higher order singularities. Let (M^n, Z) be a manifold with a distinguished hypersurface and let us assume that Z is the zero level-set of a function z. The b^m-tangent bundle, which we denote by $\,^{b^m}TM$, can be defined to be the vector bundle whose sections are generated by

$$\{z^m \frac{\partial}{\partial z}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}\}.$$

The de Rham differential can be extended to this setting. The notion of b-symplectic manifolds then naturally extends and we talk about b^m-symplectic manifolds, see [Sc, GMW]. In the same

fashion, we can extend the notion of b-contact manifolds to the b^m-setting: we say that a b^m-form $\alpha \in {}^{b^m}\Omega^n(M)$ is b^m-contact if $\alpha \wedge (d\alpha)^n \neq 0$ where the dimension of M is $2n + 1$. The proofs of the theorems of the previous sections, in particular Theorem 3 and Proposition 5.6 and the construction carried out in Section 6, generalize directly to this setting. For the sake of a clear notation, we do not write down the statements of the generalization, but only assert informally, that b can be replaced by b^m in the statements.

9. Desingularization of b^m-contact structures

In this section, we desingularize singular contact structures and consequently explain the relation to smooth contact structures. The proof is based on the idea of [GMW]. However, in contrast to the symplectic case, we need additional assumption in order to desingularize the b^m-contact form.

Recall that from Lemma 2.1, it follows that a b^m-form $\alpha \in {}^{b^m}\Omega^1(M)$ decomposes $\alpha = u\frac{dz}{z^m} + \beta$ where $u \in C^\infty(M)$ and $\beta \in \Omega^1(M)$. In order to desingularize the b^m-contact forms, we will assume that β is the pull-back under the projection of a one-form defined on Z.

Definition 9.1. We say that a b^m-contact structure $(M, \ker \alpha)$ is almost convex if $\beta = \pi^*\tilde{\beta}$, where $\pi : M \to Z$ is the projection and $\tilde{\beta} \in \Omega^1(Z)$. We will abuse notation and write $\beta \in \Omega^1(Z)$. We say that a b^m-contact structure is convex if $\beta \in \Omega^1(Z)$ and $u \in C^\infty(Z)$.

Note that the this notion is to be compared to the one of convex hypersurfaces, which we will recall in the next section. As we will see in the next lemma, almost convex b^m-contact structures are locally, in a neighbourhood around Z, isotopic to convex ones.

Lemma 9.2. Let $(M, \ker \alpha)$ be an almost convex b^m-contact manifold and let the critical hypersurface Z be compact. Then there exist a neighbourhood locally around the critical set denoted by $\mathcal{U} \supset Z$, such that α is isotopic to a convex b^m-contact manifold relative to Z on $\mathcal{U}$.

Proof. Let $\alpha = u\frac{dz}{z^m} + \beta$ where $u \in C^\infty(M)$ and $\beta \in \Omega^1(Z)$. Put $\tilde{\alpha} = u_0\frac{dz}{z^m} + \beta$, where $u_0 = u|_Z \in C^\infty(Z)$, which is convex. Take the linear path between the two b^m-contact structures, which is a path of b^m-contact structures because ξ and $\tilde{\xi}$ equal on Z. Applying Theorem 5, we obtain that there exist a local diffeomorphism f preserving Z and a non-vanishing function λ such that on a neighbourhood of Z, $f^*\alpha = \lambda\tilde{\alpha}$. $\square$

The next lemma gives intuition on this definition and gives a geometric characterization of the almost-convexity in terms of the desingularized symplectization.

Lemma 9.3. A b^m-contact manifold $(M, \ker \alpha)$ is almost-convex if and only if the vector field $\frac{\partial}{\partial t}$ is a Liouville vector field in the desingularization of the b^m-symplectic manifold obtained by the symplectization of $(M, \ker \alpha)$.

Proof. Let $(M, \ker \alpha)$ be a almost-convex b^m-contact manifold. The symplectization is given by $(M \times \mathbb{R}, \omega = d(e^t\alpha))$. The desingularization technique of Theorem 2 produces a family of symplectic forms $\omega_\epsilon = ue^t dt \wedge df_\epsilon + e^t dt \wedge \beta + e^t du \wedge df_\epsilon + e^t d\beta$. From almost-convexity follows $\frac{\partial}{\partial t}$ perserves ω_ϵ, so $\frac{\partial}{\partial t}$ is a Liouville vector field.

To prove the converse, assume that $\frac{\partial}{\partial t}$ is a Liouville vector field in (M, ω_ϵ). It follows from the fact that $\mathcal{L}_{\frac{\partial}{\partial t}}\omega_\epsilon = \omega_\epsilon$ that $\beta \in \Omega^1(Z)$. $\square$

We will see that under almost-convexity, the b^m-contact form can be desingularized.

Theorem 9. Let $(M^{2n+1}, \ker \alpha)$ a b^{2k}-contact structure with critical hypersurface Z. Assume that α is almost convex. Then there exists a family of contact forms α_ϵ which coincides with the b^{2k}-contact form α outside of an ϵ-neighbourhood of Z.

A corollary of this is that almost-convex b^m-contact manifolds admit a family of contact structures if m is even, and a family of folded-type contact structures is m is odd.

The proof of this theorem follows from the definition of convexity and makes use of the family of functions introduced in [GMW].

Proof. By the decomposition lemma, $\alpha = u\frac{dz}{z^m} + \beta$. As α is almost convex, the contact condition writes down as follows:

$$\alpha \wedge (d\alpha)^n = \frac{dz}{z^m} \wedge (u(d\beta)^n + n\beta \wedge du \wedge (d\beta)^{n-1}) \neq 0.$$

In an ϵ-neighbourhood, we replace $\frac{dz}{z^m}$ by a smooth form. The expression depends on the parity of m. Following [GMW], we introduce the family of odd smooth functions

$$f_\epsilon(x) = \begin{cases} \frac{-1}{(2k-1)x^{2k-1}} - \frac{2}{\epsilon^{2k-1}} & \text{for} \quad x < -\epsilon \\ \frac{-1}{(2k-1)x^{2k-1}} + \frac{2}{\epsilon^{2k-1}} & \text{for} \quad x > \epsilon \end{cases}$$

and such that $f'_\epsilon > 0$ in the ϵ-neighbourhood. We obtain the family of globally defined 1-forms given by $\alpha_\epsilon = udf_\epsilon + \beta$ that agrees with α outside of the ϵ-neighbourhood. Let us check that α_ϵ is contact inside of this neighbourhood. Using the almost-convexity condition, the non-integrability condition on the b^m-form α writes down as follows:

$$\alpha_\epsilon \wedge (d\alpha_\epsilon)^n = dz \wedge (f'_\epsilon(z)ud\beta + f'_\epsilon(z)\beta \wedge du - \beta \wedge \frac{\partial\beta}{\partial z}).$$

We see that $\alpha_\epsilon \wedge d\alpha_\epsilon = f'_\epsilon(z)z^m\alpha \wedge d\alpha$ and hence α_ϵ is contact. $\square$

Remark 9.4. It is possible to desingularize b^{2k+1}-contact structures following [GMW]. The resulthing one-form of this desingularization is of folded-type contact structure, as explored in [M2, JZ].

An alternative proof of this theorem would be to use the symplectization as explained in Section 6 and to use immediately Theorem 2 in the symplectization. The almost convex condition makes sure that the vector field in the direction of the symplectization is Liouville in the desingularization, see Lemma 9.3. Hence the induced structure is contact. Without the almost-convexity, the induced structure of the desingularized symplectic form on the initial manifold is not necessarily contact. This is saying that almost-convexity is a sufficient condition, but not a necessary condition to apply the desingularization method.

In the next section, we will see that in presence of convex hypersurface in contact manifolds, the inverse construction holds.

10. EXISTENCE OF SINGULAR CONTACT STRUCTURES ON A PRESCRIBED MANIFOLD

Existence of contact structures on odd dimensional manifolds has been one of the leading questions in the field. The first result in this direction was proved for open odd-dimensional manifolds by Gromov [Gr]. The case for closed manifolds turned out to be much more subtle. The 3-dimensional case was proved by Martinet–Lutz [Lu, M1]. In dimension 5, the existence problem was solved in [Et, CPP], whereas the higher dimensional case was only solved in the celebrated paper by Borman–Eliashberg–Murphy [BEM].

Theorem 10. *Let M^{2n+1} be an almost contact, compact manifold then M admits a contact structure.*

We give in this section a partial answer to the question whether or not closed manifolds also admits b^m-contact structures. The result relies on convex hypersurface theory, which was introduced by Giroux [Gi1].

Definition 10.1. Let $(M, \ker\alpha)$ be a contact manifold. A vector field X is contact if it preserves ξ, i.e. $\mathcal{L}_X\alpha = g\alpha$. A hypersurface Z in M is convex if there exists a contact vector field X that is transverse to Z.

It follows from this definition that the contact form can be written under vertically invariant form in a neighbourhood of Z, that is $\alpha = udt + \beta$, where the contact vector field X is given by $\frac{\partial}{\partial t}$, $u \in C^\infty(Z)$ and $\beta \in \Omega^1(Z)$. Note that Definition 9.1 is the analog of this definition in the b-setting. As was proved by Giroux [Gi1], in dimension 3, there are generically no obstructions to the existence of convex closed surfaces.

Theorem 11 ([Gi1]). *Let (M, ξ) be a 3-dimensional contact manifold. Then any closed surface is C^∞-close to a convex surface.*

In higher dimension, this result does not hold for generic hypersurfaces, see [Mo]. However, eventhough genericity does not hold, examples are given by boundaries of tubular neighbourhoods of Legendrian submanifolds.

In the theory of convex hypersurfaces, a fundamental role is played by the points of the convex hypersurface where the transverse contact vector field belongs to the contact distribution. It is a consequence of the non-integrability condition that Σ codimension 1 submanifold in Z. When M is of dimension 3, Σ is called the dividing curves. Loosely speaking, the dividing curves determine the germ of the contact structure on a neighbourhood of the convex surface. For a precise statement, see [Gi1, Gi2].

We will prove that convex hypersurfaces can be realized as the critical set of b^{2k}-contact structures.

Theorem 12. *Let (M, α) be a contact manifold and let Z be a convex hypersurface in M. Then M admits a b^{2k}-contact structure for all k that has Z as critical set. The codimension 2 submanifold Σ corresponds to the set where the rank of the distribution drops and the induced structures is contact.*

Using Giroux genericity result, we obtain the following Corollary in dimension 3:

Corollary 10.2. *Let M be a 3-dimensional manifold. Then for a generic surface Z, there exists a b^{2k}-contact structure on M realising Z as the critical set.*

Proof of the Corollary. Using Gromov's result in the open case and Lutz–Martinet for M closed, we can equip M with a contact form. As is proved in [Gi2], a generic surface Z is convex and the conclusion follows from Theorem 12. $\qquad\square$

Proof of Theorem 12. Using the transverse contact vector field, we find a tubular neighbourhood of Z diffeomorphic to $Z \times \mathbb{R}$ such that $\alpha = udt + \beta$, where t is the coordinate on $\mathbb{R}$, $u \in C^\infty(Z)$ and $\beta \in \Omega^1(Z)$. The non-integrability condition then is equivalent of saying that $du \wedge \beta - ud\beta$ is a volume form on Z. We will change the contact form to a b^{2k}-contact form.

Take $\epsilon > 0$. Let us take a smooth function f_ϵ such that

(1) $f_\epsilon(x) = x$ for $x \in \mathbb{R} \setminus [-2\epsilon, 2\epsilon]$
(2) $f_\epsilon(x) = -\frac{1}{x^{2m-1}}$ for $x \in [-\epsilon, 0[\cup]0, \epsilon]$
(3) $f'_\epsilon(x) > 0$ for all $x \in \mathbb{R}$.

Now consider $\alpha_\epsilon = udf_\epsilon + \beta$. By construction, α_ϵ is a b^{2m}-form that coincides with α outside of $Z \times (\mathbb{R} \setminus [-2\epsilon, 2\epsilon])$. Furthermore, α_ϵ satisfies the non-integrabilty condition on $Z \times]-2\epsilon, 2\epsilon[$ because $f'_\epsilon > 0$.

The rest of the statement follows from the discussion of Section 7. $\qquad\square$

Remark 10.3. Observe that this construction only works for an even m.

For odd m one needs to understand folded-type contact structure existence theorems on a given manifold.

Final Question: We end up this paper with a conjecture/open question. Theorem 12 provides a partial answer on the existence of b^m-contact structures as we only prove it for $m = 2k$ using contact geometry in its *full force*. In particular existence is proved for manifolds that already admit

a contact structure (used in the proof) but the existence of singularities might relax some of the obstructions existing in the realization problem in contact geometry. Specifically, it is natural to ask whether the existence of an almost contact structure is really necessary to prove the existence of a b^m-contact structure.

This is indeed a déjà vu for symplectic geometers as the topological obstructions relax when the symplectic structures in consideration are allowed to vanish on a hypersurface (as it is the case of folded-symplectic manifolds): Cannas da Silva proved in [C] that every orientable 4-manifold admits a folded-symplectic structure (in particular S^4 admits a folded-symplectic structre and no symplectic structure). In higher dimensions Cannas da Silva proved that any orientable $2n$-manifold admitting an stable almost complex structure admits a folded-symplectic manifolds. Folded-symplectic manifolds are particularly close to b^m-symplectic manifolds: Not only in [BDMOP] this apparent duality is exhibited in actual examples from Celestial Mechanics but also any given b^{2k+1}-symplectic manifold is a folded-symplectic manifolds in view of the desingularization procedure in [GMW].

We want to close up this paper conjecturing that it might be possible to prove existence of b^m-contact structures relaxing the almost-contact condition for dimensions higher than three[3].

Appendix A. Contact manifolds as Jacobi manifolds

It is well-known that every contact manifold is a particular case of Jacobi manifold, see [V]. Indeed, if (M, α) is a contact manifold, then (M, Λ, R) is a Jacobi structure, where R is the Reeb vector field and the bi-vector field Λ is defined by

$$\Lambda(df, dg) = d\alpha(X_f, X_g),$$

where X_f, X_g are the contact Hamiltonian vector fields of f and g. We give an alternative way to compute the Jacobi structure associated to the contact structure.

Let us denote the bi-vector field, dual to $d\alpha$, by Π. Furthermore, we denote by X a Liouville vector field relatively to $d\alpha$, i.e. $\mathcal{L}_X d\alpha = d\alpha$. Eventually, we define the bi-vector field

$$(A.1) \qquad\qquad\qquad\qquad \Lambda = \Pi + R \wedge X.$$

We have the following identities:

- $\mathcal{L}_X \Pi = \Pi$,
- $\mathcal{L}_R \Pi = 0$,
- $[\Pi, \Pi] = 0$.

The following lemma characterizes the Jacobi structure.

Lemma A.2. *The Jacobi structure associated to (M, α) is given by Λ and R if and only if $R \wedge [X, R] \wedge X = 0$.*

Proof. Let us check the two conditions of a Jacobi manifold, which are $[\Lambda, \Lambda] = 2R \wedge \Lambda$ and $[\Lambda, R] = 0$. The second equation writes

$$[\Lambda, R] = [\Pi + R \wedge X, R] = [\Pi, R] + [\Pi, R \wedge X] = 0 + [\Pi, R] \wedge X - R \wedge [\Pi, X] = R \wedge \Pi = 0.$$

As for the first one, we do the following computation:

$$[\Lambda, \Lambda] = [\Pi, \Pi] + 2[\Pi, R \wedge X] + [R \wedge X, R \wedge X].$$

Here, the first term is zero. The second term, using a well-known identity of the Schouten-bracket, gives us

$$2[\Pi, R \wedge X] = 2[\Pi, R] \wedge X - 2R \wedge [\Pi, X] = 0 + 2R \wedge \Pi = 0.$$

---

[3]and we believe understanding the inverse operation (if any) of the desingularization procedure in [GMW] can be relevant for that purpose.

For the third term, using the same identity, we obtain

$$[R \wedge X, R \wedge X] = R \wedge [X, R] \wedge X + [R, R] \wedge X \wedge X - R \wedge R \wedge [X, X] - R \wedge [R, X] \wedge X$$
$$= 2R \wedge [X, R] \wedge X.$$

$\square$

APPENDIX B. LOCAL MODEL OF JACOBI MANIFOLDS

We recall local structure theorems of Jacobi manifolds, proved in [DLM]. Let us first introduce some notation.

- $\Lambda_{2q} = \sum_{i=1}^{q} \frac{\partial}{\partial x^{i+q}} \wedge \frac{\partial}{\partial x^i}$
- $Z_{2q} = \sum_{i=1}^{q} x^{i+q} \frac{\partial}{\partial x^{i+q}}$
- $R_{2q+1} = \frac{\partial}{\partial x^0}$
- $\Lambda_{2q+1} = \sum_{i=1}^{q} (x^{i+q} \frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^i}) \wedge \frac{\partial}{\partial x^{i+q}}$

Theorem 13 ([DLM]). *Let (M^m, Λ, R) be a Jacobi manifold, x_0 a point of M and S be the leaf of the characteristic foliation going through x_0.*

If S is of dimension $2q$, then there exist a neighbourhood of x_0 that is diffeomorphic to $U_{2q} \times N$ where U_{2q} is an open neighbourhood containing the origin of $\mathbb{R}^{2q}$ and (N, Λ_N, R_N) is a Jacobi manifold of dimension $m - 2q$. The diffeomorphism preserves the Jacobi structure, where the Jacobi structure on $U_{2q} \times N$ is given by

$$R_{U_{2q} \times N} = \Lambda_N, \qquad R_{U_{2q} \times N} = \Lambda_{2q} + \Lambda_N - Z_{2q} \wedge R_N.$$

If S is of dimension $2q + 1$, then there exist a neighbourhood of x_0 that is diffeomorphic to $U_{2q+1} \times N$ where U_{2q+1} is an open neighbourhood containing the origin of $\mathbb{R}^{2q+1}$ and (N, Λ_N, R_N) is a homogenous Poisson manifold of dimension $m - 2q - 1$. The diffeomorphism preserves the Jacobi structure, where the Jacobi structure on $U_{2q} \times N$ is given by

$$R_{U_{2q+1} \times N} = R_{2q+1}, \qquad \Lambda_{U_{2q+1} \times N} = \Lambda_{2q+1} + \Lambda_N + E_{2q+1} \wedge Z_N.$$

REFERENCES

[BEM] M. Borman, Y. Eliashberg, E. Murphy, *Existence and classification of overtwisted contact structures in all dimensions.* Acta Math. 215 (2015), no. 2, 281–361.

[B] C. Boyer, *Completely integrable contact Hamiltonian systems and toric contact structures on $S^2 \times S^3$.* SIGMA Symmetry Integrability Geom. Methods Appl. 7 (2011), Paper 058, 22 pp.

[BDMOP] R. Braddell, A. Delshams, E. Miranda, C. Oms, A. Planas, *An Invitation to Singular Symplectic Geometry*, International Journal of Geometric Methods in Modern Physics, 2017.

[C] A. Cannas da Silva, *Fold-forms for four-folds*, J. Symplectic Geom., 8, (2010), no. 2, 189–203.

[CPP] R. Casals, D. M. Pancholi, F. Presas, *Almost contact 5-manifolds are contact*, Annals of Mathematics (2015), 429–490.

[CM] B. Chantraine, E. Murphy, *Conformal symplectic geometry of cotangent bundles.* arXiv preprint arXiv:1606.00861 (2016).

[DLM] P. Dazord, A. Lichnerowicz and Ch.-M. Marle, *Structures locales des variétés de Jacobi*, J. Math. pures et appl., 70 (1991), 101–152.

[El] Y. Eliashberg, *A few remarks about symplectic filling.* Geom. Topol. 8 (2004), 277–293.

[Et] Etnyre, John B. *Contact structures on 5-manifolds*, arXiv preprint arXiv:1210.5208 (2012).

[FMM] P. Frejlich, D. Martinez and E. Miranda, *A note on the symplectic topology of b-symplectic manifolds*, J. Symplectic Geom. 15 (2017), no. 3, 719–739.

[Ge] H. Geiges, *An introduction to contact topology*, Cambridge Studies in Advanced Mathematics, 109, Cambridge University Press, Cambridge, 2008.

[Gi1] E. Giroux, *Convexity in contact topology* Comment. Math. Helv. 66 (1991), no. 4, 637–677.

[Gi2] E. Giroux, *Structures de contact en dimension trois et bifurcations des feuilletages de surfaces.* Invent. Math., 141 (2000), 615–689.

[Gr] M. L. Gromov, *Stable mappings of foliations into manifolds*, Izv. Akad. Nauk SSSR, 33(1969), 707–734; Math. USSR–Izvestija 3(1969), 671–694.

[GL] M. Gualtieri and S. Li, *Symplectic groupoids of log symplectic manifolds*, International Mathematics Research Notices, 2014(11), 3022–3074.

CONTACT STRUCTURES WITH SINGULARITIES

[GMP] V. Guillemin, E. Miranda, A. R. Pires, *Symplectic and Poisson geometry on b-manifolds,* Adv. Math., 264, (2014), 864–896.

[GMPS] V. Guillemin, E. Miranda, A. R. Pires and G. Scott, *Toric actions on b-symplectic manifolds*, Int Math Res Notices (2015) 2015 (14): 5818-5848. doi: 10.1093/imrn/rnu108.

[GMW] V. Guillemin, E. Miranda and J. Weitsman, *Desingularizing b^m-symplectic manifolds*, International Mathematics Research Notices, rnx126, https://doi.org/10.1093/imrn/rnx126.

[JZ] B. Jakubczyk, M. Zhitomirskii, *Singularities and normal forms of generic 2-distributions on 3-manifolds*, Studia Math. 113 (1995), p.223–248.

[KM] A. Kiesenhofer and E. Miranda, *Cotangent models for integrable systems.* Comm. Math. Phys. 350 (2017), no. 3, 1123–1145.

[Le] E. Lerman, *Contact toric manifolds.* J. Symplectic Geom. 1 (2003), no. 4, 785–828.

[Lu] R. Lutz, *Sur la géométrie des structures de contact invariantes.* Ann. Inst. Fourier (Grenoble) 29 (1979), no. 1, xvii, 283–306.

[M1] J. Martinet, *Formes de contact sur les variétés de dimension 3*, in Proceedings of Liverpool Singularities Symposium, II (1969/1970), Lecture Notes in Math., 209, pp. 142–163.

[M2] J. Martinet, *Sur les singularites des formes différentielles*, Ann. Inst. Fourier, 20 (1970), No.1, 95–178.

[Mo] A. Mori, *On the violation of Thurston-Bennequin inequality for a certain nonconvex hypersurface*, preprint 2011, arXiv:1111.0383

[MO] I. Marcut, B. Osorno, Deformations of log-symplectic structures , J. Lond. Math. Soc. (2) 90 (2014), no. 1, 197–212.

[Me] R. Melrose, *Atiyah-Patodi-Singer Index Theorem (book)*, Research Notices in Mathematics, A.K. Peters, Wellesley, 1993.

[R] O. Radko, *A classification of topologically stable Poisson structures on a compact oriented surface*, Journal of Symplectic Geometry, 1, no. 3, 2002, pp. 523–542.

[Sc] G. Scott, *The Geometry of b^k Manifolds*, J. Symplectic Geom. 14 (2016), no. 1, 7195.

[Sw] R. Swan, *Vector Bundles and Projective Modules*, Transactions of the American Mathematical Society 105, (2), 264–277, (1962).

[V] I. Vaisman, *Jacobi manifolds*, Selected topics in Geometry and Mathematical Physics, Vol. 1, (2002), 81–100.

EVA MIRANDA, LABORATORY OF GEOMETRY AND DYNAMICAL SYSTEMS, DEPARTAMENT OF MATHEMATICS, EPSEB, UNIVERSITAT POLITÈCNICA DE CATALUNYA BGSMATH BARCELONA GRADUATE SCHOOL OF MATHEMATICS IN BARCELONA
E-mail address: eva.miranda@upc.edu

CÉDRIC OMS, LABORATORY OF GEOMETRY AND DYNAMICAL SYSTEMS, DEPARTAMENT OF MATHEMATICS, EPSEB, UNIVERSITAT POLITÈCNICA DE CATALUNYA BGSMATH BARCELONA GRADUATE SCHOOL OF MATHEMATICS IN BARCELONA
E-mail address: cedric.oms@upc.edu