SPLITTING OF SEPARATRICES FOR (FAST) QUASIPERIODIC FORCING

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At the end of the last century, H. Poincaré [7] discovered the phenomenon of separatrices splitting, which now seems to be the main cause of the stochastic behavior in Hamiltonian systems. He formulated a *general problem of Dynamics* as a perturbation of an integrable Hamiltonian system

$$H(I,\varphi,\varepsilon) = H_0(I) + \varepsilon H_1(I,\varphi),$$

where ε is a small parameter, $I = (I_1, I_2, \ldots, I_n)$, $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$. The values of the actions I, such that the unperturbed frequencies $\omega_k(I) = \frac{\partial H_0}{\partial I_k}$ are rationally dependent, are called *resonances*.

As a model for the motion near a resonance Poincaré studied the system, which can be described by the pendulum with high-frequency perturbation: $y^2/2+\cos x+\mu \sin x \cos(t/\varepsilon)$. His calculations of the splitting, originally validated only for μ exponentially small with respect to ε, appeared to predict correctly the splitting up to $|\mu| \le \varepsilon^p$ for any positive parameter p [3, 9]. The main problem in studying of a similar system is that the splitting is exponentially small. Namely, Neishtadt's theorem [6] implies that in a Hamiltonian

$$H\left(x,y,t/\varepsilon\right) = H_0(x,y) + H_1\left(x,y,t/\varepsilon\right),$$

where H_0 has a saddle and a corresponding homoclinic orbit, and the perturbation H_1 has zero average with respect to time, the splitting can be bounded from above by $O(e^{-\mathrm{const}/\varepsilon})$. For this estimate to be valid all the functions have to be real analytic in x and y, but even C^1 dependence on time is sufficient. Lately, the constant in the exponent was related to the position of complex time singularities of the unperturbed homoclinic orbit.

The abovementioned systems provide a realistic model for a resonance only in the case of two degrees of freedom. Near a simple resonance in a system with more degrees of freedom there are more than two phases. All of them except one can be considered as rapid variables. The analysis of such systems is quite complicated.

The simplest case is a quasiperiodic perturbation of a planar Hamiltonian systems. Neishtadt's averaging theorem was generalized to this case by C. Simó [8], as well as the upper bounds for the splitting, but these bounds essentially depend on the frequency vector of the perturbation. For a pendulum perturbation with two frequencies C. Simó [8] checked numerically that a proper modification of the Melnikov method gives a correct prediction for the splitting. For a related work, see Benettin's talk in this volume.

We consider a quasiperiodic high-frequency perturbation of the pendulum described by the Hamiltonian function

$$h = \frac{y^2}{2} + \cos x \cdot (1 + \varepsilon^p m(\theta_1, \theta_2)), \qquad \dot\theta_1 = \frac{1}{\varepsilon}, \quad \dot\theta_2 = \frac{\gamma}{\varepsilon}, \tag{1}$$

where $\varepsilon > 0$ is a small parameter, p is a real parameter and $\gamma = (\sqrt{5}+1)/2$. The function m is a 2π-periodic function of two variables θ_1 and θ_2. The unperturbed system $h_0 = y^2/2 + \cos x$ has a saddle point $(0,0)$, and a homoclinic trajectory is given by $x_0(t) = 4\arctan(e^t)$, $y_0(t) = \dot x_0(t)$.

The complete system (1) has a whiskered torus $\mathcal{T}$: $(0, 0, \theta_1, \theta_2)$. The whiskers are 3D hypersurfaces in the 4D extended phase space $(x, y, \theta_1, \theta_2)$. The stable (resp. unstable) whisker is formed by trajectories, which wind on (resp. out) the torus. These invariant manifolds are close to the unperturbed pendulum separatrix.

The standard way of studying the splitting is to calculate the *Melnikov function*

$$M(\theta_1, \theta_2; \varepsilon) = \int_{-\infty}^{\infty} \{h_0, h\}(x_0(t), y_0(t), \theta_1 + t/\varepsilon, \theta_2 + \gamma t/\varepsilon)\, dt,$$

which gives a first order approximation of the difference between the values of the unperturbed pendulum energy h_0 on the stable and unstable manifolds. The Melnikov function is 2π-periodic with respect to θ_1 and θ_2.

We assume that the function m can be represented as a Fourier series

$$m(\theta_1, \theta_2) = \sum m_{k_1 k_2} e^{i(k_1\theta_1 + k_2\theta_2)}. \tag{2}$$

Taking into account the explicit formula for $x_0(t)$ and $y_0(t)$ it is not difficult to obtain that the Fourier coefficients of the Melnikov function are given by

$$M_{k_1 k_2}(\varepsilon) = -\frac{2\pi i \varepsilon^p (k_1 + \gamma k_2)^2}{\varepsilon^2 \sinh\left(\frac{\pi(k_1+\gamma k_2)}{2\varepsilon}\right)}\, m_{k_1 k_2}. \tag{3}$$

This formula implies that all Fourier coefficients of the Melnikov function are exponentially small, i.e., $O(e^{-\mathrm{const}/|\varepsilon|})$ for small ε. However the constant in the estimate essentially depends on k_1 and k_2. In the next section we show that this implies that the Melnikov function can be of any order in ε depending on the smoothness of m.

1. First order splitting

Fix $\alpha > 0$ and consider the function m defined by

$$m(\theta_1, \theta_1) = \sum_{n=1}^{\infty} \frac{(-1)^n \sin(F_{n+1}\theta_1 - F_n\theta_2)}{F_n^{1+\alpha}}, \tag{4}$$

where F_n are Fibonacci numbers defined by the recurrence: $F_1 = 1$, $F_2 = 2$, $F_{n+1} = F_n + F_{n-1}$. The function m is $C^{1+\alpha}$ but not $C^{1+\alpha+\delta}$ for any positive δ. The Melnikov function is represented by the series

$$M(\theta_1, \theta_2; \varepsilon) = 2\pi \sum_{n=1}^{\infty} \frac{(-1)^n \varepsilon^{p-2} (F_{n+1} - \gamma F_n)^2}{F_n^{1+\alpha} \sinh\left(\frac{\pi(F_{n+1}-\gamma F_n)}{2\varepsilon}\right)} \cos(F_{n+1}\theta_1 - F_n\theta_2).$$

We will show that the maximum of the modulus of this function can be bounded from two sides by terms of the form const$\varepsilon^{p+1+\alpha}$. We note that there are two positive constants C and C_1, such that

$$\frac{C_1}{F_n} \leq |F_{n+1} - \gamma F_n| \leq \frac{C}{F_n}.$$

Then we easily obtain the upper bound for the Melnikov function:

$$\begin{aligned}
|M(\theta_1, \theta_2)| &= 2\pi \sum_{n=1}^{\infty} \frac{\varepsilon^{p-2}(F_{n+1} - \gamma F_n)^2}{F_n^{1+\alpha} \sinh\left(\frac{\pi|F_{n+1}-\gamma F_n|}{2\varepsilon}\right)} \\
&\leq 2\pi \sum_{n=1}^{\infty} \frac{C^2 \varepsilon^{p-2}}{F_n^{3+\alpha} \sinh\left(\frac{\pi C_1}{2\varepsilon F_n}\right)} \frac{F_{n+1} - F_n}{F_{n-1}} \\
&\leq 2\pi \int_0^{\infty} \frac{4\varepsilon^{p-2} C^2 \, dF}{F^{4+\alpha} \sinh\left(\frac{\pi C_1}{2\varepsilon F}\right)} = \frac{2^{6+\alpha} C^2}{\pi^{2+\alpha} C_1^{3+\alpha}} \int_0^{\infty} \frac{s^{2+\alpha}\, ds}{\sinh s} \, \varepsilon^{p+1+\alpha}.
\end{aligned}$$

The last inequality means the Melnikov function is bounded by $O(\varepsilon^{p+1+\alpha})$. To obtain the lower bound we note that

$$M(0,0) = 2\pi \sum_{n=1}^{\infty} \frac{(-1)^n \varepsilon^{p-2}(F_{n+1} - \gamma F_n)^2}{F_n^{\alpha} \sinh\left(\frac{\pi(F_{n+1}-\gamma F_n)}{2\varepsilon}\right)}.$$

All the terms are positive so

$$M(0,0) = 2\pi \sum_{n=1}^{\infty} \frac{\varepsilon^{p-2}(F_{n+1} - \gamma F_n)^2}{F_n^{1+\alpha} \sinh\left(\frac{\pi|F_{n+1}-\gamma F_n|}{2\varepsilon}\right)} \geq 2\pi \sum_{n=1}^{\infty} \frac{C_1^2 \varepsilon^{p-2}}{F_n^{3+\alpha} \sinh\left(\frac{\pi C}{2\varepsilon F_n}\right)}.$$

The largest terms in the sum correspond to number $n \sim \mathrm{const}/\varepsilon$. Since $F_{n+1} \approx \gamma F_n$ there is at least one Fibonacci number between ε^{-1} and $2\gamma\varepsilon^{-1}$. Leaving only this term in the sum we obtain a lower bound with the same power of ε as in the upper bound:

$$M(0,0) \geq \frac{2\pi C_1^2 \varepsilon^{p-2}}{F_{n(\varepsilon)}^{3+\alpha} \sinh\left(\frac{\pi C}{2\varepsilon F_{n(\varepsilon)}}\right)} \geq \frac{2\pi C_1^2 \varepsilon^{p+1+\alpha}}{2\gamma \sinh\left(\frac{\pi C}{2}\right)}.$$

The Melnikov function provides a formula for the splitting distance with the error $O(\varepsilon^{2p})$. We can choose $p > 1 + \alpha$. Then the amplitude of the Melnikov function is larger then the error and the splitting is detected in the first order of perturbation theory.

Remark 1 If we change the sine in the formula (4) by cosine and choose $\alpha > 1$, then a similar reasoning leads to an estimate of the splitting angle at a homoclinic trajectory. One only has to repeat the estimates changing the Melnikov function by its derivative.

2. Analytic perturbation

The analytic case is more difficult. Suppose that $\sup \left| m_{k_1 k_2} e^{r_1|k_1| + r_2|k_2|} \right| < \infty$ for some positive constants r_1 and r_2, and assume that there is a number k_0, such that

$$|m_{k_1 k_2}| \geq a\, e^{-r_1|k_1| - r_2|k_2|} \tag{5}$$

for some positive number a and all k_1 and k_2, such that $k_2 > k_0$ and $|k_2|$, $|k_1|$ are two consecutive Fibonacci numbers. That is, $|k_1|/|k_2|$ is a continuous fraction convergent of γ.

The function m appears to be analytic inside the product of strips $\{|\operatorname{Im}\theta_1| < r_1\} \times \{|\operatorname{Im}\theta_2| < r_2\}$ and has a singularity on the boundary.

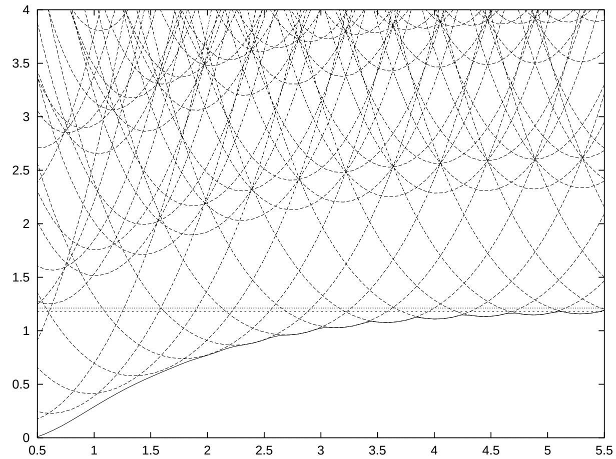

Figure 1. Each dashed line represents a Fourier coefficient of the Melnikov function as a function of ε: $-\sqrt{\varepsilon}\log_{10}|M_{k_1k_2}(\varepsilon)|$ versus $-\log_{10}\varepsilon$. The solid lines represents the maximum of the modulus of the Melnikov function in the same scale.

The dependence of Fourier coefficients (3) on ε is represented in Figure 1 in logarithmic scale. For a fixed ε lower is a point on the graph, larger is the corresponding term. The most important terms correspond to the Fibonacci numbers F_n, since only these contribute in the asymptotic for the splitting. One can see also the curves, which correspond to the sums $F_n + F_{n-2}$ and $F_n + F_{n-3}$. The contribution of these terms is very small with respect to the main ones. Like in the previous section the largest term number depends on ε. But its value now is less than any power of ε, it is $O(e^{-\mathrm{const}/\sqrt{\varepsilon}})$. Indeed, let us define a function

$$c(\delta) = C_0 \cosh\left(\frac{\delta - \delta_0}{2}\right) \quad \text{for} \quad \delta \in [\delta_0 - \log\gamma, \delta_0 + \log\gamma], \tag{6}$$

where

$$C_0 = \sqrt{\frac{2\pi(\gamma r_1 + r_2)}{\gamma + \gamma^{-1}}}, \qquad \delta_0 = \log\varepsilon^*, \quad \varepsilon^* = \frac{\pi(\gamma + \gamma^{-1})}{2\gamma^2(r_1\gamma + r_2)}, \tag{7}$$

and continue it by $2\log\gamma$-periodicity onto the whole real axis.

Lemma 1 (Properties of the Melnikov Function) *The Melnikov function is a 2π-periodic function of θ_1 and θ_2, such that*

1) *$M(\theta_1 - T/\varepsilon, \theta_2 - \gamma T/\varepsilon; \varepsilon)$ is analytic in the product of strips $|\operatorname{Im}\theta_1| < r_1$, $|\operatorname{Im}\theta_2| < r_2$ and $|\operatorname{Im} T| < \pi/2$;*
2) *the maximum of the modulus of the Melnikov function, $\max_{(\theta_1,\theta_2)\in\mathbf{T}^2}|M(\theta_1,\theta_2)|$, can be estimated from above and from below by terms of the form*

$$\mathrm{const}\,\varepsilon^{p-1}\exp\left(-\frac{c(\log\varepsilon)}{\sqrt{\varepsilon}}\right)$$

with different ε-independent constants and the function c defined by (6);

3) for a fixed small ε only 4 terms dominate in Fourier series for the Melnikov function and the rest can be estimate from above by $O(e^{-C_1/\sqrt{\varepsilon}})$, where the constant $C_1 > \max c(\delta) = C_0 \cosh(\log\sqrt{\gamma})$.

Remark 2 The number of leading terms depends on ε. In fact the largest terms correspond to $(k_1, k_2) = \pm\left(F_{n(\varepsilon)+1}, -F_{n(\varepsilon)}\right)$, where $F_{n(\varepsilon)}$ denotes Fibonacci number closest to $F^*(\varepsilon) = \sqrt{\phi_0/\varepsilon}$ and ϕ_0 is a constant, which depends on r_1, r_2 and γ. Except for a small neighborhood of $\varepsilon = \varepsilon^*\gamma^{-n}$, there is only one Fibonacci number closest to $F^*(\varepsilon)$, and then only two corresponding terms dominate in Fourier series.

Theorem 2 *For $p > 3$ and ε small enough the invariant manifolds split, and the value of the splitting is predicted correctly by the Melnikov method.*

The method used for the proof is based on the ideas proposed by Lazutkin [4] for the study of the separatrix splitting for the standard map, adapted later to differential equations [1, 2]. We use a convergent Birkhoff normal form in a neighborhood of the hyperbolic torus. The normal form theorem is similar to Moser's theorem [5] on the normal form near a periodic hyperbolic orbit, but we have to extend its domain to include points $\sqrt{\varepsilon}$-close to the singularity of the Hamiltonian in the phases θ.

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