Reduced-order Interval-observer Design for Dynamic Systems with Time-invariant Uncertainty

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Abstract: This paper addresses the design of reduced-order interval-observers for dynamic systems with time-invariant uncertainty. Because of the limitations of using the set-based approach to preserve the time dependency of parameter uncertainty and the wrapping effect to deal with interval-observers, the trajectory-based interval-observer approach is used with an appropriate observer gain. But, there could be some difficulties to satisfy the conditions for selecting a suitable gain to guarantee the positivity of the resulting observer. Then, a reduced-order observer is designed to reduce the computational complexity and to increase the degree of freedom when selecting the observer gain. Finally, a simulation example is employed for illustrating and analyzing the effectiveness of the proposed approach.

Keywords:
Uncertain dynamic system, interval observer, set-based observer, reduced-order observer

1. INTRODUCTION

Interval-observer approaches make use of the interval hull of the approximated set instead of the exact set. Moreover, in the set-based approach, the propagation of the state set is affected by several problems such as the wrapping effect, temporal variance on uncertain parameters (or uncertain parameter time dependency) and range evaluation of an interval function, especially in the case of using the interval hull of the set at each iteration. Therefore, conservative and unstable results may obtain (for even a stable system) with using the set-based approach in the simulation of the system with parametric time-invariant uncertainties [Puig et al. (2005)]. On the other hand, the approximated state set can be computed based on a set of point-wise trajectories. This type of approach is called trajectory-based approach [Puig et al. (2005)]. The advantage of trajectory-based approach in comparison with the set-based approach is to overcome the wrapping effect due to generating the real trajectories based on selecting the particular value of the uncertain parameters. Meanwhile, the uncertain parameter time dependency can be preserved if the set of point-wise trajectories are generated [Puig et al. (2005)].

Based on the literature, interval dynamic systems are those uncertain systems whose uncertain parameters are bounded by intervals [Kolev and Petrakieva (2005) and Le et al. (2012)]. Particularly, the wrapping effect appears in that kind of interval systems that are not monotonic [Puig et al. (2005)]. It means that if there are some negative elements in the state matrix related to the state-space model of the observer (i.e., it is not monotonic), the wrapping effect affects the interval-observer. As it is mentioned before, one of the possible solutions to overcome the wrapping effect is to make use of trajectory-based observers, which will be further reviewed later in this paper. But still, there are some difficulties related to the computational complexity and designing the suitable observer gain in the trajectory-based observer.

In order to reduce the computational complexity of the trajectory-based approach, the order of the observer can be reduced and unknown states of the system can be separately estimated by using a reduced-order observer, since the problem of wrapping effect appears only when those elements of the state matrix that are related to the unmeasurable states are negative values. Besides, by using the reduced-order observer not only the problem of wrapping effect can be solved with less computational complexity, but also, designing the observer gain can be done with more degree of freedom in comparison with the full-order observer.

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So far, however, several studies have reported state estimation based on trajectory-based interval observer, but there is still insufficient study for reducing the computational complexity and the design of the suitable observer gain in order to satisfy the positivity conditions. Therefore, the main contribution of this paper is to design the reduced-order interval observer to estimate the unmeasurable states. Furthermore, the observer gain can be designed with more degree of freedom to obtain the monotonic observer in spite of non-monotonicity of the system state matrix. Simultaneously, the time invariance of the uncertain parameter can be preserved.

Regarding the structure of the paper, Section 2 deals with preliminaries and the problem formulation of interval state observation. Moreover, the set-based and trajectory-based observers are introduced, respectively. The design of the interval observer is proposed in Section 3. Set-based, trajectory-based and reduced-order observers are implemented and compared by using an academic example for both monotonic and non-monotonic systems and the effect of the observer gain will be discussed in Section 4. Finally, the main conclusions are drawn in Section 5.

2. PROBLEM FORMULATION OF INTERVAL STATE OBSERVATION

2.1 Problem Statement

The uncertain discrete linear time-invariant model is represented by the following state-space form:

$$x(k+1) = A(\theta)x(k) + B(\theta)u(k), \tag{1a}$$
$$y(k) = C(\theta)x(k), \tag{1b}$$

where $k \in \mathbb{Z}_{\geq 0}$ indicates the discrete time, $x \in \mathbb{R}^{n_x}$ is the state vector, $u \in \mathbb{R}^{n_u}$ and $y \in \mathbb{R}^{n_y}$ denote the input and the output vectors, respectively. The system matrices of appropriate dimensions are $A(\theta) \in \mathbb{R}^{n_x \times n_x}$, $B(\theta) \in \mathbb{R}^{n_x \times n_u}$ and $C(\theta) \in \mathbb{R}^{n_y . \times n_x}$. Moreover, θ is the vector of time-invariant uncertain parameters with the bounded values by an compact set Θ as

$$\Theta = \left\{ \theta \in \mathbb{R}^{n_\theta} \mid \underline{\theta}_i \leq \theta_i \leq \overline{\theta}_i, \quad i = 1, \dots, n \right\}. \tag{2}$$

Furthermore, the system matrices $A(\theta)$, $B(\theta)$ and $C(\theta)$ are not exactly known and the uncertainty is included into them based on (2) as

$$\underline{A}(\theta) \leq A(\theta) \leq \overline{A}(\theta), \tag{3a}$$
$$\underline{B}(\theta) \leq B(\theta) \leq \overline{B}(\theta), \tag{3b}$$
$$\underline{C}(\theta) \leq C(\theta) \leq \overline{C}(\theta), \tag{3c}$$

where $\underline{\bullet}$ and $\overline{\bullet}$ denote the lower and upper bound of each matrix, respectively. Note that the inequalities in (3) should be understood as element-wise inequalities.

Therefore, the uncertainties in matrices $A(\theta)$, $B(\theta)$ and $C(\theta)$ can be decomposed as $A(\theta) = A_e + \Delta A(\theta)$, $B(\theta) = B_e + \Delta B(\theta)$ and $C(\theta) = C_e + \Delta C(\theta)$, where A_e, B_e and C_e are the nominal part of the system matrices $A(\theta)$, $B(\theta)$ and $C(\theta)$, respectively. Furthermore, $\Delta A(\theta)$, $\Delta B(\theta)$ and $\Delta C(\theta)$ denote the effect of uncertainties in $A(\theta)$, $B(\theta)$ and $C(\theta)$, respectively.

Remark 1. The time invariance of uncertain parameters is usually not taken into account in the literature, being this the main goal of this paper.

Assumption 1. $B(\theta)$ and $C(\theta)$ are assumed monotonic with respect to θ. It means, there exist two functions $\overline{\mathbf{F}}(y(k), u(k), \theta)$ and $\underline{\mathbf{F}}(y(k), u(k), \theta)$ where the monotonicity property is used to build them.

If (1) is observable, the Luenberger observer is written as

$$\hat{x}(k+1) = A(\theta)\hat{x}(k) + B(\theta)u(k) + L(y(k) - \hat{y}(k)), \tag{4a}$$
$$\hat{y}(k) = C(\theta)\hat{x}(k), \tag{4b}$$

where $\hat{y} \in \mathbb{R}^{n_y}$ and $\hat{x} \in \mathbb{R}^{n_x}$ are the estimated system output and state, respectively. Furthermore, the state observer (4) can be written as $\hat{x}(k+1) = (A(\theta) - LC(\theta))\hat{x}(k) + [B(\theta) \ L] \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}$. By denoting $A_{obs} = A(\theta) - LC(\theta)$, $B_{obs} = [B(\theta) \ L]$ and $u_{obs} = \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}$, the state observer can be rewritten as

$$\hat{x}(k+1) = A_{obs}(\theta)\hat{x}(k) + B_{obs}(\theta)u_{obs}(k). \tag{5}$$

Additionally, the observer gain L for $\theta \in \Theta$ should be defined such that A_{obs} was a Schur matrix. Based on the observer expression in (5), the effect of the uncertainty is introduced in the observer through A_{obs} and B_{obs} matrices. There are two main approaches to take into account the uncertainty in the output/state estimation as it will be recalled in the following sections.

2.2 Set-based vs Trajectory-based Approaches

In set-based approaches, the set of states at time instant $k+1$ is approximated by using propagation algorithms from the set of states at time k [Puig et al. (2005), Combastel (2015)]. However, in set-based approaches, the parameter time invariance is not taken into account during the propagation of the uncertainties. Uncertain parameters are unknown but bounded in their uncertainty intervals and can vary arbitrarily at each time instant within the bounded interval in the set-based approach. Moreover, some additional problems appear in the case of using set-based approaches such as wrapping effect and range evaluation of an interval function [Puig et al. (2003)] as already discussed. One way to solve the time-dependency problem of the set-based approach is to derive the relation between states and parameters, which can bring the system from the initial state to the current state. Considering this idea, the wrapping effect problem is solved while the time dependency of the uncertain parameter can be preserved. In the trajectory-based approach, the value of parameter uncertainty is unknown but bounded in the interval and the invariance of parameter uncertainty can be guaranteed at each time instant. The interval of the states can be estimated at each iteration by using the particular state trajectories corresponding to particular values of uncertainty θ. Based on Puig et al. (2003), the relation between states and uncertain parameters is derived as

$$\hat{x}(k) = (A_{obs}(\theta))^k x(0) + \sum_{j=0}^{k-1} (A_{obs}(\theta))^{k-1-j} \, B_{obs}(\theta) \, u_{obs}(j). \tag{6}$$

In (6), the uncertainty always is propagated from the initial state to avoid the wrapping effect. Furthermore, by assuming that $A_{obs}(\theta)$ at time instant $k+1$ is equal to the $A_{obs}(\theta)$ at time instant k and $B_{obs}(\theta)$ at time instant $k+1$

is equal to $B_{obs}(\theta)$ at time instant k, the time invariance is guaranteed in all iterations.

The state estimation $\hat{x}(k)$ in (6) can be bounded by the interval hull $\square\hat{\mathcal{X}}(k) = [\underline{\hat{x}}(k),\ \overline{\hat{x}}(k)]$, that can be obtained by solving the following optimization problems:

$$\overline{\hat{x}}(k) = \max_{\theta\in\Theta}\Big[(\hat{A}_{obs} + \Delta A_{obs}(\theta))^k \hat{x}(0) + \sum_{j=0}^{k-1}(\hat{A}_{obs} + \Delta A_{obs}(\theta))^{k-1-j}\ (\hat{B}_{obs} + \Delta B_{obs}(\theta))\ u_{obs}(j)\Big], \tag{7a}$$

$$\underline{\hat{x}}(k) = \min_{\theta\in\Theta}\Big[(\hat{A}_{obs} + \Delta A_{obs}(\theta))^k \hat{x}(0) + \sum_{j=0}^{k-1}(\hat{A}_{obs} + \Delta A_{obs}(\theta))^{k-1-j}\ (\hat{B}_{obs} + \Delta B_{obs}(\theta))\ u_{obs}(j)\Big], \tag{7b}$$

where $\hat{A}_{obs}$, $\hat{B}_{obs}$ and $\Delta A_{obs}(\theta)$, $\Delta B_{obs}(\theta)$ are nominal and uncertainties of the observer matrices, respectively.

3. DESIGNING THE INTERVAL OBSERVER

One possible way to avoid solving the optimization problems in (7) is to design the observer gain L in such a way to force A_{obs} into an element-wise positive matrix despite of the negative elements in $A(\theta)$. Thus, designing the observer gain in this manner imposes some additional constraints. In case the observer is not possible to be designed, a reduced-order observer can be used to estimate only those states that are affected by the negative elements of $A(\theta)$. Moreover, the observer gain L can be designed with more degree of freedom for the reduced-order observer in comparison with full-order observer. In other words, the number of conditions and constrains of selecting the observer gain can be reduced by using the reduced-order observer.

3.1 Designing Observer Gain

In the case that one element of $A(\theta)$ is negative, A_{obs} will satisfy the monotonicity property if

$$a_{ij} - (LC)_{ij} \geq 0, \tag{8}$$

where the subindices ij denote that there exist one state such $\hat{x}_i$ whose variation with respect to another state $\hat{x}_j$ is negative (e.g., $a_{ij} < 0$). Furthermore, $(LC)_{ij}$ shows the i-th row and j-th column of matrix $LC \in \mathbb{R}^{n_x\times n_x}$ that is obtained by multiplication of matrix $L \in \mathbb{R}^{n_x\times n_y}$ into the matrix $C(\theta) \in \mathbb{R}^{n_y\times n_x}$.

Therefore, the condition in (8) will be satisfied if

$$a_{ij} \geq (LC)_{ij}. \tag{9}$$

Consequently, by forcing the observer gain to eliminate the negative elements of A_{obs}, a monotonic interval observer can be obtained. Furthermore, the right side of (9) can be written as $(LC)_{ij} = L_i C_j$, where L_i shows the i-th row elements of the matrix L and C_j indicates the j-th column elements of the matrix $C(\theta)$. Obviously, the elements C_j can not be chosen freely, because it is determined by the observer model. Therefore, in order to force the condition in (9), only the elements L_i can be designed freely.

Moreover, the interval-observer convergence should be considered when designing the observer gain. Therefore, the observer gain matrix L can be divided into two matrices L^- and L^+, where

- L^- shows the elements that are chosen to force the condition (9),
- L^+ shows the elements that are chosen to guarantee the observer convergence.

Thus, the observer gain can be split into

$$L = L^+ + L^-. \tag{10}$$

Similarly, the matrix $A(\theta)$ can be written as

$$A(\theta) = A^+(\theta) + A^-(\theta), \tag{11}$$

where, $A^+(\theta)$ and $A^-(\theta)$ are determined by the positive and negative elements of matrix $A(\theta)$, respectively.

Moreover, by considering (10) and (11), A_{obs} can be written as

$$A_{obs}(\theta) = \big(A^+(\theta) - L^+C(\theta)\big) + \big(A^-(\theta) - L^-C(\theta)\big). \tag{12}$$

Furthermore, by denoting the index mn for those elements of matrices $A(\theta)$ and $LC(\theta)$ that do not have any effect on the monotonicity property of the A_{obs}, each part of (12) will be determined as

$$\text{if}\quad a_{mn} > 0 \Longrightarrow \begin{cases} \big(A^+(\theta)\big)_{mn} = a_{mn}, \\ \big(A^-(\theta)\big)_{mn} = 0, \\ \big(L^+C(\theta)\big)_{mn} = L_{m\alpha}C_{\alpha n}, \end{cases} \tag{13}$$

where $L_{m\alpha}$ are the m-th row elements of matrix L and $C_{\alpha n}$ are the n-th column of matrix $C(\theta)$ and

$$\text{if}\quad a_{ij} < 0 \Longrightarrow \begin{cases} \big(A^+(\theta)\big)_{ij} = 0, \\ \big(A^-(\theta)\big)_{ij} = a_{ij}, \\ \big(L^+C(\theta)\big)_{ij} = 0. \end{cases} \tag{14}$$

Furthermore, for the positive elements, $\big(A^-(\theta)\big)_{mn} = \big(L^-C(\theta)\big)_{mn}$. Therefore, the elements of matrix L^-C are either positive or null by assuming the condition in (9).

Taking into account that the condition (9) should be satisfied to achieve the positivity of A_{obs}, it can be written as $A_{obs} = A_\bigstar - L^+C(\theta)$, where $A_\bigstar = A(\theta) - L^-C$. Therefore, L^- should satisfy the positivity conditions and L^+ should be designed in order to guarantee the convergence of the observer. Thus, two state observers can be designed to estimate the maximum and minimum state trajectories. Therefore, A_{obs}, for each observer, is

$$\overline{A}_{obs} = \overline{A}_\bigstar - \overline{L}^+\ \overline{C}, \tag{15a}$$

$$\underline{A}_{obs} = \underline{A}_\bigstar - \underline{L}^+\ \underline{C}, \tag{15b}$$

where $\overline{\bullet}$ and $\underline{\bullet}$ denote that the matrices are related to the upper and lower trajectory observers, respectively.

In this paper, the upper $\overline{L}^+$ and lower $\underline{L}^+$ observer gains are designed using Linear Matrix Inequalities (LMIs). Therefore, $\overline{L}^+$ and $\underline{L}^+$ should be designed in such way that the eigenvalues of the $\overline{A}_\bigstar - \overline{L}^+\ \overline{C}$ and $\underline{A}_\bigstar - \underline{L}^+\ \underline{C}$ are placed in a stable LMI region. Theorem 1 can be used to designed such an observer gains.

Theorem 1. Consider the observer (5) satisfying the condition in (9) and the following LMI inequalities:

$$\begin{bmatrix} -rK & qK + K^T\overline{A}_\bigstar - \overline{W}^T\overline{C} \\ qK + \overline{A}_\bigstar^T K - \overline{C}^T\overline{W} & -rK \end{bmatrix} < 0, \quad \text{(16a)}$$

$$\begin{bmatrix} -rK & qK + K^T\underline{A}_\bigstar - \underline{W}^T\underline{C} \\ qK + \underline{A}_\bigstar^T K - \underline{C}^T\underline{W} & -rK \end{bmatrix} < 0, \quad \text{(16b)}$$

where q and r denote the center and the radius of a LMI region, respectively, K is the unknown symmetric matrix, $\overline{W} = \overline{L}^+ K$ is related to the upper observer and $\underline{W} = \underline{L}^+ K$ is related to the lower observer. Obtaining $\overline{L}^+$ and $\underline{L}^+$ as

$$\overline{L}^+ = (\overline{W}K^{-1})^T, \quad \text{(17a)}$$

$$\underline{L}^+ = (\underline{W}K^{-1})^T, \quad \text{(17b)}$$

the A_{obs} is Schur and element-wise positive matrix and hence the convergence of the upper and lower observers is guaranteed.

Proof. The proof follows from the application of results presented in Chilali et al. (1999) to the interval observer case.

3.2 Reduced-order Observer

All the states can be estimated together by using a full-order observer. In this regard, the observer gain can be designed to eliminate the effect of negative elements of A on A_{obs}. However, there exist some cases that designing the observer gain to satisfy the condition (9) is not possible or the LMI inequalities in (16) are difficult to be solved.

Consequently, the trajectory-based observer will produce wrong state estimation when the unmeasurable states are affected by the negative elements of A. As an alternative, a reduced-order observer is used to reduce the number of conditions and constrains in the LMIs in (16). Hence, it is not necessary to design the observer to estimate all the states that are directly measured by sensors. If this is the case, Theorem 2 can be used to design a reduced-order observer for estimating the unmeasured states.

Theorem 2. By separating the original states x into measurable x_a and unmeasurable x_b states such that $x(k) = [x_a(k) \; \dots \; x_b(k)]^T$, (1) can be written as

$$\begin{bmatrix} x_a(k+1) \\ \cdots \\ x_b(k+1) \end{bmatrix} = \left[\begin{array}{c|c} A_{aa}(\theta) & A_{ab}(\theta) \\ \hline A_{ba}(\theta) & A_{bb}(\theta) \end{array}\right] \begin{bmatrix} x_a(k) \\ \cdots \\ x_b(k) \end{bmatrix} + \begin{bmatrix} B_a(\theta) \\ \cdots \\ B_b(\theta) \end{bmatrix} u(k), \quad \text{(18a)}$$

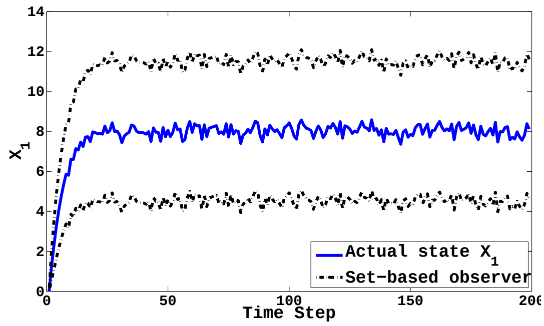
$$y(k) = \begin{bmatrix} I \;\vdots\; 0 \end{bmatrix} \begin{bmatrix} x_a(k) \\ \cdots \\ x_b(k) \end{bmatrix}, \quad \text{(18b)}$$

where $x_a \in \mathbb{R}^{n_{x_a}}$ and $x_b \in \mathbb{R}^{n_{x_b}}$. Moreover, the dimensions of the matrices are $A_{aa} \in \mathbb{R}^{n_{x_a} \times n_{x_a}}$, $A_{ab} \in \mathbb{R}^{n_{x_a} \times n_{x_b}}$, $A_{ba} \in \mathbb{R}^{n_{x_b} \times n_{x_a}}$, $A_{bb} \in \mathbb{R}^{n_{x_b} \times n_{x_b}}$, $B_a \in \mathbb{R}^{n_{x_a} \times n_u}$, $B_b \in \mathbb{R}^{n_{x_b} \times n_u}$. Considering the vector of uncertain parameters θ whose values are bounded by Θ (i.e., $\theta \in \Theta$) or by an interval $[\underline{\theta}, \overline{\theta}]$ (i.e., $\theta \in [\underline{\theta}, \overline{\theta}]$), the following reduced-order interval observer can be used for estimating x_b:

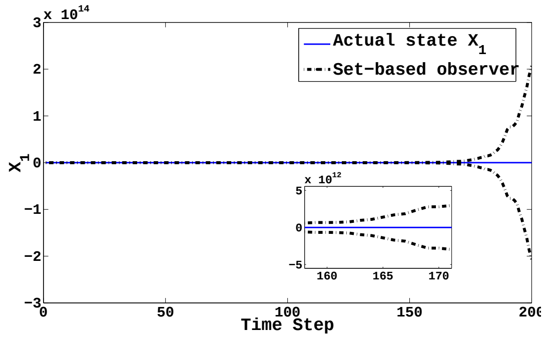
$$\hat{x}_b(k+1) = \hat{\zeta}(k+1) + Lx_a(k+1), \quad \text{(19)}$$

where

$$\hat{\zeta}(k+1) = \Big(A_{bb}(\theta) - LA_{ba}(\theta)\Big)\hat{\zeta}(k) + \Big(A_{bb}(\theta) - LA_{ab}(\theta) + A_{ba}(\theta) - LA_{aa}(\theta)\Big)y(k) + \Big(B_b(\theta) - LB_a(\theta)\Big)u(k).$$



(a) All the elements of A are positive values.



(b) A contains a negative element.

Fig. 1. Set-based interval observer.

Therefore, x_b can be estimated by (19).

Proof. The proof follows from the extension of standard reduced observer results available as e.g. in Ostertag (2011) to the reduced interval observer case.

Hence, the obtained interval observer in (19) is called reduced-order interval observer, which will be able to estimate the unmeasurable states. In other words, if A contains the negative elements, this method can be used to separate the measurable and unmeasurable states. Thus, the only concern will be the unmeasurable states. Therefore, designing the observer gain in order to compensate the effect of negative elements on the unmeasurable states can be done for this reduced-order observer with more degree of freedom as in Section 3.1.

Remark 2. If there are still some cases where selecting the observer gain in order to eliminate the effect of negative elements is not possible, a change of coordinates as proposed by Raïssi et al. (2012) may be helpful to achieve the positivity conditions and the successful interval-observer design.

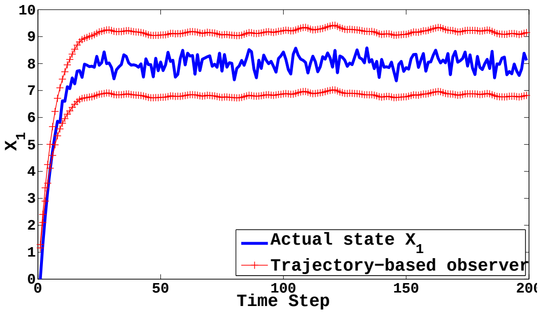
4. ILLUSTRATIVE EXAMPLE

In order to illustrate the approach proposed in previous sections, an academic example is considered based on the dynamical model (1) with

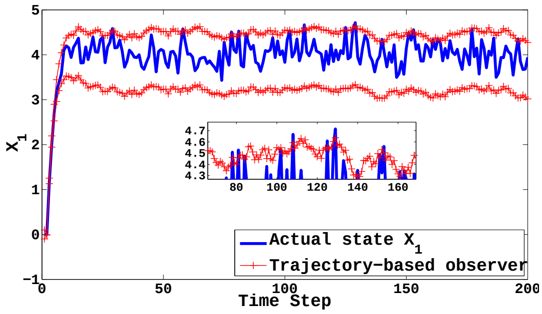
$$A(\theta) = \begin{bmatrix} 0+\theta_{11} & 0.1+\theta_{12} & 0.3+\theta_{13} \\ 0+\theta_{21} & 0.8+\theta_{22} & 0.2+\theta_{23} \\ 0.01+\theta_{31} & 0+\theta_{32} & 0.8+\theta_{33} \end{bmatrix}, \; B(\theta) = \begin{bmatrix} 0+\theta_{11} \\ 0+\theta_{21} \\ 1+\theta_{31} \end{bmatrix},$$

$$C(\theta) = \begin{bmatrix} 0+\theta_{11} & 0+\theta_{21} & 1+\theta_{31} \end{bmatrix},$$

where the time-invariant uncertain parameters are bounded by the following intervals:



(a) All the elements of A are positive values.



(b) A contains a negative element.

Fig. 2. Trajectory-based interval observer.

- $\theta_{11}, \theta_{21}, \theta_{22}, \theta_{31}, \theta_{32} \in [-0.0015, 0.0015]$,
- $\theta_{12}, \theta_{13}, \theta_{23}, \theta_{33} \in [-0.0150, 0.0150]$.

Two different scenarios are considered in this section:

(1) In the case of monotonic system that means all the elements of A are positive,
(2) In the case of a non-monotonic system that at least one of the elements in A is negative.

Therefore, the set-based, trajectory-based and reduced-order observers are designed to estimate the states for each scenario. In addition, the observer gain L is designed by the well-known LMI pole placement method for each case. Furthermore, the proposed observer-gain design method in Section 3.1 is used in the second scenario. In Fig. 1, the simulation results that are obtained by using a set-based approach based on zonotopes in both positive and negative scenarios are presented. In this type of interval-observer approach, the uncertainty is modeled by using the zonotopic set. Moreover, Fig. 1a shows the obtained estimation of the state x_1 in the case that all the elements of A are positive. On the other hand, Fig. 1b shows the result in the case that at least one of the elements of A that is related to the unmeasurable states (in this case x_1 and x_2) is negative. The wrapping effect can be observed in the non-monotonic case since the state estimations of upper and lower bounds are diverging in the second scenario.

Hence, one possible solution to avoid the wrapping effect and preserving the time invariance of the observer is to used the trajectory-based observer. In this case, the observer is constructed based on the punctual trajectories for estimating upper and lower bounds instead of using a zonotopic set. In Fig. 2, the obtained results from the simulation of the trajectory-based observer for both presented scenarios with the same observer gain as the set-based method is shown. On the one hand, the results of a monotonic system can be seen in Fig. 2a. On the other hand, Fig. 2b shows the results when A contains negative elements. The inner solution problem can be seen in Fig. 2b implying the estimated value by the observer is tighter than the correct one. The problem can be solved by designing the observer gain to eliminate the effect of the negative element of A on A_{obs}. In Fig. 3, the set-based state estimation of interval-observer approach provide wider intervals than those produced by the trajectory-based approach. In other words, the set-based approach even without wrapping effect is more conservative than the trajectory-based approach because of considering the time-varying parametric uncertainty.

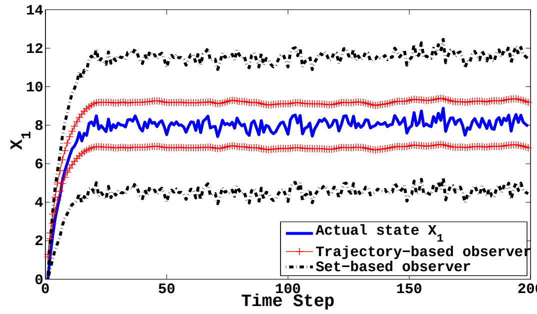


Fig. 3. Set-based vs. Trajectory-based interval observer.

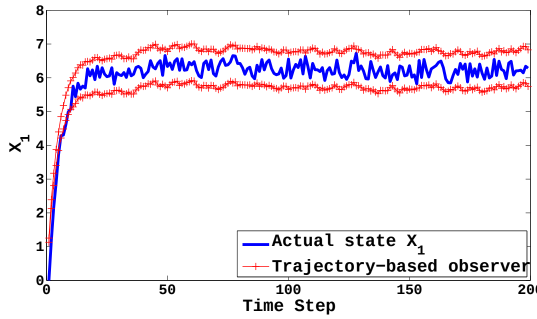
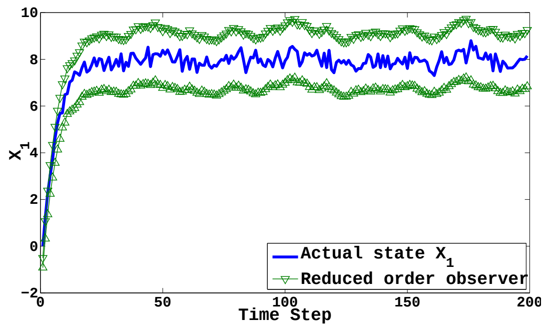
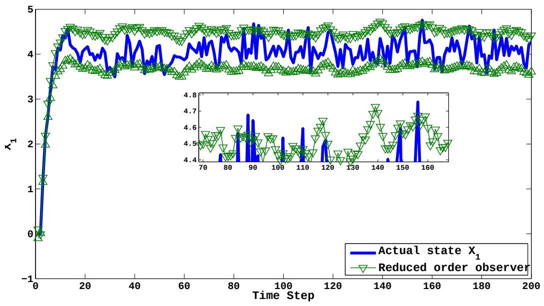


Fig. 4. Designing the trajectory-based observer gain.

On the other hand, the performance of set-based interval-observer approach and trajectory-based interval-observer approach can be compared by looking at Fig. 2b and Fig. 1b that allows seeing that the wrapping effect is avoided by using trajectory-based observer. But, this approach provides the inner solution instead of the exact solution. The clue to overcome the undesired problem in trajectory-based observer is to design the observer gain L to enforce A_{obs} into an element-wise positive matrix even in the situation that the system is non-monotonic. In Fig. 4, the state estimation is obtained by designing the observer gain according to Section 3 that forces the resulting observer to be monotonic. But, sometimes the positivity condition could be hard to be satisfied or guarantee the convergence of the observer at the same time is not possible. With considering this point in mind, the idea of this paper is to overcome this limitation by designing a reduced-order observer instead of a full-order observer that only estimates the unmeasurable states.



(a) All the elements of A are positive values.



(b) A contains a negative element.

Fig. 5. Reduced-order interval observer.

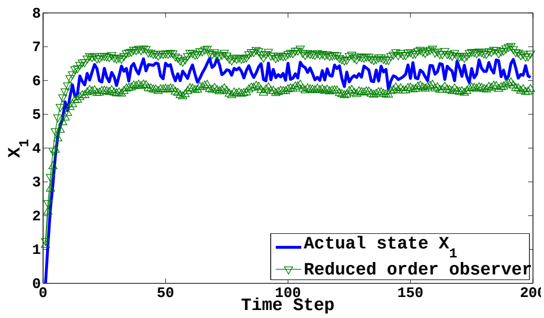


Fig. 6. Designing the reduced-observer observer gain.

Fig. 5 shows the results obtained with the designed reduced-order observer in both scenarios. Moreover, it can be seen in Fig. 5a, in the case that all the elements of the state matrix are positive, the observer is working well. But, the inner solution problem appears in Fig. 5b that indicates the state estimation of the non-monotonic system. That means, the upper and lower approximated bounds is tighter than the exact one. Therefore, the observer gain can be designed to overcome this problem as in Fig. 6, where the obtained result from the simulation that the reduced-order observer gain is designed to solve the inner solution problem is shown. Therefore, the reduced-order observer can be used instead of the full-order observer with more degree of freedom regarding the observer gain design in comparison with the full-order observer. Therefore, the wrapping effect is avoided, the inner solution problem is solved and the observer gain can be designed with more degree of freedom, simultaneously.

5. CONCLUSIONS

This paper presents the state estimation of the discrete-time linear state-space model with the time-invariant parameter uncertainty using different interval-observer approaches. First, it is shown that the time invariance of parameter uncertainty can not be preserved when the observer is designed by using the set-based method and the wrapping effect affects those systems that are not monotonic. Second, the trajectory-based observer is used to overcome the problems. But, the computational complexity is increased when solving the global optimization problem by using the classical trajectory-based observers in the case of the non-monotonic system. Therefore, a method for designing the observer gain is proposed such that a monotonic interval observer is obtained without solving the optimization problems. Moreover, in the case that the number of conditions and constrains for designing the gain does not let to compute the suitable observer gain, a reduced-order observer is proposed to increase the degrees of freedom for selecting the gain. Additionally, it is shown the set-based approach is more conservative than the trajectory-based approach in the case that the system does not affect by the wrapping effect.

As a future research, the idea of forcing the reduced-order observer matrix to satisfy the positivity condition is proposed by using a matrix transformation in order to have much more degrees of freedom to select the observer gain.

REFERENCES

M. Chilali, P. Gahinet, and P. Apkarian. Robust pole placement in LMI regions. *IEEE Transactions on Automatic Control*, 44(12):2257–2270, 1999.

C. Combastel. Zonotopes and Kalman observers: Gain optimality under distinct uncertainty paradigms and robust convergence. *Automatica*, 55:265–273, 2015.

L. Kolev and S. Petrakieva. Assessing the stability of linear time-invariant continuous interval dynamic systems. *IEEE Transactions on Automatic Control*, 50(3):393–397, 2005.

V. T. H. Le, T. Alamo, E. F. Camacho, C. Stoica, and D. Dumur. Zonotopic set-membership estimation for interval dynamic systems. In *American Control Conference (ACC)*, pages 6787–6792. Canada, 2012.

E. Ostertag. *Mono- and Multivariable Control and Estimation Linear, Quadratic and LMI Methods*. Springer-Verlag, Berlin, Germany, 2011.

V. Puig, J. Saludes, and J. Quevedo. Worst-case simulation of discrete linear time-invariant interval dynamic systems. *Reliable Computing*, 9:251–290, 2003.

V. Puig, A. Stancu, and J. Quevedo. Observers for interval systems using set and trajectory-based approaches. In *the 44th IEEE Conference on Decision and Control*, pages 6567–6572. Spain, 2005.

T. Raïssi, D. Efimov, and A. Zolghadri. Interval state estimation for a class of nonlinear systems. *IEEE Transactions on Automatic Control*, 57(1):260–265, 2012.