# ON THE NUMBER OF DEFINING RELATIONS FOR NONFIBERED KÄHLER GROUPS

J. AMORÓS, I. BAUER

The fundamental groups of complex algebraic and compact Kähler varieties have been found to form a special and remarkable class among all groups, with properties related to the geometry of the manifolds that realize them. One such property is the dichotomy between fibered and non–fibered manifolds, i.e. the fact that the existence of irregular pencils on a given complex algebraic manifold depends exclusively on its fundamental group (Beauville [5], Siu [10], [4] in the noncompact case). A measure of the complexity of the fundamental groups of nonfibered manifolds, referred to as *nonfibered groups*, is that they do not admit presentations with few defining relations, which basically means that they differ considerably from free groups. This is the so–called theorem of the few defining relations, of which several versions for compact Kähler manifolds have been proved recently by Gromov ([9]) using $L^2$ cohomology, Green and Lazarsfeld ([7]) studying deformations of line bundles, Catanese ([6]) with a simple topological argument, the first author ([1]) using Hodge theory, and by the second author ([4]) and Arapura ([2]) in the quasiprojective setting. The diversity of methods has resulted in several different bounds.

It has been our aim in this note to reach the optimal bound for the *deficiency* (=minimal difference between relations and generators in all presentations of the group) of these nonfibered groups, found in [1] in the compact Kähler case and unpublished in the quasiprojective case, to prove it in a unified way and by the simplest available method. This consists in using the Castelnuovo–de Franchis theorem, both in its classical and its logarithmic version of [4], Hodge theory for quasiprojective manifolds, and deriving from the inexistence of isotropic pencils of 1–forms bounds for the image of their cup products by a standard cone argument as in [1]. To translate the bound on the second Betti number of the manifold into a bound on the deficiency, we have used the straightforward topological argument employed in [6], [4].

*Date*: 12 January 1999.

First author supported by DGICYT grant PB96-0234.

Second author supported by A.G.E. contract ERBCHRXCT 940557.

*Acknowledgements*: This note was prepared during the Special Activity on Topology of Algebraic Varieties in honour of Boris Moishezon, held at the MPI of Mathematics in July of 1998. The authors would like to thank the organizers and the MPI for providing a very stimulating environment. We would also like to thank V. Navarro Aznar for subsequent conversations on the subject.

We start recalling the algebraic topologic tools we will require:

**Lemma 1.** *If a group $\Gamma$ admits a presentation with $n$ generators and $s$ defining relations, then*

$$s - n \geq b_2(\Gamma) - b_1(\Gamma)$$

*Proof.* If $\Gamma$ admits a presentation $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_s \rangle$, then one may build a classifying space $K = K(\Gamma, 1)$ with 0–skeleton $K^0$ a single point, 1–skeleton $K^1$ a bouquet of $n$ circumferences corresponding to the generators $x_1, \ldots, x_n$ and 2–skeleton $K^2$ formed by $s$ 2–cells with attaching maps given by the relations $r_1, \ldots, r_s$.

Let us compute $H_1, H_2(\Gamma, \mathbb{Z})$ through the cellular homology of this classifying space. All the 1–cells are cycles, and the 2-cell $e_j$ with attaching map the relation $r_j$ has as boundary the image of the relation $r_j$ in the abelianization of the free group generated by $x_1, \ldots, x_n$. so we have

$$b_1(\Gamma) = n - \dim \operatorname{Im} \partial_2$$
$$b_2(\Gamma) \leq s - \dim \operatorname{Im} \partial_2$$

from which our sought inequality follows. ☐

Given a finite type CW complex $X$, there exists a cofibration $X \hookrightarrow K(\pi_1(X), 1)$ inducing an isomorphism $H^1(\pi_1(X), \mathbb{Z}) \cong H^1(X, \mathbb{Z})$ and a monomorphism $H^2(\pi_1(X), \mathbb{Z}) \hookrightarrow H^2(X, \mathbb{Z})$. It follows that

$$s - n \geq \dim \operatorname{Im} \left( \wedge^2 H^1(X) \stackrel{\cup}{\to} H^2(X) \right) - b_1(X) \tag{1}$$

We will find lower bounds for this dimension in nonfibered manifolds. In the compact case this is done in [1] 5.4,5.5 (cf. also [3] IV) in the following way: The classical Castelnuovo–de Franchis theorem shows that if $X$ is nonfibered the cup product of holomorphic/antiholomorphic 1–forms cannot be zero. One uses this together with the Hodge decomposition of $H^*(X)$ and an elementary projective geometry argument: the cone of decomposable elements of $\wedge^2 E$, resp. $E \otimes F$, contains a linear subspace of dimension $2 \dim E - 3$, resp. $\dim E + \dim F - 1$. Thus these are the least ranks for the image of linear mappings arising from those spaces whose kernels meet the cone of decomposable elements only at the origin. In this way one arrives at:

**Proposition 2.** *Let $X$ be a nonfibered compact Kähler manifold with $b_1(X) = 2q$. Then*

(i) $\dim Im(\cup : H^{1,0}(X) \wedge H^{1,0}(X) \to H^{2,0}(X)) \geq 2q - 3$.
(ii) $\dim Im(\cup : H^{0,1}(X) \wedge H^{0,1}(X) \to H^{0,2}(X)) \geq 2q - 3$.
(iii) $\dim Im(\cup : H^{1,0}(X) \wedge H^{0,1}(X) \to H^{1,1}(X)) \geq 2q - 1$.

In the quasi–projective case we may repeat the above counting argument using the logarithmic Castelnuovo–de Franchis theorem of [4] and Deligne's mixed Hodge structure on the cohomology of the manifold $X$.

Let $\nu : X \hookrightarrow \bar{X}$ be a smooth compactification of $X$ with $Y = \bar{X} \setminus X$ a normal crossing divisor. The isomorphism $H^*(X, \mathbb{C}) \cong H^*(\Omega^*_{\bar{X}}(\log Y))$ induces the following weight filtration on $H^1(X, \mathbb{C})$:

$$W_0 = 0, W_1 = H^1(\Omega^*_{\bar{X}}), W_2 = H^1(\Omega^*_{\bar{X}}(\log Y))$$

and the Hodge filtration

$$F^0 = H^1(\Omega^*_{\bar{X}}(\log Y)), F^1 = H^0(\Omega^1_{\bar{X}}(\log Y)), F^2 = 0\,.$$

The wedge product induces a morphism of mixed Hodge structures $\cup : \wedge^2 H^1(X, \mathbb{C}) \to H^2(X, \mathbb{C})$, $\mathbb{C}$–linearly isomorphic to the graduate morphism

$$\wedge^2(\mathrm{Gr}_W H^1(X, \mathbb{C})) \stackrel{\mathrm{Gr}_W \cup}{\longrightarrow} \mathrm{Gr}_W H^2(X, \mathbb{C})$$

We will bound the rank of the image of $\cup$ on every component, and use the Hodge filtration in those where it is nontrivial. We remind the reader of the isomorphism

$$W_2/W_1(H^1(X, \mathbb{C})) \cong H^0(\Omega^1_{\bar{X}}(\log Y))/H^0(\Omega^1_{\bar{X}})\,, \tag{2}$$

and set $q = \dim H^0(\Omega^1_{\bar{X}}), h = \dim H^0(\Omega^1_{\bar{X}}(\log Y))/H^0(\Omega^1_{\bar{X}})$.

Recall that a morphism of mixed Hodge structures of type $(r, r)$ from $(A, W, F)$ to $(B, W', F')$ is a linear morphism $\varphi : A \to B$ such that $\varphi(W_m) \subset W'_{m+2r}, \varphi(F^p) \subset F'^{p+r}$. Such morphisms are *strict*. This means that

$$\varphi(W_m) = W'_{m+2r} \cap \mathrm{Im}\, \varphi$$

and

$$\varphi(F^p) = F'^{p+r} \cap \mathrm{Im}\, \varphi$$

for all $m, p$ (see [8], Lemma 1.13). In the case of cup products this means:

**Lemma 3.** (i) *Let $\omega \in H^0(\Omega^1_{\bar{X}}(\log Y)) \setminus H^0(\Omega^1_{\bar{X}})$ and $\eta \in H^0(\Omega^1_{\bar{X}})$. If $\omega \wedge \eta \in W_2 H^2(X, \mathbb{C})$ then $\omega \wedge \eta = 0$.*

(ii) *Let* $\omega_1, \omega_2 \in H^0(\Omega^1_{\bar X}(\log Y))$ *be such that* $\omega_1 \wedge \omega_2 \in W_3 H^2(X, \mathbb{C})$, *then there exists a form* $\eta \in H^0(\Omega^1_{\bar X})$ *such that* $\omega_1 \wedge (\omega_2 - \eta) = 0$.

*Proof.* (i): The morphism $\omega \wedge . : H^1(X) \to H^2(X)$ has type (1,1) so

$$W_2(H^2(X)) \cap \mathrm{Im}(\omega \wedge .) = \omega \wedge W_0(H^1(X)) = 0$$

(ii): Consider as before the morphism of type (1,1) $\omega_1 \wedge . : H^1(X) \to H^2(X)$. By strictness we get

$$W_3(H^2(X)) \cap \mathrm{Im}(\omega_1 \wedge .) = \omega_1 \wedge W_1(H^1(X))$$

Therefore there exists a cohomology class $e \in W_1(H^1(X)) \cong H^0(\Omega^1_{\bar X}) \oplus \overline{H^0(\Omega^1_{\bar X})}$ with $\omega_1 \wedge e = \omega_1 \wedge \omega_2$. If we separate the Hodge components $e = e^{1,0} + e^{0,1}$ and use that $\omega_1 \cup e \in F^2(H^2(X))$ it follows that $\omega_1 \wedge e^{0,1} = 0$, so we may choose $e = e^{1,0}$ and realize it by a form $\eta \in H^0(\Omega^1_{\bar X})$. □

The quasiprojective version of Prop. 2 is

**Proposition 4.** *If* $X$ *is a quasiprojective manifold not admitting any logarithmic irregular pencil:*

(i) $\dim Im(Gr_2 \cup : \wedge^2 W_1/W_0 H^1(X, \mathbb{C}) \to W_2/W_1 H^2(X, \mathbb{C})) \geq 4q - 6$.
(ii) *If* $h > 0$, $\dim Im(Gr_3 \cup : W_2/W_1 H^1(X, \mathbb{C}) \otimes W_1/W_0 H^1(X, \mathbb{C}) \to W_3/W_2 H^2(X, \mathbb{C})) \geq 2q + 2h - 2$.
(iii) $\dim Im(Gr_4 \cup : \wedge^2 W_2/W_1 H^1(X, \mathbb{C}) \to W_4/W_3 H^2(X, \mathbb{C})) \geq 2h - 3$.

*Proof.* (i): by the classical Castelnuovo–de Franchis theorem, if the kernel of the cup product contains any decomposable element $\eta_1 \wedge \eta_2 \in \wedge^2 H^0(\Omega^1_{\bar X})$ then $\bar X$, thus also $X$, fibers over a hyperbolic $C$. Therefore, as in the compact Kähler case, if $X$ does not admit such a fibering $\dim \cup(\wedge^2 H^0(\Omega^1_{\bar X}) \geq 2q - 3$. This image lies in the component (2,0) of the Hodge decomposition of $W_2/W_1 H^2(X)$. Conjugation shows that $\dim \cup(\wedge^2 \overline{H^0(\Omega^1_{\bar X})}) \geq 2q - 3$, and this image lies in the component (0,2) of the Hodge decomposition, so we may add our bounds.
(ii): We will require the canonical isomorphism $W_1 H^1(X, \mathbb{C}) \cong H^0(\Omega^1_{\bar X}) \oplus \overline{H^0(\Omega^1_{\bar X})}$, mapping the two terms of the latter sum to the (1,0), resp. (0,1), components of the Hodge decomposition of $W_1/W_0 H^1(X, \mathbb{C})$.

If the kernel of $\mathrm{Gr}_3 \cup$ contains any decomposable element $\omega \otimes \eta$ with $\eta \in H^0(\Omega^1_{\bar X})$, by Lemma 3 $\omega \wedge \eta = 0 \in H^2(X, \mathbb{C})$, thus also in $H^0(\Omega^2_{\bar X}(\log Y))$. It follows now from the log–CdF theorem of [4] that $X$ fibers over a log–hyperbolic curve. As a consequence, $\dim \mathrm{Gr}_3 \cup (W_2/W_1 H^1(X, \mathbb{C}) \otimes H^0(\Omega^1_{\bar X})) \geq q + h - 1$ if $X$ does not admit such a fibering. This image has Hodge type (2,1) in the pure Hodge decomposition of weight 3 of $W_3/W_2 H^2(X)$.

By conjugation in the morphism of pure Hodge structures of weight 3 $\mathrm{Gr}_3\cup$ the above argument also shows that $\dim \mathrm{Gr}_3\cup(W_2/W_1H^1(X,\mathbb{C})\otimes \overline{H^0(\Omega^1_{\bar X})}) \geq q+h-1$ if $X$ is nonfibered. This image has Hodge type (1,2) in $W_3/W_2H^2(X)$, so we may add its lower bound to that of the Hodge component (2,1).
(iii): Assume now that the cup–product $\mathrm{Gr}_4\cup$ contains a nontrivial decomposable element in its kernel. By the identification of (2) this means that there exist holomorphic 1–forms $\omega_1,\omega_2 \in H^0(\Omega^1_{\bar X}(\log Y))$ such that they are linearly independent modulo $H^0(\Omega^1_{\bar X})$ and $\omega_1\wedge\omega_2 \in W_3H^2(X,\mathbb{C})$. Strictness Lemma 3 shows that we may choose the forms so that $\omega_1\wedge\omega_2=0$, and by the log–CdF theorem of [4] $X$ fibers over a log–hyperbolic curve. Therefore, if $X$ does not admit such a fibering $\mathrm{Gr}_4$ must have image of rank at least $2h-3$. □

By the $\mathbb{C}$–linear isomorphisms with the Hodge, resp. weight, graded spaces in Props. 2 and 4 simply adding up the ranks of the images yields a lower bound for $b_2(X)$, and applying inequality (1) we get a new version of the theorem of the few defining relations that gives bounds both in the compact Kähler and in the quasi–projective case. In the cases with $h=0$, resp. $q=0$, we only consider the relevant terms (i),(ii), resp. (iii).

**Theorem 5.** *Let $X$ be a complex manifold not admitting an irregular pencil, and $\pi_1(X) = \langle x_1,\dots,x_n \,|\, r_1,\dots,r_s\rangle$ a presentation of its fundamental group with $n$ generators and $s$ relations.*

(i) *If $X$ is compact Kähler, then $s-n \geq 2b_1(X)-7$.*
(ii) *If $X$ is quasiprojective and $h,q>0$, then $s-n \geq 2b_1(X)+h-11$.*
(iii) *If $X$ is quasiprojective and $h=0, q>0$, then $s-n \geq b_1(X)-6$.*
(iv) *If $X$ is quasiprojective and $h>0, q=0$, then $s-n \geq b_1(X)-3$.*

*Remark 6.* If $b_1(X)=0$, Lemma 1 shows directly that $s-n\geq 0$.

*Remark 7.* One may replace the inequality (1) by a less straightforward study of the Malcev algebra of $\pi_1(X)$ as in [1]. This yields the same bounds as in Thm. 5, but it locates in which term of the lower central or the derived series of the free group $\langle x_1,\dots,x_n\rangle$ are found the defining relations.

## References

[1] J. Amorós, *On the Malcev completion of Kähler groups*, Comment. Math. Helv. **71** (1996) 192–212.
[2] D. Arapura, *Geometry of cohomology support loci for local systems I*, J. Alg. Geom. **6** (1997) 563–597.

[3] W. Barth,C. Peters,A. Van de Ven, *Compact Complex Surfaces*. Ergeb. Math. Grenzgeb. (3) **4**. Springer–Verlag 1984.

[4] I. Bauer, *Irrational pencils on non–compact algebraic manifolds*, Int. J. Math. **8** (1997) 441–450.

[5] A. Beauville, Appendix to F. Catanese, *Moduli and classification of irregular Kähler manifolds (and algebraic varieties) with Albanese general type fibrations*, Invent. Math. **104** (1991) 263–289.

[6] F. Catanese, *Fundamental groups with few relations* in *Higher Dimensional Complex Varieties, Trento 1994*. De Gruyter 1996, pp. 163–165.

[7] M. Green, R. Lazarsfeld, *Higher obstructions to deforming cohomology groups of line bundles*, J. Amer. Math. Soc. **4** (1991) 87–103.

[8] P. Griffiths, W. Schmid, *Recent developments in Hodge theory: a discusion of techniques and results*, in *Proceedings of the International Colloquium on Discrete Subgroups of Lie Groups. Bombay 1973*, Oxford U.P. 1975.

[9] M. Gromov, *Sur le groupe fondamental d'une variété kählérienne*, C.R. Acad. Sci. Paris **308** Série I (1989) 67–70.

[10] Y.-T. Siu, *Strong rigidity for Kähler manifolds and the construction of bounded holomorphic functions*, in R. Howe (ed.), *Discrete Groups and Analysis*, Birkhäuser Verlag 1987.

Department of Mathematics, University of Utah, Salt Lake City, UT 84112, U.S.A.

*Current address:* Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain
*E-mail address*: amoros@ma1.upc.es

Mathematisches Institut der Universität Göttingen, Bunsenstr. 3–5, 37075 Göttingen, Germany
*E-mail address*: ibauer@cfgauss.uni-math.gwdg.de