LATTICE PATH MATROIDS: STRUCTURAL PROPERTIES

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Abstract. This paper studies structural aspects of lattice path matroids, a class of transversal matroids that is closed under taking minors and duals. Among the basic topics treated are direct sums, duals, minors, circuits, and connected flats. One of the main results is a characterization of lattice path matroids in terms of fundamental flats, which are special connected flats from which one can recover the paths that define the matroid. We examine some aspects related to key topics in the literature of transversal matroids and we determine the connectivity of lattice path matroids. We also introduce notch matroids, a minor-closed, dual-closed subclass of lattice path matroids, and we find their excluded minors.

1. Introduction

A lattice path matroid is a special type of transversal matroid whose bases can be thought of as lattice paths in the region of the plane delimited by two fixed bounding paths. These matroids, which were introduced and studied from an enumerative perspective in [5], have many attractive structural properties that are not shared by arbitrary transversal matroids; this paper focuses on such properties.

The definition of lattice path matroids is reviewed in Section 2, where we also give some elementary properties of their bases and make some remarks on connectivity and automorphisms. Section 3 proves basic results that are used throughout the paper; for example, we show that the class of lattice path matroids is closed under minors, duals, and direct sums, we determine which lattice path matroids are connected, and we describe circuits and connected flats. The next section discusses generalized Catalan matroids, a minor-closed, dual-closed subclass of lattice path matroids that has particularly simple characterizations. Section 5 introduces special connected flats called fundamental flats that we use to characterize lattice path matroids and to show that the bounding paths can be recovered from the matroid. In Section 6, we describe the maximal presentation of a lattice path matroid, and we use this result to give a geometric description of these matroids as well as a polynomial-time algorithm for recognizing lattice path matroids within the class of transversal matroids. We also contrast lattice path matroids with fundamental transversal matroids and bicircular matroids. Section 7 treats higher connectivity. The final section introduces another minor-closed, dual-closed class of lattice path matroids, the notch matroids, and characterizes this class by excluded minors.

We assume familiarity with basic matroid theory (see, e.g., [16, 20]). We follow the notation and terminology of [16], with the following additions. A flat X of a matroid M is *connected* if the restriction $M|X$ is connected. A flat X is *trivial* if X is independent; otherwise X is *nontrivial*. The flats in a collection $\mathcal{F}$ of flats are

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incomparable, or *mutually incomparable*, if no flat in $\mathcal{F}$ contains another flat in $\mathcal{F}$. The *nullity*, $|X| - r(X)$, of a set X is denoted by $\eta(X)$. Recall that a matroid M of rank r is a *paving matroid* if every flat of rank less than $r-1$ is trivial.

Most matroids in this paper are transversal matroids (see [6, 12, 20]). Recall that for a transversal matroid M, a *presentation* of M is a multiset $\mathcal{A} = (D_1, D_2, \dots, D_k)$ of subsets of the ground set $E(M)$ such that the bases of M are the maximal partial transversals of $\mathcal{A}$. As is justified by the following lemma (see [6]), we always consider presentations of rank-r transversal matroids by set systems of size r.

Lemma 1.1. *Let $\mathcal{A} = (D_1, D_2, \dots, D_k)$ be a presentation of a rank-r transversal matroid M. If some basis of M is a transversal of $(D_{i_1}, D_{i_2}, \dots, D_{i_r})$, with $i_1 < i_2 < \cdots < i_r$, then $(D_{i_1}, D_{i_2}, \dots, D_{i_r})$ is also a presentation of M.*

We use $[n]$ to denote the interval $\{1, 2, \dots, n\}$ of integers, and, similarly, $[i, j]$ to denote the interval $\{i, i+1, \dots, j\}$ of integers.

2. Background

This section starts by reviewing the definition and basic properties of lattice path matroids from [5]. The notation established in this section is used throughout the paper. Also included are the basic results about matroid connectivity that we use later.

Unless otherwise stated, all lattice paths in this paper start at the point $(0,0)$ and use steps $E = (1,0)$ and $N = (0,1)$, which are called *East* and *North*, respectively. Paths are usually represented as words in the alphabet $\{E, N\}$. We say that a lattice path P has a *NE corner at h* if step h of P is North and step $h+1$ is East. An *EN corner at k* is defined similarly. A corner can also be specified by the coordinates of the point where the North and East steps meet.

A *lattice path matroid* is, up to isomorphism, a matroid of the type $M[P,Q]$ that we now define. Let P and Q be lattice paths from $(0,0)$ to (m,r) with P never going above Q. Let $\mathcal{P}$ be the set of all lattice paths from $(0,0)$ to (m,r) that go neither above Q nor below P. For i with $1 \le i \le r$, let N_i be the set

$$N_i := \{j : \text{step } j \text{ is the } i\text{-th North step of some path in } \mathcal{P}\}.$$

Thus, $N_1, N_2, \dots, N_r$ is a sequence of intervals in $[m+r]$, and both the left endpoints and the right endpoints form strictly increasing sequences; the left and right endpoints of N_i correspond to the positions of the i-th North steps in Q and P, respectively. The matroid $M[P,Q]$ is the transversal matroid on the ground set $[m+r]$ that has $(N_1, N_2, \dots, N_r)$ as a presentation. We call $(N_1, N_2, \dots, N_r)$ the *standard presentation* of $M[P,Q]$. Note that $M[P,Q]$ has rank r and nullity m.

Figure 1 shows a lattice path matroid of rank 4 and nullity 7. The intervals in the standard presentation are $N_1 = [4]$, $N_2 = [2,7]$, $N_3 = [5,10]$, and $N_4 = [6,11]$. (Section 6.3 explains how to find a geometric representation of a lattice path matroid.)

A feature that enriches the subject of lattice path matroids is the variety of ways in which these matroids can be viewed. On the one hand, the theory of transversal matroids provides many useful tools. On the other hand, the following theorem from [5] gives an interpretation of the bases that leads to attractive descriptions of many matroid concepts (see, e.g., [5, Theorem 5.4] on basis activities).

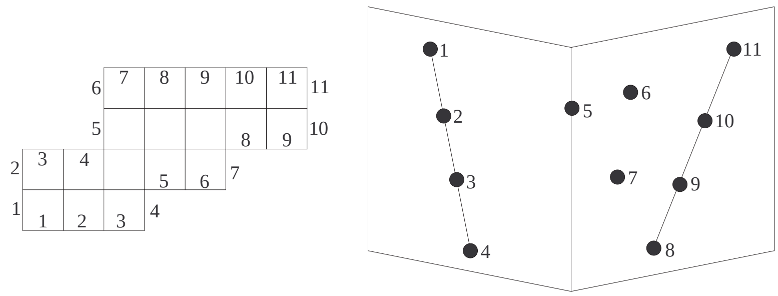

FIGURE 1. A lattice path presentation and geometric representation of a lattice path matroid.

Theorem 2.1. *The map $R \mapsto \{i$: the i-th step of R is North$\}$ is a bijection from $\mathcal{P}$ onto the set of bases of $M[P, Q]$.*

We use $\mathcal{L}$ to denote the class of all lattice path matroids. We call the pair (P, Q) a *lattice path presentation* of $M[P, Q]$, or, if there is no danger of confusion, a *presentation* of $M[P, Q]$.

Unless we say otherwise, all references to an order on the ground set $[m + r]$ of $M[P, Q]$ are to the natural order $1 < 2 < \cdots < m + r$. However, this order is not inherent in the matroid structure; the elements of a lattice path matroid typically can be linearly ordered in many ways so as to correspond to the steps of lattice paths. Also, a lattice path matroid of rank r and nullity m need not have $[m + r]$ as its ground set. These comments motivate the following definition.

Definition 2.2. *A linear ordering $s_1 < s_2 < \cdots < s_{m+r}$ of the ground set of a lattice path matroid M is a* lattice path ordering *if the map $s_i \mapsto i$ is an isomorphism of M onto a lattice path matroid of the form $M[P, Q]$.*

For some purposes it is useful to view lattice path matroids from the following perspective, which does not refer to paths. Lattice path matroids are the transversal matroids M for which $E(M)$ can be linearly ordered so that M has a presentation $(A_1, A_2, \ldots, A_r)$ where $A_i = [l_i, g_i]$ is an interval in $E(M)$ and the endpoints of these intervals form two chains, $l_1 < l_2 < \cdots < l_r$ and $g_1 < g_2 < \cdots < g_r$.

The *incidence function* of a presentation $(A_1, A_2, \ldots, A_r)$ of a transversal matroid is given by $n(X) = \{i : X \cap A_i \neq \emptyset\}$ for subsets X of $E(M)$. If no other presentation is mentioned, the incidence function of the matroid $M[P, Q]$ of rank r and nullity m is understood to be that associated with the standard presentation. For this incidence function and for any element x in $[m + r]$, the set $n(x)$ is an interval in $[r]$; if $x < y$, then $\max\big(n(x)\big) \leq \max\big(n(y)\big)$ and $\min\big(n(x)\big) \leq \min\big(n(y)\big)$.

An independent set I in a lattice path matroid $M[P, Q]$ is a partial transversal of $(N_1, N_2, \ldots, N_r)$. Typically there are many ways to match I with $N_1, N_2, \ldots, N_r$. The next two results show that I can always be matched in a natural way. The following lemma, which is crucial in the proof of Theorem 2.1, is from [5]

Lemma 2.3. *Assume $\{b_1, b_2, \ldots, b_r\}$ is a basis of a lattice path matroid $M[P, Q]$ with $b_1 < b_2 < \cdots < b_r$. Then b_i is in N_i for all i.*

Corollary 2.4 follows by extending the given independent set I to a basis and applying Lemma 2.3.

Corollary 2.4. *Assume I is an independent set of a lattice path matroid $M[P,Q]$ with $|I| = |n(I)|$; let I be $\{a_1, a_2, \dots, a_k\}$ with $a_1 < a_2 < \cdots < a_k$ and let $n(I)$ be $\{i_1, i_2, \dots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$. Then a_j is in N_{i_j} for all j with $1 \leq j \leq k$.*

We now gather several results on matroid connectivity that are relevant to parts of the paper. The first result [16, Theorem 7.1.16] gives a fundamental link between connectivity and the operation of parallel connection.

Lemma 2.5. *If M is connected and M/p is the direct sum $M_1 \oplus M_2$, then M is the parallel connection $P(M_1', M_2')$ of $M_1' := M\backslash E(M_2)$ and $M_2' := M\backslash E(M_1)$.*

In Lemma 2.5, since M is connected, both M_1' and M_2' are connected. Recall that the rank $r\big(P(M_1', M_2')\big)$ of a parallel connection whose basepoint is not a loop is $r(M_1') + r(M_2') - 1$. These observations give the following lemma.

Lemma 2.6. *If M is connected, x is not parallel to any element of M, and M/x is disconnected, then there is a pair A, B of nontrivial incomparable connected flats of M with $r(A) + r(B) = r(M) + 1$ and $A \cap B = \{x\}$.*

The following useful lemma is easy to prove by using separating sets.

Lemma 2.7. *Assume that X is a connected flat of a connected matroid M, that x is in X, and that $M|(X - x)$ is connected. Then $M\backslash x$ is connected.*

The cyclic flats of a matroid M (that is, the flats F for which $M|F$ has no isthmuses), together with their ranks, determine the matroid [8, Proposition 2.1]. As we show next, in the loopless case it suffices to consider nontrivial connected flats. Note that nontrivial connected flats are cyclic, but cyclic flats need not be connected. Thus, the next result is a mild refinement of [8, Proposition 2.1], and essentially the same idea proves both results.

Lemma 2.8. *The circuits of a loopless matroid M (and hence M itself) are determined by the nontrivial connected flats and their ranks.*

Proof. Note that if C is an i-circuit, then $\mathrm{cl}(C)$ is a connected flat of rank $i - 1$. Thus, the circuits can be recovered inductively as follows: the 2-circuits are the 2-subsets of nontrivial rank-1 flats; the 3-circuits are the 3-subsets of $E(M)$ that contain no 2-circuit and are subsets of connected lines, and so on. □

Corollary 2.9. *The automorphisms of a loopless matroid are the permutations of the ground set that are rank-preserving bijections of the collection of nontrivial connected flats.*

3. Basic Structural Properties of Lattice Path Matroids

This section treats the basic structural properties of lattice path matroids that play key roles throughout this paper. Some of these properties are shared by few other classes of matroids; for instance, every nontrivial connected lattice path matroid has a spanning circuit. Other properties, such as the closure of the class of lattice path matroids under minors and duals, while shared by many classes of matroids, do not hold for the larger class of transversal matroids. Some of the properties are more technical and their significance will become apparent only later in the paper. The topics treated are fairly diverse, so we divide the material into subsections that focus in the following issues: minors, duals, and direct sums; connectivity and spanning circuits; the structure of circuits and connected flats.

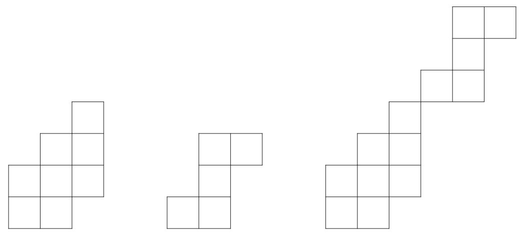

FIGURE 2. Presentations of two lattice path matroids and their direct sum.

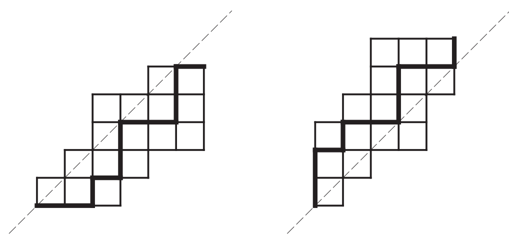

FIGURE 3. Presentations of a lattice path matroid and its dual.

3.1. **Minors, Duals, and Direct Sums.** The class of transversal matroids, although closed under deletions and direct sums, is closed under neither contractions nor duals. In contrast, we have the following result for lattice path matroids.

Theorem 3.1. *The class $\mathcal{L}$ is closed under minors, duals, and direct sums.*

Proof. Figure 2 illustrates the obvious construction to show that $\mathcal{L}$ is closed under direct sums. For closure under duality, note that, from Theorem 2.1, a basis of the dual of $M[P, Q]$ (i.e., the complement of a basis of $M[P, Q]$) corresponds to the East steps in a lattice path; the East steps of a lattice path are the North steps of the lattice path obtained by reflecting the entire diagram about the line $y = x$. This idea is illustrated in Figure 3.

For closure under minors, it suffices to consider single-element deletions. Let x be in the lattice path matroid $M = M[P, Q]$ on $[m+r]$ with standard presentation $(N_1, N_2, \dots, N_r)$. Note that $(N_1 - x, N_2 - x, \dots, N_r - x)$ is a presentation of $M\backslash x$; from this presentation, we will obtain one that shows that $M\backslash x$ is a lattice path matroid. Some set N_i is $\{x\}$ if and only if x is an isthmus of M; in this case, discard the empty set $N_i - x$ from the presentation above to obtain the required presentation of $M\backslash x$. Thus, assume x is not an isthmus of M. The sets $N_1 - x, N_2 - x, \dots, N_r - x$ are intervals in the induced linear order on $[m+r] - x$. In only two cases will the least elements or the greatest elements (or both) fail to increase strictly: (a) x is the least element of the interval N_i and $x+1$ is the least element of N_{i+1}, and (b) $x-1$ and x are the greatest elements of N_{j-1} and N_j, respectively. Assume case (a) applies. Any basis of $M\backslash x$ (that is, any basis of M that does not contain x) that contains $x+1$ can, by Lemma 2.3, be matched with $N_1, N_2, \dots, N_r$ so that $x+1$ is not matched to N_{i+1}. Thus, the set system obtained by replacing N_{i+1} by

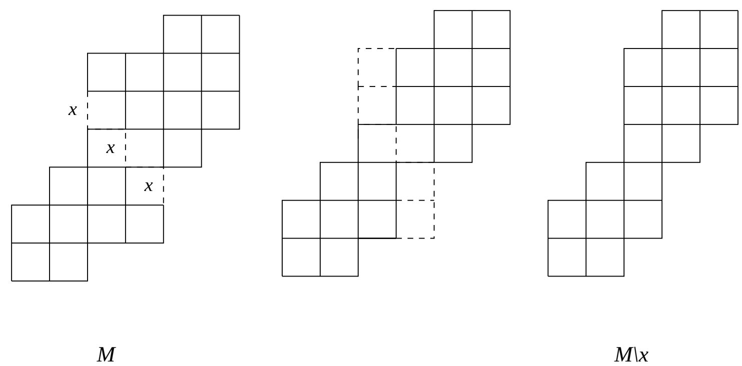

FIGURE 4. The lattice path interpretation of the shortening of intervals that yields a presentation of a single-element deletion.

$N_{i+1} - \{x+1\}$ is also a presentation of $M\backslash x$. The same argument justifies replacing N_{i+2} by $N_{i+2} - \{x+2\}$ if $x+2$ is the least element of N_{i+2}, and so on. Case (b) is handled similarly. The result is a presentation of $M\backslash x$ by intervals in which the least and greatest elements increase strictly, so $M\backslash x$ is a lattice path matroid. □

Single-element deletions and contractions can be described in terms of the bounding paths of $M = M[P,Q]$ as follows. An isthmus is an element x for which some N_i is $\{x\}$; to delete or contract x, eliminate the corresponding common North step from both bounding paths. A loop is an element that is in no set N_i; to delete or contract a loop, eliminate the corresponding common East step from P and Q. Now assume x is neither a loop nor an isthmus. The upper bounding path for $M\backslash x$ is formed by deleting from Q the first East step that is at or after step x; the lower bounding path for $M\backslash x$ is formed by deleting from P the last East step that is at or before step x. This is shown in Figure 4, where the dashed steps in the middle diagram indicate the steps that bases of $M\backslash x$ must avoid. Dually, the upper bounding path for the contraction M/x is formed by deleting from Q the last North step that is at or before step x; the lower bounding path for M/x is formed by deleting from P the first North step that is at or after step x.

Corollary 3.2 treats restrictions of lattice path matroids to intervals. The lattice path interpretation of this result is illustrated in Figure 5 on page 13.

Corollary 3.2. *Let M be the lattice path matroid $M[P,Q]$ on the ground set $[m+r]$. Let X be the initial segment $[i]$ and Y be the final segment $[j+1, m+r]$ of $[m+r]$. Let the i-th step of Q end at the point (h,k) and let the j-th step of P end at (h',k').*

(a) *The bases of the restriction $M|X$ correspond to the lattice paths that go from $(0,0)$ to (h,k) and go neither below P nor above Q.*
(b) *The bases of the restriction $M|Y$ correspond to the lattice paths that go from (h',k') to (m,r) and go neither below P nor above Q.*
(c) *If $h' \leq h$, then the bases of $M|(X \cap Y)$ correspond to the lattice paths that go from (h',k') to (h,k) and go neither below P nor above Q.*

We close this section by noting that although $U_{1,2} \oplus U_{1,2} \oplus U_{1,2}$ is a lattice path matroid, its truncation is not transversal. It follows that $\mathcal{L}$ is not closed under the following operations: truncation, free extension, and elongation.

3.2. **Connectivity and Spanning Circuits.** We begin with a rare property.

Theorem 3.3. *A connected lattice path matroid $M[P,Q]$ on at least two elements has a spanning circuit.*

Proof. Let $M[P,Q]$ have rank r, let N_j be $[l_j, g_j]$ for $1 \leq j \leq r$, and let C be the set $\{l_1, l_2, \dots, l_{r-1}, l_r, g_r\}$. Showing that each set $C - x$, for x in C, is a basis shows that C is a spanning circuit. That $C - l_r$ and $C - g_r$ are bases is clear. Since $M[P,Q]$ is not a direct sum of two matroids, l_{i+1} must be in N_i for $1 \leq i < r$, from which it follows that each set $C - l_j$, with $1 \leq j < r$, is a basis. □

It will be useful to single out the following immediate corollary of Theorem 3.3.

Corollary 3.4. *If X is a nontrivial connected flat of a matroid M and $M|X$ is a lattice path matroid, then X is $\mathrm{cl}(C)$ for some circuit C of M.*

The next theorem determines which lattice path matroids are connected. One implication follows from the description of direct sums and the other from the construction of the spanning circuit in the proof of Theorem 3.3.

Theorem 3.5. *A lattice path matroid $M[P,Q]$ of rank r and nullity m is connected if and only if P and Q intersect only at $(0,0)$ and (m,r).*

The parallel connection of two 3-point lines, which has only one spanning circuit, shows that there may be elements of a connected lattice path matroid that are in no spanning circuit. There are several ways to identify the elements of connected lattice path matroids that are in spanning circuits. The next result identifies these elements via the standard presentation.

Theorem 3.6. *An element x of a nontrivial connected lattice path matroid $M[P,Q]$ of rank r is in a spanning circuit of $M[P,Q]$ if and only if x is in at least two of the sets $N_1, N_2, \dots, N_r$, or x is in N_1 or N_r.*

Proof. Assume x is in N_i and N_{i+1}. Let C be $\{l_1, l_2, \dots, l_i, x, g_{i+1}, g_{i+2}, \dots, g_r\}$ where N_j is $[l_j, g_j]$. By connectivity, we have $l_2 \in N_1, l_3 \in N_2, \dots, l_i \in N_{i-1}$ and $g_{i+1} \in N_{i+2}, g_{i+2} \in N_{i+3}, \dots, g_{r-1} \in N_r$. An argument like that in the proof of Theorem 3.3 shows that C is a spanning circuit. Similar ideas show that x is in a spanning circuit of $M[P,Q]$ if x is in N_1 or N_r.

Assume $n(x)$ is $\{i\}$ with $1 < i < r$. Note that the basepoint is in no spanning circuit of a parallel connection of matroids of rank two or more, so to complete the proof we need only show that $M[P,Q]$ is a parallel connection of two lattice path matroids, each of rank at least two, with basepoint x. Thus, by Lemma 2.5, we need to show that $M[P,Q]/x\backslash X$ is disconnected where X is the set of loops of $M[P,Q]/x$. This statement follows from the lattice path description of contraction along with the observations that N_{i-1} contains only elements less than x while N_{i+1} contains only elements greater than x. □

The following characterizations of the elements that are in spanning circuits use structural properties rather than presentations.

Corollary 3.7. *Let x be in a nontrivial connected lattice path matroid M.*

(a) *No spanning circuit contains x if and only if M is a parallel connection of two lattice path matroids, each of rank at least two, with basepoint x.*

(b) *Some spanning circuit contains x if and only if $M/x\backslash X$ is connected, where X is the set of loops of M/x.*

Proof. Part (a) follows from the proof of Theorem 3.6. If x is in a spanning circuit C of M, then $C-x$ is a spanning circuit of M/x, so $M/x\backslash X$ is connected. Conversely, if x is in no spanning circuit of M, then, by part (a), M is a parallel connection, with basepoint x, of matroids of rank at least two, so $M/x\backslash X$ is disconnected. □

3.3. **Circuits and Connected Flats.** Our first goal in this section is to characterize the circuits of lattice path matroids. This is done in Theorem 3.9, the proof of which uses the following well-known elementary result about the circuits of arbitrary transversal matroids. This lemma follows easily from Hall's theorem.

Lemma 3.8. *Let n be the incidence function of a presentation of a transversal matroid M. If C is a rank-k circuit of M, then $|n(C)|$ is k, as is $|n(C-x)|$ for any x in C.*

Theorem 3.9. *Let $C = \{c_0, c_1, c_2, \dots, c_k\}$ be a set in the lattice path matroid $M[P,Q]$; assume $c_0 < c_1 < c_2 < \cdots < c_k$. Let $n(C)$ be $\{i_1, i_2, \dots, i_s\}$, where $i_1 < i_2 < \cdots < i_s$. Then C is a circuit of $M[P,Q]$ if and only if*

(1) $s = k$,
(2) $c_0 \in N_{i_1}$,
(3) $c_k \in N_{i_k}$, *and*
(4) $c_j \in N_{i_j} \cap N_{i_{j+1}}$ *for j with $0 < j < k$.*

Furthermore, if C is a circuit, then $i_{h+1} = i_h + 1$ for $1 \leq h < k$.

Proof. It is immediate to check that if conditions (1)–(4) hold, then C is dependent and every k-subset of C is a partial transversal and so is independent; thus C is a circuit. For the converse, assume C is a circuit. Assertion (1) follows from Lemma 3.8, which also gives the equalities $|n(C-c_0)| = k = |n(C-c_k)|$. Since $C-c_0$ is independent and $|n(C-c_0)|$ is k, it follows from Corollary 2.4 that c_j is in N_{i_j} for $1 \leq j \leq k$. A similar argument using $C-c_k$ shows that c_j is in $N_{i_{j+1}}$ for $0 \leq j \leq k-1$. This proves assertions (2)–(4). To prove the last assertion, assume there were an h not in $n(C)$ with $i_j < h < i_{j+1}$. From statement (4), we have that c_j is in both N_{i_j} and $N_{i_{j+1}}$. The inequalities

$$\min(N_h) < \min(N_{i_{j+1}}) \leq c_j \leq \max(N_{i_j}) < \max(N_h)$$

imply that c_j is in N_h, which contradicts the assumption that h is not in $n(C)$. □

By Lemma 3.8, if x is parallel to some element, then $|n(x)| = 1$. By property (4) of Theorem 3.9, at most two elements x in a circuit of a lattice path matroid can satisfy the equality $|n(x)| = 1$. This observation proves the next result.

Corollary 3.10. *At most two elements in any circuit of a lattice path matroid are in nonsingleton parallel classes.*

The following result gives two useful properties of connected flats.

Theorem 3.11. *Let $M[P,Q]$ have rank r and nullity m. Any nontrivial connected flat X of $M[P,Q]$ is an interval in $[m+r]$ and $n(X)$ is an interval of $r(X)$ elements in $[r]$.*

Proof. The second assertion follows from Corollary 3.4 and Theorem 3.9. For the first statement, let $n(X)$ be $[s,t]$ and assume $i<j<k$ with $i,k \in X$. That j is in X follows from the inequalities

$$s \leq \min\bigl(n(i)\bigr) \leq \min\bigl(n(j)\bigr) \leq \max\bigl(n(j)\bigr) \leq \max\bigl(n(k)\bigr) \leq t.$$

□

Theorem 3.11 has many implications for the connected flats of lattice path matroids, of which we mention four.

Corollary 3.12. *Assume $M[P,Q]$ has rank r.*

(i) *For $0 \leq k \leq r-1$, there are at most $k+1$ nontrivial connected flats of rank $r-k$ in $M[P,Q]$. In particular, $M[P,Q]$ has at most two connected hyperplanes and at most $r-1$ connected lines.*
(ii) *A flat of positive rank of $M[P,Q]$ is covered by at most two connected flats.*
(iii) *The nontrivial connected flats of $M[P,Q]$ that are not contained in a fixed connected hyperplane H of $M[P,Q]$ are linearly ordered by inclusion.*
(iv) *If H and H' are connected hyperplanes of $M[P,Q]$, then every nontrivial connected flat of $M[P,Q]$ is contained in at least one of H and H'.*

The matroid $M[(E^2N)^{r-1}EN, NE(NE^2)^{r-1}]$, which is a parallel connection of $r-1$ three-point lines in which elements have been added parallel to the "joints" and the "ends", shows that all upper bounds in parts (i) and (ii) of Corollary 3.12 are optimal.

The next result is another corollary of Theorem 3.9.

Corollary 3.13. *Let C be the circuit $\{c_0, c_1, \ldots, c_k\}$ of $M[P,Q]$ with $c_0 < c_1 < \cdots < c_k$. If x is not in C and $Z \cup x$ is a circuit of $M[P,Q]$ for some subset Z of C, then Z is either an initial segment $\{c_0, c_1, \ldots, c_i\}$ or a final segment $\{c_j, c_{j+1}, \ldots, c_k\}$ of C.*

Proof. The result follows from Lemma 3.8 and this simple corollary of Theorem 3.9: for any proper subset X of C that is neither an initial nor final segment of C, the inequality $|n(X)| > |X|$ holds. □

We conclude this section with a result we will use to show that certain matroids are not lattice path matroids.

Theorem 3.14. *Assume a rank-r matroid M has two nontrivial connected flats X and X' such that*

(1) $X \cap X' \neq \emptyset$,
(2) $r(X \cup X') = r$, *and*
(3) $X \cup X'$ *is a proper subset of the ground set $E(M)$ of M.*

Then M is not a lattice path matroid.

Proof. Assume, to the contrary, that M is $M[P,Q]$. Fix x in $X \cap X'$ and y in $E(M) - (X \cup X')$. By Theorem 3.11, along with assumptions (1) and (2), up to switching X and X' we would have $n(X) = [k]$ and $n(X') = [k', r]$ for some k and k' with $k' \leq k$. The inequality $y < x$ would give $\max\bigl(n(y)\bigr) \leq \max\bigl(n(x)\bigr) \leq k$, so y would be in $\mathrm{cl}(X)$. The inequality $x < y$ would give $\min\bigl(n(y)\bigr) \geq k'$, so y would be in $\mathrm{cl}(X')$. That these conclusions contradict the hypothesis proves the lemma. □

4. Generalized Catalan Matroids

Our next aim is to characterize lattice path matroids; this will be done in Section 5. This section focuses on an important subclass of $\mathcal{L}$ that has particularly simple characterizations and many interesting properties.

Definition 4.1. *The n-th* Catalan matroid M_n *is* $M[E^nN^n, (EN)^n]$. *A* generalized Catalan matroid *is, up to isomorphism, a matroid of the form* $M[E^mN^r, Q]$.

For generalized Catalan matroids, the notation $M[P, Q]$ is simplified to $M[Q]$. We use $\mathcal{C}$ to denote the class of generalized Catalan matroids.

Generalized Catalan matroids have arisen in different contexts with a corresponding variety of names and perspectives. We gather here the references currently known to us. Crapo [9, Section 8] introduced these matroids to show that there are at least $\binom{n}{r}$ nonisomorphic matroids of rank r on n elements. His perspective was rediscovered in [5, Theorem 3.14]: generalized Catalan matroids are precisely the matroids that are obtained from the empty matroid by repeatedly applying the operations of adding an isthmus and forming the free extension (this result is generalized in Theorem 6.7 below). By using "nested" presentations, Welsh [19] proved that Crapo's lower bound on the number of matroids holds within the smaller class of transversal matroids. These matroids arose again in [17] in connection with matroids defined in terms of integer-valued functions on finite sets. They were studied further in [18], where they were called Schubert matroids and were shown to have the rapid mixing property. In [1] they were rediscovered and related to shifted complexes, and so acquired the name shifted matroids. The link that was established in [5] between generalized Catalan matroids and an enumerative problem known as the tennis ball problem influenced the techniques used in [15] to solve that problem. In [10], under the name of freedom matroids, generalized Catalan matroids were used to construct a free algebra of matroids.

Catalan matroids have rich enumerative properties (see [5]). Their name comes from the fact that the number of bases of M_n is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$; several other invariants of M_n are also Catalan numbers. Although there is only one Catalan matroid of each rank, these matroids generate the entire class $\mathcal{C}$, in the sense of the following theorem.

Theorem 4.2. *The smallest minor-closed class of matroids that contains all Catalan matroids is $\mathcal{C}$.*

Proof. It follows from the lattice path interpretation of deletion and contraction given after the proof of Theorem 3.1 that $\mathcal{C}$ is closed under minors. To see that any generalized Catalan matroid $M[Q]$ is a minor of a Catalan matroid, simply insert East and North steps into Q so that the result is a Catalan matroid $M[(EN)^t]$. From $M[(EN)^t]$, delete the elements that correspond to the added East steps and contract the elements that correspond to the added North steps; by the lattice path interpretation of these operations, the resulting minor of $M[(EN)^t]$ is $M[Q]$. □

It is easy to see that $\mathcal{C}$, in addition to being closed under minors, is closed under duals and (unlike $\mathcal{L}$) free extension; therefore $\mathcal{C}$ is closed under truncation and elongation. However, $\mathcal{C}$ is not closed under direct sums.

By Theorem 3.5, a generalized Catalan matroid with at least two elements is connected if and only if it has neither loops nor isthmuses. The rest of this section focuses mainly on connected generalized Catalan matroids since some results are slightly easier to state with this restriction and, by what we just noted, there is essentially no loss of generality.

The feature that makes generalized Catalan matroids easy to characterize is the structure of the connected flats, as described in the following lemma.

Lemma 4.3. *Assume $M[Q]$ has rank r, nullity m, and neither loops nor isthmuses. Let the EN corners of Q be at steps $i_1, i_2, \ldots, i_k$ with $i_1 < i_2 < \cdots < i_k$. The proper nontrivial connected flats of $M[Q]$ are the initial segments $[i_1] \subset [i_2] \subset \cdots \subset [i_k]$ of $[m+r]$. The rank (resp. nullity) of $[i_h]$ is the number of North (resp. East) steps among the first i_h steps of Q.*

Proof. The lemma follows easily once we show that any proper nontrivial connected flat F of $M[Q]$ is an initial segment of $[m+r]$. By Theorem 3.11, F is an interval, say $[u, v]$, in $[m+r]$. By Corollary 3.2, the restriction of $M[Q]$ to $[v]$ is $M[Q_v]$ where Q_v consists of the first v steps of Q. Since v is not an isthmus of $M[Q]|F$, it is not an isthmus of $M[Q_v]$, so the v-th step of Q must be East. Let $M[Q_v]$ have rank k. Note that $[v-k, v]$ is a spanning circuit of $M[Q_v]$ that is contained in F and has closure $[v]$. Thus, F is the initial segment $[v]$. □

The following result (which is essentially Lemma 2 of [17]) is an immediate corollary of Lemmas 2.8 and 4.3.

Corollary 4.4. *A connected matroid is a generalized Catalan matroid if and only if its nontrivial connected flats are linearly ordered by inclusion.*

The following excluded-minor characterization of $\mathcal{C}$ from [17] is not difficult to prove from Corollary 4.4 and the results in Section 3. Let P_n be the truncation $T_n(U_{n-1,n} \oplus U_{n-1,n})$ to rank n of the direct sum of two n-circuits. Thus, P_n is the paving matroid of rank n whose only nontrivial proper flats are two disjoint circuit-hyperplanes whose union is the ground set. It follows that P_n is isomorphic to $M[E^{n-1}NEN^{n-1}, N^{n-1}ENE^{n-1}]$ and, by Corollary 4.4, that P_n is not in $\mathcal{C}$.

Theorem 4.5. *A matroid is in $\mathcal{C}$ if and only if it has no minor isomorphic to P_n for any $n \geq 2$.*

5. Fundamental Flats and a Characterization of Lattice Path Matroids

While the structure of the connected flats of arbitrary connected lattice path matroids is not as simple as that for generalized Catalan matroids (Corollary 4.4), this structure is still easy to describe. We analyze this structure in this section and we use it to characterize connected lattice path matroids. We also show that if $M[P, Q]$ is connected, then the paths P and Q are determined, up to a 180° rotation, by any matroid isomorphic to $M[P, Q]$. The flats of central interest for these results are those we define now.

Definition 5.1. *Let X be a connected flat of a connected matroid M for which $|X| > 1$ and $r(X) < r(M)$. We say that X is a* fundamental flat *of M if for some spanning circuit C of M the intersection $X \cap C$ is a basis of X.*

The first lemma shows how fundamental flats of lattice path matroids reflect the order of the elements.

Lemma 5.2. *Assume $M[P,Q]$ is connected and has rank r and nullity m. Let X be a connected flat of $M[P,Q]$ with $|X|>1$ and $r(X)<r$. Then X is a fundamental flat of $M[P,Q]$ if and only if X is an initial or final segment of $[m+r]$.*

Proof. Let N_i be $[l_i,g_i]$ for $1\le i\le r$. If X is an initial segment $[h]$ of $[m+r]$, then the spanning circuit $C=\{l_1,l_2,\dots,l_r,g_r\}$, constructed in the proof of Theorem 3.3, has the property that $X\cap C$ is a basis of X. Similarly, for a final segment X of $[m+r]$, a spanning circuit with the required property is $\{l_1,g_1,g_2,\dots,g_r\}$.

Conversely, assume C is a spanning circuit of $M[P,Q]$ and $X\cap C$ is a basis of X; say C is $\{c_0,c_1,\dots,c_r\}$ with $c_0<c_1<\cdots<c_r$. By Theorem 3.11, it suffices to show that either 1 or $m+r$ is in X. Let x be in $X-C$ and let C' be the unique circuit in $(X\cap C)\cup x$. By Corollary 3.13, C' has the form $\{x,c_0,c_1,\dots,c_u\}$ or $\{x,c_v,c_{v+1},\dots,c_r\}$. We will show that in the first case, 1 is in X; a similar argument gives $m+r$ in X in the second case. Thus, let C' be $\{x,c_0,c_1,\dots,c_u\}$. Note that $n(1)$ is $\{1\}$ and 1 is in $n(c_0)$. Note also that $\{c_0,c_1,\dots,c_u\}$ is an independent set and, by Lemma 3.8 applied to C', we have $|n(\{c_0,c_1,\dots,c_u\})|=u+1$. Thus,

$$r(\{1,c_0,c_1,\dots,c_u\})\le|n(\{1,c_0,c_1,\dots,c_u\})|=u+1=r(\{c_0,c_1,\dots,c_u\}).$$

It follows that 1 is in $\mathrm{cl}(\{c_0,c_1,\dots,c_u\})$, so 1 is in X, as claimed. □

Hence, to determine the fundamental flats of $M[P,Q]$, it suffices to know which initial and final segments of $[m+r]$ are connected flats. Note that the initial segment $[h]$ of $[m+r]$ is a proper nontrivial connected flat, and hence a fundamental flat, if and only if the upper path Q has an EN corner at h. Similarly, the final segment $[k,m+r]$ of $[m+r]$ is a fundamental flat of $M[P,Q]$ if and only if P has a NE corner at $k-1$. These observations prove the following theorem.

Theorem 5.3. *Assume $M[P,Q]$ is connected and has rank r and nullity m. Let the EN corners of Q be at $i_1,i_2,\dots,i_h$, with $i_1<i_2<\cdots<i_h$, and the NE corners of P be at $j_1-1,j_2-1,\dots,j_k-1$, with $j_1<j_2<\cdots<j_k$. The fundamental flats of $M[P,Q]$ are $[i_1]\subset[i_2]\subset\cdots\subset[i_h]$ and $[j_k,m+r]\subset\cdots\subset[j_2,m+r]\subset[j_1,m+r]$.*

Corollary 5.4 follows immediately from Theorem 5.3. Note that for generalized Catalan matroids, the fundamental flats are precisely the flats given in Lemma 4.3, so they form one chain under inclusion.

Corollary 5.4. *The fundamental flats of a connected matroid M in $\mathcal{L}-\mathcal{C}$ form two chains under inclusion; no set in one chain contains a set in the other chain. Furthermore, for each pair X,Y of incomparable fundamental flats,*

(a) *if $X\cap Y\neq\emptyset$, then $X\cup Y=E(M)$, and*
(b) *if $r(X)+r(Y)\ge r(M)$, then $r(X\cup Y)=r(M)$.*

While a connected lattice path matroid of rank r has at most $k+1$ connected flats of rank $r-k$ (Corollary 3.12), it has at most two fundamental flats of any given rank.

Theorem 5.3 and the lattice path interpretation of duality give the next result.

Corollary 5.5. *For any lattice path matroid M, the fundamental flats of the dual M^* are the set complements, $E(M)-F$, of the fundamental flats F of M.*

A key observation that follows from Theorem 5.3 is that although which flats are fundamental is independent of the order of the elements that is inherent in any particular lattice path presentation of a lattice path matroid, such a presentation

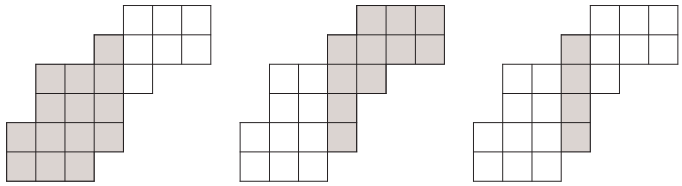

FIGURE 5. The shaded regions show the presentations of $M|F_i$, $M|G_j$, and $M|(F_i \cap G_j)$.

makes it easy to identify the fundamental flats. Conversely, the chains of fundamental flats give the bounding paths. More precisely, the paths P and Q associated with $M[P,Q]$ are determined by the NE corners of P and the EN corners of Q, and these corners are determined by the ranks and nullities of the fundamental flats. Typically there are two possible pairs of paths, according to which chain of fundamental flats contains the least element of the ground set. These observations give the following theorem, which is one of the main results of this section.

Theorem 5.6. *The bounding paths P and Q of a connected lattice path matroid $M[P,Q]$ are determined by the matroid structure, up to a $180°$ rotation. That is, the only matroids $M[P^*,Q^*]$ isomorphic to $M[P,Q]$ are $M[P,Q]$ and $M[Q^\rho,P^\rho]$ where $(s_1s_2\cdots s_{m+r})^\rho$ is $s_{m+r}\cdots s_2s_1$.*

Theorem 5.3 and its corollaries (including Theorem 5.6) show that a connected lattice path matroid is determined by its fundamental flats and their ranks. The next several results further develop this idea. The following theorem describes all connected flats of a connected lattice path matroid in terms of its fundamental flats.

Theorem 5.7. *Let M be the connected lattice path matroid $M[P,Q]$ of rank r and nullity m and let $F_1 \subset F_2 \subset \cdots \subset F_h$ and $G_1 \subset G_2 \subset \cdots \subset G_k$ be the chains of fundamental flats of M. The proper nontrivial connected flats of M are*

(i) *$F_1, F_2, \ldots, F_h, G_1, G_2, \ldots, G_k$, and*
(ii) *the intersections $F_i \cap G_j$ for which the inequality $m < \eta(F_i) + \eta(G_j)$ holds.*

A nontrivial connected flat of the form $F_i \cap G_j$ has rank $r(F_i) + r(G_j) - r$.

Proof. The flats $F_1, F_2, \ldots, F_h, G_1, G_2, \ldots, G_k$, being fundamental, are connected. The element 1 is in either F_1 or G_1; we may assume it is in F_1. For part (ii), we use Corollary 3.2 to find a lattice path presentation that shows that $F_i \cap G_j$ is connected. Using the notation in that corollary, let X be F_i, so the point (h,k) on Q is $\big(\eta(F_i), r(F_i)\big)$; let Y be G_j, so the point (h',k') on P is $\big(m-\eta(G_j), r-r(G_j)\big)$. The inequality in part (ii) along with part (c) of Corollary 3.2 give a presentation of $M|(F_i \cap G_j)$ (illustrated in Figure 5) that, together with the fact that P and Q meet only at $(0,0)$ and (m,r), implies that $F_i \cap G_j$ is connected and nontrivial.

Now assume X is a proper nontrivial connected flat. By Theorem 3.11, X is an interval, say $[u,v]$, in $[m+r]$. As in the proof of Lemma 4.3, it follows that the u-th step of P and the v-th step of Q are East steps. Since X is a flat, both $r(X \cup \{u-1\})$ and $r(X \cup \{v+1\})$ exceed $r(X)$, so step $u-1$ of P and step $v+1$ of Q, if there are such steps, are North steps. From these observations and Theorem 5.3, it follows that X is of the form F_i, G_j, or $F_i \cap G_j$. We need to show that if $F_i \cap G_j$

is connected, then the inequality $m - \eta(G_j) < \eta(F_i)$ holds. This inequality follows by viewing $M|(F_i \cap G_j)$ as a restriction of $M|F_i$ and using the path presentations of these matroids given in Corollary 3.2. Indeed, from the lattice path diagrams (Figure 5) it follows that $M|(F_i \cap G_j)$ is either free or connected, and the latter holds precisely when $\big(m - \eta(G_j), r - r(G_j)\big)$ is strictly to the left of $\big(\eta(F_i), r(F_i)\big)$.

Lastly, let the connected flat X be $F_i \cap G_j$. From lattice path diagrams, we get

$$r(M) = \big(r(F_i) - r(X)\big) + \big(r(G_j) - r(X)\big) + r(X),$$

from which the last assertion follows. □

It follows from Theorem 5.7 that any intersection of connected flats is either a fundamental flat or an intersection of two fundamental flats. From this observation and the second paragraph of the proof, it follows that a nonempty intersection of connected flats is either connected or trivial. Despite what the last part of Theorem 5.7 might suggest, it is easy to construct examples in which the fundamental flats of lattice path matroids are not modular.

The image, under an automorphism, of a fundamental flat of any matroid is also fundamental. This observation, Corollary 2.9, and Theorem 5.7 give the following result.

Corollary 5.8. *The automorphisms of a connected lattice path matroid are the permutations of the ground set that are rank-preserving bijections of the collection of fundamental flats.*

The proof of the second main result of this section, Theorem 5.10, uses the following basic notions about ordered sets. A *strict partial order* is an irreflexive, transitive relation. Thus, strict partial orders differ from partial orders only in whether each element is required to be unrelated, or required to be related, to itself. Given a strict partial order $<$ on S, elements x and y of S are *incomparable* if neither $x < y$ nor $y < x$ holds. *Weak orders* are strict partial orders in which incomparability is an equivalence relation. Thus, linear orders are weak orders in which the incomparability classes are singletons. Two weak orders $<_1$ and $<_2$ on S are *compatible* if whenever elements x and y of S are comparable in both $<_1$ and $<_2$, and $x <_1 y$, then $x <_2 y$.

Lemma 5.9. *Any two compatible weak orders have a common linear extension.*

Proof. Let $<_1$ and $<_2$ be compatible weak orders on S and let the relation $<$ on S be defined as follows: $x < y$ if either $x <_1 y$ or $x <_2 y$. It is easy to check that $<$ is a weak order. The lemma follows since $<$, like any strict partial order, can be extended to a linear order. □

We now turn to the second main result of the section. This theorem shows that the properties we developed above for the fundamental flats and the connected flats of connected lattice path matroids characterize these matroids.

Theorem 5.10. *A connected matroid M is a lattice path matroid if and only if the following properties hold.*

(i) *The fundamental flats form at most two disjoint chains under inclusion, say $F_1 \subset F_2 \subset \cdots \subset F_h$ and $G_1 \subset G_2 \subset \cdots \subset G_k$.*
(ii) *If $F_i \cap G_j \neq \emptyset$, then $F_i \cup G_j = E(M)$.*
(iii) *The proper nontrivial connected flats of M are precisely the following sets:*

(a) $F_1, F_2, \ldots, F_h, G_1, G_2, \ldots, G_k$, *and*

(b) *intersections* $F_i \cap G_j$ *for which the inequality* $m < \eta(F_i) + \eta(G_j)$ *holds.*

(iv) *The rank of the flat* $F_i \cap G_j$ *of item* (iii:b) *is* $r(F_i) + r(G_j) - r(M)$

Proof. By Theorem 5.3, Lemma 4.3, and Corollary 4.4, M is a generalized Catalan matroid if and only if properties (i)–(iv) hold where there is at most one chain of fundamental flats. By Theorems 5.3 and 5.7, the fundamental flats of a lattice path matroid that is not a generalized Catalan matroid satisfy properties (i)–(iv) with neither chain of fundamental flats being empty. Hence we need only prove the converse in the case that neither chain of fundamental flats is empty.

Assume M has rank r and nullity m. To show that M is a lattice path matroid, we construct lattice paths P and Q and an isomorphism of M onto $M[P, Q]$. To show that P stays strictly below Q except at $(0, 0)$ and (m, r), we will use the following statements about fundamental flats.

(A) If $F_i \cap G_j \neq \emptyset$, then $r(F_i) + r(G_j) > r$.

(B) If $F_i \cap G_j = \emptyset$, then $\eta(F_i) + \eta(G_j) < m$.

To prove statement (A), note that we have the inequality

$$r(F_i) + r(G_j) \geq r(F_i \cup G_j) + r(F_i \cap G_j) = r(M) + r(F_i \cap G_j)$$

by semimodularity and property (ii). Since M has no loops, $r(F_i \cap G_j)$ is positive, so the desired inequality follows. To prove statement (B), first recall that η is nondecreasing, i.e., if $X \subseteq Y$, then $\eta(X) \leq \eta(Y)$. Since F_i and G_j are disjoint, we have $\eta(F_i)+\eta(G_j) = |F_i \cup G_j| - r(F_i) - r(G_j)$. Thus, if $r(F_i)+r(G_j) > r(F_i \cup G_j)$, then we have $\eta(F_i) + \eta(G_j) < \eta(F_i \cup G_j) \leq m$. If $r(F_i) + r(G_j) = r(F_i \cup G_j)$, then $M|(F_i \cup G_j)$ is disconnected and we have the equality $\eta(F_i) + \eta(G_j) = \eta(F_i \cup G_j)$. Since M is connected, we have $\eta(F_i \cup G_j) < \eta(M)$, which gives the desired inequality.

Let lattice paths P and Q from $(0, 0)$ to (m, r) be given as follows.

(a) The NE corners of P are at the points $\big(m - \eta(G_j), r - r(G_j)\big)$ for j in $[k]$.

(b) The EN corners of Q are at the points $\big(\eta(F_i), r(F_i)\big)$ for i in $[h]$.

Note that P stays strictly below Q except at the endpoints if and only if for every NE corner (x_P, y_P) of P and every EN corner (x_Q, y_Q) of Q, at least one of the inequalities $x_Q < x_P$ and $y_Q > y_P$ holds. These inequalities are those in statements (A) and (B), so P stays strictly below Q except at $(0, 0)$ and (m, r).

To construct an isomorphism of M onto $M[P, Q]$, we define a linear order on $E(M)$ that we use to map $E(M)$ onto $[m + r]$, the ground set of $M[P, Q]$. We first define two relations $<_F$ and $<_G$ on $E(M)$. Let F_{h+1} and G_{k+1} be $E(M)$. Define $<_F$ as follows: $x <_F y$ for $x, y \in E(M)$ if there is an integer i in $[h]$ with $x \in F_i$ and $y \in F_{i+1} - F_i$. Note that $<_F$ is a weak order whose incomparability classes are F_1 and the set differences $F_{i+1} - F_i$. Define $<_G$ similarly: $x <_G y$ for $x, y \in E(M)$ if there is an integer j in $[k]$ with $x \in G_{j+1} - G_j$ and $y \in G_j$. Thus, $<_G$ is also a weak order and the incomparability classes are G_1 and the differences $G_{i+1} - G_i$. Note that if we had $x <_F y$ and $y <_G x$, then there would be fundamental flats F_i and G_j that both contain x and not y, contrary to hypothesis (ii). Thus, the weak orders $<_F$ and $<_G$ are compatible, so by Lemma 5.9 there is a linear order, say $x_1 < x_2 < \cdots < x_{m+r}$, of $E(M)$ that extends both $<_F$ and $<_G$.

Let $\phi : E(M) \to [m + r]$ be given by $\phi(x_i) = i$. By construction, ϕ is a bijection of $E(M)$ onto $[m+r]$ that is a rank-preserving bijection of the fundamental flats of M onto the fundamental flats of $M[P, Q]$. Furthermore, by assumptions (iii) and

(iv) and Theorem 5.7, ϕ is a rank-preserving bijection of the set of connected flats of M onto those of $M[P, Q]$. By Lemma 2.8, it follows that ϕ is an isomorphism of M onto $M[P, Q]$; thus, M is a lattice path matroid. □

We close this section by giving a pair of six-element matroids that have the same collection of fundamental flats, yet only one of which is in $\mathcal{L}$; thus, conditions (i) and (ii) in Theorem 5.10 are not enough to characterize lattice path matroids. The uniform matroid $U_{4,6}$ is a lattice path matroid with no fundamental flats since the bounding paths are $P = E^2N^4$ and $Q = N^4E^2$. The prism (the matroid $C_{4,2}$ of Figure 11 on page 26) is not a lattice path matroid (condition (iii) of Theorem 5.10 fails) and, since it has no spanning circuits, it too has no fundamental flats.

6. Lattice Path Matroids as Transversal Matroids

The aspects of lattice path matroids treated in this section relate to important topics in the theory of transversal matroids. We start by characterizing the set systems that are maximal presentations of lattice path matroids. This result plays a key role in an algorithm for determining whether a transversal matroid is in $\mathcal{L}$. By combining the result on maximal presentations with Brylawski's affine representation of transversal matroids, we get a geometric description of lattice path matroids. We conclude the section by comparing $\mathcal{L}$ with the dual-closed class of fundamental transversal matroids and the minor-closed class of bicircular matroids.

6.1. **Maximal and Minimal Presentations.** Two types of presentations are of interest in this section. A presentation $\mathcal{A} = (A_1, A_2, \ldots, A_r)$ of a transversal matroid M is *minimal* if the only presentation $(A'_1, A'_2, \ldots, A'_r)$ of M with A'_i contained in A_i for all i is $\mathcal{A}$. The presentation $\mathcal{A}$ is *maximal* if the only presentation $(A'_1, A'_2, \ldots, A'_r)$ of M with A_i contained in A'_i for all i is $\mathcal{A}$. It is well known that while each transversal matroid has a unique maximal presentation, it typically has many minimal presentations. (See, e.g., [2, 6, 12].)

Theorem 6.1. *Standard presentations of lattice path matroids are minimal.*

Proof. Let $(N_1, N_2, \ldots, N_r)$ be the standard presentation of the matroid $M[P, Q]$ and let $(N'_1, N'_2, \ldots, N'_r)$ be any presentation of $M[P, Q]$ with $N'_i \subseteq N_i$ for $1 \leq i \leq r$. To prove the theorem, we must show the inclusion $N_i \subseteq N'_i$ for $1 \leq i \leq r$. Let x be in N_i. Let B consist of the least elements of $N_1, N_2, \ldots, N_{i-1}$, the greatest elements of $N_{i+1}, N_{i+2}, \ldots, N_r$, and x. Thus, B is a basis of $M[P, Q]$. Note that for B to be a transversal of $(N'_1, N'_2, \ldots, N'_r)$, the element x must be in N'_i, as needed. □

With the following result of Bondy [2], we will get a simple description, in terms of intervals, of the maximal presentation of a lattice path matroid.

Lemma 6.2. *Given a presentation $(A_1, A_2, \ldots, A_r)$ of a rank-r transversal matroid M, the maximal presentation of M is $(A_1 \cup I_1, A_2 \cup I_2, \ldots, A_r \cup I_r)$ where I_j is the set of isthmuses of the deletion $M \backslash A_j$.*

Together with Lemma 6.2, the following result from [11] implies that from any presentation of a transversal matroid, the maximal presentation can be found in polynomial time in the size of the ground set. This observation will be important in the algorithm for recognizing lattice path matroids among transversal matroids.

Lemma 6.3. *The maximal size of a matching in a bipartite graph can be found in polynomial time in the number of vertices.*

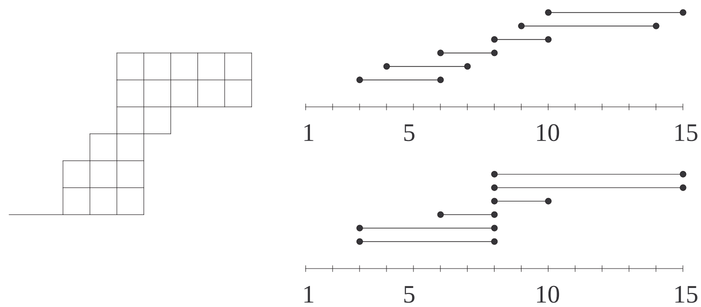

FIGURE 6. The standard and maximal presentations of a lattice path matroid.

The discussion below focuses on matroids that have no isthmuses. This restriction is justified by noting that the isthmuses of a transversal matroid are in all sets in the maximal presentation, and so are easy to deal with.

Let $(N_1, N_2, \dots, N_r)$ be the standard presentation of the lattice path matroid $M = M[P, Q]$ on $[m+r]$, where M has no isthmuses. Let N_i be $[l_i, g_i]$. Theorem 3.9 implies that each connected component of $M \backslash N_i$ is a subset of either $[g_i+1, m+r]$ or $[l_i - 1]$. Thus, the set of isthmuses of $M \backslash N_i$ is the union of the sets I_i^+ and I_i^- of isthmuses of the restrictions of M to $[g_i + 1, m + r]$ and $[l_i - 1]$, respectively. Corollary 3.2 implies that I_i^+ and I_i^- are given as follows:

(1) $$I_i^+ = \{g_i + j \,:\, g_i + j \text{ is the greatest element of } N_{i+j}, j > 0\},$$

(2) $$I_i^- = \{l_i - j \,:\, l_i - j \text{ is the least element of } N_{i-j}, j > 0\}.$$

This proves the following theorem.

Theorem 6.4. *Let $(N_1, N_2, \dots, N_r)$ be the standard presentation of the lattice path matroid $M[P, Q]$ that has no isthmuses. The maximal presentation of $M[P, Q]$ is $(N_1', N_2', \dots, N_r')$ where N_i' is $N_i \cup I_i^+ \cup I_i^-$ and I_i^+ and I_i^- are given by Eqs. (1)–(2).*

The sets in the maximal presentation of a lattice path matroid have a simple graphical interpretation, as Figure 6 illustrates. While there are no containments among intervals in the standard presentation, this figure shows that there may be containments (even equalities) among intervals in the maximal presentation.

Theorem 6.5, which characterizes the multisets of intervals in $[m + r]$ that are maximal presentations of lattice path matroids, uses the following notation. For an indexed multiset $(T_1, T_2, \dots, T_r)$ of nonempty intervals in $[m+r]$ with $T_i = [a_i, b_i]$, write $T_i \prec T_j$ if either $a_i < a_j$ or $b_i < b_j$. Thus, two intervals are unrelated if and only if they are equal. For arbitrary multisets of intervals, both $T_i \prec T_j$ and $T_j \prec T_i$ may hold; in contrast, if $(T_1, T_2, \dots, T_r)$ is the maximal presentation of a lattice path matroid, then $\prec$ is a weak order. If $\prec$ is a weak order, then we assume that the set system $(T_1, T_2, \dots, T_r)$ is indexed so that we can have $T_i \prec T_j$ only for $i < j$. In this case, let $d(T_h)$ be $|\{i \,:\, i < h, a_i = a_h\}|$ and let $d'(T_h)$ be $|\{j \,:\, h < j, b_h = b_j\}|$.

Theorem 6.5. *A set system* $(T_1, T_2, \ldots, T_r)$ *of nonempty intervals in* $[m+r]$ *is the maximal presentation of a rank-r lattice path matroid on* $[m+r]$ *that has no isthmuses if and only if*

(i) *the relation* $\prec$ *is a weak order,*
(ii) *for all pairs* T_i *and* T_j*, neither* $|T_i - T_j|$ *nor* $|T_j - T_i|$ *is* 1*, and*
(iii) $d(T_i) + d'(T_i) + 2 \leq |T_i|$ *for every* i*.*

Proof. For the maximal presentation of a lattice path matroid $M[P, Q]$ with no isthmuses, properties (i)–(iii) follow from Theorem 6.4. For the converse, note that removing from T_i its least $d(T_i)$ elements and its greatest $d'(T_i)$ yields the standard presentation of a lattice path matroid that, by property (iii), has no isthmuses and for which $(T_1, T_2, \ldots, T_r)$ is, by Theorem 6.4, the maximal presentation. □

6.2. **Recognizing Lattice Path Matroids.** When treating algorithmic questions about matroids, it is usual to assume that a matroid is given by an *independence oracle*, that is, a subroutine that outputs, in constant time, whether a subset of the ground set is independent. While there are algorithms that recognize transversal matroids within the class of all matroids (see [7]), Jensen and Korte [13] have shown that there is no polynomial-time algorithm to decide if a matroid is transversal from an independence oracle. The same proof as in [13] shows that there is no such algorithm to decide whether a matroid is a lattice path matroid. Transversal matroids are more conveniently specified by set systems than by independence oracles. This section gives a polynomial-time algorithm that, given a set system, decides whether the corresponding transversal matroid is a lattice path matroid.

We start with some simplifications. A presentation $\mathcal{A}$ of M can be represented by a bipartite graph $\Delta[\mathcal{A}]$ in the obvious way [16, Section 1.6]. Therefore, by Lemma 6.3, the isthmuses of a transversal matroid can be identified and deleted in polynomial time. If M has no isthmuses, then the connected components of M come from those of $\Delta[\mathcal{A}]$. These observations and Theorem 3.1 justify focusing on connected transversal matroids. As noted in Section 6.1, the maximal presentation can be found from any presentation in polynomial time, so we focus on maximal presentations.

The key to the recognition algorithm below is to efficiently recover lattice path orderings from the maximal presentation. We begin with some observations that relate these notions. Assume $\mathcal{A} = (A_1, A_2, \ldots, A_r)$ is the maximal presentation of the connected lattice path matroid $M[P, Q]$ on the ground set $[m+r]$ and let n be the incidence function of $\mathcal{A}$. Let $C_1, C_2, \ldots, C_k$ be the equivalence classes of the relation on $[m+r]$ in which x and y are related if and only if $n(x) = n(y)$. Each set C_i is an interval in $[m+r]$. We may assume that $C_1, C_2, \ldots, C_k$ are indexed so that $x_1 < x_2 < \cdots < x_k$ for any elements $x_1, x_2, \ldots, x_k$ with x_i in C_i. Since $M[P, Q]$ is connected, we have $n(C_i) \cap n(C_{i+1}) \neq \emptyset$ for i with $1 \leq i < k$. Any permutation σ of $[m+r]$ with $\sigma(C_i) = C_i$ for $1 \leq i \leq k$ is clearly an automorphism of $M[P, Q]$, so the linear order $\sigma(1) < \sigma(2) < \cdots < \sigma(m+r)$ is a lattice path order, as is $\sigma(m+r) < \cdots < \sigma(2) < \sigma(1)$. Relative to any of these linear orders, the sets in $\mathcal{A}$ are intervals and the properties in Theorem 6.5 hold. These lattice path orderings of $[m+r]$ are essentially equivalent to the orderings $C_1 < C_2 < \cdots < C_k$ and $C_k < C_{k-1} < \cdots < C_1$ of $C_1, C_2, \ldots, C_k$. Observe that $C_1, C_2, \ldots, C_k$ and $C_k, C_{k-1}, \ldots, C_1$ are the only permutations $X_1, X_2, \ldots, X_k$ of $C_1, C_1, \ldots, C_k$ that satisfy the following property.

(P) *For all i and j with $1 < i < j \leq k$,*
(a) $n(X_{i-1}) \cap n(X_j) \subseteq n(X_{i-1}) \cap n(X_i)$, *and*
(b) $n(X_i) - n(X_{i-1}) \subseteq n(X_j) - n(X_{i-1})$ *whenever* $n(X_j) \cap n(X_{i-1})$ *is nonempty.*

Thus, to determine whether a transversal matroid M with a given presentation is a lattice path matroid, carry out the following steps.

(1) Detect and delete the isthmuses.
(2) Determine the connected components.
(3) Find the maximal presentation for each connected component.
(4) For each component, find the classes defined above relative to the maximal presentation.
(5) For each component, determine whether there is a linear order of these classes that satisfies property (P).
(6) If there is such a linear order of these classes for each component, then use the criterion in Theorem 6.5 to determine whether, with respect to any corresponding linear order of a component, the intervals in the maximal presentation of that component are those of a maximal presentation of a lattice path matroid.

If, in step (5), there is no suitable order for some connected component, then M is not a lattice path matroid. If there is such an order for each connected component, then M is a lattice path matroid if and only if step (6) yields only positive results. Each of these steps can be done in polynomial time in the size of the ground set, so we get the following result.

Theorem 6.6. *Whether a transversal matroid is a lattice path matroid can be determined from any presentation in polynomial time in the size of the ground set.*

6.3. **A Geometric Description of Lattice Path Matroids.** Brylawski [8] (see also [16, Proposition 12.2.26]) gave a geometric description of arbitrary transversal matroids. This section applies his result to lattice path matroids.

Let M be a transversal matroid on the set $\{x_1, x_2, \ldots, x_k\}$ with presentation $(A_1, A_2, \ldots, A_r)$. Brylawski showed that M can be realized geometrically as follows. Start with the free matroid M_0 on a set $\{e_1, e_2, \ldots, e_r\}$ disjoint from $E(M)$. For i from 1 to k, form M_i from M_{i-1} by taking the principal extension of M_{i-1} defined by the flat $\mathrm{cl}_{M_{i-1}}(\{e_j : x_i \in A_j\})$, with the element added being x_i. The matroid M is $M_k \backslash \{e_1, e_2, \ldots, e_r\}$. Thus, a rank-$r$ matroid is transversal if and only if it can be realized by placing the elements freely on the faces of the r-simplex.

The next theorem, which is illustrated in Figure 1, shows how lattice path matroids can be constructed by successively adding isthmuses and loops, and by taking principal extensions by certain flats. To motivate this result, consider a lattice path matroid $M[P, Q]$ that has rank r and nullity m in which $m + r$ is neither a loop nor an isthmus. Let l be the length of the longest final segment of North steps in P. By Theorem 6.4, the sets of the maximal presentation of $M[P, Q]$ that contain $m + r$ are the last l (those arising from $N_{r-l+1}, \ldots, N_r$). By Brylawski's result, $m + r$ is added freely to the flat spanned by $e_{r-l+1}, \ldots, e_r$ in the notation above; note that this flat is also spanned by the last l elements of $[m + r - 1]$, since they are independent in $M[P, Q] \backslash (m + r)$. Thus, we have the following result.

Theorem 6.7. *A matroid M is a lattice path matroid if and only if the ground set can be written as $\{x_1, x_2, \dots, x_k\}$ so that each restriction $M_i := M|\{x_1, x_2, \dots, x_i\}$ is formed from M_{i-1} by either*

(i) *adding x_i as an isthmus,*
(ii) *adding x_i as a loop, or*
(iii) *adding x_i via the principal extension of M_{i-1} generated by the closure of an independent set of the form $\{x_h, x_{h+1}, \dots, x_{i-1}\}$ for some h with $h < i$.*

6.4. **Relation to Other Classes of Transversal Matroids.** We have seen that the class of lattice path matroids is closed under taking both minors and duals. While [4] develops a dual-closed, minor-closed class of transversal matroids that properly contains $\mathcal{L}$, and while there are infinitely many dual-closed, minor-closed classes contained in $\mathcal{L}$ (see Sections 4 and 8 for two such classes), few other known classes of transversal matroids are either dual-closed or minor-closed. In this section, we make some remarks about two important classes of transversal matroids, each of which has one of these properties.

Fundamental transversal matroids (called principal transversal matroids in [8]) were introduced by Bondy and Welsh [3] and they play an important role in the study of transversal matroids. A transversal matroid M is a *fundamental transversal matroid* if it can be represented on the simplex with an element of M at each vertex of the simplex. Thus, transversal matroids are the restrictions of fundamental transversal matroids. While the class $\mathcal{F}$ of fundamental transversal matroids is closed under neither deletion nor contraction, it is well-known and not hard to prove that $\mathcal{F}$ is dual-closed. The class $\mathcal{F}$ is much larger than $\mathcal{L}$: Brylawski [8] showed that there are on the order of c^{n^2} simple fundamental transversal matroids on n elements, for some constant c; in contrast, 4^n is an upper bound on the number of lattice path matroids on n elements since there are 4^n pairs of paths of length n (see [5] for a formula for the number of connected lattice path matroids). Both $\mathcal{F}$ and $\mathcal{L}$ contain all transversal matroids of rank two. However, a fundamental transversal matroid of rank three or more cannot have a pair of disjoint connected hyperplanes, but such hyperplanes can occur in lattice path matroids, such as the matroid $P_n = T_n(U_{n-1,n} \oplus U_{n-1,n})$ of Theorem 4.5. On the other hand, the number of connected hyperplanes of a fundamental transversal matroid, such as the n-whirl $\mathcal{W}^n$, can exceed two (see Corollary 3.12).

Let us call a matroid *bitransversal* if both the matroid and its dual are transversal. It is easy to prove that the class of bitransversal matroids is closed under direct sums, free extensions, and free coextensions. Hence by starting with the union of the classes $\mathcal{L}$ and $\mathcal{F}$, and using these three operations, we can construct a larger class of bitransversal matroids; let $\mathcal{LF}$ denote this class. For instance, the free extension $(P_n \oplus \mathcal{W}^n) + e$ of $P_n \oplus \mathcal{W}^n$ is in $\mathcal{LF}$ but not in $\mathcal{L} \cup \mathcal{F}$. There are bitransversal matroids, such as the identically self-dual matroids of [3, Section 4], that are not in $\mathcal{LF}$. The problem of characterizing all bitransversal matroids, which was posed by Welsh, currently remains open (see [16, Problem 14.7.4]).

Bicircular matroids [14] form another important class of transversal matroids. The notion of a bicircular matroid we consider is a mild extension of that in [14] (as originally defined, bicircular matroids have no loops). A transversal matroid M is *bicircular* if it has a presentation $\mathcal{A}$ so that each element of M is in at most two sets in $\mathcal{A}$ (counting multiplicity). Thus, bicircular matroids are the transversal matroids that have a representation on the simplex in which all nonloops are on vertices or lines of the simplex. It follows that minors of bicircular matroids are bicircular. On the other hand, the class of bicircular matroids is not dual-closed: the prism (the matroid $C_{4,2}$ of Figure 11) is bicircular, but its dual (the matroid $B_{2,2}$ in the same figure) is not transversal. Among the matroids that are both bicircular and lattice path matroids are all transversal matroids of rank two as well as iterated parallel connections of rank-2 uniform matroids, $M_1 := U_{2,n_1}$ and $M_i := P(M_{i-1}, U_{2,n_i})$, where the basepoint used to construct M_i is not in M_{i-2}. A bicircular matroid, unlike a lattice path matroid, can have more than two connected hyperplanes. Also, while most uniform matroids are not bicircular (for instance, $U_{3,n}$ is bicircular if and only if $n \leq 6$), all uniform matroids are in $\mathcal{L}$. Thus, the class of bicircular matroids differs significantly from $\mathcal{L}$ in all ranks greater than two.

7. Higher Connectivity

In this section, we show how to find the connectivity $\lambda(M)$ of a lattice path matroid in a simple way from the path presentation of M. We also show that at least one exact $\lambda(M)$-separation of M is given by a fundamental flat and its complement. We start by recalling the relevant definitions; for more information on higher connectivity, see [16, Chapter 8].

For a positive integer k, a *k-separation* of a matroid M is a partition of the ground set into two sets X and Y, each with at least k elements, such that the inequality $r(X)+r(Y) \leq r(M)+k-1$ holds. A k-separation for which the equality $r(X) + r(Y) = r(M) + k - 1$ holds is an *exact k-separation.* The *connectivity*, or *Tutte connectivity*, $\lambda(M)$ of M is the least positive integer k such that M has a k-separation; if there is no such k, then $\lambda(M)$ is taken to be ∞. The connectivity of uniform matroids is well known (see [16, Corollary 8.1.8]), so we consider only lattice path matroids that are not uniform. Also, as justified by Theorem 3.5, we focus exclusively on lattice path matroids that are connected.

Let M be a connected lattice path matroid, say $M[P, Q]$, that is not uniform. Let the integer k_M be defined as follows:

$$k_M := \min\{|n(j)| : P \text{ has a } NE \text{ corner at } j \text{ or } Q \text{ has an } EN \text{ corner at } j-1\}.$$

Figure 7 (a) illustrates a lattice path matroid M in which the relevant values of j are $7, 9, 14, 16$ (for which $|n(j)|$ is 3) and 21 (for which $|n(j)|$ is 4), so k_M is 3. The main result of this section, Theorem 7.4, is that the connectivity $\lambda(M)$ of M is k_M. Several lemmas enter into the proof of this result. The first lemma reflects the equality $\lambda(M) = \lambda(M^*)$ that holds for any matroid.

Lemma 7.1. *The number k_M is invariant under duality, that is, $k_M = k_{M^*}$.*

Proof. Recall that the lattice path diagram for the dual of $M[P, Q]$ is obtained by reflecting the lattice path diagram for $M[P, Q]$ about the line $y = x$ (Figure 3). Equivalently, the dual of $M[P, Q]$ is $M[Q', P']$ where P' and Q' are obtained from P and Q by switching East and North steps. Let n and n' be the incidence functions of the standard presentations of $M[P, Q]$ and $M[Q', P']$, respectively. Note that P has a NE corner at j if and only if P' has an EN corner at j; also, Q has an EN corner at $j - 1$ if and only if Q' has a NE corner at at $j - 1$. Thus, the lemma follows once we show the following statements: if Q has an EN corner at $j - 1$, then $|n(j)| = |n'(j-1)|$; if P has a NE corner at j, then $|n(j)| = |n'(j+1)|$. These assertions hold since we can pair off the relevant East and North steps that share a lattice point, as suggested in Figure 7 (b). □

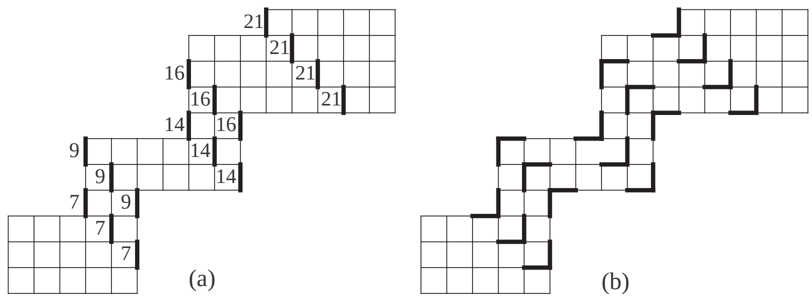

FIGURE 7. (a) A lattice path matroid M with $k_M = 3$. (b) A pairing that shows the equality $k_M = k_{M^*}$.

Recall that in a matroid of connectivity at least n with at least $2(n-1)$ elements, circuits and cocircuits have at least n elements [16, Proposition 8.1.6]. The next lemma will be used to show that circuits and cocircuits of a lattice path matroid M have at least k_M elements.

Lemma 7.2. *Every element of M is in at least $k_M - 1$ sets in the maximal presentation of M.*

Proof. Let M be $M[P, Q]$ and let n and n' be the incidence functions of its standard and maximal presentations. Let steps $q_N + 1$, $q_E - 1$, and $p_E + 1$ be, respectively, the first East step of Q, the last North step of Q, and the first North step of P. By the symmetry given by the order-reversing isomorphism of $M[P, Q]$ onto $M[Q^\rho, P^\rho]$ (see Theorem 5.6), it suffices to prove (a) if $i \le q_N$, then $|n'(i)| \ge k_M - 1$ (b) if $q_N < i < q_E$, then $|n(i)| \ge k_M - 1$ and (c) if $q_E \le i \le p_E$, then $|n(i)| = r(M)$. Theorem 6.4 and the observation that q_N is at least $k_M - 1$ prove part (a). Part (c) is trivial. The proof of part (b) uses the following easily-verified statements.

- (i) If the j-th and $(j+1)$-st steps of Q are East, then $n(j+1)$ is either $n(j)$ or $n(j) - \min\big(n(j)\big)$, so we have $|n(j)| - 1 \le |n(j+1)| \le |n(j)|$.
- (ii) If the j-th step of Q is North, then $n(j-1)$ is either X or $X - \min(X)$ where X is $\{h - 1 : h \in n(j)\}$, so $|n(j)| - 1 \le |n(j-1)| \le |n(j)|$.

First assume that steps $i, i+1, \ldots, h$ of Q are East and that step $h+1$ is North. Thus, Q has an EN corner at h. Statements (i) and (ii) give the inequalities

$$|n(i)| \ge |n(i+1)| \ge \cdots \ge |n(h)| \ge |n(h+1)| - 1.$$

Since $|n(h+1)| \ge k_M$, we have $|n(i)| \ge k_M - 1$. Finally, if the i-th step of Q is North, a similar application of statement (ii) completes the proof of part (b). $\square$

From Lemmas 3.8 and 7.2, the rank of any circuit of M is at least $k_M - 1$. The next lemma follows from this observation and Lemma 7.1. The generalized Catalan matroid $M[(NE)^2]$ shows that M can have circuits of rank $k_M - 1$.

Lemma 7.3. *Any set of $k_M - 1$ element of $[m + r]$ is independent in both M and M^*. Circuits of M have at least k_M elements, as do circuits of M^*.*

We now prove that k_M is the connectivity of the lattice path matroid M.

Theorem 7.4. *Let M be a connected lattice path matroid of rank r and nullity m, say $M[P,Q]$, that is not uniform. The connectivity $\lambda(M)$ of M is k_M, where k_M is*

$$\min\{|n(j)| \,:\, P \text{ has a } NE \text{ corner at } j \text{ or } Q \text{ has an } EN \text{ corner at } j-1\}.$$

Furthermore, at least one exact k_M-separation of M consists of some fundamental flat and its complement.

Proof. We first show that M has an exact k_M-separation that consists of a fundamental flat and its complement. Assume first that k_M is $|n(j)|$ where P has a NE corner at j. Let X and Y be $[j]$ and $[j+1, m+r]$, respectively. Thus, Y is a fundamental flat of M. Note that both X and Y have at least $|n(j)|$ elements. It follows from the path presentations of restrictions given in Corollary 3.2 that $r(X)$ is $r(M) - r(Y) + |n(j)| - 1$, that is, $r(X) + r(Y) = r(M) + k_M - 1$, so X, Y is an exact k_M-separation of M. Similarly, if k_M is $|n(j)|$ where Q has an EN corner at $j-1$, then $[j-1]$ and $[j, m+r]$ give an exact k_M-separation of M.

It remains to show that M has no h-separation for any positive integer h less than k_M. Let h be such an integer and assume X and Y partition $[m+r]$, where both X and Y have at least h elements. We need to prove the inequality

$$r(X) + r(Y) \geq r(M) + h. \tag{3}$$

If an element y in X is in the closure of Y, and if X has more than h elements, then we have $|X - y| \geq h$, $|Y \cup y| \geq h$, and $r(X) + r(Y) \geq r(X - y) + r(Y \cup y)$. Thus, it suffices to prove inequality (3) when $|X|$ is h or Y is a nontrivial flat of M. By Lemma 7.3, each nontrivial connected component of the restriction $M|Y$ to a flat Y of M has more than h elements; with an argument similar to the one above, it follows that if Y is a nontrivial flat of M, then we may assume Y is connected.

Assume $|X|$ is h. By Lemma 7.3, X is an independent set that does not contain a cocircuit, so Y spans M. Thus, $r(X) + r(Y)$ is $r(M) + h$.

Now assume Y is a nontrivial connected flat of M. If Y is a fundamental flat, then inequality (3) follows as in the first paragraph. If Y is not a fundamental flat, then, by Theorem 5.7, Y is the intersection of two incomparable fundamental flats, say $Y \cup A$ and $Y \cup B$ where A and B partition X. We may assume 1 is in A, so $m+r$ is in B. Since $A \cup Y$ is a fundamental flat and B is the complement of $A \cup Y$, we have $r(A \cup Y) + r(B) \geq r(M) + k_M - 1$. Thus, since k_M exceeds h, to prove inequality (3), it suffices to prove $r(X) + r(Y) \geq r(A \cup Y) + r(B)$, that is,

$$r(A \cup B) + r(Y) \geq r(A \cup Y) + r(B). \tag{4}$$

Observe that $r(A \cup B)$ is $|n(A \cup B)|$; the inequality $r(A \cup B) \leq |n(A \cup B)|$ is obvious and the inequality $r(A \cup B) \geq |n(A \cup B)|$ follows by matching each set N_i, for i in $n(A)$, with its first element, which must be in A, and each set in N_j, for j in $n(B) - n(A)$, with its last element, which must be in B. A similar argument gives the equality $r(B) = |n(B)|$. From Theorem 3.11, we also have $r(A \cup Y) = |n(A \cup Y)|$ and $r(Y) = |n(Y)|$. Thus, inequality (4) is equivalent to

$$|n(A \cup B)| + |n(Y)| \geq |n(A \cup Y)| + |n(B)|. \tag{5}$$

Note that $|n(A \cup B)|$ is $|n(A)| + |n(B)| - |n(A) \cap n(B)|$. Substituting this and the analogous formula for $|n(A \cup Y)|$ into inequality (5) and simplifying gives that this inequality is equivalent to the inequality $|n(A) \cap n(Y)| \geq |n(A) \cap n(B)|$, which clearly holds. Thus, inequality (3) holds, as needed to complete the proof. □

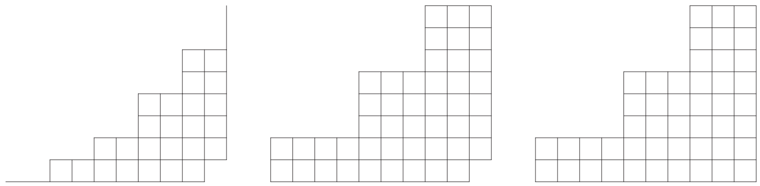

Figure 8. Lattice path presentations of three notch matroids.

As the matroid E_3 of Figure 14 shows, not every exact k_M-separation of a lattice path matroid M has a fundamental flat as one of the sets.

8. Notch Matroids and their Excluded Minors

There are infinitely many minor-closed, dual-closed classes of transversal matroids within the class of lattice path matroids. One way to define such classes is to impose certain requirements on the bounding paths; for example, the lower bounding path of a generalized Catalan matroid must have the form $E^m N^r$. In this section we introduce the minor-closed, dual-closed class of notch matroids, which is defined by special forms for the bottom bounding path. We relate notch matroids to generalized Catalan matroids via circuit-hyperplane relaxations. The main result is the characterization of notch matroids by excluded minors. We include some remarks on the excluded minors for lattice path matroids.

Definition 8.1. *A* notch matroid *is, up to isomorphism, a lattice path matroid of the form* $M[E^m N^r, Q]$ *or* $M[E^{m-1}NEN^{r-1}, Q]$.

As Figure 8 illustrates, notch matroids are either in $\mathcal{C}$ or their lattice path presentations differ from those of generalized Catalan matroids by the "notch" in the lower right corner. It follows from the lattice path descriptions of minors and duals, along with Theorem 5.6, that the class $\mathcal{N}$ of notch matroids is minor-closed and dual-closed. Note that $\mathcal{N}$, like its subclass $\mathcal{C}$, is not closed under direct sums. In contrast to $\mathcal{C}$, the class $\mathcal{N}$ is not closed under any of the following operations, as can be seen from the matroid D_3 of Figure 14: free extension, truncation, and the dual operations. The first lemma gives a basic property that $\mathcal{N}$ shares with $\mathcal{C}$.

Lemma 8.2. *Adding loops and isthmuses to a notch matroid yields a notch matroid.*

Note that a connected notch matroid either is in $\mathcal{C}$ or has a circuit-hyperplane relaxation in $\mathcal{C}$. Not every matroid that has a circuit-hyperplane relaxation in $\mathcal{C}$ is a notch matroid; for instance, the matroids A_3 and A_4 of Figure 10 each have two circuit-hyperplane relaxations that are in $\mathcal{C}$, yet neither is a lattice path matroid since condition (ii) of Theorem 5.10 fails. However, we have the following result.

Theorem 8.3. *A connected matroid in $\mathcal{L} - \mathcal{C}$ is a notch matroid if and only if it has a circuit-hyperplane. Relaxing any circuit-hyperplane of a lattice path matroid yields a generalized Catalan matroid.*

Proof. The last r elements of a connected notch matroid $M[E^{m-1}NEN^{r-1}, Q]$ obviously form a circuit-hyperplane. For the converse, assume that H is a circuit-hyperplane of the rank-r, nullity-m matroid $M = M[P, Q]$. Since H is an r-circuit of M, by Theorem 3.11 the set $n(H)$ is an interval of $r - 1$ elements in $[r]$; we may

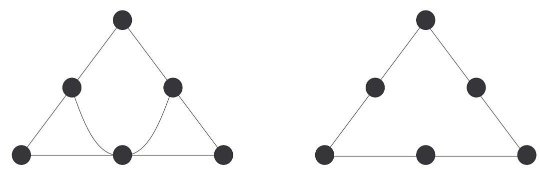

FIGURE 9. The 3-wheel $\mathcal{W}_3$ and the 3-whirl $\mathcal{W}^3$.

assume that $n(H)$ is $[2, r]$. Since H is a flat, H is an interval of r elements in the ground set $[m+r]$ of M, so $[m+r]$ consists of an initial interval, the interval H, and a final interval Y. Since H is a hyperplane, Y must be empty, so H consists of the last r elements of $[m+r]$. From these conclusions, it is immediate that M is a notch matroid. The last assertion follows from part (iii) of Corollary 3.12. □

Similar ideas yield the following result.

Lemma 8.4. *Let M' be $M[Q]$, a connected rank-r, nullity-m matroid in $\mathcal{C}$. If the basis B of M' is mapped onto the final segment $[m+1, m+r]$ by some automorphism of M', then there is a unique matroid M in which B is a circuit-hyperplane and from which M' is obtained by relaxing B. Furthermore, M is in $\mathcal{N}$.*

The following two lemmas will be used heavily in the proof of the excluded-minor characterization of $\mathcal{N}$.

Lemma 8.5. *If X and Y are nontrivial incomparable connected flats of a notch matroid M that has no isthmuses, then either X or Y is a circuit-hyperplane.*

Proof. The incomparable flats X and Y show that M is not in $\mathcal{C}$, so M has a circuit-hyperplane, say H. Either X or Y must be H since H cannot properly contain either X or Y and, by part (iii) of Corollary 3.12, nontrivial connected flats that are not contained in H are comparable. □

Lemma 8.6. *Three nontrivial connected flats X, Y, and Z of a notch matroid M cannot be mutually incomparable.*

Proof. We may assume that M has no isthmuses and that X and Y are incomparable. From Lemma 8.5, either X or Y, say X, is a circuit-hyperplane of M. Part (iii) of Corollary 3.12 implies that Y and Z are comparable. □

We turn to the excluded-minor characterization of $\mathcal{N}$. Let $ex(\mathcal{N})$ and $ex(\mathcal{L})$ denote the sets of excluded minors for $\mathcal{N}$ and $\mathcal{L}$, respectively. We first discuss the matroids in $ex(\mathcal{N})$ that are not lattice path matroids and so are in $ex(\mathcal{N}) \cap ex(\mathcal{L})$. In each case, we show that the matroids are not in $\mathcal{L}$; it is easy to check that all their proper minors are in $\mathcal{N}$, so we omit this part.

Among the self-dual matroids in $ex(\mathcal{N}) \cap ex(\mathcal{L})$ are the 3-wheel $\mathcal{W}_3$ and the 3-whirl $\mathcal{W}^3$, which are shown in Figure 9. Since all 3-point lines of $\mathcal{W}_3$ and $\mathcal{W}^3$ are fundamental flats, condition (i) of Theorem 5.10 fails, so $\mathcal{W}_3$ and $\mathcal{W}^3$ are not in $\mathcal{L}$.

For $n \geq 3$, let A_n be the rank-n paving matroid with only two nontrivial hyperplanes, $\{x, a_2, a_3, \ldots, a_n\}$ and $\{x, b_2, b_3, \ldots, b_n\}$, and with only one point, y, in neither circuit-hyperplane (Figure 10). The two circuit-hyperplanes violate condition (ii) of Theorem 5.10, so A_n is not in $\mathcal{L}$. Note that A_n is self-dual.

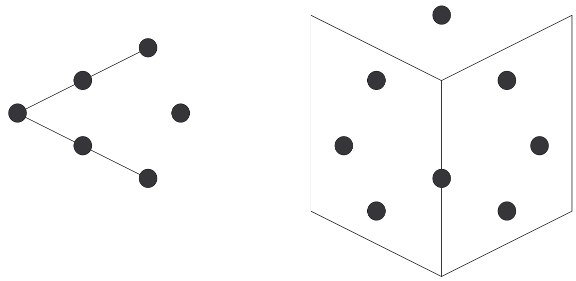

FIGURE 10. The matroids A_3 and A_4.

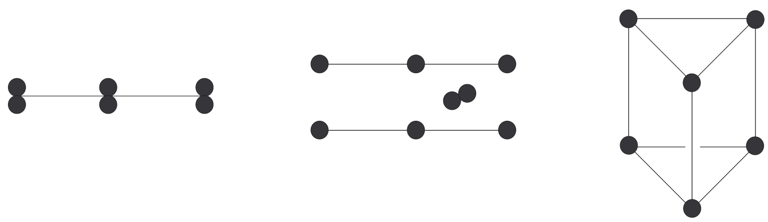

FIGURE 11. The matroids $B_{2,2}$, $B_{3,2}$, and $C_{4,2}$.

We next consider two doubly-indexed families in $ex(\mathcal{N}) \cap ex(\mathcal{L})$ that are related by duality; three of these matroids are shown in Figure 11. Let n and k be integers with $2 \leq k \leq n$. Let $B_{n,k}$ be the truncation $T_n(U_{n-1,n} \oplus U_{n-1,n} \oplus U_{k-1,k})$ to rank n of the direct sum of two n-circuits and a k-circuit. The three disjoint circuits are fundamental flats of $B_{n,k}$, so condition (i) of Theorem 5.10 shows that $B_{n,k}$ is not in $\mathcal{L}$. The dual $C_{n+k,k}$ of $B_{n,k}$ is the rank-$(n+k)$ paving matroid $C_{n+k,k}$ for which the ground set can be partitioned into sets X, Y, Z with $|X| = |Y| = n$ and $|Z| = k$ so that the only nontrivial hyperplanes are $X \cup Y$, $X \cup Z$, and $Y \cup Z$.

The remaining matroids in $ex(\mathcal{N}) \cap ex(\mathcal{L})$, two of which are shown in Figure 12, form two infinite families that are related by duality. Recall that $M + y$ denotes the free extension of M by the point y. For $n \geq 3$, let D_n be the rank-n matroid

$$\big(T_{n-1}(U_{n-2,n-1} \oplus U_{n-2,n-1}) \oplus U_{1,1}\big) + y.$$

That D_n is not in $\mathcal{L}$ for $n \geq 4$ follows since the two $(n-1)$-circuits, as well as their union, are fundamental flats of D_n, contrary to condition (i) of Theorem 5.10. In the dual E_n of D_n, the element y is parallel to an element x, and the deletion $E_n \backslash y$ is a rank-n paving matroid whose only nontrivial hyperplanes are two circuit-hyperplanes that intersect in x. (The matroids D_3 and E_3, which are shown in Figure 14, are lattice path matroids.)

We have proven the easy part of the following theorem; the more substantial part of this result follows from the excluded-minor characterization of notch matroids, which is given in Theorem 8.8.

Theorem 8.7. *The matroids in $ex(\mathcal{L}) \cap ex(\mathcal{N})$ are:*

(1) *the three-wheel $\mathcal{W}_3$ and the three-whirl $\mathcal{W}^3$,*
(2) *A_n for $n \geq 3$,*

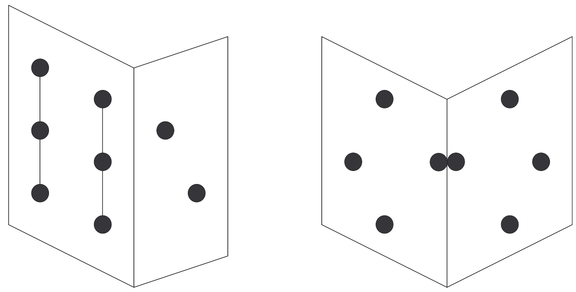

FIGURE 12. The matroids D_4 and E_4.

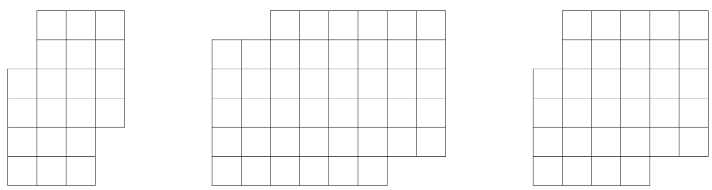

FIGURE 13. Lattice path presentations of F_6, G_6, and H_6.

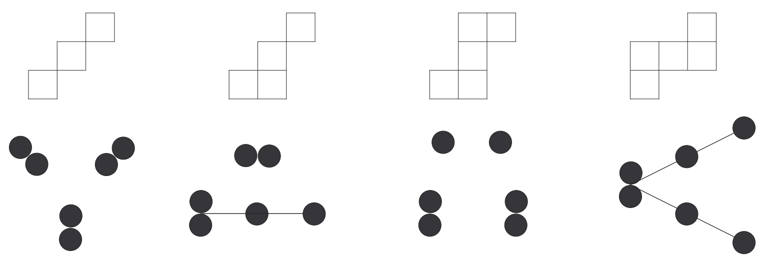

FIGURE 14. Path presentations and geometric representations of $U_{1,2} \oplus U_{1,2} \oplus U_{1,2}$, $T_2(U_{1,2} \oplus U_{1,1} \oplus U_{1,1}) \oplus U_{1,2}$, D_3, and E_3.

(3) *$B_{n,k}$ and $C_{n+k,k}$ for n and k with $2 \leq k \leq n$, and*
(4) *D_n and E_n for $n \geq 4$.*

We now turn to the excluded-minor characterization of notch matroids. The excluded minors are those in Theorem 8.7 together with the three types of lattice path matroids illustrated in Figure 13 and the four matroids in Figure 14.

Theorem 8.8. *The excluded minors for the class of notch matroids are:*

(1) *$U_{1,2} \oplus U_{1,2} \oplus U_{1,2}$ and $T_2(U_{1,2} \oplus U_{1,1} \oplus U_{1,1}) \oplus U_{1,2}$,*
(2) *the three-wheel, $\mathcal{W}_3$, and the three-whirl, $\mathcal{W}^3$,*
(3) *A_n for $n \geq 3$,*
(4) *$B_{n,k}$ and $C_{n+k,k}$ for n and k with $2 \leq k \leq n$,*
(5) *D_n for $n \geq 3$,*
(6) *E_n for $n \geq 3$,*
(7) *for $n \geq 4$, the rank-n matroid $F_n := T_n(U_{n-2,n-1} \oplus U_{n-2,n-1})$,*

(8) *for* $n \geq 2$*, the rank-*n *matroid* $G_n := T_n(U_{n-1,n+1} \oplus U_{n-1,n+1})$*, and*
(9) *for* $n \geq 3$*, the rank-*n *matroid* $H_n := T_n(U_{n-2,n-1} \oplus U_{n-1,n+1})$.

To make the proof of Theorem 8.8 less verbose, we will use abbreviations such as the following: from Theorem 3.14 applied to M, X_1, X_2, and y, we get $M \notin \mathcal{L}$. By this we mean that the matroid M and the flats X_1 and X_2 satisfy the hypotheses of Theorem 3.14, with the point y showing the validity of the third condition.

Proof of Theorem 8.8. The remarks before Theorem 8.7 show that of the matroids in the theorem, only D_3, E_3, and those in items (1) and (7)–(9) are in $\mathcal{L}$. The presentations of these matroids, illustrated in Figures 13 and 14, make it clear that they are not in $\mathcal{N}$. It is easy to check that all proper minors of these matroids are in $\mathcal{N}$. Note that H_n is self-dual, and that F_n and G_{n-2} are dual to each other.

The proof that Theorem 8.8 gives all excluded minors is intricate, so we first outline the argument. Part (8.8.1) proves that the disconnected excluded minors are $U_{1,2} \oplus U_{1,2} \oplus U_{1,2}$, $T_2(U_{1,2} \oplus U_{1,1} \oplus U_{1,1}) \oplus U_{1,2}$, F_4, G_2, and H_3. The rest of the proof revolves around three properties a connected excluded minor M may have:

(a) $r(X_1 \cup X_2) < r(M)$ for some nontrivial incomparable connected flats X_1, X_2,
(b) M contains three mutually incomparable connected flats,
(c) M has no circuit-hyperplane.

In (8.8.2), we show that if M has property (a), then M is D_n for some $n \geq 3$. Part (8.8.3) gives a key property of all connected excluded minors. In (8.8.4), we show that if property (b) but not (a) holds, then M is one of the matroids in items (2) and (4). Part (8.8.5) shows that if only property (c) holds, then M is one of the matroids in items (6)–(9). If none of the properties holds, then for any mutually incomparable connected flats $X_1, X_2, \dots, X_k$, we have $k \leq 2$, and if k is 2, then at least one of X_1 or X_2 is a circuit-hyperplane. Since restrictions to proper subsets of circuit-hyperplanes are free, it follows that relaxing a circuit-hyperplane of such an excluded minor yields a matroid M' in which the connected flats are linearly ordered by inclusion, that is, M' is in $\mathcal{C}$. The proof of Theorem 8.8 is completed in (8.8.6) by showing that the only rank-n excluded minor that has a circuit-hyperplane relaxation in $\mathcal{C}$ is A_n.

Throughout the proof, M denotes a rank-n excluded minor for the class of notch matroids. By Lemma 8.2, M has neither loops nor isthmuses.

> **(8.8.1)** If M is disconnected, then M is one of $U_{1,2} \oplus U_{1,2} \oplus U_{1,2}$, $T_2(U_{1,2} \oplus U_{1,1} \oplus U_{1,1}) \oplus U_{1,2}$, F_4, G_2, and H_3.

Proof of (8.8.1). Assume M has at least three components. Each component has a circuit of two or more elements, so M has $U_{1,2} \oplus U_{1,2} \oplus U_{1,2}$ as a minor, which is itself an excluded minor. Thus, M is $U_{1,2} \oplus U_{1,2} \oplus U_{1,2}$.

Now assume M has exactly two components, M_1 and M_2. Being proper minors of M, both M_1 and M_2 are notch matroids. Observe that if $r(M_i) \geq 2$, then, by Theorem 3.3 and Corollary 3.10, there is an element x for which M_i/x is connected. Dually, if $\eta(M_i) \geq 2$, then $M_i \backslash y$ is connected for some y.

Assume M_1 is $U_{1,2}$. From lattice path presentations and from the statements $M_2 \in \mathcal{N}$ and $U_{1,2} \oplus M_2 \notin \mathcal{N}$, it follows that $r(M_2)$ and $\eta(M_2)$ are both at least 2. Similarly, if M_2' is a connected minor of M_2 for which $r(M_2')$ and $\eta(M_2')$ are both 2, then $U_{1,2} \oplus M_2' \notin \mathcal{N}$. These observations, together with those in the last paragraph, imply that $r(M_2)$ and $\eta(M_2)$ are both 2. From lattice path presentations, we see that only two connected lattice path matroids have rank and nullity 2, namely $U_{2,4}$ and $T_2(U_{1,2} \oplus U_{1,1} \oplus U_{1,1})$, so M is either H_3 or $T_2(U_{1,2} \oplus U_{1,1} \oplus U_{1,1}) \oplus U_{1,2}$.

Now assume $M_1 = U_{1,k}$ with $k \geq 3$. Since $M \notin \mathcal{N}$, the nullity of M_2 is at least 2. Arguments like those in the last paragraph imply that k is 3, that $\eta(M_2)$ is 2, and that $r(M_2)$ is 1; therefore M_2 is $U_{1,3}$, so M is G_2.

Finally, if M_1 and M_2 have rank 2 or greater, then, by the same types of arguments, both M_1 and M_2 have rank 2 and nullity 1, so M is F_4. □

From now on, we assume M is connected.

> **(8.8.2)** If M has nontrivial incomparable connected flats X_1 and X_2 with $r(X_1 \cup X_2) < n$, then M is D_n.

Proof of (8.8.2). Choose such a pair of flats X_1, X_2 so that $r(X_1) + r(X_2)$ is as small as possible. Lemma 8.5 applied to $M|(X_1 \cup X_2)$, X_1, and X_2 implies that either X_1 or X_2 is a circuit-hyperplane of $M|(X_1 \cup X_2)$.

Assume $M|(X_1 \cup X_2)$ is disconnected. This disconnected notch matroid has neither loops nor isthmuses, so one component, say X_1, has rank 1 and the other, X_2, has nullity 1; thus, X_1 is a parallel class and X_2 is a circuit. If $|X_1| > 2$ and $y \in X_1$, then $M\backslash y$, $X_1 - y$, and X_2 contradict Lemma 8.5. If $|X_2| > 2$ and $z \in X_2$, then M/z, $\mathrm{cl}_{M/z}(X_1)$, and $X_2 - z$ contradict Lemma 8.5. Thus, $|X_1| = |X_2| = 2$. Since M has neither $B_{2,2}$ nor $U_{1,2} \oplus U_{1,2} \oplus U_{1,2}$ as a proper minor, X_1 and X_2 are the only nontrivial parallel classes of M. Let x and y be in $E(M) - \mathrm{cl}(X_1 \cup X_2)$. By Lemma 8.5, the rank-1 flats $\mathrm{cl}_{M/x}(X_1)$ and $\mathrm{cl}_{M/x}(X_2)$ are hyperplanes of M/x, so $r(M)$ is 3. It follows that $M|(X_1 \cup X_2 \cup \{x, y\})$, and so M, is one of the excluded minors $T_2(U_{1,2} \oplus U_{1,1} \oplus U_{1,1}) \oplus U_{1,2}$ or D_3; since M is connected, M is D_3.

Now assume $M|(X_1 \cup X_2)$ is connected. We show that M is D_n by proving the following statements:

(i) M is simple,
(ii) X_1 and X_2 are disjoint circuits, and $X_1 \cup X_2$ is a flat of M,
(iii) $E(M) - (X_1 \cup X_2)$ contains only two elements, say x and y,
(iv) the only nonspanning circuits of $M\backslash x, y$ are X_1 and X_2,
(v) $|X_1| = |X_2|$, so both X_1 and X_2 are circuit-hyperplanes of $M\backslash x, y$, and
(vi) the only circuits of M that contain x and y are spanning circuits.

To prove statement (i), note that since $M|(X_1 \cup X_2)$ is connected, and since X_1 and X_2 are incomparable flats, neither X_1 nor X_2 is a parallel class. If elements x and y of M were parallel, then $M\backslash y$, $X_1 - y$, and $X_2 - y$ (which may be X_1 and X_2) would contradict Lemma 8.5.

For statement (ii), we first show that both $M|X_1/x$ and $M|X_2/x$ are connected for any x in $X_1 \cap X_2$. If, say, $M|X_1/x$ were disconnected, then by Lemma 2.6, there would be nontrivial incomparable connected flats A and B of $M|X_1$ with $r(A)+r(B) = r(X_1)+1$. Since M is simple, $r(X_2)$ exceeds 1, so the flats A and B of M would contradict the choice of X_1 and X_2 as minimizing the sum $r(X_1)+r(X_2)$. Since $M|X_1/x$ and $M|X_2/x$ are connected, M/x, $X_1 - x$, and $X_2 - x$ contradict Lemma 8.5. Thus, X_1 and X_2 are disjoint. The connected notch matroids $M|X_1$ and $M|X_2$ have spanning circuits; this observation and the minimality of M show that X_1 and X_2 are circuits. For any x in $\mathrm{cl}(X_1 \cup X_2) - (X_1 \cup X_2)$, the deletion $M\backslash x$ is connected, so $M\backslash x$, X_1, and X_2 would violate Lemma 8.5. Thus, $\mathrm{cl}(X_1 \cup X_2)$ is $X_1 \cup X_2$, so statement (ii) holds.

Let y be in $E(M) - (X_1 \cup X_2)$. The contraction M/y has neither loops nor isthmuses. By Lemma 8.5, at least one of $\mathrm{cl}_{M/y}(X_1)$ and $\mathrm{cl}_{M/y}(X_2)$ is a circuit-hyperplane of the notch matroid M/y, so $r(X_1 \cup X_2)$ is $n-1$. For $M\backslash y$, X_1, and X_2 to not contradict Lemma 8.5, $M\backslash y$ must have an isthmus. From these conclusions, statement (iii) follows.

Assume C is a nonspanning circuit of $M\backslash x, y$ other than X_1 and X_2. Recall that either X_1 or X_2, say X_1, is a circuit-hyperplane of $M\backslash x, y$. Thus, X_1 and $\mathrm{cl}(C)$ are incomparable and $X_1 \cup C$ spans the flat $X_1 \cup X_2$. Let z be in the difference $X_2 - C$ of circuits. Note that $M\backslash z$ is connected. That $M\backslash z$, X_1, and $\mathrm{cl}(C) - z$ contradict Lemma 8.5 proves statement (iv). Statement (v) follows since if $|X_2| < |X_1|$ and z is in X_1, then M/z, $X_1 - z$, and X_2 would contradict Lemma 8.5.

From statements (i) and (v) we have $n \geq 4$. Assume x and y are in a nonspanning circuit C. At least one of X_1 and X_2 is not contained in $\mathrm{cl}(C)$, so we may assume that X_1 and $\mathrm{cl}(C)$ are incomparable. Let z be in the difference $X_2 - C$ of circuits. Note that $X_1 \cup (X_2 - z)$ is a connected hyperplane of $M\backslash z$ since $n \geq 4$, so $M\backslash z$ is connected. Lemma 8.5 applied to $M\backslash z$, X_1 and $\mathrm{cl}_{M\backslash z}(C)$ implies that $\mathrm{cl}_{M\backslash z}(C)$ must be a circuit-hyperplane of $M\backslash z$, so $\mathrm{cl}(C)$ is a hyperplane of M. Note that $\mathrm{cl}(C)$ is either $\mathrm{cl}_{M\backslash z}(C)$ or $\mathrm{cl}_{M\backslash z}(C) \cup z$, that is, either C or $C \cup z$, so $|\mathrm{cl}(C)| \leq n+1$. Thus, if $X_2 \subseteq \mathrm{cl}(C)$, then $\mathrm{cl}(C)$ is $X_2 \cup \{x, y\}$. However, if $\mathrm{cl}(C)$ is $X_2 \cup \{x, y\}$ and w is in X_1, then $M\backslash w$, $(X_1 - w) \cup X_2$, $X_2 \cup \{x, y\}$ contradict Lemma 8.5. Therefore X_2 and $\mathrm{cl}(C)$ are incomparable. By switching X_1 and X_2 if necessary, we may assume $C \cap X_1 \neq \emptyset$. Since $r(C) = n - 1$, we have $r(C \cup X_1) = n$; however, there are at least two elements, say a and b, in $X_2 - \big(\mathrm{cl}(C) \cup X_1\big)$, that is, in $X_2 - \mathrm{cl}(C)$, so by Theorem 3.14, $M\backslash a$ is not a lattice path matroid, contrary to the minimality of M. Thus, statement (vi) holds, so M is D_n. □

> **(8.8.3)** If X is a proper nontrivial connected flat of M and the element x of X is not parallel to any element, then $X - x$ is a connected flat of M/x.

Proof of (8.8.3). If $X - x$ were a disconnected flat of M/x, then, by Lemma 2.6 applied to $M|X$, we would have $r(X_1 \cup X_2) \leq r(X) < r(M)$ for some nontrivial incomparable connected flats X_1, X_2 of $M|X$. Since X_1 and X_2 would also be flats of M, by (8.8.2), M would be D_n. That D_n has no such flat X and element x provides the contradiction that proves the result. □

> **(8.8.4)** If M has three mutually incomparable connected flats X_1, X_2, X_3, then M is $\mathcal{W}_3$, $\mathcal{W}^3$, $B_{n,k}$, or $C_{n,k}$.

Proof of (8.8.4). The minimality of M and Lemma 8.6 imply that the ground set of M is $X_1 \cup X_2 \cup X_3$ and that any pair x, y of parallel elements can be in only one of X_1, X_2, X_3. If an element x were in $X_1 \cap X_2 \cap X_3$, then by (8.8.3), M/x, $X_1 - x$, $X_2 - x$, and $X_3 - x$ would contradict Lemma 8.6, so $X_1 \cap X_2 \cap X_3 = \emptyset$. Note that M is not D_n, so we have $r(X_i \cup X_j) = n$ for $\{i, j\} \subset \{1, 2, 3\}$.

First assume $X_1 \cap X_2 = \emptyset$. There are at least two points x and y in $X_2 - X_3$, so if $X_1 \cap X_3$ were nonempty, then $M\backslash y$, X_1, and X_3 would contradict Theorem 3.14. Thus, $X_1 \cap X_3 = \emptyset$. Similarly $X_2 \cap X_3 = \emptyset$. The minimality of M implies that X_1, X_2, and X_3 are circuits. Let $\{i, j, k\}$ be $\{1, 2, 3\}$. Since $r(X_i \cup X_j)$ is n, for any x in X_k the notch matroid $M\backslash x$ has no isthmuses; thus, from Lemma 8.5, either X_i or X_j is a circuit-hyperplane of $M\backslash x$ and so of M. It follows that at least two of

X_1, X_2, X_3, say X_1 and X_2, are circuit-hyperplanes of M. Let $|X_3|$ be k. Note that M is $B_{n,k}$ if X_1, X_2, and X_3 are the only nonspanning circuits of M. If C were another nonspanning circuit, then for any z in the difference $X_3 - C$ of circuits, the flat $\mathrm{cl}_{M\backslash z}(C)$ would be contained in neither of the hyperplanes X_1 and X_2 of $M\backslash z$, contrary to part (iv) of Corollary 3.12. Thus, M is $B_{n,k}$.

Now assume $X_i \cap X_j \neq \emptyset$ for all sets $\{i,j\} \subset \{1,2,3\}$. We claim that X_1, X_2, and X_3 are hyperplanes and the union of any two contains all but at most one point of M. Let $\{i,j,k\}$ be $\{1,2,3\}$ and let x be in $X_i \cap X_j$. The equality $r(X_i \cup X_k) = n$ and Theorem 3.14 give the inequality $|E(M) - (X_i \cup X_k)| \leq 1$, so the second claim holds. To see that X_k is a hyperplane, note that Lemma 8.6 applied to M/x, $\mathrm{cl}_{M/x}(X_k)$, $X_i - x$, and $X_j - x$ implies that there is a containment among at least two of these sets. Of the two possible containments, we may assume $X_i - x \subseteq \mathrm{cl}_{M/x}(X_k)$. Thus, $X_i \subseteq \mathrm{cl}(X_k \cup x)$. This containment, the inequality $|E(M) - (X_i \cup X_k)| \leq 1$, and that X_j is connected imply that $\mathrm{cl}(X_k \cup x)$ is $X_1 \cup X_2 \cup X_3$, so X_k is a hyperplane of M.

If x and y are in $X_i \cap X_j$, then x is in the nontrivial connected hyperplanes $X_i - y$ and $X_j - y$ of the notch matroid M/y, so, by Theorem 3.14, $E(M/y)$ is $(X_i - y) \cup (X_j - y)$. Thus, if $|X_i \cap X_j| \geq 2$, then $E(M) = X_i \cup X_j$.

Assume $|X_1 \cap X_2|$ is 1. Since X_1 is connected and at most one point of X_1 is in neither $X_1 \cap X_2$ (one point) nor $X_1 \cap X_3$ (a flat), there is one point in $X_1 - (X_2 \cup X_3)$. Similarly, there is one point in $X_2 - (X_1 \cup X_3)$. These conclusions, and that in the last paragraph, give the equality $|X_1 \cap X_3| = |X_2 \cap X_3| = 1$. Therefore X_1, X_2, and X_3 are 3-point lines. It follows easily that M is either $\mathcal{W}_3$ or $\mathcal{W}^3$.

Assume $|X_i \cap X_j| \geq 2$ for $\{i,j,k\} = \{1,2,3\}$. Thus, $X_i = (X_i \cap X_j) \cup (X_i \cap X_k)$. Let x and y be in $X_i \cap X_j$. Lemma 8.5 applied to M/y, $X_i - y$, and $X_j - y$ implies that either $X_i - y$ or $X_j - y$ is a circuit-hyperplane of M/y. Since, in addition, X_i and X_j are connected hyperplanes of M, either X_i or X_j is a circuit-hyperplane of M. It follows that at least two hyperplanes, say X_1 and X_2, are circuit-hyperplanes of M. Assume $|X_1 \cap X_2| = k$. That X_1 and X_2 are circuit-hyperplanes of M gives the equality $|X_1 \cap X_3| = n - k = |X_2 \cap X_3|$. To prove that M is $C_{n,k}$, we need only show that the only proper nontrivial connected flat X other than X_1 and X_2 is X_3. Clearly X is incomparable to the circuit-hyperplanes X_1 and X_2. As we deduced for X_1, X_2, X_3, we get $X \cap X_1 \cap X_2 = \emptyset$, so $X \subseteq X_3$. Since $X_1 \cap X_3$ and $X_2 \cap X_3$ are independent, both $X \cap X_1$ and $X \cap X_2$ are nonempty. With this, the claim in the third paragraph shows that X is a hyperplane. Since $X \subseteq X_3$, it follows that X is X_3, as needed. □

> **(8.8.5)** If M has no circuit-hyperplane and is not D_n, then M is one of E_n, F_n, G_n, or H_n.

Proof of (8.8.5). Since M is not a generalized Catalan matroid, there is a pair X_1, X_2 of incomparable connected flats. Since M is not D_n, part (8.8.2) gives the equality $r(X_1 \cup X_2) = n$ for any such pair of flats.

Assume there were an element x in $E(M) - (X_1 \cup X_2)$. Since $r(X_1 \cup X_2)$ is n, the deletion $M\backslash x$ would have no isthmuses. Therefore either X_1 or X_2 would be a circuit-hyperplane of $M\backslash x$ and so of M. Since M has no circuit-hyperplane, the equality $E(M) = X_1 \cup X_2$ follows.

First assume M has two incomparable connected flats X_1 and X_2 that are not disjoint. We show that M is E_n by proving the following statements:

(i) each element in $X_1 \cap X_2$ is parallel to another element of M,
(ii) $X_1 \cap X_2$ contains just two elements, say x and y, and at least one of $X_1 - x$ and $X_2 - x$, say $X_1 - x$, is a circuit-hyperplane of $M \backslash x$,
(iii) $X_2 - x$ is a circuit,
(iv) $|X_1| = |X_2|$, and
(v) the nonspanning circuits of M are $X_1 - x$, $X_1 - y$, $X_2 - x$, $X_2 - y$, and $\{x, y\}$.

Assume statement (i) failed for some x in $X_1 \cap X_2$. From (8.8.3) and Lemma 8.5, either $X_1 - x$ or $X_2 - x$, say $X_1 - x$, would be a circuit-hyperplane of M/x. It follows that X_1 would be a circuit-hyperplane of M. This contradiction to the hypotheses of (8.8.5) proves statement (i). It follows that for each $x \in X_1 \cap X_2$, the deletion $M \backslash x$ is a connected notch matroid, so by Lemma 8.5, either $X_1 - x$ or $X_2 - x$, say $X_1 - x$, is a circuit-hyperplane of $M \backslash x$. Since the circuit $X_1 - x$ of $M \backslash x$ cannot contain parallel elements, statement (ii) follows. By (8.8.3) the minor $M|X_2/y\backslash x$ is connected, so by part (b) of Corollary 3.7 there is a spanning circuit X_2' of $M|X_2$ that contains y. Lemma 8.5 and the minimality of the excluded minor M imply that X_2 is $X_2' \cup x$, so statement (iii) holds. For statement (iv), note that if $|X_1| > |X_2|$ and $z \in X_1 - X_2$, then M/z, $X_1 - z$, and $\mathrm{cl}_{M/z}(X_2)$ contradict Lemma 8.5. Statement (v) follows from part (iv) of Corollary 3.12 since each of the notch matroids $M \backslash x$ and $M \backslash y$ has two circuit-hyperplanes.

Now assume any two incomparable nontrivial connected flats are disjoint. We showed that the union of any two such flats is $E(M)$. Let X_1, X_2 be such flats. It follows that all nonspanning circuits of M span either $M|X_1$ or $M|X_2$, so M is $T_n(M|X_1 \oplus M|X_2)$; also, $M|X_1$ and $M|X_2$ are uniform matroids. If X_1 is not a circuit and x is in X_1, then $M \backslash x$ is a connected notch matroid in which X_2 is not a circuit-hyperplane, so $X_1 - x$ is a circuit-hyperplane of $M \backslash x$; it follows that $M|X_1$ is $U_{n-1,n+1}$. Assume that X_1 is a circuit, and so not a hyperplane of M; let x be in X_2. Note that X_1 and $X_2 - x$ are incomparable connected flats of the notch matroid M/x, which has no isthmuses. Since X_2 is not a circuit-hyperplane of M, it follows that $X_2 - x$ cannot be a circuit-hyperplane of M/x. Therefore by Lemma 8.5, X_1 is a circuit-hyperplane of M/x. Thus, $M|X_1$ is $U_{n-2,n-1}$. In this manner, we see that there are, up to switching X_1 and X_2, three possibilities: $M|X_1$ and $M|X_2$ are both $U_{n-2,n-1}$; $M|X_1$ is $U_{n-2,n-1}$ and $M|X_2$ is $U_{n-1,n+1}$; both $M|X_1$ and $M|X_2$ are $U_{n-1,n+1}$. These possibilities give, respectively, F_n, H_n, and G_n. □

> **(8.8.6)** If relaxing some circuit-hyperplane C of M gives a generalized Catalan matroid M', then M is A_n.

Proof of (8.8.6). We show that M is A_n by proving the following statements.

(i) There is a nonspanning circuit $C' \neq C$ of M with $C \cap C' \neq \emptyset$.

Fix such a circuit C' of least cardinality.

(ii) There is at least one element y in $E(M) - \big(C \cup \mathrm{cl}(C')\big)$.
(iii) The ground set of M is $C \cup C' \cup y$; also $|C \cap C'| = 1$.
(iv) The circuit C' is a hyperplane of M.
(v) The only nonspanning circuits of M are C and C'.

Let the chain of proper nontrivial connected flats of M' be $X_1 \subset \cdots \subset X_k$. If $C \cap X_k$ were empty, then, by Corollary 5.8, there would be an automorphism of

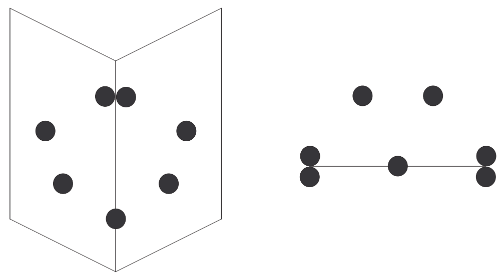

FIGURE 15. Two more excluded minors for the class of lattice path matroids.

M' that maps C to a final segment; by Lemma 8.4 we would get the contradiction that M is a notch matroid. Thus, $C \cap X_k$ is not empty, which gives statement (i). Among all circuits that intersect C, choose C' with smallest cardinality. The closure $\mathrm{cl}(C')$ is one of the connected flats X_j, and by the choice of C', the basis C of M' is disjoint from X_i for $i < j$. To prove statement (ii) we must show that C does not contain the complement of X_j; if this were false, then by Corollary 5.8 and Lemma 8.4 we would get, as before, that M is a notch matroid.

By Theorem 3.14, $M|(C \cup C' \cup y)$ is not a lattice path matroid. This observation and the minimality of M prove the first part of statement (iii). The second part holds since if $|C \cap C'| \geq 2$ and $x \in C \cap C'$, then, by Theorem 3.14, M/x would not be a lattice path matroid. Let $C \cap C'$ be x.

To prove statement (iv), first note that $M|\mathrm{cl}_M(C')$ is a uniform matroid since, by the choice of C', any nonspanning circuit Z of $M|\mathrm{cl}_M(C')$ would be disjoint from C, which gives the contradiction that the circuit C' properly contains the circuit Z. Since $M|\mathrm{cl}_M(C')$ is a uniform matroid that consists of C' and a subset of C, and since, by statement (iii), any circuit $C'' \neq C$ with $|C''| = |C'|$ that intersects C contains just one element of C, it follows that $C \cap \mathrm{cl}_M(C')$ is x, so C' is closed. If C' is not a hyperplane of M, then there is an element z in $C - \mathrm{cl}_M(C' \cup y)$, so y is not in $\mathrm{cl}_M(C' \cup z)$. However, for such a z, Theorem 3.14 applied to M/z, $\mathrm{cl}_{M/z}(C')$, $C - z$, and y shows that M/z is not in $\mathcal{L}$, contrary to M being an excluded minor for $\mathcal{N}$.

Since C' is a circuit-hyperplane of M and of the generalized Catalan matroid M', it follows that C' is the only nonspanning circuit of M', so C and C' are the only nonspanning circuits of M, as needed to complete the proof. □

□

Figure 15 shows two excluded minors for $\mathcal{L}$ that are not among those given in Theorem 8.7. Presently we do not know whether these two matroids complete the list of excluded minors for the class of lattice path matroids.

We close by noting that a lattice path matroid is graphic if and only if it is the cycle matroid of an outerplanar graph in which each inner face shares edges with at most two other inner faces. One implication follows since $\mathcal{W}_3$ and $C_{4,2}$ (i.e., the cycle matroids of the two excluded minors, K_4 and $K_{2,3}$, for outerplanar graphs) are excluded minors for lattice path matroids, as is $B_{2,2}$, which is the cycle matroid of the graph formed by adding an edge parallel to each edge of K_3. The other implication follows since by adding edges any graph of the stated type can be extended to a graph of this type in which each face is bounded by at most three edges, and the cycle matroids of such graphs, which are certain parallel connections of 3-point lines, are easily seen to be lattice path matroids.

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