LEIBNIZ AND THE INFINITE

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1.-Introduction.

On the 5^{th} (15^{th}) of September, 1695 Leibniz wrote to Vincentius Placcius:

> *"But I have so many new insights in mathematics, so many thoughts in philosophy, so many other literary observations that I am often irresolutely at a loss which as I wish should not perish*[1]*".*

Leibniz's extraordinary creativity especially concerned his handling of the infinite in mathematics. He was not always consistent in this respect. This paper will try to shed new light on some difficulties of this subject mainly analysing his treatise *On the arithmetical quadrature of the circle, the ellipse, and the hyperbola* elaborated at the end of his Parisian sojourn.

2.- Infinitely small and infinite quantities.

In the Parisian treatise Leibniz introduces the notion of infinitely small rather late. First of all he uses descriptions like: *ad differentiam assignata quavis minorem sibi appropinquare* (to approach each other up to a difference that is smaller than any assigned difference)[2], *differat quantitate minore quavis data* (it differs by a quantity that is smaller than any given quantity)[3], *differentia data quantitate minor reddi potest* (the difference can be made smaller than a

1 "Habeo vero tam multa nova in Mathematicis, tot cogitationes in Philosophicis, tot alias litterarias observationes, quas vellem non perire, ut saepe inter agenda anceps haeream." (LEIBNIZ, since 1923: II, 3, 80).

2 LEIBNIZ (2016), 18.

3 *Ibid*., 20.

given quantity)[4]. Such a difference or such a quantity necessarily is a variable quantity. Its value depends on the assigned or given quantity without ever becoming equal to zero.

Yet, when Leibniz speaks about errors he uses another terminology: *error minor quovis errore assignabili* (the error is smaller than any assignable error)[5], *ostendendo errorem quovis assignabili esse minorem, adeoque nullum* (by demonstrating that the error is smaller than any assignable error and therefore zero)[6], *intervallo ab eo aberunt infinite parvo, sive error quovis assignabili errore minor erit* (they - that is, the terms of an infinite, convergent, geometrical sequence - will be distant from it - that is, the limit - by an infinitely small interval or the error will be smaller than any assignable error)[7]. The notion of *assigned* has been replaced by the notion of *assignable*. Such an error necessarily is equal to zero as Leibniz rightly states. For if we assume that such an error is unequal to zero it would have a certain value. But this implies a contradiction against the postulate that the error has to be smaller than any assignable quantity, that is, also smaller than this certain value.

Yet, Leibniz explicitly calls such errors infinitely small: We should not try to make things seem better. There is an inconsistency in Leibniz's terminology. This especially applies to his example of the terms of a convergent, geometrical sequence: The interval between its terms and its limit does not become equal to zero.

In his treatise *On the arithmetical quadrature of the circle* etc. Leibniz does not give an explicit definition of the notion of infinitely small. The first occurrence is to be found in the *scholium* after proposition VII. He explains why certain quadratures cannot be carried out *sine quantitatibus fictitiis, infinitis scilicet vel infinite parvis assumtis* (without the fictitious quantities, namely those assumed to be infinite or infinitely small)[8].

Zero is no infinitely small quantity, in contrast to Euler. *Infinitely small* means: larger than zero and smaller than any given quantity. "larger than zero" is never explicitly postulated anywhere. We might assume that the error is the value assigned to the variable, infinitely small quantity. But we

4 *Ibid*., 26.

5 *Ibid*., 28.

6 *Ibid*., 46.

7 *Ibid*., 142.

8 *Ibid*., 36.

have to accept that Leibniz also says: *(μ)μ quolibet assignabili intervallo μ4B minor est* ((μ)μ is smaller than an arbitrary assignable interval μ4B)[9].

What does the fictionality of these quantities mean? It means that we behave as if such a quantity had a numerical value so that we can calculate with it. Hence the statement is not true: "His view of infinitesimals as useful fictions seems to have taken shape in the mid-1690s, although there are certainly traces of it as early as the 1670s"[10]. From the very beginning the fictionality of these variable quantities was an unavoidable, fundamental property of them. *Infinite parvum* (the infinitely small) is an *ens mathematicum* (mathematical being), Leibniz says in 1695[11]. This is in perfect agreement with his classification of scientific disciplines elaborated in 1696/1697[12]: *philosophia imaginabilium seu mathesis* (philosophy of imaginable objects or mathematics).

There is a similar double terminology regarding the *infinitum* (infinitely large): *rationem omni assignata majorem* (a ratio that is larger than any assigned ratio)[13], *ordinata potest fieri major recta quavis data…sive infinita* (the ordinate can become larger than any given straight line…or infinite)[14]. Such a ratio or ordinate necessarily is a variable quantity. Its value depends on the assigned or given quantity without ever becoming actually infinite. Such a terminology can be found for example in Kepler's *New solid geometry* published in 1615 and known to Leibniz: *proportio quacunque data proportione maior* (a ratio that is larger than any arbitrary given ratio)[15].

Yet, Leibniz uses also another terminology: *ordinata (μ)λ erit longitudine infinita, major qualibet assignabili 4B4D* (the ordinate will be of infinite length, larger than any assignable ordinate 4B4D)[16], *infinitum est, sive majus plano quovis assignabili* (the plane will be infinite or larger than any assignable plane)[17]. The notion of *assigned* has been replaced by the notion of *assignable*. Hence there is an unavoidable consequence. The set of all finite cardinal numbers 1, 2, 3, … is a transfinite set. Its cardinal number is Alef_0. This is the least

9 *Ibid*., 58.

10 JESSEPH (2015), 195.

11 LEIBNIZ (1695), 238.

12 LEIBNIZ (since 1923), IV, 6, 517.

13 LEIBNIZ (2016), 120.

14 *Ibid*., 220.

15 KEPLER (1615), 100.

16 LEIBNIZ (2016), 58.

17 *Ibid*., 220.

cardinal number being larger than any finite cardinal number. Leibniz's terminology implies actual infinity though he rejects the existence of an infinite number, and that again in contrast to Euler. In his *Elements of the differential calculus* Euler explains:

> *"…∞, by this sign a quantity is denoted that is larger than any finite or assignable quantity…But an infinitely small quantity is nothing but a vanishing quantity and therefore in reality it will be equal to 0. This definition of infinitely small quantities corresponds also with that definition by which they are called smaller than any assignable quantity. For if a quantity should be so small that it is smaller than any assignable quantity, this quantity cannot be unequal to zero. For if it were unequal to zero, a quantity could be assigned that is equal to it. This is against the assumption*[18]*".*

There is still an utmost important distinction that Leibniz makes between two types of the infinite, that is, betwe the bounded and the unbounded infinite: *interminatum…voco in quo nullum punctum ultimum sumi potest, saltem ab una parte, infinitum vero, quantitatem sive terminatam, sive interminatam, modo qualibet nobis assignabili, numerisve designabili, majorem intelligamus.* (I call something unbounded in which no last point can be taken, at least on one side, but infinite, a bounded or an unbounded quantity, provided that we understand it as a quantity that is larger than any quantity that is assignable by us or can be designated by numbers.)[19]. Hence we get the following dichotomy:

infinite
bounded unbounded

It is worth mentioning that Galileo used the notion of *terminata* in his *Discorsi*[20], that is, in a work Leibniz was well acquainted with because it contains Galileo's discussion of indivisibles. We come back to this issue later on.

18 «…∞, quo denotatur quantitas omni quantitate finita seu assignabili maior…Sed quantitas infinite parva nil aliud est nisi quantitas evanescens adeoque revera erit = 0. Consentit quoque ea infinite parvorum definitio, qua dicuntur omni quantitate assignabili minora, si enim quantitas tam fuerit parva, ut omni quantitate assignabili sit minor, ea certe non poterit non esse nulla; namque nisi esset = 0, quantitas assignari posset ipsi aequalis, quod est contra hypothesin ». (EULER, 1755: 69).

19 LEIBNIZ (2016), 60.

20 GALILEI (1638), 83.

3.- How did Leibniz demonstrate that a quantity is infinitely small or infinite?

We shall discuss three methods used by Leibniz in order to show that a certain quantity is infinitely small or infinite.

3.1.- The first method is based on the trichotomy law.

This law states that there is exactly one of three possibilities for a quantity, that is, it can be finite, infinitely small or infinite. Always two possibilities have to be excluded in order to demonstrate that one of these three possibilities is true.

Let us consider proposition 20 of the treatise *On the arithmetical quadrature of the circle* etc.: Three quantities X, Z, V are given. Let V+X have a finite ratio to V+Z which is unequal to 1, that is, (V+X):(V+Z)≠1. If X and Z are finite, V will also be finite. If X or Z is infinite, V will also be infinite.

Let us prove the second statement and let us assume without restriction of generality that Z is infinite, X is finite. Leibniz has to demonstrate that V must be infinite, that is, neither finite nor infinitely small. He refutes only the first possibility:

Let us assume that V is finite. Then V+X is finite, V+Z is infinite. Hence (V+X):(V+Z) is infinitely small. This is a contradiction against the presupposition.

We complete the proof without any difficulty: Let us assume that V is infinitely small. Then V+X is finite, V+Z is infinite. Hence we get the same contradiction as before.

3.2.- The second method determines the third proportional in a proportionality.

Let us consider proposition 21 of the mentioned treatise: Let the curve $_0C_1C_2C$ be a hyperboloid (a hyperbola of an arbitrary degree) $x^n y^m = a$. The rectangle under the infinitely small abscissa A0B and the infinitely large ordinate $_0B_0C$ is

1) an infinite quantity, if n>m;
2) an infinitely small quantity, if n<m;
3) a finite quantity, if n=m.

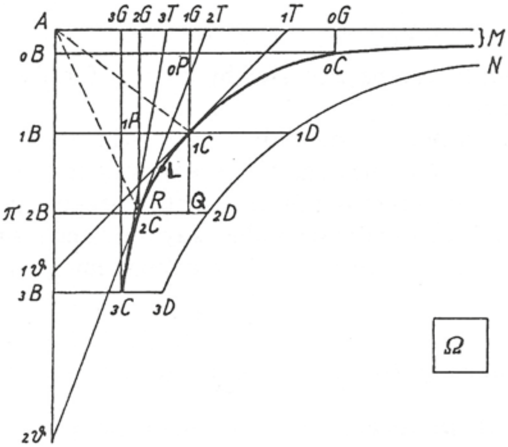

Illustration 1. Hyperboloid ${}_0C{}_1C{}_2C$ etc. of proposition 21 (LEIBNIZ, 2016: 78)

Leibniz applies the rule: If x : finite = finite : infinitely small, x is infinite. Arthur has shown how such a rule can be proved[21]. First of all Leibniz demonstrates the proposition for n=2, m=1. Then he adds a general proof that is based on powers of 0. Both proofs are worth discussing.

We generalize Leibniz's ideas regarding his special case n=2, m=1. Let us assume that n>m≥1. The size of the rectangle $A{}_0B{}_0C{}_0G$ is looked for. According to the equation of the hyperboloid ${}_0B{}_0C^m : {}_1B{}_1C^m = A{}_1B^n : A{}_0B^n$. Hence rectangle $A{}_0B{}_0C{}_0G$: rectangle $A{}_1B{}_1C{}_1G = (A{}_0B^m : A{}_1B^m)\cdot({}_0B{}_0C^m : {}_1B{}_1C^m) = (A{}_0B^m : A{}_1B^m)\cdot(A{}_1B^n : A{}_0B^n) = A{}_1B^{n-m} : A{}_0B^{n-m}$.

Hence rectangle x: finite rectangle = finite: infinitely small, hence x is infinite.

Leibniz's general proof considers the rectangle xy and $y^m = x^{-n}$ or $y^m x^n = 1$. Hence $y = x^{\frac{n}{m}}$ or $yx = x^{\frac{m-n}{m}}$. If m>n, m-n:m will be positive. If m<n, m-n:m will be negative. Leibniz argues with x=0: a positive power of 0 is infinitely small, a negative power of 0 is infinite. In other words, 0 is used an as infinitely small quantity.

21 ARTHUR (2008), 28f.

3.3.- The third method is based on the verification of the definition of infinite.

An infinite quantity is larger than any given quantity. It is applied in proposition 45 in order to prove the divergence of the harmonic series. This problem will be dealt with in chapter 4.

4.- Asymptotic spaces.

Especially interesting considerations about the infinite concern Leibniz's handling of asymptotic spaces. First of all let us consider his pointwise construction of the versiera.

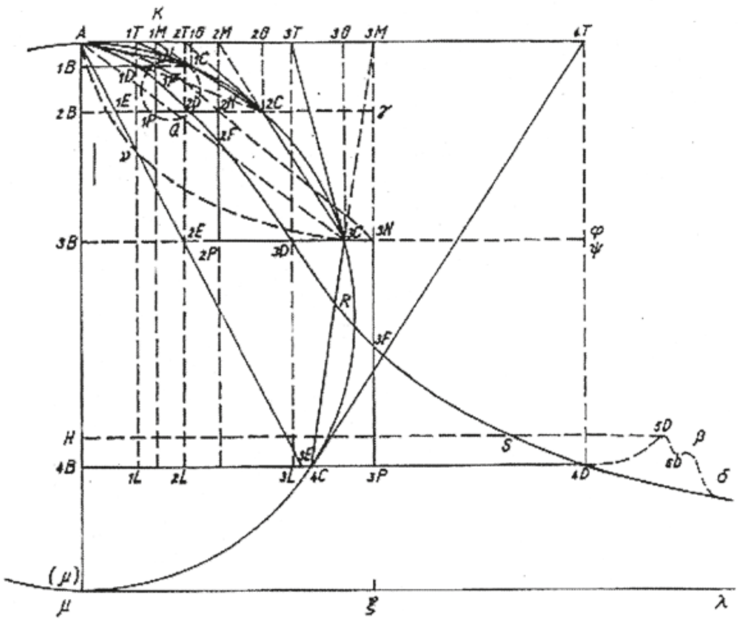

Illustration 2. A semicircle and the construction of the versiera (LEIBNIZ, 2016: 20).

His original curve is a semicircle through the points 1C, 2C, 3C etc. The tangents in these points cut the horizontal axis of ordinates in 1T, 2T, 3T etc. The straight lines A1T, A2T, A3T etc. are transferred to the ordinates 1B1C, 2B2C, 3B3C etc. thus supplying the points 1D, 2D, 3D etc. The curve through these points is the versiera. The straight line μλ is its asymptote as Leibniz demonstrates.

To that end the transmutation theorem (proposition 7) is needed:

The quadrilateral area 1D1B3B3D2D1D (between 1B3B, the ordinates 1B1D, 3B3D and the new curve 1D2D3D) = 2 times the trilateral area 1CA3C2C1C (between the two straight lines A1C, A3C and the curve through 3C, 2C, 1C)

Proposition 11 explains how a finite area can be transformed into an infinitely long, but finite area: To cut in infinitely many ways a portion from an arbitrarily small curvilinear figure; to exhibit an infinitely long figure that is equal to it.

Leibniz constructs the lower part of illustration 2 in the following way:

Let μ be an arbitrary point of the curve, draw the tangent μλ. 2Bμ is the perpendicular on the tangent in μ. Let A be an arbitrary point of 2Bμ. AT is the perpendicular on 2Bμ in A and therefore parallel to the tangent μλ. The tangents in the points 3C, 4C of the curve 2Cμ cut AT in 3T, 4T. The perpendiculars 3T3D, 4T4D are drawn from 3T, 4T on the ordinates 3B3D, 4B4D.

Hence we get an infinite space comprehended by the two finite straight lines 2B2D, 2Bμ and by two infinite lines, that is, the curve 2D3D4D etc. and the asymptote μλ. Leibniz demonstrates three assertions.

Assertion 1: The curve (versiera) is infinite.

Demonstration: One can chose 4C in such a way that A4T is larger than any given finite straight line.

Assertion 2: It will never meet μλ.

Demonstration: Otherwise μλ would be the ordinate of the curve. It would meet a portion of AT cut by the tangent in μ. Yet, μλ cannot meet the parallel AT. Hence μλ meets 4Dδ nowhere. It is an asymptote. Though this is a clear statement we will see in chapter 4 that Leibniz gives also another answer to the question: What happens in the neighbourhood of an asymptote?

Assertion 3: The infinitely long area 2D2Bμλ…δ4D3D2D = 2 times the trilateral area 2C2Bμ3C2C.

The text of the demonstration is worth citing literally:

> *"This can only happen in such a way (so that nobody errs here) that one replaces μλ by (μ)λ, whereby the point (μ) is chosen a little bit over μ in an infinitely small distance (μ)μ. Thus the ordinate (μ)λ will be infinitely long or larger than any assignable 4B4D because μ(μ) is smaller than any assig-*

> *nable distance μ4B. Hence (μ)λ will not be an asymptote of the curve Dδ but will meet it somewhere, for example in λ though λ is distant by an infinite interval, that is, the straight line (μ)λ is indeed infinite or larger than any designable straight line, but not unbounded*[22]*".*

Two things are worth emphasizing: Leibniz uses the definitions of infinite and infinitely small that lead to actual infinity or zero. He applies his distinction between *infinite* and *unbounded*.

Gaston Pardies enthusiastically commented upon the matter of fact that an infinitely long area could be finite:

> *"There one will find the nature and the measure of asymptotic areas the knowledge of which is the most admirable thing of the world and which let see in the clearest way the dignity and spirituality of our soul. For only by the light of its mind, penetrating beyond the infinite, it discovers so clearly things that no sensible experience can teach it…These areas are of an actually infinite extension…Though the infinite is unmeasurable and innumerable, it is reduced to calculation and to the measure of geometry which our mind, still greater than it, is able to include…May I dare to go even further and say that in this demonstration one finds also the invincible proof of the existence of God?*[23]*".*

Reading this text Leibniz remained remarkably business-like. In the replaced version of the *scholium* to proposition 11 he commented:

> *"Pardies…attributed so much to considerations of this kind that he believed*

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22 "Hoc non aliter fieri potest, (ne quis hic erret) nisi pro recta μλ ponatur (μ)λ, puncto (μ) paulo supra punctum μ sumto, intervallo (μ)μ infinite parvo, ita ordinata (μ)λ erit longitudine infinita; major qualibet assignabili 4B4D, quia etiam μ(μ) quolibet assignabili intervallo μ4B minor est. Proinde (μ)λ non erit curvae Dδ asymptotos, sed ei occurrens alicubi ut in λ, licet λ absit infinito abhinc intervallo. Id est recta (μ)λ erit quidem infinita, sive quavis designabili major, sed non interminata." (LEIBNIZ, 2016 : 20).

23 "C'est là qu'on trouvera la nature et la mesure des espaces asymptotiques, dont la connaissance est la chose du monde la plus admirable, et qui fait voir le plus clairement la grandeur et la spiritualité de notre âme, puisque par la seule lumière de son esprit, pénétrant au-delà de l'infini, elle découvre si clairement des choses, que nulle expérience sensible ne lui peut apprendre…Ces espaces sont d'une étendue actuellement infinie…L'infini même tout immense et tout innombrable qu'il est, se réduit néanmoins au calcul et à la mesure de la géométrie, et que notre esprit, encore plus grand que lui, est capable de le comprendre… Oserai-je passer encore plus avant, et dire, que dans cette même démonstration on trouve aussi la preuve invincible de l'existence de Dieu?" (PARDIES, 1671 : A7-A8 (préface)).

> *that they supply a sufficiently effective argument for the spirituality of the soul…As to this action of mind by which we measure infinite areas it does not contain any extraordinary because it is based upon a certain fiction. After supposing a certain line, bounded indeed, yet infinite, it easily proceeds. Hence it is no more difficult than if we measured an area finite by its length*[24]*".*

In 1705 Leibniz evaluated Pardies's explanations more positively. On the 27th of December, 1705 he wrote to Johann Bernoulli[25]: "*Scio philosophos inter alia ex infiniti cognitione pro animae immortalitate argumentari, et quidem non male*" (I know that philosophers argue for the immortality of the soul among other things on the basis of the recognition of the infinite, and that not badly).

This statement reminds us of his interpretation of the binary arithmetic as an image of the creation of the world.

In the replaced version of the *scholium* to proposition 11 he continued[26]:

Magis mirarer, si quis ipsum spatium absolute interminatum inter curvam atque perfectam asymptoton interjectam ad finitum spatium reducere posset (I would be more surprised if anybody could reduce an absolutely unbounded area that lies between a curve and a perfect asymptote to a finite area.)

He emphasizes the difference between indivisible and infinitely small and between infinite and unbounded, respectively, saying:

Lineae interminatae magnitudo nullo modo geometricis considerationibus subdita est (The magnitude of an unbounded line is by no means subject to geometrical considerations).

His explanations remain utmost interesting though it must not be forgotten that he has deleted them:

> *"It cannot be said that the bounded line is the mean proportional between a point or minimal line and the unbounded or maximal line. But it can be said that a finite line truly and exactly is the mean proportional between a certain*

24 "Pardies…tantum hujusmodi meditationibus tribuebat, ut credret efficax satis argumentum praebere ad evincendam animae immaterialitatem…Quod hanc vero attinet mentis actionem qua spatia infinita metimur, ea nihil extraordinarium continent, cum fictione quadam nitatur, et supposita quadam linea terminata quidem, infinita tamen, nullo negotio procedat, unde non plus habet difficultatis, quam si finitum longitudine spatium metiremur." (LEIBNIZ, 2016: 60).

25 GM III, 778.

26 LEIBNIZ (2016), 60.

infinitely small and a certain infinite line[27]*"*.

Two examples, that is, the figure of angles and the conic hyperbola, may illustrate Leibniz's statement. Both examples have to do with Leibniz's distinction between the two types of infinite and an absolutely unbounded area.

The first example is given by proposition 14:

Figuram angulorum exhibere…ad quam figura constituatur, cujus portiones parallelis comprehensae sint ut anguli, modo portiones axis abscissae sive altitudines, sint ut sinus (To exhibit a figure of angles… constitute a figure with regard to it so that its portions comprehended by parallels are as the angles on condition that the portions cut off of the axis or heights are as the sines).

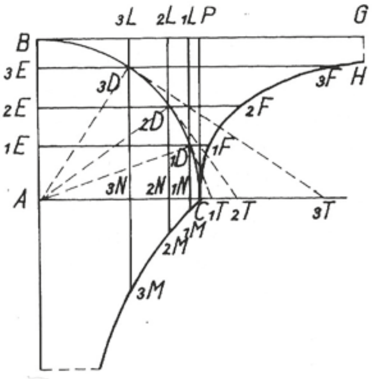


Illustration 3. The figure of angles (LEIBNIZ, 2016: 70).

Illustration 3 corresponds to illustration 2. This time the original curve is a quadrant through the points 1D, 2D, 3D etc. The constructed points 1F, 2F, 3F etc. form the new curve. According to the transmutation theorem the following equation holds:

Quadrilateral area CAnEnFC = 2 times trilateral area (sector) nDACnD
Hence (2x1DAC1D):(2x2DAC2D) = arc 1DC: arc 2DC = angle CA1D: angle CA2D
We get the corollary:
CABG…HFC : finite part CAEFC = right angle BAC : acute angle DAC

27 "Dici non potest lineam terminatam esse proportione mediam inter punctum seu lineam minimam et interminatam seu lineam maximam. At dici potest lineam finitam esse mediam proportione…vere exacteque inter quandam infinite parvam et quandam infinitam". (LEIBNIZ, 2016: 60).

Therefore Leibniz argued:

> *"The right angle seems to correspond to the absolutely unbounded area. Yet, I do not dare to affirm that therefore this area is reduced to a finite area for the reason cited in the scholium of proposition 11. Nevertheless that is certain that just the right angle either does not correspond to any area of the figure of angles or to the absolutely unbounded[28]".*

The second example concerns the paradox of the conic hyperbola based on proposition 18:

If $y^n x^m = a$, the ratio of the zone between the two ordinates…to the conjugated zone between the two corresponding abscissas is the same as n: m.

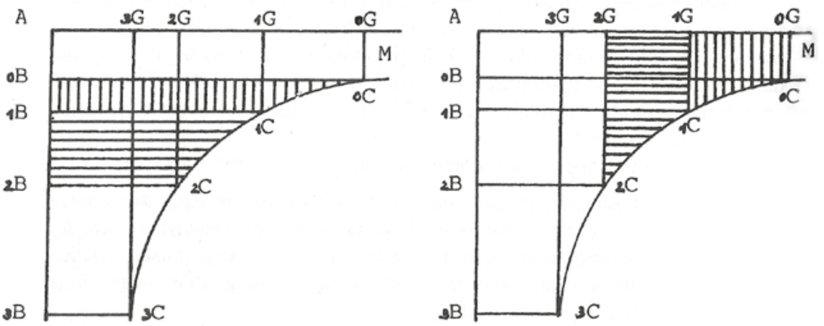


Illustration 4. The paradox of the conic hyperbola (KNOBLOCH, 1994: 277)

If n=m=1, that is, if it is a matter of the conic hyperbola, the equally hatched areas are equal. All horizontal zones up to A fill the area 2C2BAM2C, all vertical corresponding zones fill only the area 2C2GM2C. A part equals the whole. This is absurd, because Leibniz always presupposed the validity of the axiom: The whole is greater than its parts.

Leibniz commented upon this seeming paradox in the scholium belonging to proposition 22 and in his letter to Johann Bernoulli written on the 9th of July, 1698 which is of special interest here[29]. First of all Leibniz emphasizes

28 "Videtur angulus rectus respondere spatio absolute interminato, idque proinde reductum esse ad finitum attamen ob rationem prop. 11 schol. adductam id asserere non ausim. Illud tamen certum est ipsum angulum rectum aut nulli respondere spatio figurae angulorum, aut absolute interminato." (LEIBNIZ, 2016: 72).

29 GM III, 523-524.

the difference between indivisible and infinitely small and between unbounded and infinitely large, respectively, saying:

Respondi…neque sermonem hic fieri debere de spatio absolute interminato (I have answered that here one must not speak about the absolutely unbounded area). This area is comprehended by the two finite straight lines 2C2B, 2BA, the unbounded asymptote, and the unbounded curve. The last abscissa A0B does not equal 0 as if 0 falls upon A, the last ordinate 0B0C is not unbounded as if 0B0C falls upon the asymptote. A0B is infinitely small, 0B0C is infinitely large, but bounded. Leibniz literally continued:

> *"For here not the two unbounded areas mentioned above are equated with each other or are produced of the quadrilateral areas but both infinite areas must be quadrilateral and bounded. Yet, among us I add what I have already written long ago in the mentioned unpublished treatise, that one might doubt whether straight lines are really existent that are infinite by their magnitude but nevertheless bounded. That it suffices for the calculation in the meantime that they are imagined like the imaginary roots in algebra*[30]*."*

5.- Divergence of the harmonic series.

Leibniz's dealing with the hyperbola and logarithms is strongly influenced by Grégoire de St. Vincent's monograph about the *Quadrature of the circle*[31] :

30 "Neque enim duo spatia interminata supra dicta…aut sibi aequantur aut a quadrilineis… conflantur, sed spatia infinita ambo debent esse quadrilinea et terminata. Inter nos autem haec addo, quod et jam olim in dicto Tractatu inedito adscripsi, dubitari posse an lineae rectae infinitae magnitudine et tamen terminatae revera dantur. Interim sufficere pro calculo, ut fingantur, ut imaginariae radices in algebra."(POSER, 2016: 343 erroneously writes *determinatae*).

31 GRÉGOIRE (1647).

Illustration 5. Title page of Grégoire's monograph on the quadrature of the circle (GREGOIRE, 1647: Title page).

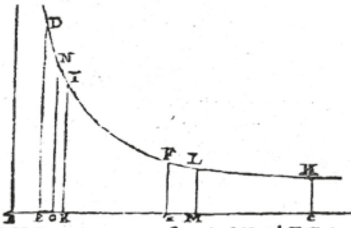
The copper plate engraving shows Archimedes demonstrating his theorem that the area of the circle is equal to the area of a certain triangle. The light rays passing through a square frame form a circle on the ground and illustrate the quadrature of the circle. Grégoire's fundamental result regarding the hyperbola was the insight that the ordinates form a geometric sequence if the hyperbolic areas form an arithmetic sequence.

Hence Leibniz introduces logarithms by means of a correspondence between a geometric and an arithmetic sequence[32] and refers to the relevant theorem 129 of Grégoire's book on the hyperbola[33] and uses the following theorem 130.

PROPOSITIO CXXIX.

SInt A B, B C aſymptoti hyperbolæ D F H, & DE, FG, H C parallelę aſymptoto: plano autem D E G F incommenſurabile ſit planum F G C H.

Dico rationem D E ad F G, toties multiplicare rationem F G, ad HC, quoties quantitas D E C F, continet quantitatem F G C H.



Demonſtratio.

SI enim ita non ſit; igitur ratio DE ad F G, ſæpius multiplicat rationem FG ad HC, quam DEFG planum, continet planum F GHC; vel contra: ſi primum, ponatur ratio IK ad FG, toties multiplicans rationem F G ad H C, quoties planum DEFG continet planum F G H C: minor ergo eſt ratio I K ad F G, b quam D E ad F G; adeoque & IK minor D E. Vlterius diuidatur per æquales partes planum F G H C, vt pars eius F L G M plano F G H C commenſurabilis, c minor ſit parte D E K I: tum F M ablatâ ex D E I K quoties poteſt, relinquat partem D N O E: quæ minor eſt parte D E K I, adeoque N O linea cadit inter DE, IK: ſi enim D E N O non minor ſit D E K I, ſed illi æqualis, vel illâ maior; poterit ex D E N O iterum auferri F G L E planum quod minus eſt D E I K ex plano D E F G, quod eſt contra ſuppoſitum. minor igitur eſt quantitas D E N O, quantitate D E I K, & N O linea eſt inter D E, I K. igitur ratio N O ad F G maior eſt ratione d I K ad F G. ac proinde ratio N O ad F G, magis multiplicat rationem F G ad H C, quam eamdem multiplicet ratio I K ad F G: quia verò F M quantitas communis eſt menſura planorum F G H C. N O F G. commenſurabilia ſunt plana e F G H C, adeoque toties multiplicat ratio N O ad F G, rationem F G ad H C, quoties planum N O F G continet planum F G H C: ſed & planum D E F G toties continet planum F G H C, quoties ratio I K ad F G, multiplicat rationem F G ad H C; ſæpius igitur planum N O F G continet planum F G H C, quam D E F G idem F G H C contineat: quod fieri non poteſt,

Illustration 6. Theorem 129 of book VI of Grégoire's *Quadratura circuli* (GRÉGOIRE, 1647: 596).

32 LEIBNIZ (2016), 198.

33 *Ibid.*, 216.

Let AB, BC be the asymptotes of the hyperbola DFH. Let DE, FG, HC be parallels to the asymptote. Let the area FGCH be incommensurable with the area DEGF. Then

$$\frac{DE}{FG} = \left(\frac{FG}{HC}\right)^{\frac{DEGF}{FGCH}}$$

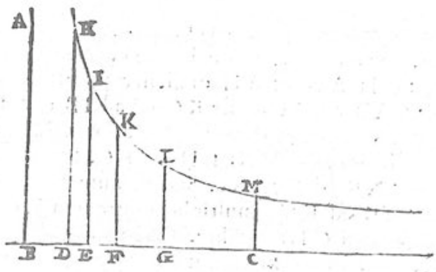
Theorem 130 is closely connected to this theorem:

PROPOSITIO CXXX.

SInt AB, BC aſymptoti hyperbolæ, & ponantur parallelæ aſymptoto DH, EI, FK, GL, CM, auferentes ſegmenta æqualia HE, IF, KG, LC.

Dico lineas HD, IE, KF, LG, MC eſſe in continua analogia.

Demonſtratio.



Ratio enim DH ad IE, toties multiplicat rationem IE ad MC, quoties ſuperficies HE a continetur in ſuperficie IC. ſed ſuperficies HE eſt quarta pars, verbi cauſa, plani HC; igitur ratio HD ad MC quadruplicata eſt rationis HD ad IE. Similiter ratio IE ad MC, oſtenditur eſſe triplicata rationis KF ad MC; Igitur ratio HD ad IE, eſt eandem cum ratione IE ad KF. eodem pacto demonſtratur, rationem IE ad KF, eſſe eandem cum ratione KF ad LG, & LG ad MC. Igitur continuant eandem rationem lineæ HD, IE, KF, LG, MC. Quod fuit demonſtrandum.

a 125. & 129. *huius.*

Illustration 7. Theorem 130 of book VI of Grégoire's *Quadratura circuli* (GRÉGOIRE, 1647:597).

Let AB, BC be the asymptotes of the hyperbola. Let DH, EI, KF, GL, CM be parallels to the asymptote subtracting equal segments HE, IF, KG, LC. Then the straight line HD, IE, KF, LG, MC form a continuous proportion:

HD:IE=IE:KF=KF:LG=LG:MC or $\frac{HD}{MC} = \left(\frac{HD}{IE}\right)^4$

Leibniz's own figure combines the hyperbola through V, P, N, M etc. with the construction of the corresponding logarithmic curve through A, R, S, T etc.:

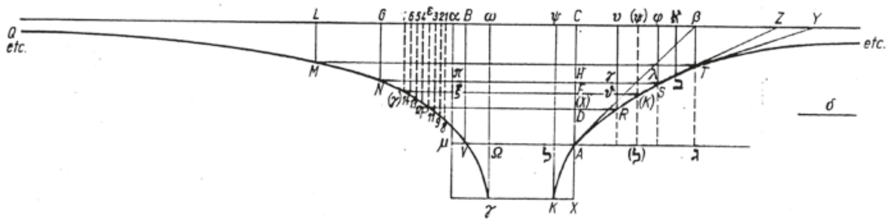


Illustration 8. Hyperbola and corresponding logarithmic curve through A, R, S, T etc. (LEIBNIZ, 2016: 202).

The logarithmic curve is defined by the equation: $\frac{CX}{CA} = \left(\frac{CD}{CA}\right)^{\frac{KX}{RD}}$. CX, CA are the abscissas, KX, RD are the corresponding ordinates.

By proposition 45 of his treatise *On the arithmetical quadrature of the circle* etc. Leibniz demonstrates the divergence of the harmonic series:

The infinitely long area of the conic hyperbola VACQ etc. MV is also infinite with regard to the area or larger than any assignable plane and hence the sum of the series of numbers of he harmonic progression decreasing to infinity, $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$ etc. that expresses the area of the space is also infinite.

It is worth emphasizing that Leibniz again says "larger than any assignable plane". We already know that this definition inevitably leads to actual infinity rejected by him in mathematics.

The hyperbolical areas are proportional to the logarithmic straight lines:

VADPV is proportional to RD, VAFNV is proportional to SF, VAHMV is proportional to TH etc. In order to get equality one has to multiply by AV, If CA=1 one gets perfect equality. There are two assertions that Leibniz demonstrates one after the other in order to demonstrate proposition 45.

First assertion: The complete hyperbolic, infinitely long area is proportional to the asymptote Cδ etc.

For VACQ…MV : finite area (for example) VADPV = infinite straight line Cβ etc. (or log 0 or log of the infinitely small) : finite straight line RD. This proportion is of the type:

x : finite = infinite : finite. We have discussed this demonstration method in chapter 2. Hence the first quantity x sought must be infinitely large.

What is interesting here is again Leibniz's use of zero as infinitely small quantity and his statement about the asymptote that seems to be incompatible with his explanations dealt with in chapter 3. Yet, the contradiction disappears of Leibniz considered the asymptote as an unbounded, infinite straight line in chapter 3, while here he is speaking about a bounded, infinite straight line.

Second assertion: The straight line Cβ etc. is an asymptote, which is (*seu*), it can meet (*occurrere posse*) the logarithmic curve ARST only after an infinite interval (*infinito abhinc intervallo*).

Demonstration: Leibniz shows that the logarithmic ordinate can become larger than any given linear segment *a*. In other words if RD is an arbitrary logarithmic ordinate, one finds another linear segment FS (ordinate) between RD and Cβ etc. that is larger than any given straight line *a*.

The demonstration consists of four steps.

(1) Let δ be larger than a, δ:RD is an arbitrary ratio. There is a line CF=φS so that

$$\left(\frac{CD = vR}{CA}\right)^{\frac{\delta}{RD}} = \frac{CF}{CA}$$

(2) RD = log $\frac{CD}{CA}$, hence δ=log $\frac{CF}{CA}$. FS is drawn from F and meets the logarithmic curve, FS=δ.

(3) FS=δ > a according to the presupposition. The largest of all ordinates Cβ etc. is the asymptote or infinite.

(4) The area of the infinitely long hyperbolic space VACQ…MV that corresponds to this straight line Cβ etc. is infinite.

Leibniz was rather proud about his "very clear demonstration" (*liquidissima demonstratio*)[34]. He did not yet know Pietro Mengoli's far simpler, more elegant proof of the divergence of the harmonic series that the Italian mathematician had already published in 1650[35].

34 LEIBNIZ (2016), 222.

35 MENGOLI (1650), 3-4 (preface).

Mengoli based his demonstration on the characteristic property of three neighbouring terms A, B, C of the harmonic series: $\frac{A}{C} = \frac{A\text{-}B}{B\text{-}C}$. A+C<2B, hence A+B+C>3B. This inequality can be used for an estimation of the sum of three neighbouring terms: $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}>1, \frac{1}{5}+\frac{1}{6}+\frac{1}{7}>\frac{1}{2}, \frac{1}{8}+\frac{1}{9}+\frac{1}{10}>\frac{1}{3}$ etc. This estimation reproduces the terms of the harmonic series. Hence Mengoli could conclude:

The first three elements are larger than 1, the next nine elements are larger than1, the next 27 elements are larger than 1, the next 81 elements are larger than 1 etc. If the sum should be larger than 4, one needs $3^1+3^2+3^3+3^4=120$ elements. Any given quantity can be surpassed. The series is divergent.

Leibniz became acquainted with Mengoli's proof during his second visit to London between the 18th and the 29th of October, 1676. John Collins had sent the so-called *Historiola* or *Collectio* to Henry Oldenburg in May/June 1676 containing excerpts from his correspondence with James Gregory[36]. Therein he explains Mengoli's method saying[37]: *so that it will not be difficult to say how many such fractions shall be greater than any number assigned.*

Leibniz met Collins and made Latin or English written excerpts from the correspondence between Collins and Gregory known as *Excerpta ex commercio epistolico inter Collinium et Gregorium*[38] . Leibniz translated Collins's explanations of Mengoli's proof nearly completely into Latin[39]: *Ergo simul jungendo semper dici poterit quotnam fractiones simul sumtae sint numero quovis dato majores*. (Hence combining them it will be always possible to say how many fractions taken together are larger than any given number.) Leibniz commented: *Ingeniose* (ingenious).

36 LEIBNIZ (since 1923), III,1, 433-484.

37 *Ibid.*, III, 1, 437.

38 *Ibid.*, III, 1, 485-503.

39 *Ibid.*, III, 1, 486-487.

6.-Epilogue.

I would like to conclude with a citation taken from Godefrey Harold Hardy's autobiography[40]:

> *"Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not. 'Immortality' may be a silly word, but probably a mathematician has the best chances of achieving whatever it may mean".*

7.- Acknowledgement.

I would like to thank my colleague and friend John C. Stillwell who carefully read this paper.

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40 HARDY (1969), 81.

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