

ON GEOMETRIC QUANTIZATION OF b -SYMPLECTIC MANIFOLDS

VICTOR W. GUILLEMIN, EVA MIRANDA, AND JONATHAN WEITSMAN

ABSTRACT. We study a notion of pre-quantization for b -symplectic manifolds. We use it to construct a formal geometric quantization of b -symplectic manifolds equipped with Hamiltonian torus actions with nonzero modular weight. We show that these quantizations are finite dimensional T -modules.

1. INTRODUCTION

Let (M, ω) be an integral symplectic manifold, and let (\mathcal{L}, ∇) be a line bundle \mathcal{L} with connection ∇ of curvature ω . The quadruple $(M, \omega, \mathcal{L}, \nabla)$ is called a *prequantization* of (M, ω) , which morally should give rise to a geometric quantization $Q(M)$ of M . A complication arises in that all known constructions of $Q(M)$ require additional data, a polarization of M ; such a polarization may be real, a foliation of M by Lagrangian subvarieties, or else complex, a complex or almost complex structure on M compatible with ω . It is generally believed, and in many cases verified, that the quantization $Q(M)$ should be independent of the polarization. However there is no theorem guaranteeing that this should be the case.

Work of Kontsevich [Kon] extending deformation quantization to Poisson manifolds raises the issue as to whether any of the constructions above has any relevance in the Poisson setting. If (M, π) is a Poisson manifold, it is not clear what the analog of (\mathcal{L}, ∇) should be, let alone what one would mean by a polarization. The purpose of this paper is to try to begin developing some examples, guided by symplectic geometry, where a sensible theory of geometric quantization of Poisson manifolds can be proposed. Hopefully the repertoire of examples may be a guide to a theory of geometric quantization of Poisson manifolds.

To do this we focus on a special class of Poisson manifolds that have two helpful properties. First, we require that the Poisson structure be symplectic on the complement of a real hypersurface $Z \subset M$ and have a simple zero on Z . Such b -symplectic manifolds have been the subject of intensive study [GMP, GMPS2] and are by now well understood geometrically. And second, we require that the manifold have a Hamiltonian action of a torus with a certain nondegeneracy condition (nonzero modular weight; see Theorem 3.5 below for the precise definition). The presence of these two conditions allows us to bring tools from symplectic geometry to bear on the problem. One concept which, as far as we know, has not been investigated at all in the b -symplectic setting and which will play an essential ingredient in describing how to quantize these manifolds, is the concept of “integrality” for the b -symplectic form ω (or, alternatively of “pre-quantizability” for the pair, (M, ω)); and one of the main goals of this paper will be to provide an appropriate definition of this concept and explore some of its consequences. We then show that a natural functoriality condition for quantization (“formal geometric quantization”) determines what the quantization of the manifold should be.

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Formal geometric quantization was studied in [W] in the context of the quantization of Hamiltonian T -spaces with proper moment map. We will see that where M is a compact b -symplectic manifold, with a Hamiltonian torus action of nonzero modular weight, the manifold $M - Z$ is such a space, and that an analog of formal geometric quantization for b -symplectic manifolds yields essentially the quantization of $M - Z$. However, in the b -symplectic case, there is a surprise; unlike in the case of noncompact manifolds with proper moment map, where the quantization is always infinite dimensional, though with finite multiplicities, in the case of a b -symplectic manifold, the quantization is always a *finite dimensional* virtual T -module. This raises the question of whether it is the index of a Fredholm operator.

2. b -SYMPLECTIC MANIFOLDS

Let M be a compact, connected, oriented n -dimensional manifold, $Z \subseteq M$ a closed hypersurface and $f : M \rightarrow \mathbb{R}$, $f|_Z = 0$, a defining function for Z . We recall (see [GMP]) that a b -symplectic form on M is a 2-form of the form

$$(2.1) \quad \omega = \frac{df}{f} \wedge \mu + \gamma$$

with $\mu \in \Omega^1(M)$ and $\gamma \in \Omega^2(M)$, which is symplectic in the usual sense on $M - Z$, and is symplectic at $p \in Z$ as an element of $\wedge^2({}^bT_p^*)$ where ${}^bT_p^*$ is the span of T_p^*Z and the “ b -form” $\left(\frac{df}{f}\right)_p$.

Some properties of the form (2.1) which we will need below are:

- (1) Let $\iota : Z \rightarrow M$ be the inclusion map. Then $\iota^*\mu =: \mu_Z$ is an intrinsically defined one-form on Z
- (2) f is not intrinsically defined but replacing f by $f = hg$ with $h > 0$ on Z , $\frac{df}{f} = \frac{dg}{g} + \frac{dh}{h}$ so $\frac{df}{f}$ is intrinsically defined mod $\Omega^1(M)$. Moreover

$$(2.2) \quad \omega = \frac{dg}{g} \wedge \mu + \gamma'$$

where

$$(2.3) \quad \gamma' = \gamma + \frac{dh}{h} \wedge \mu$$

- (3) Since $d\omega = 0 = -\frac{df}{f} \wedge d\mu + d\gamma$ the forms $\iota_Z^*\mu = \mu_Z$ and $\iota_Z^*\gamma = \gamma_C$ are closed.
- (4) For $\omega_p, p \in Z$, to be symplectic in the sense described above, μ_Z has to be nonvanishing on Z and hence, by item 3, defines a foliation of Z . Moreover it also requires that if L is a leaf of this foliation $\iota_L^*\gamma$ is a symplectic form on L . In addition, by (2.2) $\iota_L^*\gamma' = \iota_L^*\gamma$ so this symplectic structure on L is intrinsically defined.

Turning next to “pre-quantization” we note that, since μ_Z is intrinsically defined, so is its cohomology class, $[\mu_Z]$ and by (2.3) the cohomology class $[\gamma]$ is intrinsically defined as well. Moreover the Melrose-Mazzeo isomorphism

$$(2.4) \quad {}^bH^2(M, \mathbb{R}) \rightarrow H^2(M, \mathbb{R}) \oplus H^1(Z, \mathbb{R})$$

maps $[\omega]$ onto $[\gamma] \oplus [\mu_Z]$, hence a natural definition of “integrality” for ω , i.e. of “ $[\omega] \in {}^b H^2(M, \mathbb{Z})$ ” is to require that $[\mu_Z]$ be in $H^1(Z, \mathbb{Z})$ and $[\gamma]$ be in $H^2(M, \mathbb{Z})$. We will list a few consequences of this assumption.

(1) The integrality of γ implies that there exists a circle bundle,

$$(2.5) \quad \pi : V \rightarrow M$$

and a one form α on V such that

$$(2.6) \quad d\alpha = \pi^*\gamma,$$

$$(2.7) \quad \iota(X)\alpha = 1$$

and

$$(2.8) \quad \mathcal{L}_X\alpha = 0$$

where X is the generator of the circle action on V .

(2) The integrality of μ_Z implies that there exists a map, $\varkappa : Z \rightarrow S^1$, with the property

$$(2.9) \quad \mu_Z = \varkappa^* d\theta$$

Therefore, in particular, the foliation of Z that we described above is that defined by the level sets of \varkappa , and hence since Z is compact, the leaves of this foliation are compact as well. Moreover if we let v be the vector field on Z defined by

$$(2.10) \quad \iota_v \mu_Z = 1 \text{ and } \iota_v \gamma_t = 0$$

then v and $\frac{\partial}{\partial \theta}$ are \varkappa -related. Therefore if we let $\phi : Z \rightarrow Z$ be the map $\exp 2\pi v$ and let L be a leaf of the foliation defined by \varkappa , Z can be identified with the mapping torus

$$(2.11) \quad L \times [0, 2\pi] / \sim$$

where “ \sim ” is the identification

$$(2.12) \quad (p, 0) \sim (\phi(p), 2\pi)$$

3. GROUP ACTIONS

As in §2 let $f : (M, Z) \rightarrow (\mathbb{R}, 0)$ be a defining function for Z and let $Z_i, i = 1, 2, \dots, k$, be the connected components of Z . We will denote by ${}^b C^\infty(M)$ the space of functions which are C^∞ on $M - Z$ and near each Z_i can be written as a sum,

$$(3.1) \quad c_i \log |f| + g$$

with $c_i \in \mathbb{R}$ and $g \in C^\infty(M)$. Now let T be a torus and $T \times M \rightarrow M$ an action of T on M .¹ We will say that this action is *Hamiltonian* if the elements, $X \in \mathfrak{t}$ of the Lie algebra of T satisfy

$$(3.2) \quad \iota(X_M)\omega = d\phi, \phi \in {}^b C(M),$$

in other words:

¹The material in this section is taken more or less verbatim from [GMPS2].

$$(3.3) \quad \iota(X_M)\omega = c_i(X)d(\log|f|) + dg$$

in a tube neighborhood of Z_i for $g \in C^\infty(M)$.

The map

$$(3.4) \quad v_i : X \in \mathfrak{t} \rightarrow c_i(X)$$

is called the modular weight of Z_i and depends on i ; however, one can show ([GMPS2], §2.3)

Theorem 3.5 ([GMPS2]). *The v_i 's are either zero for all i or non-zero for all i .*

In this paper we will assume that the latter is the case, in which case, for fixed i we can choose $X_i \in \mathfrak{t}$ such that $c_i(X_i) = 1$. Hence, by (2.11) $\exp 2\pi X_i$ maps the leaves, L_i , of the null foliation of Z_i onto themselves, and thus $\exp 2\pi X_i = \exp 2\pi Y_i$ where $Y_i \in \mathfrak{t}$ is tangent to the leaves of this foliation. Thus, replacing X_i by $X_i - Y_i$ we can assume that $\exp 2\pi X_i$ is the identity map on Z . In other words the map

$$S^1 \times L \rightarrow Z, (\theta, p) \rightarrow (\exp \theta X_i)p$$

is a diffeomorphism, and hence the mapping tori (2.11) are all products: $L \times S^1$. Moreover if we split T into a product

$$T = T_i \times S^1$$

where T_i is the subgroup of T fixing the leaves of the null-foliation of Z_i , the action of T on Z_i is just the product of the canonical action of S^1 on S^1 and of T_i on L .

4. FORMAL GEOMETRIC QUANTIZATION

4.1. Compact symplectic manifolds. Let (M, ω) be a compact symplectic manifold and let (\mathcal{L}, ∇) be a line bundle with connection of curvature ω . Choose an almost complex structure J compatible with the symplectic structure. Then this almost complex structure gives \mathcal{L} the structure of a complex line bundle, and by twisting the spin- \mathbb{C} Dirac operator on M by \mathcal{L} we obtain an elliptic operator $\bar{\partial}_{\mathcal{L}}$. Since M is compact, $\bar{\partial}_{\mathcal{L}}$ is Fredholm, and we define the geometric quantization $Q(M)$ by

$$Q(M) = \text{ind}(\bar{\partial}_{\mathcal{L}})$$

as a virtual vector space.

If M is equipped with a Hamiltonian action of a torus T , the action lifts to \mathcal{L} , and one can choose the almost complex structure to be T -invariant. Then the quantization $Q(M)$ is a finite-dimensional virtual T -module, and it satisfies the following principle.

For $\xi \in \mathfrak{t}^*$, we denote by $M//_{\xi}T$ the reduced space of M at ξ . Also, for α a weight of T , and V a virtual T -module, denote by V^{α} the submodule of V of weight α .

Theorem 4.1 (Quantization Commutes with Reduction ([Mei], [V1, V2])). *Let α be a weight of T . Then*

$$(4.2) \quad Q(M)^{\alpha} = Q(M//_{\alpha}T).$$

In other words,

$$(4.3) \quad Q(M) = \bigoplus_{\alpha} Q(M//_{\alpha}T)\alpha$$

as virtual T -modules.

Remark 4.4. Both Theorem 4.1 and equation (4.3) are strictly speaking valid only for regular values of the moment map. In the case where α is a singular value of the moment map, the singular quotient must be replaced by a slightly different construction using a shift of α . For details, we refer the interested reader to [Mei] and [V1, V2]. For a review of this subject, see [S]. A similar caution applies in the case of noncompact Hamiltonian T -spaces and of b -symplectic manifolds below.

Remark 4.5. If (M, ω) is a compact, integral symplectic manifold, one can always find a line bundle \mathcal{L} with connection ∇ of curvature ω , and the quantization $Q(M)$ is independent of this choice. We therefore suppress the line bundle and connection and simply write $Q(M)$ for the quantization.

4.2. Noncompact Hamiltonian T -spaces. If we now consider the case where M is not compact, the analysis above cannot be carried out, since the operator $\bar{\partial}_{\mathcal{L}}$ is elliptic, but no longer Fredholm. Instead, in [W] (see also [P]), equation (4.2) is used to *define* the quantization of such Hamiltonian T -spaces, where the moment map is proper, so that the reduced spaces are compact and the right hand side of equation (4.2) makes sense.²

Definition 4.6 ([W]). Let M be a Hamiltonian T -space with integral symplectic form. Suppose the moment map for the T -action is proper. Let V be an infinite-dimensional virtual T -module with finite multiplicities. We say

$$V = Q(M)$$

if for any compact Hamiltonian T -space N with integral symplectic form, we have

$$(4.7) \quad (V \otimes Q(N))^T = Q((M \times N) //_0 T).$$

In other words, as in (4.3),

$$Q(M) = \bigoplus_{\alpha} Q(M //_{\alpha} T) \alpha,$$

where the sum is taken over all weights α of T .³

Note that the fact that the moment map is proper implies that the reduced space $(M \times N) //_0 T$ is compact for any compact Hamiltonian T -space N , so that the right hand side of equation (4.7) is well-defined.

In other words, we have used Theorem 4.1 to give us enough functoriality to force a definition of the quantization in this case, despite the fact that the elliptic operator $\bar{\partial}_{\mathcal{L}}$ is not Fredholm.

4.3. b -symplectic manifolds. Suppose now that M is a compact b -symplectic manifold, with integral b -symplectic form as above. Suppose that it is equipped with a Hamiltonian action of a torus T with *nonzero modular weight*. Then, in analogy with Definition 4.6, we define

Definition 4.8. Let V be a virtual T -module with finite multiplicities. We say

$$V = Q(M)$$

if for any compact Hamiltonian T -space N with integral symplectic form, we have

²It is also possible in this case to use index theory to define the quantization; see [B, P]

³Again, care must be taken about singular values

$$(4.9) \quad (V \otimes Q(N))^T = Q((M \times N)//_0 T).$$

In other words,

$$Q(M) = \bigoplus_{\alpha} Q(M//_{\alpha} T),$$

where the sum is taken over all weights α of T .⁴ In this b -symplectic case the condition that the modular weight be nonzero guarantees that the reduced space $(M \times N)//_0 T$ is a compact and symplectic (and in the generic case, a manifold) for any compact Hamiltonian T -space N ; so that as in the case of noncompact Hamiltonian T -spaces, the right hand side of equation (4.9) is well-defined.

Another way to say this is to note that

$$Q(M) = Q(M - Z)$$

where $Q(M - Z)$ is the quantization of the noncompact Hamiltonian T -space $M - Z$. The fact that the modular weights on M are nonzero insures that the moment map on $M - Z$ is proper.

The main result of this paper is that $Q(M)$ is a *finite* virtual T -module. To see this, we must return to the geometry of the manifold M in the vicinity of the hypersurface Z .

5. SYMMETRY PROPERTIES

We have shown above that if the modular weight v_i of Z_i is non-zero then, in the vicinity of Z_i , M is just a product

$$(5.1) \quad Z_i \times (-\epsilon, \epsilon)$$

and $Z_i = S^1 \times L$.

We will show below that this can be slightly strengthened (see also [GMPS1]): the b -symplectic form on $Z \times (-\epsilon, \epsilon)$ can be taken to be the two-form

$$(5.2) \quad -d\theta \wedge \frac{dt}{t} + \gamma_L$$

where γ_L is the symplectic form on L and $-d\theta \wedge \frac{dt}{t}$ the standard b -symplectic form on $S^1 \times (-\epsilon, \epsilon)$.

To see this, we note that, under the hypotheses above, we can assume that the symplectic form (2.1) has the form

$$(5.3) \quad d\theta \wedge \frac{dt}{t} + \gamma_L + d\theta \wedge \beta$$

Moreover if $\iota \left(\frac{\partial}{\partial \theta} \right) \beta = h$ we can replace β by $\beta - h d\theta$ in the expression above and arrange that $\iota \left(\frac{\partial}{\partial \theta} \right) \beta = 0$. Hence since the action of S^1 on M is Hamiltonian

$$\iota \left(\frac{\partial}{\partial \theta} \right) \omega = d(\log |t| + \rho)$$

⁴Again, adjusting for singular values as described in Remark 4.4.

for some $\rho \in C^\infty(M)$ and hence

$$(5.4) \quad \beta = d\rho$$

Consider now the one parameter family of forms

$$(5.5) \quad d\theta \wedge \frac{dt}{t} + \gamma_L - sd(\rho d\theta)$$

for $0 \leq s \leq 1$. For $s = 1$ this form is ω and for $s = 0$ the form (5.2). Moreover for ϵ small and $-\epsilon < t < \epsilon$ the first summand of (5.5) is much larger than the third so the form (5.5) is b -symplectic and for all s

$$[\omega_s] = [\omega_0]$$

so we can apply b -Moser theorem (see [GMP]) to conclude that ω_0 and ω_1 are equivariantly symplectomorphic.

Finally note that the 2-form, γ_L , depends in principle on t .
However the inclusion map

$$\iota : L \rightarrow L \times (-\epsilon, \epsilon), p \rightarrow (p, 0)$$

and the projection map

$$\pi : L \times (-\epsilon, \epsilon) \rightarrow L, (p, e) \rightarrow p$$

induce isomorphisms on cohomology; hence $[\mu] = [\pi^* \iota^* \mu]$. Therefore, by Moser, we can assume that $\mu = \pi^* \iota^* \mu$ i.e. that μ is just a symplectic 2-form on L itself.

6. FORMAL QUANTIZATION OF b -SYMPLECTIC MANIFOLDS

We now prove the main result of this paper.

Theorem 1. *Let M be an integral b -symplectic manifold equipped with a Hamiltonian T -action with nonzero modular weight. Then the formal geometric quantization $Q(M)$ is a finite dimensional T -module.*

Proof We will show that if we take for the quantization of $N = Z_u \times (-\epsilon, \epsilon)$ the sum:

$$(6.1) \quad \bigoplus Q(N//_\alpha T)_\alpha, \alpha \in \mathbb{Z}(T)$$

then the virtual vector spaces $Q(N//_\alpha T)$, are all zero and hence so is this sum.

To define $Q(N//_\alpha T)$ one has to define orientations on the $N//_\alpha T$ and to do this consistently one has to define an orientation on N . The natural way to do so would be to assign to each connected component of $N-Z$ the orientation defined by the symplectic form, ω ; however, because of the factor $\frac{df}{f}$, in the formula (2.1) the symplectic orientations on adjacent components of the space $N-Z$ don't agree; and, in particular, if $N = Z_i \times (-\epsilon, \epsilon)$ and $\mathbb{R} \times \mathfrak{t}_i$ is the Lie algebra of $S^1 \times T_i$ the moment map

$$\phi : N \rightarrow \mathbb{R} \times \mathfrak{t}_i^*$$

associated with the action of $S^1 \times T_i$ on N is the map

$$(\theta, p, t) \in S^1 \times L \times (-\epsilon, \epsilon) \rightarrow (\log |t|, \phi_i(p))$$

where $\phi_i : L \rightarrow \mathfrak{t}_i^*$ is the moment map associated with the T_i action on L . Thus for a weight, (c, α_i) of $S^1 \times T_i$, the reduced space

$$\phi^{-1}(c, \alpha_i)/S^1 \times T_i$$

consists of two copies of the reduced space

$$\phi_i^{-1}(\alpha_i)/T_i$$

with opposite orientations, so the quantization of this space is a virtual vector space $V_+ \oplus V_-$ with $V_- = -V_+$. \square

We end the paper with a conjecture.

Conjecture 6.2. *There exists a natural Fredholm operator on M whose index gives $Q(M)$.*

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DEPARTMENT OF MATHEMATICS, MIT, CAMBRIDGE, MA 02139
E-mail address: vwg@math.mit.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITAT POLITÈCNICA DE CATALUNYA AND BGSMATH, BARCELONA, SPAIN
E-mail address: eva.miranda@upc.edu

DEPARTMENT OF MATHEMATICS, NORTHEASTERN UNIVERSITY, BOSTON, MA 02115
E-mail address: j.weitsman@neu.edu