

INTEGRABLE SYSTEMS AND CLOSED ONE FORMS

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ABSTRACT. In the first part of this paper we revisit a classical topological theorem by Tischler [12] and deduce a topological result about compact manifolds admitting a set of independent closed forms proving that the manifold is a fibration over a torus. As an application we reprove the Liouville theorem for integrable systems asserting that the invariant sets or compact connected fibers of a regular integrable system is a torus. We give a new proof of this theorem (including the non-commutative version) for symplectic and more generally Poisson manifolds.

1. INTRODUCTION

A Liouville integrable system on a symplectic manifold is given by n functions f_i (constants of motion) which pairwise commute $\{f_i, f_j\} = 0$ with respect to the Poisson bracket defined as $\{f, g\} = \omega(X_f, X_g)$ where X_f and X_g denote the Hamiltonian vector fields associated to the smooth functions f and g . The set of first integrals $F = (f_1, \dots, f_n)$ is often referred to as *moment map*. This notion is related to classical integration of the equations of motion and can be generalized to other geometrical settings such as that of Poisson manifolds but also to non-geometrical ones like non-Hamiltonian integrable systems.

Liouville-Mineur-Arnold theorem on integrable systems asserts that a neighborhood of a compact invariant subset (Liouville torus) of an integrable system on a symplectic manifold (M^{2n}, ω) is fibred by other Liouville tori. Furthermore, the symplectic form can be described as the Liouville symplectic structure on $T^*(\mathbb{T}^n)$ in adapted coordinates to the fibration which can be described using the cotangent lift of translations of the base torus. In particular in adapted coordinates (action-angle) the moment map is indeed a moment map of a Hamiltonian toric action. This theorem admits generalizations to the Poisson setting [7, 6].

The fact that the fibers of the moment map are tori is a key point in the theory and it is a purely topological result. This topological result often attributed to Liouville [8] was indeed probably first observed by Einstein [3]¹. Probably, the best well-known proof of this fact (see for instance [2]) uses the existence of a toric action associated to the joint flow of the distribution of the Hamiltonian vector fields of the first integrals and the identification of the Liouville tori as orbits of this action. The classical proof is rich because it describes not only the Liouville torus but closeby tori but somehow diverts from the topological nature of Liouville fibers.

Given an integrable system the set of n 1-forms associated to the first integrals df_i defines a set of closed 1-forms. These closed 1-forms are constant on the fibers of the associated *moment map*. In this paper we pay attention to the following fact, the regular fibers of an integrable system are

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¹We thank Alain Albouy for pointing this out.

naturally endowed with n independent closed 1-forms defined using symplectic duality from the constants of motion. This fact together with a generalization of a result of Tischler for compact manifolds admitting k closed forms yields a new proof of the classical Liouville theorem. This new topological proof of Liouville theorem works in the Poisson and non-commutative setting as well. In this paper, we concentrate on topological aspects of foliations associated to a set of k -closed one forms in the framework of integrable systems. We suspect that a detailed dual proof of the action-angle theorem by Liouville-Mineur-Arnold theorem may be obtained as a consequence of our dual viewpoint². We plan to apply these techniques to generalization of cosymplectic manifolds [5] in the future.

Organization of this paper: In Section 1 we revisit a theorem by Tischler concerning manifolds endowed with non-vanishing closed one-forms and prove that a manifold admitting k independent one-forms fibers over a torus. We apply these results in Section 2 to reprove Liouville theorem for integrable systems in the symplectic and Poisson settings for commutative and non-commutative integrable systems.

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2. A TOPOLOGICAL RESULT FOR MANIFOLDS WITH CLOSED FORMS

In this first section we prove a generalization of a result of Tischler on manifolds with closed forms. We start recalling the following theorem by Tischler:

Theorem 2.1 (Tischler theorem). *Let M^n be a compact manifold admitting a nowhere vanishing closed 1-form ω , then M^n is a fibration over S^1 .*

In this first section we prove the following generalization of Tischler's theorem which is stated without proof for foliations without holonomy in his paper [12]:

Theorem 2.2. *Let M^n be a compact connected manifold admitting k closed 1-forms $\beta_i, i = 1, \dots, k$ which are linearly independent at every point of the manifold, then M^n fibers over a torus \mathbb{T}^k .*

We will need the following lemma:

Lemma 2.3 (Ehresmann lemma [4]). *A smooth mapping $f : M^m \rightarrow N^n$ between smooth manifolds M^m and N^n such that:*

- (1) *f is a surjective submersion, and*
- (2) *f is a proper map*

is a locally trivial fibration.

²We thank Alain Chenciner for inspiring discussions in this direction.

Proof. (of Theorem 2.2) We start by proving that the cohomology classes in $H^1(M^n, \mathbb{R})$, $\{[\beta_i]\}_{i=1}^k$ are all different. Assume the opposite β_i and β_j with $i \neq j$ such that $[\beta_i] = [\beta_j]$. Then there exists $f \in C^\infty(M^n)$ such that

$$(1) \quad \beta_i = \beta_j + df.$$

Since the 1-forms β_i are linearly independent the k -form $\beta_1 \wedge \dots \wedge \beta_k$ is nowhere vanishing. Using equation 1 we obtain

$$(2) \quad \beta_i \wedge \beta_j = \beta_i \wedge (\beta_i + df) = \beta_i \wedge df.$$

But note that due to Weierstrass theorem f has a maximum and a minimum on a compact manifold, thus $\beta_i \wedge \beta_j$ vanishes at these points (where $df = 0$). This contradicts the fact that $\beta_1 \wedge \dots \wedge \beta_k$ is nowhere vanishing.

Denote b_1 the first Betti number of M^n and θ the usual angular coordinate in S^1 . It is well known that there exist b_1 maps $g_j : M^n \rightarrow S^1$ such that the set of 1-forms $g_j^*(d\theta)$ define a set of cohomology classes $[g_j^*(d\theta)]$ which is a basis of $H_{DR}^1(M^n, \mathbb{R})$. With this basis, we can express β_i as:

$$\beta_i = \sum_{j=1}^{b_1} a_{ij} \nu_j + dF_i, \text{ for } i = 1, \dots, k.$$

Using the argument on Tischler theorem proof [12], we can choose appropriate $q_{ij} \in \mathbb{Q} \forall i, j$, such that $\tilde{\beta}_i = \sum_{j=1}^p q_{ij} \nu_j + dF_i$ are still non-singular and independent. Taking suitable $N_i \in \mathbb{Z}$ we obtain forms $\beta'_i = N_i \tilde{\beta}_i$ such that

$$\beta'_i = \sum_{j=1}^p k_{ij} \nu_j + dH_i,$$

where $k_{ij} = N_i q_{ij} \in \mathbb{Z}$ and $H_i = N_i F_i \in C^\infty(M)$. Of course, they are also non singular and independent.

Without loss of generality we can assume $dH_i = 0$. Indeed, the image $H_i \in C^\infty(M^n)$ is contained in a closed interval because M^n is compact. Functions H_i quotients to S^1 with a projection π , and we can redefine $g_i := g_i + \pi \circ H_i$ for $i = 1, \dots, k$.

Recall that the basis ν_j is defined as $\nu_j = g_j^*(d\theta) = d(\tilde{g}_j)$, with $\tilde{g}_j = \theta \circ g_j$. Hence the forms β'_i can be written

$$\beta'_i = d\left(\sum_{j=1}^p p_{ij} \tilde{g}_j\right).$$

If we define the functions $\theta_i = \sum_{j=1}^p p_{ij} \tilde{g}_j$, then the induced mappings on the quotient $\tilde{\theta}_i : M^n \rightarrow S^1$ are k submersions of M^n to S^1 . Consider

$$\begin{aligned} \Theta : M^n &\longrightarrow S^1 \times \dots \times S^1 = \mathbb{T}^k \\ p &\longmapsto (\tilde{\theta}_1(p), \dots, \tilde{\theta}_k(p)). \end{aligned}$$

Since the forms β'_i are independent in $H^1(M^n, \mathbb{R})$ this implies $d\theta_i$ are independent seen as one-forms from M^n to \mathbb{R}^k and so $d\tilde{\theta}_i$ are also independent into \mathbb{T}^k , this implies that Θ is a surjective submersion. Since M^n is compact, we can apply Ehresmann lemma (Lemma 2.3) and Θ defines a locally trivial fibration. \square

When $k = n$ we obtain the following as a corollary:

Corollary 2.4. *Let M^n be a compact connected manifold admitting n closed 1-forms $\beta_i, i = 1, \dots, n$ which are linearly independent at every point of the manifold, then M^n is diffeomorphic to a torus \mathbb{T}^n .*

Proof. Applying Theorem 2.3, M^n fibers over a torus \mathbb{T}^n . From the invariance of domain theorem it is an immersion because the target space is n -dimensional too. Thus Θ defines a covering map but since M^n is connected it defines a diffeomorphism

$$M^n \cong \mathbb{T}^n.$$

□

3. APPLICATIONS TO REGULAR INTEGRABLE SYSTEMS

One of the best well-known theorem of integrable systems is *Liouville-Mineur-Arnold theorem* which roughly speaking asserts that the fibers of the map F defined by the first integrals describe a fibration by tori (if the ambient manifold is compact and the fibers are regular) and also that there exists privileged coordinates (called action-angle coordinates) in which the symplectic form can be expressed in a unique Darboux chart in a neighborhood of one of these tori.

The first statement of Liouville-Mineur-Arnold theorem says that a compact connected regular set of an integrable system is in fact a torus of dimension n . This theorem has been attributed to Liouville for a long time but it was indeed probably first proved by Einstein [3]. The theorem remains valid when we consider an integrable system on a Poisson manifold but also for the so-called non-commutative systems.

In this section we apply Corollary 2.4 to reprove that the fibers are tori for integrable systems on symplectic and Poisson manifolds. The tools used for this new proof differ from the classical tools where a torus action is used [2, 7].

3.1. Liouville tori of integrable systems on symplectic manifolds. Recall the definition of an integrable system as well as the Liouville-Mineur-Arnold theorem.

Definition 3.1. An **integrable system** on a symplectic manifold (M^{2n}, ω) is a set of n functions f_1, \dots, f_n generically functionally independent (i.e. $df_1 \wedge \dots \wedge df_n \neq 0$ on a dense set) and pairwise commuting with respect to the Poisson bracket $\{f_i, f_j\} = \omega(X_{f_i}, X_{f_j}) = 0, \forall i, j$.

A point p is called regular point for the integrable system if $df_1 \wedge \dots \wedge df_n(p) \neq 0$.

Theorem 3.2. *Let (M^{2n}, ω) be a symplectic manifold and $F = (f_1, \dots, f_n)$ an integrable system. Let p be a regular point denote $F(p) = c$ and assume $L_c = F^{-1}(c)$ is compact and connected, then*

- (1) $L_c \cong \mathbb{T}^n$,
- (2) A neighborhood U of the torus L_c is the direct product of \mathbb{T}^n and the disc D^n , and the fibration given by F coincides with the projection on the disc.
- (3) in a neighborhood of L_c , $U(L_c)$, there exist coordinates $(\theta_1, \dots, \theta_n, p_1, \dots, p_n)$ such that ω is written $\omega = \sum_{i=1}^n dp_i \wedge d\theta_i$ and F only depends of p_1, \dots, p_n .

Denote by L^n any connected component of $F^{-1}(c)$ (or all of it if assumed connected) and we also assume it is compact. Denote by X_i the Hamiltonian vector associated to f_i . Observe that

$$\begin{aligned} 0 &= \{f_i, f_j\} \\ &= \omega(X_i, X_j) \\ &= \iota_{X_i}\omega(X_j) \\ &= -df_i(X_j) = -X_j(f_i) \quad \forall i, j = 1, \dots, n. \end{aligned}$$

and the vector fields X_1, \dots, X_n are tangent to L^n for all $p \in L^n$. So we can indeed write $T(L^n)_p = \langle X_{f_1}, \dots, X_{f_n} \rangle_p$. Take now in \mathbb{R}^n the canonical basis of vector fields $\{\partial_i = \frac{\partial}{\partial x_i}\}_{i=1}^n$ on \mathbb{R}^n and consider their pullbacks by F , $S_i := F^*(\partial_i)$, which are vector fields on M^{2n} which are transverse to L^n . They satisfy:

$$S_i(f_j) = \delta_{ij}.$$

They are determined by this condition modulo $T_p L^n$.

Lemma 3.3. *Let $j : L^n \rightarrow M^{2n}$ be the inclusion of the regular level set L^n into M^{2n} . Define the one-forms $\alpha_i = \iota_{S_i}\omega$. Then the one-forms $\beta_i = j^*\alpha_i$ are closed.*

Proof. By definition of S_i , we have $S_i(f_j) = \delta_{ij}$. Applying it $\forall i, j$:

$$\begin{aligned} \alpha_i(X_j) &= \omega(S_i, X_j) \\ &= -\omega(X_j, S_i) \\ &= -\iota_{X_j}\omega(S_i) \\ &= df_j(S_i) = S_i(f_j) = \delta_{ij}. \end{aligned}$$

To prove that β_i is closed, we just have to check that $d\alpha_i(X_i, X_j) = 0$ for all X_i, X_j sections of TL^n .

$$\begin{aligned} d\alpha_i(X_j, X_k) &= X_k(\alpha_i(X_j)) - X_j(\alpha_i(X_k)) - \alpha_i([X_j, X_k]) \\ &= X_k(\delta_{ij}) - X_j(\delta_{ik}) - \alpha_i(0) = 0. \end{aligned}$$

We conclude that $d(j^*\alpha_i) = d\beta_i = 0$ and so our forms β_i are closed in L^n . \square

Lemma 3.4. *The 1-forms β_1, \dots, β_n are linearly independent at all points of L^n .*

Proof. As seen in the previous lemma, we have that $\beta_i(X_j) = \delta_{ij}$. We deduce that $\beta_i = X_i^*$, by definition of dual basis. Since X_1, \dots, X_n form a basis of the tangent space at every point in L^n , we have that β_1, \dots, β_n form a basis of the cotangent space at every point in L^n . In particular all β_i are independent at all points of L^n . \square

We now prove Liouville theorem using Corollary 2.4.

Theorem 3.5 (Liouville's theorem). *The compact regular fiber of an integrable system on (M^{2n}, ω) is diffeomorphic to a torus \mathbb{T}^n .*

Proof. The forms β_i in the preceding lemma are n closed 1-forms which are independent at all points of L^n . Applying theorem 2.4, $L^n \cong \mathbb{T}^n$. \square

3.2. Commutative and non-commutative integrable systems on Poisson manifolds.

A Poisson manifold is a pair (M, Π) where Π is a bi-vector field with an associated bracket on functions

$$\{f, g\} := \Pi(df, dg), \quad f, g : M \rightarrow \mathbb{R}$$

satisfying the Jacobi identity. This is equivalent to the integrability equation $[\Pi, \Pi] = 0$. The Hamiltonian vector field of a function f in this context is defined as $X_f := \Pi(df, \cdot)$.

Poisson manifolds constitute a generalization of symplectic manifolds and it generalizes very natural structures such that of linear Poisson structures associated to the dual of a Lie algebra. Integrability of Hamiltonian systems in the Poisson setting is a rich field which is naturally connected to representation theory (Gelfand-Ceitlin systems on $\mathfrak{u}(n)^*$). We recall the notion of integrable system for Poisson manifolds.

Definition 3.6. Let (M, Π) be a Poisson manifold of (maximal) rank $2r$ and of dimension n . An s -tuple of functions $\mathbf{F} = (f_1, \dots, f_s)$ on M is said to define a *Liouville integrable system* on (M, Π) if

- (1) f_1, \dots, f_s are independent (i.e., their differentials are independent on a dense open set),
- (2) f_1, \dots, f_s are pairwise in involution and $r + s = n$.

The map $\mathbf{F} : M \rightarrow \mathbb{R}^s$ is called the *moment map* of (M, Π, \mathbf{F}) .

A point m is called regular whenever $d_m f_1 \wedge \dots \wedge d_m f_s \neq 0$. Observe that complete integrability in the Poisson context also implies that the distribution generated by X_{f_1}, \dots, X_{f_s} is integrable in a neighborhood of a regular point in the sense of Frobenius because $[X_{f_i}, X_{f_j}] = X_{\{f_i, f_j\}}$.

For integrable systems on Poisson manifolds it is possible to prove that the compact leaves of this distribution are tori and indeed to prove an action-angle theorem in a neighborhood of a regular torus (see [7]).

Theorem 3.7. Let (M, Π) be a Poisson manifold of dimension n of maximal rank $2r$. Suppose that $\mathbf{F} = (f_1, \dots, f_s)$ is an integrable system on (M, Π) , i.e., $r + s = n$ and the components of \mathbf{F} are independent and in involution. Suppose that $m \in M$ is a point such that

- (1) $d_m f_1 \wedge \dots \wedge d_m f_s \neq 0$;
- (2) The rank of Π at m is $2r$;
- (3) The integral manifold \mathcal{F}_m of the distribution generated by X_{f_1}, \dots, X_{f_s} , passing through m , is compact.

Then there exists \mathbb{R} -valued smooth functions $(\sigma_1, \dots, \sigma_s)$ and \mathbb{R}/\mathbb{Z} -valued smooth functions $(\theta_1, \dots, \theta_r)$, defined in a neighborhood U of \mathcal{F}_m such that

- (1) The manifold \mathcal{F}_m is a torus \mathbb{T}^r .
- (2) The functions $(\theta_1, \dots, \theta_r, \sigma_1, \dots, \sigma_s)$ define an isomorphism $U \simeq \mathbb{T}^r \times B^s$;
- (3) The Poisson structure can be written in terms of these coordinates as

$$\Pi = \sum_{i=1}^r \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial \sigma_i},$$

in particular the functions $\sigma_{r+1}, \dots, \sigma_s$ are Casimirs of Π (restricted to U);

- (4) The leaves of the surjective submersion $\mathbf{F} = (f_1, \dots, f_s)$ are given by the projection onto the second component $\mathbb{T}^r \times B^s$, in particular, the functions $\sigma_1, \dots, \sigma_s$ depend on the functions f_1, \dots, f_s only.

The functions $\theta_1, \dots, \theta_r$ are called **angle coordinates**, the functions $\sigma_1, \dots, \sigma_r$ are called **action coordinates** and the remaining functions $\sigma_{r+1}, \dots, \sigma_s$ are called **transverse coordinates**.

We can apply Tichler's trick to prove the orbits of a non-commutative integrable systems are tori in the Poisson setting (and deduce the result for the commutative particular case). We start recalling some definitions from [7].

Definition 3.8. Let (M, Π) be a Poisson manifold. An s -tuple of functions $F = (f_1, \dots, f_s)$ on M is a **non-commutative (Liouville) integrable system** of rank $r \leq s$ on (M, Π) if

- (1) f_1, \dots, f_s are independent (i.e. their differentials are independent on a dense open subset of M) and the Hamiltonian vector fields of the functions f_1, \dots, f_r are linearly independent at some point of M .
- (2) The functions f_1, \dots, f_r are in involution with the functions f_1, \dots, f_s and $r + s = \dim M$.

Remark 3.9. As a consequence the maximal rank of the Poisson structure is $2r$.

Remark 3.10. When $r = s$ and thus all the first integrals commute we obtain Liouville integrable systems as a particular case.

Some notation: We denote the subset of M where the differentials df_1, \dots, df_s are independent by U_F and the subset of M where the vector fields X_{f_1}, \dots, X_{f_r} are independent by $M_{F,r}$.

On the open subset $M_{F,r} \cap U_F$ of M , the Hamiltonian vector fields X_{f_1}, \dots, X_{f_r} define an involutive distribution of rank r . Let us denote by \mathcal{F} its foliation with r -dimensional leaves, see [7]. When \mathcal{F}_m is a compact r -dimensional manifold, the action-angle coordinate theorem proved in [7] (Theorem 1.1) proves that \mathcal{F}_m is a torus and gives a semilocal description of the Poisson structure in a neighborhood of a compact invariant set:

Theorem 3.11. *Let (M, Π, F) be a non-commutative integrable system of rank r , where $F = (f_1, \dots, f_s)$ and suppose that \mathcal{F}_m is compact, where $m \in M_{F,r} \cap U_F$. Then there exist \mathbb{R} -valued smooth functions $(p_1, \dots, p_r, z_1, \dots, z_{s-r})$ and \mathbb{R}/\mathbb{Z} -valued smooth functions $(\theta_1, \dots, \theta_r)$, defined in a neighborhood U of \mathcal{F}_m , and functions $\phi_{kl} = -\phi_{lk}$, which are independent of $\theta_1, \dots, \theta_r, p_1, \dots, p_r$, such that*

- (1) \mathcal{F}_m is a torus \mathbb{T}^r .
- (2) The functions $(\theta_1, \dots, \theta_r, p_1, \dots, p_r, z_1, \dots, z_{s-r})$ define a diffeomorphism $U \simeq \mathbb{T}^r \times B^s$;
- (3) The Poisson structure can be written in terms of these coordinates as,

$$\Pi = \sum_{i=1}^r \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial p_i} + \sum_{k,l=1}^{s-r} \phi_{kl}(z) \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_l};$$

- (4) The leaves of the surjective submersion $F = (f_1, \dots, f_s)$ are given by the projection onto the second component $\mathbb{T}^r \times B^s$ and as a consequence the functions f_1, \dots, f_s depend on $p_1, \dots, p_r, z_1, \dots, z_{s-r}$ only.

Let us now prove the first part of the theorem above using Corollary 2.4.

Proof. (of first item above)

Consider $F = (f_1, \dots, f_s)$ the set of first integrals of the non-commutative integrable system. Consider the span of the Hamiltonian vector fields $X_i := \Pi(df_i, \cdot)$. From definition of the non-commutative integrable system at each point on the regular set, dimension of the vector space is r . Denote by α_i the 1-forms such that $\alpha_i(X_j) = \delta_{ij}$. They are not uniquely determined, but if we

consider the inclusion j of the orbit into the manifold, then the 1-forms $\beta_i = j^*\alpha_i$ are uniquely determined. We can easily check that the forms β_i are closed:

$$\begin{aligned} d\beta_i(X_j, X_k) &= X_k(\beta_i(X_j)) - X_j(\beta_i(X_k)) - \beta_i([X_j, X_k]) \\ &= X_k(\delta_{ij}) - X_j(\delta_{ik}) - \beta_i(0) = 0 \end{aligned}$$

where in the last equality we have used that $[X_j, X_k] = X_{\{f_j, f_k\}}$ and from the definition of non-commutative integrable system $X_{\{f_j, f_k\}} = X_0 = 0$.

From the definition the dimension of the orbit is r and we have exactly r forms thus applying Corollary 2.4 we conclude that the orbit is a torus. From the regular value theorem, observe also that this orbit is the connected component through the point of the mapping given by $F = (f_1, \dots, f_s)$. \square

Finally, when $r = s$ we obtain as corollary the first statement of Theorem 3.7 of the commutative case.

Corollary 3.12. *Given an integrable system on a Poisson manifold $\mathbf{F} = (f_1, \dots, f_s)$, the regular integral manifold \mathcal{F}_m of the distribution generated by X_{f_1}, \dots, X_{f_s} , passing through m , is a torus of dimension r , \mathbb{T}^r .*

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