

# On the Notion of Dimension and Codimension of Simple Games

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**Abstract** This paper presents novel concept of a *dimension* and *codimension* for the class of simple games. It introduces a dual concept of a *dimension* which is obtained by considering the union instead of the intersection as the basic operation, and several other extensions of the notion of dimension. It also shows the existence and uniqueness of a minimum subclasses of games, with the property that every simple game can be expressed as an intersection, or respectively, the union of them. We show the importance of these subclasses in the description of a simple game, and give a practical interpretation of them.

**Keywords:** Simple games; Hypergraphs; Boolean algebra; Dimension and codimension; dimensionally minimum class of simple games

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## 1. Introduction

In this paper we introduce a novel approach of viewing a simple game by its *dimension* or *codimension* according to the *minimal dimensional class of simple games* or respectively *minimal codimensional class of simple games*. We introduce and study the properties of these notions. We show how they can be referred to as a *basis* or *dual basis* for the class of simple games. We also provide a completeness in the sense that any simple game can be measured in this terms, ie. any simple game has a *dimension* and *codimension*, as defined in this paper. We also describe many important corollaries, following from these results, and we give some practical interpretations of these notions. In particular we show the relations of these notions to the winning and losing coalitions of the game - more precisely to the set of *minimal winning coalitions* and *maximal losing coalitions*, or *minimal blocking coalitions* in the game.

To recall a classical concept of dimension we refer to the class of *weighted simple games* which are probably the most important subclass of simple games. It is well known that every simple game can be represented as an intersection of weighted games. It nevertheless becomes of interest to ask how efficiently this can be done for a given simple game. The concept of dimension known in the literature is based on the fact that each simple game can be expressed as a finite intersection

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of weighted simple games, (Taylor and Zwicker, 1993), (Taylor and Zwicker, 1995), (Taylor and Zwicker, 1999). The question of efficiency leads to the definition of the classical *dimension*. A simple game is said to be of *dimension*  $k$  if and only if it can be represented as the intersection of exactly  $k$  weighted games, but not as the intersection of  $k - 1$  weighted games. In this sense, we can regard weighted simple games as a generating system of the class of simple games with the *intersection* as the basic operation. There exist several real-world examples of voting systems in use today where laws are passed by a method known as ‘count and account,’ which are examples of dimension 1 or 2. For analysis and some examples on this voting system, we refer the reader to (Peleg, 1992) and (Carreras and Freixas, 2004).

The definition of dimension was introduced for graphs in the late 1970s; its extension to hypergraphs (simple games not necessarily monotonic) is due to Jereslow (Jereslow, 1975). Nevertheless, this notion for simple games is reminiscent of the dimension (Dushnik and Miller (Dushnik and Miller, 1941)) of a partially ordered set, which was defined as the minimum number of linear orders whose intersection is the given partial ordering. In the present paper we restrict the contents to simple games, disregarding the possibility of abstention or absence.

The first goal of this paper consists in extending the concept of dimension, originally defined for weighted games, to other subclasses of simple games, and in particular finding a ‘basic’ subclass of simple games for which the associated notion of dimension exists. We show in this paper that, in fact, such a smallest class exists and we show its uniqueness as well. We show the same for the dual case, i.e. of the *codimension*.

The notion of a *codimension* which we introduce gives the completeness to this theory, and provides us with the tools of studying the dual cases. We answer a natural question which arises in this context – whether it is feasible to get an analogous concept of dimension, namely codimension, by using the *union* instead of the intersection as the basic operation. Indeed, we show that the codimension is well defined for several subclasses of simple games. Moreover, we show that there exists a unique smallest subclass of simple games for which the associated codimension exists.

The results presented here emphasize the extent to which the study of the dimension constitutes a bridge between the theory of simple games and hypergraphs and other theories like Reliability and Circuits Theory see, for example, Ramamurthy, 1990 or (Freixas and Puente, 2001).

The paper is organized as follows. After a section on preliminaries, revising the terminology and recalling some significant results on classical dimension of simple games, we extend in Section 3 the concept of dimension to several classes of simple games and introduce the notion of codimension of a simple game. Section 4 contains some results on dimension and codimension and establishes a relationship between the two notions. Section 5 shows the existence and uniqueness of subclasses of simple games being (co)dimensionally minimum. Some computational aspects of dimension and codimension are provided in Section 6. Concluding remarks are provided in Section 7.

## 2. Preliminaries

To revise and establish notations we recall the important definitions.

A *simple game* is a pair  $(N, W)$  where  $N = \{1, 2, \dots, n\}$  and  $W$  is an arbitrary collection of subsets of  $N$ . The simple game is *monotonic* if, moreover,  $S \in W$  and  $S \subseteq T$ , then  $T \in W$ . From now on we only deal with monotonic simple games with the two additional assumptions  $\emptyset \notin W$  and  $N \in W$ .

Simple games can be viewed as models of voting systems in which a single alternative, such as a bill or an amendment, is pitted against the status quo. The set  $N$  is called the *grand coalition*, its members are called *players* or *voters*, and the sets in  $W$  are called *winning coalitions*. The intuition here is that a set  $S$  is a winning coalition *if and only if* the bill or amendment passes when the players in  $S$  are precisely the ones who voted for it. A subset of  $N$  that is not in  $W$  is called a *losing coalition* and the collection of losing coalitions is denoted by  $L$ . If each proper subcoalition of a winning coalition is losing, this winning coalition is called *minimal*. The set of minimal winning coalitions is denoted by  $W^m$ . It should be noted that a simple game is completely determined by its minimal winning coalitions. If each proper supra-coalition of a losing coalition is winning, this losing coalition is called *maximal*. The set of maximal losing coalitions is denoted by  $L^M$ . A player  $i \in N$  is *null* in  $(N, W)$  if  $i \notin S$  for every  $S \in W^m$ . A player  $i \in N$  is a *winner* if  $\{i\} \in W$ . A simple game  $(N, W)$  is *proper* if  $S \in W$  implies  $N \setminus S \in L$ . A simple game  $(N, W)$  is *strong* if  $S \in L$  implies  $N \setminus S \in W$ . If  $D$  is a finite set, then  $|D|$  denotes the cardinality of  $D$ .

A simple game  $(N, W)$  is called a *weighted game* if it admits a representation by means of the  $n + 1$  non-negative real numbers  $[q; w_1, \dots, w_n]$  such that  $S \in W$  iff  $w(S) \geq q$ , where for each coalition  $S \subseteq N$ ,  $w(S) = \sum_{i \in S} w_i$ . The number  $q$  is called the *quota* of the game and  $w_i$  the *weight* of player  $i$ . A *k-out-of-n* game is a simple game where the minimal winning coalitions are those with  $k$  members. It is obvious that *k-out-of-n* game is weighted and admits the representation  $[k; \underbrace{1, \dots, 1}_n]$ .

Two simple games  $(N, W)$  and  $(N', W')$  are said to be *isomorphic* if there exists a bijective map  $f : N \rightarrow N'$  such that  $S \in W$  iff  $f(S) \in W'$ .

Let  $(N, W)$  be a simple game. Set  $W^i = \{S \in W : i \in S\}$  and let  $\tau_{ij} : N \rightarrow N$  denote the *transposition* of players  $i, j \in N$  (i.e.,  $\tau_{ij}(i) = j$ ,  $\tau_{ij}(j) = i$  and  $\tau_{ij}(k) = k$  for  $k \neq i$  and  $k \neq j$ ). The individual *desirability relation* considered in (Isbell, 1956) and (Isbell, 1958) and later on generalized in (Maschler and Peleg, 1966), is the binary relation  $\succsim$  on  $N$ :

$$i \succsim j \quad \text{iff} \quad \tau_{ij}(W^j) \subseteq W^i,$$

and say that  $i$  is *at least as desirable as*  $j$  as a coalition partner. It is easy to see that  $\succsim$  is a preorder, and the lack of antisymmetry is then solved by introducing the associated equivalence relation  $\approx$  the indifferent part of the desirability relation which is defined by

$$i \approx j \quad \text{iff} \quad i \succsim j \text{ and } j \succsim i;$$

hence  $i \approx j$  means that  $i$  and  $j$  are equi-desirable as coalitional partners. The other basic problem with desirability is that it is not always complete (total). Then, if any two players are comparable by  $\succsim$ ,  $(N, W)$  is said to be a *complete simple game* (complete games are also known in the literature of simple games as linear, directed or ordered games); in this case the  $\approx$ -classes are linearly ordered. Note that each weighted game is complete because  $w_i \geq w_j$  implies  $i \succsim j$ .

### 2.1. Preliminaries on the classical dimension

Revising the results and possible motivations related to the concept of the classical dimension, we find that in the literature only one concept of dimension has been proposed, and as far as we know only this notion of dimension has been considered in the theory of simple games. Following (Taylor and Zwicker, 1999), a simple game  $W$  is said to be of *dimension*  $k$  if it can be represented as the intersection of  $k$  weighted games, but cannot be represented as the intersection of  $k - 1$  weighted games. Equivalently, the *dimension* of  $W$  is the least  $k$  such that there exists weighted simple games  $W_1, \dots, W_k$  such that

$$W = W_1 \cap \dots \cap W_k.$$

The possible motivation to introduce this concept of dimension has been based on the observation that most naturally occurring, real-life voting systems in use are of small dimension. As an example we can consider, the Electoral College of the United States, which is a weighted game and hence has dimension 1. Another example is the United Nations Security Council which has dimension 1 (for weighted a representation see (Taylor, 1995) and (Taylor and Zwicker, 1999) taking into account the possibility of abstention). Interesting examples of dimension 2 are the United States federal system, also the procedure for amending the Canadian Constitution (see (Taylor, 1995), (Kilgour, 1983) and (Levesque and Kilgour, 1984)) and the current European Union Council of Ministers for motions not coming from the European Commission. Two conspicuous examples of dimension 3 can be found in the Union Council of Ministers after the enlargement to 27 members agreed in Nice in December 2000, (Freixas, 2004). We do not know about the existence of some real-world voting system with dimension greater than 3. Whenever the dimension of a simple game is not a huge number an efficient decomposition of the simple game as intersection of weighted games can be used to compute power indices with generating functions as was illustrated in (Algaba et.al., 2001) for the Banzhaf power index for the European Union after the enlargement to 27 countries.

In Deĭneko and Woeginger (Deĭneko and Woeginger, 2006) it is proven that the following problem ‘Given  $k$  weighted games, decide whether the dimension of their intersection exactly equals  $k$ ’ is NP-hard and hence computationally intractable, thus their result indicates that the computation of the dimension of simple games is a combinatorially complicated concept.

An important property of the concept of dimension is that:

**Theorem 1 (Theorem 1.7.2, (Taylor and Zwicker, 1999)).** *Every simple game has a finite dimension.*

The proof in (Taylor and Zwicker, 1999) of Theorem 1 uses the following argument (further details on the proof are omitted here). For a simple game  $W$  on  $N$ , let  $L^M = \{T_1, \dots, T_k\}$ . Clearly,  $W = W_1 \cap \dots \cap W_k$  where each  $W_i$  admits the weighted representation  $[1; w_1^i, \dots, w_n^i]$  where

$$w_j^i = \begin{cases} 0, & \text{if } j \in T_i; \\ 1, & \text{otherwise.} \end{cases}$$

Hence,  $W$  is the intersection of  $|L^M|$  weighted games and, therefore it has dimension. As a consequence of Theorem 1 the dimension of a simple game is bounded above

by the number of maximal losing coalitions. The following result shows that there are games of every dimension.

**Theorem 2 (Theorem 1.7.4, (Taylor and Zwicker, 1999)).** *For every  $m \geq 1$ , there is a simple game  $W$  of dimension  $m$ .*

The proof in (Taylor and Zwicker, 1999) of Theorem 2 involves the following game  $W$  of dimension  $m$  defined on a set  $M$  (further details on the proof are omitted here). Fix  $m \geq 1$  and let  $M = \{1, \dots, 2m\}$ . Let  $W$  be the collection of sets defined by

$$S \in W \text{ iff } S \cap \{2i - 1, 2i\} \neq \emptyset \quad \text{for } i = 1, \dots, m.$$

However, these games are highly ‘non-complete.’ Thus, one is left with the hope that for games with high dimension, the complexity is caused by their non-completeness. However, in (Freixas and Puente, 2008) it is proved that this conjecture is not true, so that there exist complete games of every dimension. Exponential dimension is also achieved for some simple games, see Theorem 1.7.5 in (Taylor and Zwicker, 1999) and Theorem 1 in (Freixas and Puente, 2001) for games with a higher dimensional behavior.

### 3. Dimension and codimension

#### 3.1. Duality and basic subsets of simple games

To introduce a notion of dimension and codimension we recall the basic structures related to a simple game. First we recall a notions of blocking coalitions and the dual game.

Let  $\mathcal{S}$  be the class of simple games on  $N$ . With every simple game  $W \in \mathcal{S}$ , we can associate a *dual* game  $W^* = \{S \subseteq N : N \setminus S \notin W\}$  whose elements are the *blocking* coalitions in  $W$ , i.e., those that can prevent an issue from being passed. It is straightforward to check that the dual game is idempotent, i.e.,  $W^{**} = W$  and satisfies the two de Morgan’s laws (see, e.g. Proposition 1.4.3 in (Taylor and Zwicker, 1999)):

$$\begin{aligned} (W_1 \cup W_2)^* &= W_1^* \cap W_2^*, \\ (W_1 \cap W_2)^* &= W_1^* \cup W_2^*. \end{aligned} \tag{1}$$

Let  $\mathcal{C} \subseteq \mathcal{S}$  be a subset of simple games on  $N$ , and let  $\mathcal{C}^* = \{W^* \in \mathcal{S} : W \in \mathcal{C}\}$ . We say that  $\mathcal{C}$  is *closed* under duality if  $\mathcal{C} = \mathcal{C}^*$ . Particularly,  $\mathcal{S}$  is itself closed under duality.

An example of a proper subset of  $\mathcal{S}$  which is closed under duality is the class of complete games, briefly denoted  $\mathcal{L}$ , because  $i \succsim j$  in  $W \in \mathcal{L}$  iff  $i \succsim j$  in  $W^* \in \mathcal{L}$ . A proper subset of  $\mathcal{L}$  which is closed under duality is the class of weighted games, briefly denoted  $\mathcal{M}$ , because if  $W \in \mathcal{M}$  admits  $[q; w_1, \dots, w_n]$  as a weighted representation then  $[w - q + 1; w_1, \dots, w_n]$  is a weighted representation for  $W^* \in \mathcal{M}$  where  $w$  denotes hereafter the total sum of weights  $\sum_{i=1}^n w_i$ .

A proper subset of  $\mathcal{M}$  which is not closed under duality is the class of *homogeneous* games. Let  $\mathcal{H}_1$  be the class of *homogeneous* games, i.e.  $W \in \mathcal{H}_1$  if  $W$  admits a representation  $[q; w_1, \dots, w_n]$  for which the weight of all minimal winning coalitions equals the quota. Note that  $\mathcal{H}_1$  is not closed under duality, but  $\sum_{i \in S} w_i = q$  for all  $S \in W^m$  implies that the dual game,  $W^* \in \mathcal{S}$  admits the weighted representation

$[w - q + 1; w_1, \dots, w_n]$  with  $w(T) = w - q$  for all  $T \in (L^*)^M$ . In words, the dual of a homogeneous game is a game that admits a representation as a weighted game wherein the weight of all maximal losing coalitions coincide. Let  $\mathcal{H}_2$  denote that class of games, then  $\mathcal{H}_1 \neq \mathcal{H}_2$  and  $\mathcal{H}_1^* = \mathcal{H}_2$ , and by idempotency  $\mathcal{H}_1 = \mathcal{H}_2^*$ . Of course, the set  $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$  is closed under duality, although neither  $\mathcal{H}_1$  nor  $\mathcal{H}_2$  are.

### 3.2. The class of unitary vs. unanimity games

In this subsection we recall the dual analogy between the classes of *unitary* and *unanimity* games.

We say that a game  $W$  is *unitary* if it admits a weighted representation  $[q; w_1, \dots, w_n]$  where  $q = 1$ ,  $w \geq 1$ , and  $w_i$  is either 0 or 1 for all  $i = 1, \dots, n$ . If  $w_i = 1$  then  $i$  is a winner in  $W$ , whereas  $w_i = 0$  implies that  $i$  is null in  $W$ . Equivalently,  $W$  is unitary iff  $W^m$  is uniquely formed by singletons (the winners of the game). Let  $\mathcal{U}_1$  denote the class of unitary games then  $|\mathcal{U}_1| = 2^n - 1$ .

The dual representation of a unitary game  $[1; w_1, \dots, w_n]$  is  $[w; w_1, \dots, w_n]$  which is a weighted representation of the *unanimity* game  $W$  of coalition  $\{k \in N : w_k = 1\}$ , equivalently,  $W^m$  is formed by a single coalition whenever the game is unanimous. Let  $\mathcal{U}_2$  denote the class of unanimity games, then  $\mathcal{U}_1 \neq \mathcal{U}_2$  and  $\mathcal{U}_1^* = \mathcal{U}_2$ , and by idempotency  $\mathcal{U}_1 = \mathcal{U}_2^*$ . Of course, the set  $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$  is closed under duality, although neither  $\mathcal{U}_1$  nor  $\mathcal{U}_2$  are. It is clear that  $|\mathcal{U}_2| = 2^n - 1$  and  $|\mathcal{U}| = 2^{n+1} - (n+2)$ .

### 3.3. New notions of dimension

Here we introduce the new concepts of dimension and codimension for simple games.

Let  $\mathcal{C} \subsetneq \mathcal{S}$  be a proper subset of simple games on  $N$ ,  $W \in \mathcal{S}$  is said to be of  $\mathcal{C}$ -*dimension*  $k$  if it can be represented as the intersection of  $k$  games in  $\mathcal{C}$ , but cannot be represented as the intersection of  $k - 1$  games in  $\mathcal{C}$ . Equivalently, the  $\mathcal{C}$ -*dimension* of  $W$  is the least  $k$  such that there exists games  $W_1, \dots, W_k$  in  $\mathcal{C}$  such that

$$W = W_1 \cap \dots \cap W_k.$$

Briefly, we will denote  $\dim_{\mathcal{C}}(W) = k$ . Notice that  $W \in \mathcal{C}$  iff  $\dim_{\mathcal{C}}(W) = 1$ .

$W \in \mathcal{S}$  is said to be of  $\mathcal{C}$ -*codimension*  $k$  if it can be represented as the *union* of  $k$  games in  $\mathcal{C}$ , but cannot be represented as the union of  $k - 1$  games in  $\mathcal{C}$ . Equivalently, the  $\mathcal{C}$ -*codimension* of  $W$  is the least  $k$  such that there exists games  $W_1, \dots, W_k$  in  $\mathcal{C}$  such that

$$W = W_1 \cup \dots \cup W_k.$$

Briefly, we will denote  $\text{codim}_{\mathcal{C}}(W) = k$ . The  $\mathcal{M}$ -codimension is simply called *codimension*.

### 3.4. Practical interpretation

The meaning in practice of the introduced notions is as follows. Let us consider the new game “*blocking the law to pass*” for each of the games with dimension 2 or 3 mentioned in the Preliminaries. These games are the respectively the dual games of the original ones e.g. (United States federal system, the procedure for amending the Canadian Constitution or the current European Union Council of Ministers for motions not coming from the European Commission), thus they decompose as union of two or three weighted games (which are at the same time the duals of the weighted games appearing in the original decomposition as intersection of them). As we shall

see in Theorem 3-(ii) the dimension of a game coincides with the codimension of its dual game. Thus, *every real-world example of a voting system with dimension  $p$  provides a real-world example of a voting system with codimension  $p$  and conversely.* Note that for each of the voting systems considered in the Preliminaries (with dimension greater than 1), it is even easier to win in the blocking game (with codimension greater than 1) than to win in the original one. This is motivated since voting systems are usually demanded to be proper so that  $|W| \leq 2^{n-1}$ , which implies that the dual game is strong and hence  $|W^*| \geq 2^{n-1}$ .

A natural source of examples with small codimension but with exponential dimension (see (Freixas and Puente, 2001)) appears for a special type of *compound simple games* considered by Shapley (Shapley, 1962). For compound games each player belongs to one of  $m$  chambers. The bill is previously accepted or refused in each one of the chambers and finally a rule of global decision (which includes all the possible results for the chambers) is applied. A particular game of this type uses unanimity in each one of the chambers, while the global decision is an individualist game played by all the chambers. The resulting game is a *composition of unanimity games via individualism* which models voting systems in which a lobby in a committee can impose its criteria to other lobbies whenever a full agreement of all its members is reached.

#### 4. Relationship between dimension and codimension

##### 4.1. A minimum (co)dimensional class of simple games

A subset  $\mathcal{C} \subsetneq \mathcal{S}$  is *dimensionally minimal* if it has  $\mathcal{C}$ -dimension for all  $W \in \mathcal{S}$ , but for all  $\mathcal{B} \subsetneq \mathcal{C}$  there exists a  $W \in \mathcal{S}$  without  $\mathcal{B}$ -dimension. Analogously, a subset  $\mathcal{C} \subsetneq \mathcal{S}$  is *codimensionally minimal* if it has  $\mathcal{C}$ -codimension for all  $W \in \mathcal{S}$ , but for all  $\mathcal{B} \subsetneq \mathcal{C}$  there exists a  $W \in \mathcal{S}$  without  $\mathcal{B}$ -codimension.

A subset  $\mathcal{C} \subsetneq \mathcal{S}$  is *dimensionally minimum* if it is dimensionally minimal and  $\mathcal{C} \subseteq \mathcal{D}$  for all set  $\mathcal{D}$  being dimensionally minimal. Analogously, a subset  $\mathcal{C} \subsetneq \mathcal{S}$  is *codimensionally minimum* if it is codimensionally minimal and  $\mathcal{C} \subseteq \mathcal{D}$  for all set  $\mathcal{D}$  being codimensionally minimal.

It is an important issue to determine whether there exists a set of games  $\mathcal{C}$  being (co)dimensionally minimum because  $\mathcal{C}$  would be the most reduced class of simple games with (co)dimension. In this section we are concerned about the existence of a minimum (co)dimensional class of simple games.

##### 4.2. Dimension vs. Codimension

The following lemma is an easy consequence obtained from observing relations between previous definitions.

**Lemma 1.** *Let  $\mathcal{B} \subsetneq \mathcal{C} \subsetneq \mathcal{S}$  and let  $W \in \mathcal{S}$  which has a finite  $\mathcal{B}$ -dimension ( $\mathcal{B}$ -codimension). Then:*

- (i)  $W$  has a finite  $\mathcal{C}$ -dimension ( $\mathcal{C}$ -codimension), and
- (ii)  $\dim_{\mathcal{C}}(W) \leq \dim_{\mathcal{B}}(W)$  ( $\text{codim}_{\mathcal{C}}(W) \leq \text{codim}_{\mathcal{B}}(W)$ ).

*Proof.* (i) If  $W = W_1 \cap \dots \cap W_k$  ( $W = W_1 \cup \dots \cup W_k$ ) with all  $W_i \in \mathcal{B}$ . Then  $\mathcal{B} \subsetneq \mathcal{C}$  implies  $W = W_1 \cap \dots \cap W_k$  ( $W = W_1 \cup \dots \cup W_k$ ) where all  $W_i \in \mathcal{C}$ .

(ii) From (i) it directly follows that  $\dim_{\mathcal{C}}(W) \leq k$  ( $\text{codim}_{\mathcal{C}}(W) \leq k$ ) if  $\dim_{\mathcal{B}}(W) = k$  ( $\text{codim}_{\mathcal{B}}(W) = k$ ).

Let  $\mathcal{C}^*$  be the subset of simple games on  $N$  such that  $W \in \mathcal{C}^*$  iff  $W^* \in \mathcal{C}$ , i.e.  $\mathcal{C}^*$  is the dual of  $\mathcal{C}$ .

**Theorem 3.** *Let  $\mathcal{C} \subseteq \mathcal{S}$  and let  $\mathcal{C}^*$  be its dual. Let  $W \in \mathcal{S}$ . Then:*

- (i)  *$W$  has a finite  $\mathcal{C}$ -dimension iff  $W^*$  has a finite  $\mathcal{C}^*$ -codimension, and*
- (ii)  *$\text{codim}_{\mathcal{C}^*}(W^*) = \text{dim}_{\mathcal{C}}(W)$  if  $W$  has a finite  $\mathcal{C}$ -dimension.*

Proof of the theorem can be found in (Freixas and Marciniak, 2009).

Here we concentrate on the important consequences of these results:

**Remark 1.** (i) The games appearing in the proof of Theorem 1 are unitary. Hence, it follows that each simple game has a finite  $\mathcal{U}_1$ -dimension. Moreover, all these notions of dimension are bounded above by the number of maximal losing coalitions.

- (ii) Analogously, taking the game considered in the proof of Theorem 2 and applying Theorem 3 it follows that each simple game has a finite  $\mathcal{U}_2$ -codimension. Moreover, if we apply de Morgan's laws (1) to the unitary game  $W$  considered in the proof of Theorem 1 we obtain that

$$W^* = U_{S_1} \cup \dots \cup U_{S_k}$$

where every  $U_S$  is the unanimity game of coalition  $S$ , which is weighted and admits the representation  $[w; w_1, \dots, w_n]$  where

$$w_i = \begin{cases} 1, & \text{if } i \in S \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the  $\mathcal{U}_2$ -codimension is bounded above by the number of minimal winning coalitions and by Lemma 1 it also has  $\mathcal{H}_2$ -codimension,  $\mathcal{M}$ -codimension or  $\mathcal{L}$ -codimension. Moreover, all these notions of codimension are bounded by *the number of minimal winning coalitions*.

- (iii) Taking the game considered in the proof of Theorem 2 and applying Theorem 3 we obtain that for every  $m \geq 1$  there is a game of codimension  $m$ . By applying Morgan's laws to the game in the proof of Theorem 2 it may be deduced that if  $M = \{1, \dots, 2m\}$  and  $W$  be the collection of sets defined by the minimal winning coalitions  $\{1, 2\}, \{3, 4\}, \dots, \{2m-1, 2m\}$ , then  $W$  is the union of the  $m$  unanimity games. Let  $W_i$  be the game that gives weight 1 to each people in  $\{2i-1, 2i\}$  and gives weight zero to everyone else. For each of these games, let the quota  $q$  be 2. Notice that  $S \in W$  iff  $\{2i-1, 2i\} \subseteq S$  for some  $i$  iff  $w_i(S) \geq 2$  for some  $i$  iff  $S \in W_i$  for some  $i$  iff  $S \in \bigcup_{i=1}^m W_i$ .
- (iv) Theorem 4.3 in (Freixas and Puente, 2008) shows the existence of complete games, i.e.  $W \in \mathcal{L}$  with  $\mathcal{M}$ -dimension  $m$  for every  $m \geq 1$ . By considering the dual games of these complete games we obtain complete games of every codimension  $m$ . Further, Theorem 1.7.5 in (Taylor and Zwicker, 1999) and also Theorem 2.1 in (Freixas and Puente, 2001) show games with exponential  $\mathcal{M}$ -dimension (or simply, dimension). From Remark 1-(i) these games have  $\mathcal{U}_1$ -dimension and by Lemma 1-(ii)  $\text{dim}_{\mathcal{M}}(W) \leq \text{dim}_{\mathcal{U}_1}(W)$ , for all  $W \in \mathcal{S}$ . Hence, those games have exponential  $\mathcal{U}_1$ -dimension. Taking the dual games of those games we get games with exponential  $\mathcal{U}_2$ -codimension.

- (v) If we desire to compute a power index of a simple game that satisfies the transfer axiom and the game has a large number of players, a large  $\mathcal{M}$ -dimension but a reduced  $\mathcal{M}$ -codimension. Then we can compute the power index by using generating functions (see e.g. (Brams and Affuso, 1976)) and applying them to each game appearing in the decomposition obtained by applying the union-exclusion principle to an efficient union of weighted games.

The next result allows computing the  $\mathcal{U}_1$ -dimension and the  $\mathcal{U}_2$ -codimension of every simple game.

- Theorem 4.** (i) *The  $\mathcal{U}_1$ -dimension of a simple game is the number of maximal losing coalitions.*  
 (iii) *The  $\mathcal{U}_2$ -codimension of a simple game is the number of minimal winning coalitions.*

Proof of the theorem can be found in (Freixas and Marciniak, 2009). Here we present a corollary following from this result:

Theorem 4 together with Sperner's theorem on independent families of subsets of a given finite set allows providing upper bounds for the  $\mathcal{U}_1$ -dimension and  $\mathcal{U}_2$ -dimension of a given simple game.

**Corollary 1.** *Both  $\mathcal{U}_1$ -dimension and  $\mathcal{U}_2$ -codimension for a simple game  $W$  with  $n$  players have upper bounds:*

$$\dim_{\mathcal{U}_1} W \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

$$\text{codim}_{\mathcal{U}_2} W \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

*In the case  $n$  even there is a unique such game, which is the  $(\frac{n}{2} + 1)$ -out-of- $n$  game for the dimension and its dual game, i.e. the  $\frac{n}{2}$ -out-of- $n$  game for the codimension. If  $n$  is odd there are two such games which are: the  $(\lfloor \frac{n}{2} \rfloor + 1)$ -out-of- $n$  game and the  $(\lfloor \frac{n}{2} \rfloor + 2)$ -out-of- $n$  game for the dimension, and their dual games, the  $(\lfloor \frac{n}{2} \rfloor + 1)$ -out-of- $n$  game (it is autodual) and  $\lfloor \frac{n}{2} \rfloor$ -out-of- $n$  game for the codimension.*

If a simple game is given by either the set of winning coalitions or the set of losing coalitions, then obtaining the set of minimal winning coalitions or the set of maximal losing coalitions requires polynomial time. Hence, the  $\mathcal{U}_1$ -dimension or the  $\mathcal{U}_2$ -codimension of a simple game can be computed in polynomial time whenever the game is defined by either the set of winning coalitions or the set of losing coalitions. Instead, if the game is defined by the set of minimal winning coalitions then it is required exponential time to get the maximal losing coalitions, and therefore computing the  $\mathcal{U}_1$ -dimension requires exponential time. Analogously, if the game is defined by the set of maximal losing coalitions then it is required exponential time to get the minimal winning coalitions, and therefore computing the  $\mathcal{U}_2$ -codimension requires exponential time. See (Freixas et.al.,2008) for details on these results.

We conclude this section by giving a practical interpretation of the codimension of the game. Codimension with respect to the class of unanimity games is equal to the number of chambers, i.e. the number of minimal winning coalitions, thus it can be seen as the number of possible strategies to win in this game, i.e. the number of choices that an outsider can regard the game to force win. Let us consider a

Parliament and a businessman, who wants to change a law in some way, then the  $\mathcal{U}_2$ -codimension shows how many essential choices has the law to be passed. In fact, the main objectives for the businessman to exert a great deal of influence are the minimal winning coalitions, which are different alternatives to pass the law.

## 5. Minimum (co)dimension

Here we point out the classes which are the minimum dimensionally and respectively minimum codimensionally classes for simple games. They exist and moreover they are unique.

**Theorem 5.** (i) *The class,  $\mathcal{U}_1$ , of unitary games is dimensionally minimum.*  
(ii) *The class,  $\mathcal{U}_2$ , of unanimity games is codimensionally minimum.*

Proof of the theorem can be found in (Freixas and Marciniak, 2009).  
The uniqueness of this class is shown in a following theorem.

**Theorem 6.** *The set of all the unitary games or equivalently those which have exactly one maximal losing coalition is the smallest set of games, with respect to which every game has dimension.*

*Proof.* We already know, from the previous theorem that it is a minimal one. Every set of games for which the dimension of any games is finite has to contain all the unitary games. Thus in fact the set of unitary games is the smallest one among the sets of games, with respect to which the dimension of any simple game exists.

**Corollary 2.** *From the above results the dual statement can be obtained: the set of all games with exactly one minimal winning coalition (unanimity games) is the smallest set of games, with respect to which the codimension of any game exists.*

*Proof.* Let  $T$  be a class of games for which the  $\text{codim}_T$  is finite for any simple game, then  $T^*$  has the property that  $\text{dim}_{T^*}$  is finite for any game, thus  $T^*$  contains the class of unitary games, so  $T$  must contain dual of unitary games.

It is important to note the significant role played by unanimity games in the axiomatizations of several power indices. Indeed, Theorem 5-(ii) confirms  $\mathcal{U}_2$  to be the minimum class for which simple games can be expressed as union of its elements. The decomposition of a game as a union of unanimity games has been used in some proofs for the most well-known power indices, among them, Shapley–Shubik, Banzhaf, Johnston, Deegan–Packel, Holler.

### 5.1. Some properties of unitary and unanimity games

We finally provide a coalitional property to test whether a given game is unitary and do the same for unanimity games.

Let us consider two classes of simple games. A simple game  $W$  is *primitive* iff  $S \in W$  and  $S = S_1 \cup S_2$  then  $S_1$  or  $S_2$  belongs to  $W$ . A simple game  $W$  is *smooth* iff for all  $S, T \in W$  implies  $S \cap T \in W$ .

**Proposition 1.** (i) *A simple game is unitary if and only if it is primitive.*  
(ii) *A simple game is unanimous if and only if it is smooth.*

*Proof.* (i) ( $\Rightarrow$ ) Assume that  $W$  is unitary, i.e.  $W^m$  is uniquely formed by singletons (the winners of the game). Let  $S \in W$  and  $S = S_1 \cup S_2$ .  $S \in W$  implies that  $i \in S$  for a winner  $i$ , therefore either  $i \in S_1$  or  $i \in S_2$ , and thus either  $S_1 \in W$  or  $S_2 \in W$  which implies primitiveness.

( $\Leftarrow$ ) Assume that  $W$  is primitive but not unitary. Then there is a coalition  $S$  in  $W^m$  containing at least two players. Let  $S_1 \subsetneq S$  and  $S_2 \subsetneq S$  a partition of  $S$ , i.e.  $S_1 \cup S_2 = S$  and  $S_1 \cap S_2 = \emptyset$ . As  $S_1$  and  $S_2$  are losing coalitions we have a contradiction with the hypothesis of primitiveness for the game.

(ii) ( $\Rightarrow$ ) Assume that  $W$  is unanimous, i.e.  $W^m$  is formed by a single coalition  $R$ . Let  $S$  and  $T$  be two arbitrary winning coalitions, then there are minimal winning coalitions  $R_1$  and  $R_2$  such that  $R_1 \subseteq S$  and  $R_2 \subseteq T$ . As  $R$  is the unique element in  $W^m$  it yields  $R_1 = R_2 = R$  and, hence  $R \subseteq S \cap T$  which implies that  $S \cap T \in W$  and so the game is smooth.

( $\Leftarrow$ ) Assume that  $W$  is smooth but not unanimous. Then  $W^m$  contains at least two minimal winning coalitions  $S$  and  $T$  with  $S \cap T \subsetneq S$ ,  $S \cap T \subsetneq T$ . Hence,  $S \cap T \notin W$  which is a contradiction with the assumption of smoothness.

## 6. Computational aspects of dimension and codimension

Unanimity games have a natural description – these are the games with unique minimal winning coalitions. Dually the unitary games are precisely those which has unique maximal losing coalition. The unanimity game which minimal winning coalition is  $A$  will be denoted as  $\lfloor A \rfloor$  and the unitary game which maximal losing coalition is  $A$  we will denote by  $\lceil A \rceil$ .

**Lemma 2.** *Let  $A \subseteq N$ . Then*

$$\dim_{\mathcal{U}_1} \lfloor A \rfloor = |A|, \quad \text{codim}_{\mathcal{U}_2} \lceil A \rceil = |N \setminus A|$$

where  $\mathcal{U}_1$  is the class of unitary games and  $\mathcal{U}_2$  is the class of unanimity games.

*Proof.* For the dimension of  $\lfloor A \rfloor$  it is enough to see that

$$\lfloor A \rfloor = \bigcap_{a \in A} \lceil N \setminus \{a\} \rceil \quad (2)$$

A coalition  $W$  of the game  $\lfloor A \rfloor$  is winning if and only if  $A \subseteq W$ .  $W$  is winning in the game  $\lceil N \setminus \{a\} \rceil$  if and only if  $W \not\subseteq N \setminus \{a\}$ . So  $W$  is a winning coalition of the game  $\bigcap_{a \in A} \lceil N \setminus \{a\} \rceil$  if and only if  $\forall a \in A W \not\subseteq N \setminus \{a\}$ , which holds if and only if  $A \subseteq W$ . Thus the equality (2) holds. Moreover, none of the games  $\lceil N \setminus \{a\} \rceil$  for  $a \in A$  can be omitted. It follows that  $\dim_{\mathcal{U}_1} \lfloor A \rfloor = |A|$ . Using this result and Theorem 3 we obtain that  $\text{codim}_{\mathcal{U}_2} \lceil A \rceil = |N \setminus A|$ . One just have to observe that  $\lceil A \rceil = \lfloor N \setminus A \rfloor^*$ .

**Proposition 2.** *Let  $G$  be a game such that  $\text{codim}_{\mathcal{U}_2} G = k$ . Let  $\{A_1, \dots, A_k\}$  be the set of all minimal winning coalitions of  $G$  ( $G = \lfloor A_1 \rfloor \cap \lfloor A_2 \rfloor \cap \dots \cap \lfloor A_k \rfloor$ ). Let  $A$  be a subset of the set of players. Then*

$$A \text{ is admissible if and only if } \forall_{m \in \{1, \dots, k\}} \exists!_{a \in A} a \in A_m \quad (3)$$

We claim that there is a 1-1 correspondence between admissible subsets and maximal losing coalitions of the game  $G$  given by  $\{x_1, x_2, \dots\} \mapsto N \setminus \{x_1, x_2, \dots\}$ . Thus  $G$  can be expressed as

$$G = \bigcap_{A \subseteq N: A \text{-admissible}} \lfloor N \setminus A \rfloor \quad (4)$$

*Proof.* All admissible subsets can be constructed in the following manner:

We choose a sequence of players with respect to the conditions:

$$\begin{aligned}
 x_1 &\in A_1 \cup \dots \cup A_k & (5) \\
 x_2 &\in \bigcup_{i=1}^k A_i \setminus \left( \bigcup_{i: x_1 \in A_i} A_i \right) \\
 x_3 &\in \bigcup_{i=1}^k A_i \setminus \left( \bigcup_{i: x_1 \in A_i} A_i \cup \bigcup_{i: x_2 \in A_i} A_i \right) \\
 &\vdots \\
 x_l &\in \bigcup_{i=1}^k A_i \setminus \left( \bigcup_{i: x_1 \in A_i \vee x_2 \in A_i \vee \dots \vee x_{l-1} \in A_i} A_i \right) \\
 &\vdots
 \end{aligned}$$

The sequence  $(x_1, x_2, \dots)$  has at most  $k$  elements. Then the set  $\{x_1, x_2, \dots\}$  is an admissible subset.

One can easily see that if  $A$  is admissible then any sequence made of its element satisfies the conditions (5). Let us fix an admissible subset  $A = \{x_1, x_2, \dots, x_l\}$ . Then the coalition  $N \setminus A$  is loosing. A coalition of  $G$  is loosing if and only if it doesn't contain any of the sets  $A_1, A_2, \dots, A_k$ . By the construction for each  $A_m, m = 1, \dots, k$  there exists corresponding  $x_{i_m}$  for some  $i_m \in \{1, 2, \dots, l\}$  such that  $x_{i_m} \in A_m$  and thus  $N \setminus A$  do not contain  $A_m$  as  $m$  was arbitrary  $N \setminus A$  is a loosing coalition.

Now let us show that  $N \setminus A$  is a maximal loosing coalition. It is so, because for each set  $A_m, m = 1, \dots, l$  there exists exactly one element of  $A$  which belongs to  $A_m$ .

Clearly, if  $L$  is a maximal loosing coalition then  $N \setminus L$  posses the above property and thus it is defined by an admissible subset. The equation (4) follows from the fact that every game  $G$  is equal to the intersection of unitary games whose unique maximal loosing coalition is a maximal loosing coalition of  $G$ .

**Corollary 3.** *Let  $G$  be a game such that  $\text{codim}_{\mathcal{U}_2} G = k$  then its  $\mathcal{U}_1$  dimension is equal to the number of admissible subsets of players and for small  $k$  it is given by the formulas:*

$$k = 1 \dim_{\mathcal{U}_1} G = |A_1|$$

$$k = 2 \dim_{\mathcal{U}_1} G = |A_1 \cap A_2| + |A_1 \setminus A_2| \cdot |A_2 \setminus A_2|$$

$$k = 3 \left\{ \begin{array}{l}
 \dim_{\mathcal{U}_1} G = |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \setminus A_3| \cdot |A_3 \setminus (A_1 \cup A_2)| \\
 \quad + |A_1 \cap A_3 \setminus A_2| \cdot |A_2 \setminus (A_1 \cup A_3)| \\
 \quad + |A_2 \cap A_3 \setminus A_1| \cdot |A_1 \setminus (A_2 \cup A_3)| \\
 \quad + |A_1 \setminus (A_2 \cup A_3)| \\
 \quad + |A_2 \setminus (A_1 \cup A_3)| \\
 \quad + |A_3 \setminus (A_1 \cup A_2)|
 \end{array} \right.$$

*Proof.* We already know that  $\mathcal{U}_2$ -codimension of a game is equal to the number of maximal losing coalitions. It remains to show the formulas, for  $k = 1$  we did it in lemma 2. For the case  $k = 2$  and  $k = 3$  it is straightforward using the characterisation of admissible subsets given in (3).

**Corollary 4.** *Let  $G$  be a game of  $\mathcal{U}_1$ -dimension equal to  $k$ , and let  $A_1, \dots, A_k$  denote the maximal losing coalitions of  $G$ , i.e.  $G = \bigcap_{i=1}^k [A_i]$ . Then the dual game  $G^*$  has  $\mathcal{U}_2$ -codimension equal to  $k$ . All its minimal winning coalitions are  $N \setminus A_1, \dots, N \setminus A_k$ .*

$$\text{codim}_{\mathcal{U}_2} G = \text{dim}_{\mathcal{U}_1} G^* = \text{dim}_{\mathcal{U}_1} \bigcup_{i=1}^k [N \setminus A_i]$$

And the above theorem provides a direct formula for the  $\mathcal{U}_2$ -codimension of a game of a given  $\mathcal{U}_1$ -dimension. To obtain the formulas like in the preceding corollary one just have to replace each  $A_i$  by its complement. Thus in particular admissible subsets of  $G^*$ , or equivalently minimal winning coalitions of  $G$ , in this setting are the one with respect to the condition:

$$A \text{ is admissible if and only if } \forall_{m \in \{1, \dots, k\}} \exists!_{a \in A} a \notin A_m \quad (6)$$

*Example 1.* Let  $G$  be a game with three players  $\{1, 2, 3\}$ . Let the maximal losing coalitions be  $A_1 = \{1\}$  and  $A_2 = \{2, 3\}$ . Then all its minimal winning coalitions are  $\{\{1, 3\}, \{1, 2\}\}$  and thus its  $\mathcal{U}_2$ -codimension is 2. It is easy to check that the given formula for the  $\mathcal{U}_2$ -codimension gives the same result:  $A_1 \cap A_2 = \emptyset$  and thus the formula reduces to  $\text{dim}_{\mathcal{U}_1} G = |A_1| \cdot |A_2| = 2$ . The admissible subsets of players are  $\{2, 3\}$  and  $\{1\}$ . In this example they are the same as maximal losing coalitions, this is just because this game has the property that a coalition is a maximal losing coalition if its complement is a maximal losing coalition.

## 7. Conclusions

The notions of dimension and codimension introduced in this paper extend the classical notion of dimension for simple games. The paper proposes several subclasses of simple games with associated dimension or codimension. The most relevant result is the existence of a smallest subclass of games with dimension and the existence of a smallest subclass of games with codimension. For these smallest classes we found a closed formula to compute the dimension or codimension of a given simple game.

There is an analogy between dimension or codimension and basis of vector spaces. In fact, a basis of a vector space is a maximal collection of linearly independent vectors. A class of simple games is a ‘basis’ with the intersection as a basic operation iff each game has dimension for this class and it is a maximal class of games in which none of them is intersection of some other members of the family (maximal independent set of games). We have proved that the class of unitary games is a ‘basis’ (Theorem 5-(i)) and its constructive proof shows that the space of all simple games with the intersection as operation has *only one basis*. An analogous reasoning follows for the class of unitary games with the union as operation. Analogously, a class of simple games is a ‘basis’ with the union as a basic operation iff each game has codimension with for this class and it is a maximal class of games in which none of them is union of some other members of the family (maximal

independent set of games). We have proved that the class of unanimity games is a ‘basis’ (Theorem 5-(ii)) and its constructive proof shows that the space of all simple games with the union as operation has *only one basis*.

Some hints for future research concern, for example, the classes of homogeneous games, complete (or linear) games or weakly linear games (introduced in (Carreras and Freixas, 2008)). It becomes of interest to study whether there exist a sequence of weighted games  $G_m$  with  $\mathcal{H}_1$ -dimension equal to  $m$  for all positive integer  $m$ , and further, whether there exist a sequence of weighted games  $G_m$  with exponential  $\mathcal{H}_1$ -dimension. It becomes of interest to investigate whether there exist sequences of simple games  $G_m$  with  $\mathcal{L}$ -dimension equal to  $m$  for all positive integer  $m$ , and further, whether there exist sequences of simple games  $G_m$  with exponential  $\mathcal{L}$ -dimension. If the answers to the previous questions for complete games are affirmative then we could think of similar questions for weakly linear games instead of complete games. More general, which properties need to fulfill a proper subclass of simple games, namely  $\mathcal{C}$ , in such a way that the  $\mathcal{C}$ -dimension of every simple game is bounded above. Analogous questions may be asked for codimension.

It also deserves attention to extend the notions of dimension and codimension to voting games with abstention see e.g. (Fishburn, 1973), (Rubenstein, 1980), (Felsenthal and Machover, 1997), (Tchantcho, et. al. 2008). Indeed, a general notion of weighted voting system for voting systems with several alternatives in the input and output has been developed in (Freixas and Zwicker, 2003).

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