Consensus, Dissension and Precision in Group Decision Making by means of an Algebraic Extension of Hesitant Fuzzy Linguistic Term Sets

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Abstract

Present measures of the degree of agreement in group decision-making using hesitant fuzzy linguistic term sets allow consensus or agreement measurement when decision makers' assessments involve hesitance. Yet they do not discriminate with different degrees of consensus among situations with discordant or polarized assessments. The visualization of differences among groups for which there is no agreement but different possible levels of disagreement is an important issue in collective decision-making situations. In this paper, we propose new collective and individual consensus measures that explicitly consider the hesitance of the decision makers' hesitance in giving an opinion and also the gap between non-overlapping assessments, thus allowing the measurement of the polarization present within the group's opinions. In addition, an expert's profile is defined by considering the expert's behavior in previous assessments in group decision-making processes in terms of precision and dissension.

Keywords: Hesitant fuzzy linguistic term sets, Group decision making, Consensus measures, Decision maker's profile

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Introduction

Several studies have shown that, in general, people do not use purely quantitative models when expressing preferences and interests and are more comfortable using global or abstract forms, that can be understood as models based on qualitative or linguistic information [1, 2, 3]. Analogously, in Group Decision-Making (GDM) environments, the design of systems to facilitate decision-making processes is considered suitable for describing alternatives to be made in terms of non-numerical values and reflect the uncertainty inherent in human reasoning [4, 5, 6, 7, 8]. In the literature, this impreciseness has been modeled with intervals or fuzzy values through a linguistic approach [9, 10, 11].

Rodríguez *et al.* in [9] introduced the Hesitant Fuzzy Linguistic Term Sets (HFLTSs) over a well-ordered set of linguistic labels to deal with decision-making situations through hesitant fuzzy linguistic assessments. In this way, one can express not only the uncertainty but also the hesitance inherent in human reasoning. There are several contributions in the literature that have studied HFLTSs, their properties, aggregation functions, preference relations, distances and so on [12, 13, 14, 15, 16]. These approaches have contributed either from a theoretical point of view or by proposing different applications. An algebraic extension of the set of HFLTSs is presented in [17] to take into account the gap between non-overlapping assessments.

In recent times, consensus in GDM problems through HFLTSs has been studied by several approaches [12, 18, 19, 20, 21, 22]. While some of them focus on the aim of quantifying the level of agreement, some others focus on the consensus reaching process. The problem is set, for all of them, with a group of experts or Decision Makers (DMs) evaluating a set of several alternatives by means of HFLTSs. Despite this, some differences emerge among the approaches that try to quantify the consensus level. A first key difference between them is that, while some approaches study, for each alternative, the consensus of an expert with respect to the rest of the group [12, 20], others study the consensus of the whole group on each alternative [18, 19, 21]. Both types of consensus

approaches might be useful under different kinds of situations: while approaches of the first type can be used to evaluate the relation of each expert with respect to the group, approaches of the second type can be used to evaluate the available alternatives. For instance, when in a GDM process the most dissenting decision makers are asked to reconsider their opinion, a measure of the first kind should be used. On the contrary, when everyone is asked to reconsider his or her assessment on the most controversial alternative, a second type measure should be used instead. In this paper, we propose a new measure of consensus that can be adapted to the measurement of both individual and collective consensus.

The second main difference among approaches lies in whether the definition of the measure of consensus is based on the concept of distance or on the concept of similarity. On the one hand, the consensus level presented in [12] is a distance-based measure. According to the distance that it is used in [12], if two opinions do not overlap, the consensus level is always zero, regardless how far apart the opinions are. This is because the distance used does not take into consideration the gap between HFLTSs in the cases in which the intersection is the empty set. In this paper we define more accurate agreement measures, based on the distance presented in [13] that does take into consideration this gap. On the other hand, the measures presented in [18, 19, 20, 21] are not distance-based but similarity-based. The concept of similarity between HFLTSs is presented in [18], and later used in [21], based on the comparison, between two experts, of their preferences of a given alternative over another one and extended in [19] as a comparison, between two experts, of their assessment of a specific alternative. In any case, this similarity concept neither takes into consideration how distant non-overlapping assessments are nor the level of hesitance used by the experts when assessing an alternative. The measures presented in this paper solve these issues by considering both the hesitance of the assessments and the gap between them if they do not overlap.

Selecting or prioritizing suitable experts or DMs is a frequent problem in GDM applications in real situations [23, 24]. This paper introduces the concepts of preciseness and dissent of an expert assessing a set of alternatives. This allows

the definition of an expert's profile, which keeps track of how experts have made his/her previous assessments with respect to how precise or how dissenting they are. These profiles characterize the up-to-date behavior of experts in GDM processes and can be useful for the task of selecting the appropriate experts to form part of future committees or decision groups.

The rest of this paper is structured as follows: first, Section 1 presents a summary of the basic concepts in the literature that are used throughout the paper. A new degree of consensus for the whole group on each alternative is introduced in Section 2 with a further comparison study with other similar measures. Section 3 defines a different degree of consensus for an expert with respect to the group and it is also compared with the similar existing measures. A precision-dissension profile is presented in Section 4 to keep track of the assessments of a DM within several groups. Finally, Section 5 presents the main conclusion and lines of future research.

1. Theoretical framework

The aim of this section is to provide a summary of basic concepts related to HFLTSs that appear throughout this paper. In particular, a special focus on the distance between HFLTSs that is used in this work is required.

From this point onwards, let $\mathcal{S}$ denote a finite total ordered set of linguistic terms, $\mathcal{S} = \{a_1, \ldots, a_n\}$ with $a_1 < \cdots < a_n$.

Definition 1. ([9]) A *hesitant fuzzy linguistic term set (HFLTS)* over $\mathcal{S}$ is a subset of consecutive linguistic terms of $\mathcal{S}$, i.e., $\{x \in \mathcal{S} \,|\, a_i \leq x \leq a_j\}$, for some $i, j \in \{1, \ldots, n\}$ with $i \leq j$.

Following the concept of uncertain linguistic term introduced by Xu in [25], in this paper we denote HFLTSs by linguistic intervals. Thus, for the rest of this article, the HFLTS $\{x \in \mathcal{S} \,|\, a_i \leq x \leq a_j\}$ is denoted as $[a_i, a_j]$ or, if $j = i$, $\{a_i\}$. In addition, $\mathcal{H}_\mathcal{S}$ represents the set of all the possible HFLTSs over $\mathcal{S}$ including the empty HFLTS, $\emptyset$.

In order to define a suitable distance between two HFLTSs that takes into consideration not just the intersection of them, but also the gap between them if they do not intersect, an algebraic extension of the set $\mathcal{H}_{\mathcal{S}}^{*} = \mathcal{H}_{\mathcal{S}} - \{\emptyset\}$ is presented in [17] as $\overline{\mathcal{H}_{\mathcal{S}}}$ different than the extension presented in [14] that includes HFLTS with non-consecutive linguistic terms from $\mathcal{S}$. This algebraic extension includes the concepts of the *negative HFLTSs*, $-\mathcal{H}_{\mathcal{S}}^{*} = \{-H | H \in \mathcal{H}_{\mathcal{S}}^{*}\}$, the *zero HFLTSs*, $\mathcal{A} = \{\alpha_0, \ldots, \alpha_n\}$ and the *positive HFLTSs*, $\mathcal{H}_{\mathcal{S}}^{*}$. The graph of this set is presented in Figure 1.

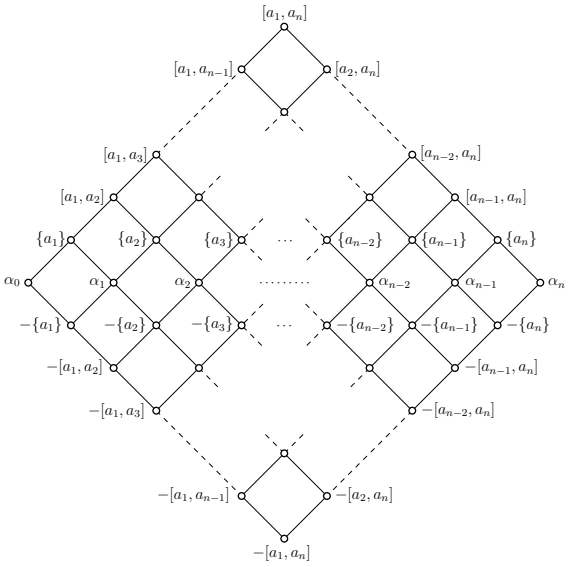


Figure 1: Graph of the extended set of HFLTSs, $\overline{\mathcal{H}_{\mathcal{S}}}$.

In the frame of $\overline{\mathcal{H}_{\mathcal{S}}}$, an *extended inclusion relation* is introduced based on the graph of $\overline{\mathcal{H}_{\mathcal{S}}}$ (Figure 1) and the usual inclusion relation between HFLTSs. Figure 2 shows, as an example, all the elements of $\overline{\mathcal{H}_{\mathcal{S}}}$ included in $[a_1, a_2]$ according to the extended inclusion relation. Additionally, this extended inclusion relation is used to extend the connected union and the intersection of HFLTSs to an operation between elements of $\overline{\mathcal{H}_{\mathcal{S}}}$.

Definition 2. ([17]) Given $H_1, H_2 \in \overline{\mathcal{H}_{\mathcal{S}}}$, then:

a) The *extended connected union of* H_1 *and* H_2, $H_1 \sqcup H_2$, is defined as the least element that contains H_1 and H_2, according to the extended inclusion relation.

b) The *extended intersection of* H_1 *and* H_2, $H_1 \sqcap H_2$, is defined as the largest element being contained in H_1 and H_2, according to the extended inclusion relation.

As an example, Figure 3 shows the extended connected union and the extended intersection of $[a_1, a_2]$ and $\{a_4\}$.

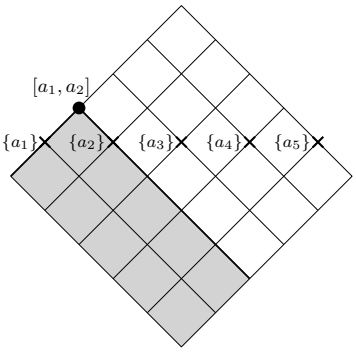


Figure 2: Elements of $\overline{\mathcal{H}_\mathcal{S}}$ included in $[a_1, a_2]$.

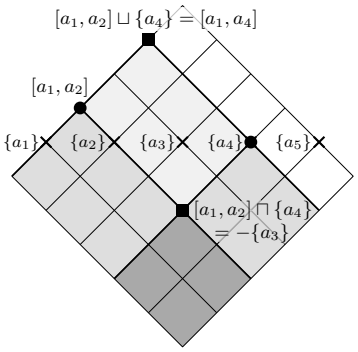


Figure 3: Extended connected union and extended intersection of $[a_1, a_2]$ and $\{a_4\}$.

The negative and zero HFLTSs appear only as a result of the extended intersection of two elements H_1 and H_2 from $\mathcal{H}^*_\mathcal{S}$. If $H_1 \sqcap H_2 = -[a_i, a_j]$ with $i \leq j$, then there is a gap of $[a_i, a_j]$ between them. Whilst, if $H_1 \sqcap H_2 = \alpha_i$, then H_1 and H_2 are consecutive, with one of them ending at a_i and the other one starting at a_{i+1}.

Finally, given $H \in \overline{\mathcal{H}_\mathcal{S}}$, the *width* of H, $\mathcal{W}(H)$, is defined as the cardinal of H if $H \in \mathcal{H}^*_\mathcal{S}$, $-card(-H)$ if H is a negative HFLTS or 0 if H is a zero HFLTS. All these concepts are used to introduce the following distance between HFLTSs:

Definition 3. ([17]) Let $H_1, H_2 \in \overline{\mathcal{H}_\mathcal{S}}$, then $D(H_1, H_2) := \mathcal{W}(H_1 \sqcup H_2) - \mathcal{W}(H_1 \sqcap H_2)$ provides a distance in $\overline{\mathcal{H}_\mathcal{S}}$.

Remark 1. Notice that since the $\mathcal{W}$ operator is based on the concept of cardinal, it works under the assumption of a uniformly distributed set of linguistic terms $\mathcal{S}$. If this is not the case, the cardinal operator should be replaced in the definition of *width* by a measure μ on $\mathcal{H}_{\mathcal{S}}$, such that $\mu(H)$ represents the *size* of the semantic content of H, for all $H \in \mathcal{H}_{\mathcal{S}}$.

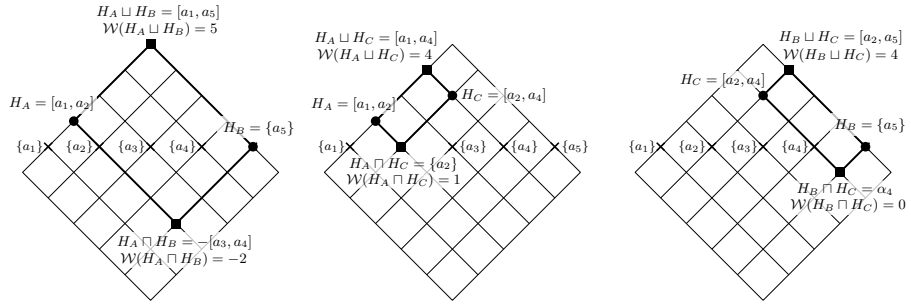
The distance provided by Definition 3 has three main advantages with respect to other measures between HFLTSs existing in the literature [15]: first of all, this new measure takes explicitly into consideration the gap between two non-overlapping HFLTSs; secondly, it is simply computed even between HFLTSs with different cardinalities and, finally, this measure satisfies the triangle inequality and, therefore, it is a distance. From here on, all computations of distances between HFLTSs appearing in this article are done based on this definition. For this reason, and for the sake of comprehensiveness, let us present the following example to illustrate all the foregoing concepts:

Example 1. Let $a_1 =$ *very bad*, $a_2 =$ *bad*, $a_3 =$ *regular*, $a_4 =$ *good* and $a_5 =$ *very good* be 5 linguistic labels defining the set $\mathcal{S} = \{a_1, a_2, a_3, a_4, a_5\}$. Then, three possible assessments by means of $\mathcal{S}$ are $A =$ *"below regular"*, $B =$ *"very good"* and $C =$ *"neither very good nor very bad"* and their corresponding HFLTS by means of $\mathcal{S}$ are $H_A = [a_1, a_2]$, $H_B = \{a_5\}$ and $H_C = [a_2, a_4]$ respectively. The extended connected union and extended intersection of all the possible pairs among H_A, H_B and H_C are shown in Figure 4.

According to these results, $D(H_A, H_B) = 5 - (-2) = 7$, $D(H_A, H_C) = 4 - 1 = 3$ and $D(H_B, H_C) = 4 - 0 = 4$.

Remark 2. In order to ease future computations, it is important to note that, as proved in [17], the presented distance is equivalent to the taxicab metric in the graph of $\overline{\mathcal{H}_{\mathcal{S}}}$. Therefore, if $H_1 = [a_{i_1}, a_{j_1}]$ and $H_2 = [a_{i_2}, a_{j_2}]$, then $D(H_1, H_2)$ can be calculated as $|i_1 - i_2| + |j_1 - j_2|$. This fact can be easily seen in the previous example and in Figure 1.

The next step in any GDM situation is to assess not just one single alternative, but a set of them. With the aim of dealing with this kind of situations,



(a) H_A and H_B. (b) H_A and H_C. (c) H_B and H_C.

Figure 4: Extended connected union and extended intersection of two HFLTSs.

Montserrat-Adell *et al.* in [13] developed the concept of *Hesitant Fuzzy Linguistic Description (HFLD)* of a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ as a function F_H on Λ such that for all $\lambda \in \Lambda$, $F_H(\lambda) \in \mathcal{H}_{\mathcal{S}}^*$. For the rest of this article, each DM or expert is modeled by a HFLD.

Following this definition, the distance D between HFLTSs is extended to a distance, $D^{\mathcal{F}}$, between HFLDs as the addition of the distances between the corresponding HFLTSs for each alternative in Λ. Formally,

Definition 4. ([17]) Let F_H^1 and F_H^2 be two HFLDs of a set $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of $\mathcal{H}_{\mathcal{S}}$, with $F_H^1(\lambda_i) = H_i^1$ and $F_H^2(\lambda_i) = H_i^2$, for all $i \in \{1, \ldots, r\}$. Then, the *distance* $D^{\mathcal{F}}$ *between* F_H^1 *and* F_H^2 is defined as:

$$D^{\mathcal{F}}(F_H^1, F_H^2) = \sum_{t=1}^{r} D(H_t^1, H_t^2).$$

Finally, the distance $D^{\mathcal{F}}$ is used to propose a central opinion (or centroid) of a group of DMs about a set of alternatives Λ as the HFLD that minimizes the addition of distances to the opinion of each expert.

Definition 5. ([17]) Let Λ be a set of r alternatives, G a group of k DMs and $F_H^1, \ldots, F_H^k$ the HFLDs of Λ provided by the DMs in G. Then, a *centroid of the group* is:

$$F_H^C = arg \min_{F_H^x \in (\mathcal{H}_{\mathcal{S}}^*)^r} \sum_{i=1}^{k} D^{\mathcal{F}}(F_H^x, F_H^i).$$

Notice that this centroid does not have to be unique and this might lead us to some issues when working with the centroid. To fix this problem, let us consider the following remark.

Remark 3. In order to ease the calculation of the centroid, it is proved in [17] that, for each specific alternative $\lambda \in \Lambda$, if $F_H^p(\lambda) = [a_{i_p}, a_{j_p}]$ for $p \in \{1, \ldots, k\}$, then the set of all the HFLTS associated to the centroid of the group for λ is:

$$\{[a_i, a_j] \in \mathcal{H}_{\mathcal{S}}^* \mid i \in \mathcal{M}(i_1, \ldots, i_k), j \in \mathcal{M}(j_1, \ldots, j_k)\},$$

where $\mathcal{M}(\ )$ is the set that contains just the median of the index values if k is odd or any integer number between the two central index values sorted from smallest to largest if k is even. Therefore, if k is odd, the centroid is unique, while if k is even, the controid might be not unique. Henceforth, to avoid possible problems with a non-unique centroid, when there are more than one possible centroid of the group, the one with a highest cardinality, which can be understood as the most hesitant one, is considered as $F_H^C(\lambda)$. Thus, $F_H^C(\lambda) = [a_{i^*}, a_{j^*}]$, where $i^* = \min(\mathcal{M}(i_1, \ldots, i_k))$ and $j^* = \max(\mathcal{M}(j_1, \ldots, j_k))$.

Example 2. Let G be a group of 5 DMs assessing a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_4\}$ by means of HFLTSs over the set $\mathcal{S} = \{a_1, \ldots, a_5\}$ from Example 1, and let $F_H^1, F_H^2, F_H^3, F_H^4, F_H^5$ be the HFLDs modeling their corresponding assessments shown in the Table 1. Then, the centroid of the group, F_H^C, can be easily computed by median calculations as stated in Remark 3 providing the results shown in the same table.

Note that, contrary to some other common aggregation operators such as the union, the centroid of the group is robust with respect to extreme hesitances in one expert. When aggregating with the union, a big hesitance in the opinion of one of the experts implies a big hesitance in the central opinion. That is not the case with the centroid from Definition 5. This can be seen, for instance, in alternative λ_1, where the assessment of one of the experts is $[a_1, a_5]$, but the centroid is $[a_2, a_3]$. That it to say that a large hesitance of a DM does not necessarily imply a lack of precision of the centroid.

	F_H^1	F_H^2	F_H^3	F_H^4	F_H^5	$\boldsymbol{F_H^C}$
λ_1	$[a_1, a_2]$	$\{a_2\}$	$[a_1, a_5]$	$[a_4, a_5]$	$[a_2, a_3]$	$\boldsymbol{[a_2, a_3]}$
λ_2	$[a_2, a_4]$	$\{a_3\}$	$[a_1, a_5]$	$[a_3, a_4]$	$[a_2, a_3]$	$\boldsymbol{[a_2, a_4]}$
λ_3	$[a_4, a_5]$	$\{a_5\}$	$[a_4, a_5]$	$[a_1, a_2]$	$[a_4, a_5]$	$\boldsymbol{[a_4, a_5]}$
λ_4	$\{a_3\}$	$\{a_3\}$	$[a_2, a_3]$	$[a_3, a_4]$	$\{a_3\}$	$\boldsymbol{\{a_3\}}$

Table 1: Centroid of the group G for Λ from Example 2.

Note that, since in this example there are 5 DMs, which is an odd number, the centroid of the group obtained from Definition 5 is unique.

2. Collective consensus

In this section, a new degree of consensus of the whole group on a specific alternative or a set of alternatives is introduced based on the distance proposed in [17]. This new measure seeks to quantify the level of agreement within a group of DMs on a specific alternative or a set of alternatives. A further study on the properties of the introduced measure and a comparison with the similar existing measures in the literature are also presented in this section. Finally, an example is provided to illustrate the commented properties.

2.1. A collective degree of consensus

The idea of this new degree of consensus arises with the need of finding a measure that depends neither on the number of DMs assessing the alternatives nor on the number of linguistic labels used in $\mathcal{S}$. Thus, the degree of consensus presented in this section is a normalization of the addition of distances between the centroid of the group and each of the HFLDs given by the DMs. In order to define this normalization, the first step is to study the maximum value that this addition of distances can take.

Lemma 1. Let F_H^1, F_H^2 be two HFLDs of the set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of $\mathcal{S} = \{a_1, \ldots, a_n\}$. Then,

$$D^{\mathcal{F}}(F_H^1, F_H^2) \leq r \cdot (2n-2).$$

Proof. According to Definition 3, the most distant HFLTSs are $H_1 = \{a_1\}$ and $H_2 = \{a_n\}$. In this case, $H_1 \sqcup H_2 = [a_1, a_n]$ and $H_1 \sqcap H_2 = -[a_2, a_{n-1}]$. Thus, $D(H_1, H_2) = \mathcal{W}([a_1, a_n]) - \mathcal{W}(-[a_2, a_{n-1}]) = n - (-(n-2)) = 2n-2$. Consequently, the most distant HFLDs are those that for all the alternatives, the corresponding two HFLTSs used by each HFLD are the most distant ones. In such case,

$$D^{\mathcal{F}}(F_H^1, F_H^2) = \sum_{i=1}^{r}(2n-2) = r \cdot (2n-2).$$

□

Therefore, Lemma 1 can be used to find an upper bound for the addition of distances between the centroid of a group and each of the DMs of the assessing group.

Proposition 1. Let $F_H^1, \ldots, F_H^k$ be the HFLDs of a group of k DMs of the set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of $\mathcal{S} = \{a_1, \ldots, a_n\}$, and let F_H^C be the centroid of the group. Then,

$$\sum_{i=1}^{k} D^{\mathcal{F}}(F_H^C, F_H^i) \leq k \cdot r \cdot (n-1).$$

Proof. If k is even, the worst-case scenario is given when, for each of the alternatives $k/2$ of the DMs have assessed it with $\{a_1\}$, and the other $k/2$ of the DMs have assessed it with $\{a_n\}$. In such case, calculating the corresponding medians, we get that any HFLD could be considered as the centroid of the group given that all of them give the same addition of distances, but, according to Remark 3, $\mathcal{F}_H^C(\lambda_i) = [a_1, a_n]$ for $i = 1, \ldots, r$, then:

$$\sum_{i=1}^{k} D^{\mathcal{F}}(F_H^C, F_H^i) = \frac{k}{2} \cdot r \cdot (n-1) + \frac{k}{2} \cdot r \cdot (n-1) = k \cdot r \cdot (n-1).$$

If k is odd, the worst-case scenario is met when, for each of the alternatives, $\lfloor k/2 \rfloor$ of the DMs have assessed it with $\{a_1\}$ and $\lfloor k/2 \rfloor$ of the DMs have assessed it with $\{a_n\}$, regardless what is the last HFLTS. If so, based on the median calculations, the centroid of the group is equal, for each alternative, to this last HFLTS, and the addition of distances is equal to $(k-1) \cdot r \cdot (n-1)$. Choosing, for example, the last HFLTS to be $\{a_1\}$ for all the alternatives, then:

$$\sum_{i=1}^{k} D^{\mathcal{F}}(F_H^C, F_H^i) = \left(\left\lfloor \frac{k}{2} \right\rfloor + 1 \right) \cdot 0 + \left\lfloor \frac{k}{2} \right\rfloor \cdot r \cdot (2n-2)$$

$$= (k-1) \cdot r \cdot (n-1) \leq k \cdot r \cdot (n-1).$$

□

Corollary 1. Under the same conditions as in Property 1, in the particular case where $r = 1$, just one single alternative to be assessed, the upper bound results to be $k \cdot (n-1)$.

The upper bounds provided in Proposition 1 and Corollary 1 for the total addition of distances between the centroid of the group and all the HFLD of the group enables us to proceed with the normalization that leads us to the definition of a measure of agreement within the group, in a similar way to [12, 19], as follows:

Definition 6. Let $F_H^1, \ldots, F_H^k$ be the HFLDs given by a group G of k DMs about the set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of $\mathcal{S} = \{a_1, \ldots, a_n\}$ and let F_H^C the centroid of the group, being $H_j^i = F_H^i(\lambda_j)$ for $i \in \{1, \ldots, k, C\}$. Then, the *degree of consensus of G on λ_j* is defined as:

$$\delta_{\lambda_j}(G) = 1 - \frac{\sum_{i=1}^{k} D(H_j^C, H_j^i)}{k \cdot (n-1)}.$$

Analogously, the *degree of consensus of G on Λ* is defined as:

$$\delta_{\Lambda}(G) = 1 - \frac{\sum_{i=1}^{k} D^{\mathcal{F}}(F_H^C, F_H^i)}{k \cdot r \cdot (n-1)}.$$

Note that, by Proposition 1, $0 \leq \delta_\Lambda(G) \leq 1$. The closer to 0 $\delta_\Lambda(G)$ is, the closer to its maximum value the addition of distances is, which implies a lot of disagreement. On the contrary, the closer to 1 $\delta_\Lambda(G)$ is, the smaller the addition of distances is, and that means a high level of agreement. The same reasoning is valid for the degree of consensus of one specific alternative.

Notice also that, the upper bound given by Proposition 1 can be reached only when k is even. Thus, if k is odd, the degree of consensus cannot be zero. This fact is coherent given that situations with maximum disagreement arise when half of the experts assess an alternative with the worst linguistic label and the other half do it with the best linguistic label. Obviously, this situation is only possible with an even number of opinions.

Property 1. Let G be a group of k DMs assessing a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of $\mathcal{S} = \{a_1, \ldots, a_n\}$. Then,

$$\delta_\Lambda(G) = \frac{\sum_{j=1}^{r} \delta_{\lambda_j}(G)}{r}.$$

Proof. Let $F_H^1, \ldots, F_H^k$ be the HFLDs given by the DMs and F_H^C the centroid of the group, being $H_j^i = F_H^i(\lambda_j)$ for $i \in \{1, \ldots, k, C\}$. Then,

$$\frac{\sum_{j=1}^{r} \delta_{\lambda_j}(G)}{r} = \frac{\sum_{j=1}^{r} 1 - \frac{\sum_{i=1}^{k} D(H_j^C, H_j^i)}{k \cdot (n-1)}}{r} = \frac{r - \frac{\sum_{i=1}^{k} \sum_{j=1}^{r} D(H_j^C, H_j^i)}{k \cdot (n-1)}}{r}$$

$$= 1 - \frac{\sum_{i=1}^{k} D^{\mathcal{F}}(F_H^C, F_H^i)}{k \cdot r \cdot (n-1)} = \delta_\Lambda(G).$$

□

This property states the consistency between the degree of consensus on each alternative and on the whole set of alternatives.

2.2. Comparison with existing measures

This section presents a comparison of the degree of consensus defined in Section 2.1 with similar existing measures. Out of all the agreement measures

for GDM by means of HFLTSs summarized in the Introduction, the ones defined as a degree of consensus on the alternatives are those presented by Rodriguez *et al.* in [18] and by Wu *et al.* in [19] and in [21]. When calculating the agreement on an alternative λ_j, there is a main difference between these two measures: the first and third degrees are defined based on the preference of said alternative over another alternative $\lambda_k, \forall k \neq j$, while the second one is based just on the assessment of λ_j, regardless the assessment of the rest of alternatives. This leads us to the automatic conclusion that the most similar measure to the one presented in this paper is the second one. For this reason, we proceed with a further study to compare the results provided by both measures.

To begin with, let us summarize the measure presented in [19]. Wu *et al.* defined the consensus level within all the DMs for an alternative as the average of all the similarity degrees between any pair of DMs about this alternative. This similarity degree is based on what they call the mean (or expected value) of a HFLTS, which is just the center of the HFLTS in the case of a set $\mathcal{S}$ with uniform and symmetric linguistic labels. Translated to the notation used in this paper, in which $H^i = [a_{x_i}, a_{y_i}]$ for $i \in \{1, \ldots, k\}$ are the assessments given by a group of k DMs about an alternative λ by means of $\mathcal{S} = \{a_1, \ldots, a_n\}$, the consensus level within all the DMs for λ defined in [19] can be calculated as

$$ca_\lambda = \frac{\displaystyle\sum_{i=1}^{k}\sum_{j>i}^{k}\left(1 - \frac{\left|\frac{x_j+y_j}{2} - \frac{x_i+y_i}{2}\right|}{(n-1)}\right)}{\binom{k}{2}}. \tag{1}$$

Remark 4. The fact that this measure ignores the width of the HFLTSs and, in the case with uniform and symmetric linguistic labels, is based just on the mean of the HFLTSs, implies that the hesitance of each expert is not taken into consideration. Therefore, the similarity degree of two experts assessing an alternative with HFLTSs with the same expected value but with different levels of hesitance would be 1, the maximum.

On the other hand, the degree of consensus presented in Section 2.1 can be

rewritten in a similar way as shown in the following lemma:

Lemma 2. Let $H^1, \ldots, H^k$ be the assessments of a group G of k DMs about an alternative λ, and let H^C be the centroid of G for λ, where $H^i = [a_{x_i}, a_{y_i}]$ for $i \in \{1, \ldots, k, C\}$. Then,

$$\delta_\lambda(G) = 1 - \frac{\sum_{i=1}^{k} |x_i - x_C| + |y_i - y_C|}{k \cdot (n-1)}.$$

Proof. The proof is straightforward by Definition 6 and Remark 3. □

In order to compare the two consensus measures, we first need the following definition:

Definition 7. Let $H^1, \ldots, H^k$ be a collection of HFLTSs over $\mathcal{S}$, where $H^i = [a_{x_i}, a_{y_i}]$ for $i \in \{1, \ldots, k\}$. Then,

(a) H^i is *lower* than H^j, $H_i \preccurlyeq H_j$, if $x_i \leq x_j$ and $y_i \leq y_j$.

(b) $H^1, \ldots, H^k$ are *sorted* if $H^1 \preccurlyeq H^2 \preccurlyeq \ldots \preccurlyeq H^k$.

(c) $H^1, \ldots, H^k$ are *sortable* if there exists a permutation of them which is sorted.

Property 2. Let $H^1, \ldots, H^k$ be the assessments of a group G of k DMs about an alternative λ. Then,

$$\delta_\lambda(G) \leq ca_\lambda$$

and the equality is met when $H^1, \ldots, H^k$ are sortable and the $k-2$ central opinions are the same.

Proof. For this proof, let us assume $H^i = [a_{x_i}, a_{y_i}]$ for $i \in \{1, \ldots, k, C\}$. Thus, beginning with Equation 1,

$$ca_\lambda = \frac{\sum_{i=1}^{k} \sum_{j>i}^{k} \left(1 - \frac{\left|\frac{x_j+y_j}{2} - \frac{x_i+y_i}{2}\right|}{(n-1)}\right)}{\binom{k}{2}}$$

$$= \frac{\binom{k}{2} - \frac{1}{2\cdot(n-1)} \cdot \sum_{i=1}^{k}\sum_{j>i}^{k} (|x_j + y_j - x_i - y_i|)}{\binom{k}{2}}$$

$$= 1 - \frac{\sum_{i=1}^{k}\sum_{j>i}^{k} |x_j + y_j - x_i - y_i|}{\frac{k\cdot(k-1)}{2}\cdot 2\cdot(n-1)}$$

$$\geq 1 - \frac{\sum_{i=1}^{k}\sum_{j>i}^{k} |x_j - x_i| + |y_j - y_i|}{k\cdot(k-1)\cdot(n-1)}$$

$$= 1 - \frac{\sum_{i=1}^{k}\sum_{j>i}^{k} |x_j - x_C - x_i + x_C| + |y_j - y_C - y_i + y_C|}{k\cdot(k-1)\cdot(n-1)}$$

$$\geq 1 - \frac{\sum_{i=1}^{k}\sum_{j>i}^{k} |x_j - x_C| + |x_i - x_C| + |y_j - y_C| + |y_i - y_C|}{k\cdot(k-1)\cdot(n-1)}$$

$$= 1 - \frac{\sum_{i=1}^{k} (k-1)\cdot|x_i - x_C| + (k-1)\cdot|y_i - y_C|}{k\cdot(k-1)\cdot(n-1)}$$

$$= 1 - \frac{(k-1)\left(\sum_{i=1}^{k} |x_i - x_C| + |y_i - y_C|\right)}{k\cdot(k-1)\cdot(n-1)}$$

$$= 1 - \frac{\sum_{i=1}^{k} |x_i - x_C| + |y_i - y_C|}{k\cdot(n-1)} = \delta_\lambda(G).$$

Additionally, for the first inequality to be an equality $(x_j - x_i)$ and $(y_j - y_i)$ have to have the same sign for any $j > i$, which means that $H^1, \ldots, H^k$ have to be sorted. Since the order of the DMs is not important, it is enough for $H^1, \ldots, H^k$ to be sortable. On the other hand, for the equality to be met in the second inequality, $(x_i - x_C)$ and $(x_j - x_C)$ have to have opposite signs or be zero for any $j > i$, and analogously for $(y_i - y_C)$ and $(y_j - y_C)$. Given that, because of the previous condition, we can assume $H^1, \ldots, H^k$ to be sorted, this happens only if $x_2 = \ldots = x_{k-1} = x_C$ and $y_2 = \ldots = y_{k-1} = y_C$. □

The reason why $\delta_\lambda(G) \leq ca_\lambda$ is explained by the fact that ca_λ does not take into account the hesitance of the experts and, therefore, for some alternatives the consensus level is higher that what it would be expected.

Additionally, if these degrees of consensus are applied to to end-users of a product instead of a set of experts, then the number of DMs might be very large, and the time complexity of calculating the consensus level for an alternative becomes a crucial point. Given that the degree of consensus presented in [18] and in [21] compute the similarity between each pair of DMs about the preference of the studied alternative over all the other ones one by one, its time complexity is $O(rk^2)$, where k is the number of DMs within the group and r is the number of alternatives to be assessed. The consensus level in [19] studies the similarity between each pair of DMs on a specific alternative, without comparing it with the rest of alternatives. Thus, its time complexity is $O(k^2)$. On the contrary, the degree of consensus presented in Section 2.1 only makes one comparison with the central opinion. Therefore, its time complexity is $O(1)$ once the centroid of the group for the studied alternative is computed. Since this centroid, as staten before, is based on the median calculation, which is known to be done in linear time, the time complexity of $\delta_{\lambda_i}(G)$ is $O(k)$.

Table 2 summarizes the main characteristics of the different collective degrees of consensus using HFLTSs.

2.3. An illustrative example on collective consensus

For an easier understanding of the introduced degree of consensus, in this subsection we present a clarifying example to illustrate its computation. The same example is also used to point out its properties commented in Section 2.2 with respect to similar existing measures.

Example 3. Following Example 2, where G is a group of 5 DMs assessing a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_4\}$ by means of HFLTSs over the set $\mathcal{S} = \{a_1, \ldots, a_5\}$, with the assessments provided in Table 1, we can now proceed to compute the degree of consensus on each of the alternatives in Λ as shown in

	Rodríguez *et al.* [18]	Wu *et al.* [19]	Wu *et al.* [21]	Montserrat-Adell *et al.*
Year	2015	2016	2016	2017
Groupal consensus	✓	✓	✓	✓
Individual consensus				
Distance-based				✓
Similarity-based	✓	✓	✓	
Preference similarity	✓		✓	
Alternative similarity		✓		
Pairwise comparison	✓	✓	✓	
Central opinion comparison				✓
Considers gap				✓
Considers hesitance				✓
Time complexity [a,b]	$O(rk^2)$	$O(k^2)$	$O(rk^2)$	$O(1)+\mathcal{T}_C$

[a] $\mathcal{T}_C$ stands for the time complexity of calculating the central opinion.

[b] For the overall degree of consensus of a set of r alternatives, all times are multiplied by r.

Table 2: Comparison of the presented collective degrees of consensus.

Table 3, where D_j^i stands for $D(H_j^C, H_j^i)$, as well as the degree of consensus for the whole set Λ.

In order to illustrate the properties presented in Section 2.2, we can now use the methodology introduced by Wu *et al.* in [19] to calculate their degree of consensus on each alternative λ_j, ca_j, for $j = 1, \dots 4$. To this end, the first step is to calculate the similarity matrix for each alternative, showing, in a scale from 0 to 1, the agreement between each pair of experts on the corresponding alternative. These similarity coefficients are calculated as one minus the difference between the middle points of the corresponding HFLTSs over $n-1$, where n is the cardinality of $\mathcal{S}$ ($n = 5$ in this example). These similarity matrices are shown in Figure 5.

	D_j^1	D_j^2	D_j^3	D_j^4	D_j^5	$\sum_{i=1}^{5} D_j^i$	$\delta_{\lambda_j}(G)$
λ_1	2	1	3	4	0	10	0.5
λ_2	0	2	2	1	1	6	0.7
λ_3	0	1	0	6	0	7	0.65
λ_4	0	0	1	1	0	2	0.9
Λ	2	4	6	12	1	25	0.6875

Table 3: Degree of consensus on each alternative and on the set Λ.

$$\begin{pmatrix} - & 0.875 & 0.625 & 0.25 & 0.75 \\ & - & 0.75 & 0.375 & 0.875 \\ & & - & 0.625 & 0.875 \\ & & & - & 0.5 \\ & & & & - \end{pmatrix}$$

(a) λ_1

$$\begin{pmatrix} - & 1 & 1 & 0.875 & 0.875 \\ & - & 1 & 0.875 & 0.875 \\ & & - & 0.875 & 0.875 \\ & & & - & 0.75 \\ & & & & - \end{pmatrix}$$

(b) λ_2

$$\begin{pmatrix} - & 0.875 & 1 & 0.25 & 1 \\ & - & 0.875 & 0.125 & 0.875 \\ & & - & 0.25 & 1 \\ & & & - & 0.25 \\ & & & & - \end{pmatrix}$$

(c) λ_3

$$\begin{pmatrix} - & 1 & 0.875 & 0.875 & 1 \\ & - & 0.875 & 0.875 & 1 \\ & & - & 0.75 & 0.875 \\ & & & - & 0.875 \\ & & & & - \end{pmatrix}$$

(d) λ_4

Figure 5: Similarity matrices for each alternative.

Once these matrices are calculated, the next step to get ca_j is, for each alternative, to compute the average of the similarity between each pair of experts. Table 4 presents a comparison of the results of $\delta_{\lambda_j}(G)$ and ca_j on each alternative.

As staten in Property 2, $\delta_{\lambda_j}(G) \leq ca_j$ for all the alternatives, being the equality met in alternatives λ_3 and λ_4. In Table 1, it can be seen that, for λ_3, the assessments are sortable as $F_H^4(\lambda_3) \preccurlyeq F_H^1(\lambda_3) = F_H^3(\lambda_3) = F_H^5(\lambda_3) \preccurlyeq F_H^2(\lambda_3)$, while for λ_4, they are sortable as $F_H^3(\lambda_4) \preccurlyeq F_H^1(\lambda_4) = F_H^2(\lambda_4) = F_H^5(\lambda_4) \preccurlyeq F_H^4(\lambda_4)$. On the contrary, the assessments for alternatives λ_1 and λ_2 are not

	$\delta_{\lambda_j}(G)$	ca_j
λ_1	0.5	0.65
λ_2	0.7	0.9
λ_3	0.65	0.65
λ_4	0.9	0.9

Table 4: δ_{λ_j} and ca_j for the alternatives in Λ.

sortable, for instance $F_H^3(\lambda_1)$ and $F_H^5(\lambda_1)$ or $F_H^1(\lambda_2)$ and $F_H^3(\lambda_2)$, and, therefore, $\delta_{\lambda_j}(G) < ca_j$.

Additionally, alternatives λ_2 and λ_4 are a clear example for Remark 4. Again, in Table 1, it can be seen that the HFLTSs used by the experts to assess the two alternatives have the same mean, but the level of hesitance of the answers is different in the two cases. Given that there is much more hesitance on λ_2, it seems intuitive that the degree of consensus on this alternative is lower than the one on λ_4, where there is much more coincidence of opinions. Table 4 shows that $ca_2 = ca_4$ given that this measure does not take into account the hesitance of the experts while $\delta_{\lambda_2}(G) < \delta_{\lambda_4}(G)$.

This leads us to the conclusion that, under the HFLTSs-based GDM framework, $\delta_{\lambda_j}(G)$ provides a measure of the consensus of a group of experts on a set of alternatives closer to common-sense reasoning.

3. Individual consensus

This section studies the idea of consensus within a group of DMs as the agreement of an expert with respect to the group instead of the agreement of the whole group on an alternative as in Section 2. To this end, a convenient degree of consensus is defined for each expert. Even though there are some other measures already defined in the literature, the convenience of a new measure is explained by the fact that the previous ones present some issues like not considering the hesitance of the assessments or not considering the gap between

non-overlapping assessments. Additionally, this degree is compared with similar already existing measures and also exemplified to point out its properties.

3.1. An individual degree of consensus

As in Definition 6, this new measure is thought to be on a scale from 0 to 1 independently from the number of linguistic labels used in $\mathcal{S}$ and the number of DMs in the group. The degree of consensus presented in this section is a normalization of the distance between the opinion of the expert and the centroid of the group as follows:

Definition 8. Let G be a group of DMs, $\epsilon_1, \ldots, \epsilon_k$, assessing a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of HFLTSs over $\mathcal{S} = \{a_1, \ldots, a_n\}$, and let F_H^i and F_H^C be the HFLDs of ϵ_i for $i = 1, \ldots, k$ and the centroid of the group respectively, with $H_j^i = F_H^i(\lambda_j)$ for $i \in \{1, \ldots, k, C\}$. Then, the *degree of consensus of* ϵ_i *with respect to* G *on* λ_j is defined as:

$$\delta_{\lambda_j}^G(\epsilon_i) = 1 - \frac{D(H_j^C, H_j^i)}{2n-2}.$$

Analogously, the *degree of consensus of* ϵ_i *with respect to* G *on* Λ is defined as:

$$\delta_{\Lambda}^G(\epsilon_i) = 1 - \frac{D^{\mathcal{F}}(F_H^C, F_H^i)}{r \cdot (2n-2)}.$$

By Lemma 1, the upper bound for the distance between two HFLTSs is $2n-2$ and the one for the distance between two HFLDs is $r \cdot (2n-2)$. Thus, it can be easily seen that both $\delta_{\lambda_j}^G(\epsilon_i)$ and $\delta_{\Lambda}^G(\epsilon_i)$ range between 0 and 1. The closer to 1 these coefficients are, the more similar the opinion of ϵ_i is to the centroid, while the closer to 0 the more dissidence there is.

Note that this degree of consensus is 1 only when the opinion of the expert coincides with the centroid of the group and it is 0 if and only if the opinion of the expert is $\{a_1\}$ and the centroid is $\{a_n\}$ or vice versa.

Property 3. Let G be a group of DMs, $\epsilon_1, \ldots, \epsilon_k$, assessing a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of $\mathcal{S} = \{a_1, \ldots, a_n\}$. Then, for $i = 1, \ldots, k$,

$$\delta_{\Lambda}^G(\epsilon_i) = \frac{\sum_{j=1}^{r} \delta_{\lambda_j}^G(\epsilon_i)}{r}.$$

Proof. Let $F_H^1, \ldots, F_H^k$ be the HFLDs given by the DMs and F_H^C the centroid of the group, being $H_j^i = F_H^i(\lambda_j)$ for $i \in \{1, \ldots, k, C\}$. Then,

$$\frac{\sum_{j=1}^{r} \delta_{\lambda_j}^G(\epsilon_i)}{r} = \frac{\sum_{j=1}^{r} 1 - \frac{D(H_j^C, H_j^i)}{(2n-2)}}{r} = \frac{r - \frac{\sum_{j=1}^{r} D(H_j^C, H_j^i)}{(2n-2)}}{r}$$

$$= 1 - \frac{D^{\mathcal{F}}(F_H^C, F_H^i)}{r \cdot (2n-2)} = \delta_\Lambda^G(\epsilon_i).$$

□

In the same way than Property 1, this property provides consistency to the definition of the degree of consensus of an expert with respect to a group on an alternative and on a set of alternatives.

3.2. Comparison with existing measures

As staten before, the degree of consensus for experts introduced in Section 3.1 is similar to some of the measures presented in the literature. The aim of this section is to compare the degree of consensus defined in Section 3.1 with the most similar existing ones.

From the agreement measures by GDM by means of HFLTSs presented in the Introduction, those defined as degrees of consensus for an expert are the ones introduced by Dong *et al.* in [12] and by Wu *et al.* in [20].

On the one hand, Dong *et al.* defined the consensus level of ϵ_i on an alternative based on the intersection and the union of the opinion of ϵ_i and a central opinion as:

$$CL_i = \frac{\text{card}(H^i \cap H^C)}{\text{card}(H^i \cup H^C)}, \tag{2}$$

being H^i the opinion of ϵ_i and H^C the central opinion. The main issue with this consensus level is that, in the case of an empty intersection between the opinion of the expert and the central opinion, the result is always 0, without taking into consideration how far H^i is from H^C. The reason that explains this is the fact that CL_i is based on a distance between HFLTSs, that contrarily to the one from Definition 3, does not take into account the gap between two HFLTSs with null intersection.

Because of this reason, we have considered more interestingly to proceed with a further study to compare the results provided by the consensus measure for experts introduced in [20] with the one given by the degree of consensus for experts presented in this article.

In order to carry on this comparison, we first need to introduce the consensus level proposed by Wu *et al.* It is based on the same idea of similarity than ca_j in Equation 1, but in this case, between the opinion of the expert and a central opinion. In this case, we use the centroid from Definition 5 as central opinion. Therefore, if $F_H^i(\lambda) = [a_{x_i}, a_{y_i}]$ is the opinion of expert ϵ_i on λ and $F_H^C(\lambda) = [a_{x_C}, a_{y_C}]$ is the centroid of the group on λ, then the degree of consensus presented by Wu *et al.* is defined as:

$$SM_\lambda^i = 1 - \frac{\left|\frac{x_i+y_i}{2} - \frac{x_C+y_C}{2}\right|}{n-1}, \tag{3}$$

where n is the cardinal of $\mathcal{S}$. Additionally, they defined the overall consensus level for expert ϵ_i on the set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$, SM_i, as the average of $SM_1^i, \ldots, SM_r^i$.

On the other hand, the following lemma rewrites the degree of consensus from Section 3.1 in a similar way.

Lemma 3. Let G be a group of DMs, $\epsilon_1, \ldots, \epsilon_k$, whose assessments about alternative λ are $H^i = [a_{x_i}, a_{y_i}]$ for $i = 1, \ldots, k$, and let $H^C = [a_{x_C}, a_{y_C}]$ be the centroid of the group for λ. Then,

$$\delta_\lambda^G(\epsilon_i) = 1 - \frac{|x_i - x_C| + |y_i - y_C|}{(2n-2)}.$$

Proof. The proof is straightforward from Definition 8 and Remark 2. □

With the foregoing lemma, we can proceed to compare the two measures.

Property 4. Let G be a group of DMs, $\epsilon_1, \ldots, \epsilon_k$, whose assessments about alternative λ are $H^1, \ldots, H^k$ respectively. Then,

$$\delta_\lambda^G(\epsilon_i) \leq SM_\lambda^i$$

and the equality is met when H^i and H^C are sortable, being H^C the centroid of group G for λ.

Proof. For this proof, let us assume $H^i = [a_{x_i}, a_{y_i}]$ for $i \in \{1, \ldots, k\}$. Thus, beginning with Equation 3,

$$SM^i_\lambda = 1 - \frac{\left|\frac{x_i+y_i}{2} - \frac{x_C+y_C}{2}\right|}{n-1} = 1 - \frac{\frac{1}{2}\left|x_i + y_i - x_C - y_C\right|}{n-1}$$

$$= 1 - \frac{|x_i - x_C + y_i - y_C|}{2n-2} \geq 1 - \frac{|x_i - x_C| + |y_i - y_C|}{2n-2} = \delta^G_\lambda(\epsilon_i).$$

In addition, for the inequality to be an equality, $x_i - x_C$ and $y_i - y_C$ must have the same sign or at least one of them has to be 0, which is equivalent to $x_i \leq x_C$ and $y_i \leq y_C$, i.e. $H^i \preccurlyeq H^C$, or $x_i \geq x_C$ and $y_i \geq y_C$, i.e. $H^C \preccurlyeq H^i$. Therefore, H^i and H^C have to be sortable. □

Corollary 2. Let G be a group of k DMs assessing a set of alternatives Λ. Then, for any expert ϵ_i, $i \in \{1, \ldots, k\}$, $\delta^G_\Lambda(\epsilon_i) \leq SM_i$. In addition, the equality is met when, for any alternative $\lambda_j \in \Lambda$, $F^i_H(\lambda_j)$ and $F^C_H(\lambda_j)$ are sortable, being F^i_H and F^C_H the HFLDs of ϵ_i and the centroid of the group respectively.

Proof. The proof is straightforward from Properties 3 and 4 and the definition of SM_i. □

In an analogous way to Property 2 in Section 2, this property and its corollary show that the degree of consensus for experts introduced in Section 3.1 can capture differences among situations in which the measure presented in [20] cannot.

Lastly, referring to the time complexity, measures presented in [12] and in [20] have the same time complexity than the one presented in Section 3.1, which is a constant time plus the time of computing the central opinion for λ_j. Using the centroid from Definition 5, which is computed in linear time as commented in the previous section, the time complexity for $\delta^G_{\lambda_j}(\epsilon_i)$ is $O(k)$ where k is the number of DMs within the group.

Table 5 summarizes the main characteristics of the presented individual consensus measures.

	Dong *et al.* [12]	Wu *et al.* [20]	Montserrat-Adell *et al.*
Year	2015	2016	2017
Groupal consensus			
Individual consensus	✓	✓	✓
Distance-based	✓		✓
Similarity-based		✓	
Preference similarity			
Alternative similarity		✓	
Pairwise comparison			
Central opinion comparison	✓	✓	✓
Considers gap			✓
Considers hesitance	✓		✓
Time complexity [a,b]	$O(1) + \mathcal{T}_C$	$O(1) + \mathcal{T}_C$	$O(1) + \mathcal{T}_C$

[a] $\mathcal{T}_C$ stands for the time complexity of calculating the central opinion.

[b] For the overall degree of consensus of a set of r alternatives, all times are multiplied by r.

Table 5: Comparison of the presented individual degrees of consensus.

3.3. An illustrative example on individual consensus

For the seek of clarifying the calculation of the degree of consensus for each expert, let us present an example. In the same example, the foregoing properties can also be checked.

Example 4. Following Example 2, where G is a group of 5 DMs assessing a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_4\}$ by means of HFLTSs over the set $\mathcal{S} = \{a_1, \ldots, a_5\}$, with the assessments provided in Table 1, we can now use the presented methodology to compute the degree of consensus for each expert. For instance,

$$\delta^G_{\lambda_1}(\epsilon_1) = 1 - \frac{D(H_1^C, H_1^1)}{2n-2} = 1 - \frac{D([a_2, a_3], [a_1, a_2])}{2n-2} = 1 - \frac{2}{8} = 0.75.$$

Following the same steps for all the experts and alternatives, we get the results shown in Table 6.

$\delta^G_{\lambda_j}(\epsilon_i)$	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5
λ_1	0.75	0.875	0.625	0.5	1
λ_2	1	0.75	0.75	0.875	0.875
λ_3	1	0.875	1	0.25	1
λ_4	1	1	0.875	0.875	1
Λ	0.9375	0.875	0.8125	0.625	0.96875

Table 6: Degrees of consensus $\delta^G_{\lambda_j}(\epsilon_i)$ and $\delta^G_{\Lambda}(\epsilon_i)$.

Analogously, we can calculate the consensus level presented in [20] following Equation 3, as for instance,

$$SM_1^1 = 1 - \frac{\left|\frac{1+2}{2} - \frac{2+3}{2}\right|}{n-1} = 1 - \frac{|-1|}{4} = 0.75.$$

In the same way, we can compute all the consensus levels as shown in Table 7.

SM^i_j	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5
λ_1	0.75	0.875	0.875	0.5	1
λ_2	1	1	1	0.875	0.875
λ_3	1	0.875	1	0.25	1
λ_4	1	1	0.875	0.875	1
Λ	0.9375	0.9375	0.9375	0.625	0.96875

Table 7: Consensus levels SM^i_j and SM_i.

Property 4 can be easily checked by comparing results from Tables 6 and 7. It is clear that $\delta^G_{\lambda_j}(\epsilon_i) = SM^i_j$ except for expert ϵ_2 on alternative λ_2 and expert ϵ_3 on alternatives λ_1 and λ_2, where $\delta^G_{\lambda_j}(\epsilon_i) < SM^i_j$. In this three cases, the opinion of the expert is not sortable with the centroid of the group, while in any other case, it is.

Notice also that, in the cases where the two consensus measures are different, the one presented in [20] is greater given the fact that it only cares about the

center of the HFLTS without taking into consideration either the hesitance of the DMs or the existing gaps between opinions. For this reason, for instance, $SM_2^2 = SM_2^3 = 1$, even if the opinions of experts ϵ_2 and ϵ_3 are not the same than the centroid of the group for alternative λ_2. This leads us to a situation in which, experts ϵ_1, ϵ_2 and ϵ_3 share the same overall consensus level, $SM_1 = SM_2 = SM_3$, but, comparing F_H^1, F_H^2 and F_H^3 with respect to F_H^C, it seems quite intuitive that their coincidence with the central opinion should not be the same. By contrast, in Table 6 we can see that this problem is fixed given that $\delta_\Lambda^G(\epsilon_3) < \delta_\Lambda^G(\epsilon_2) < \delta_\Lambda^G(\epsilon_1)$.

4. A precision-dissension profile

Sometimes, when choosing DMs to assess a set of alternatives, a more precise expert is preferable to a more hesitant one. Sometimes a more dissenting expert is interesting to open a door to innovation, or sometimes it is just the other way around. The aim of this section is to present an expert's profile that keeps track of how experts have done their previous assessments to know how precise or how dissenting they are.

This profile might be useful to whoever has to choose among several decision makers to be part of a GDM situation because he or she can know beforehand the main characteristics of each expert's assessments. For instance, if we want to have a committee where common decisions are easily taken, we will choose un certain decision makers whose opinions are always close to the average opinion, which means a low precision and a low dissension. On the contrary, if we prefer a committee where polarized opinions are strongly defended, we should choose determined decision makers whose opinions tend to be far away from the central opinion, which means a high precision as well as a high dissension.

To this end, we present two numerical descriptors that characterize the assessment of a decision maker. Firstly, similarly to the notion of determinacy presented in [26], we introduce the concept of preciseness of an expert assessing a set of alternatives as a discrete version of determinacy. Both the preciseness

and the determinacy seek to quantify the certainty of an expert but, while the determinacy is based on areas calculated as fuzzy integrals, the preciseness is based on the number of linguistic labels from $\mathcal{S}$ that the experts uses.

Definition 9. Let ϵ_i be a DM assessing a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of HFLTSs over $\mathcal{S} = \{a_1, \ldots, a_n\}$, and let F_H^i be his HFLD about Λ, being $H_j^i = F_H^i(\lambda_j)$. Then, the *preciseness of* ϵ_i *on* Λ is defined as:

$$\pi_\Lambda(\epsilon_i) = \frac{\sum_{j=1}^{r} \frac{n-\text{card}(H_j^i)}{n-1}}{r}.$$

Note that, given that $\text{card}(H_j^i)$ is between 1 and n for any $j \in \{1, \ldots, r\}$, $\pi_\Lambda(\epsilon_i)$ ranges from 0 to 1, being 0 when $\text{card}(H_j^i) = n$ for any j and being 1 when $\text{card}(H_j^i) = 1$ for any j. Thus, the closer to 1 $\pi_\Lambda(\epsilon_i)$ is, the more precise ϵ_i has been with his assessments. Whilst, if $\pi_\Lambda(\epsilon_i)$ is close to 0, it means that there is more hesitance in the assessments of ϵ_i about Λ.

Secondly, we also introduce the concept of dissent of an expert with respect to a group as follows:

Definition 10. Let $\epsilon_1, \ldots, \epsilon_k$ be a group G of DMs assessing a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of HFLTSs over $\mathcal{S} = \{a_1, \ldots, a_n\}$, and let F_H^i be the HFLD of ϵ_i for $i = 1, \ldots, k$ and F_H^C the centroid of the group. Then, the *dissent of* ϵ_i *on* Λ *with respect to* G is defined as:

$$\sigma_\Lambda^G(\epsilon_i) = 1 - \delta_\Lambda^G(\epsilon_i).$$

Notice that, again, $\sigma_\Lambda^G(\epsilon_i)$ moves between 0 and 1 for any $i \in \{1, \ldots, k\}$. The smaller $\sigma_\Lambda^G(\epsilon_i)$ is, the closer the opinion of the expert ϵ_i and the central opinion are, being exactly 0 if $F_H^i = F_H^C$.

With these two measures, a profile for each expert assessing a set of alternatives can be defined as:

Definition 11. Let $\epsilon_1, \ldots, \epsilon_k$ be a group G of DMs assessing a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of HFLTSs over $\mathcal{S} = \{a_1, \ldots, a_n\}$. Then, the *precision-dissension profile of* ϵ_i *on* Λ *with respect to* G is defined as:

$$\phi_\Lambda^G(\epsilon_i) = (\pi_\Lambda(\epsilon_i), \sigma_\Lambda^G(\epsilon_i)).$$

For the seek of a better understanding, let us present the following example illustrating the previous concepts.

Example 5. Following Example 2, with the assessments about the set of alternatives Λ shown in Table 1, the preciseness and the dissent of each expert can be calculated, as, for instance,

$$\pi_\Lambda(\epsilon_1) = \frac{\frac{5-2}{4} + \frac{5-3}{4} + \frac{5-2}{4} + \frac{5-1}{4}}{4} = 0.75$$

and

$$\sigma^G_\Lambda(\epsilon_1) = 1 - 0.9375 = 0.0625,$$

given that $\delta^G_\Lambda(\epsilon_1)$ was already calculated in Example 4. Thus,

$$\phi^G_\Lambda(\epsilon_1) = (0.75, 0.0625).$$

Repeating this process for all the experts, we get the results shown in Table 8.

	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5
$\pi_\Lambda(\epsilon_i)$	0.75	1	0.375	0.75	0.8125
$\sigma^G_\Lambda(\epsilon_i)$	0.0625	0.125	0.1875	0.375	0.03125
$\phi^G_\Lambda(\epsilon_i)$	$(0.75, 0.0625)$	$(1, 0.125)$	$(0.375, 0.1875)$	$(0.75, 0.375)$	$(0.8125, 0.03125)$

Table 8: Preciseness and dissent of each expert on Λ.

It can be seen that ϵ_2 has a preciseness of 1 given that he has assessed all the alternatives with just one linguistic label without hesitation. In contrast, ϵ_3 has a very low preciseness due to a a big hesitance on his assessments. For instance, he has assessed two alternatives with all the possible linguistic labels.

On the other hand, ϵ_4 has the highest dissent of the whole group. This fact can be corroborated by having a look at Figure 6, which is a graphical representation of the assessments provided in Table 1, where it is clear than F^4_H is the most distant assessment to the central opinion in almost all the alternatives. On the contrary, F^1_H and F^5_H are equal to the central opinion in almost all the alternatives, and that is why ϵ_1 and ϵ_5 have the lowest dissent of the group.

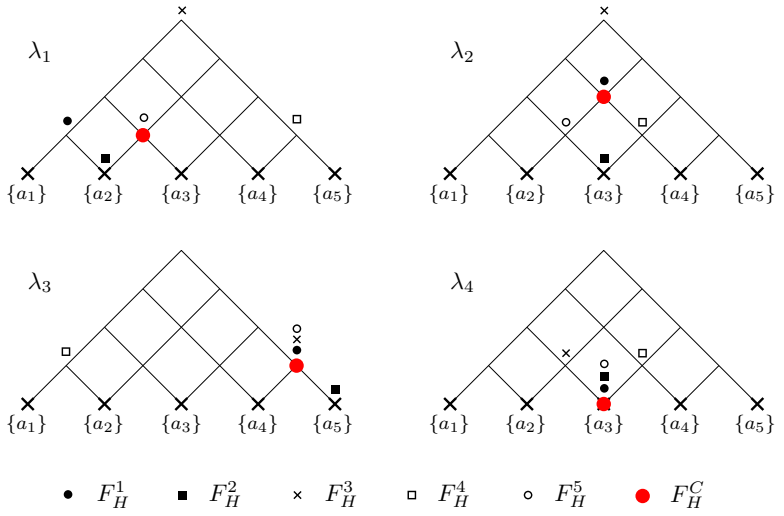


Figure 6: HFLDs modeling the DMs' assessments and the centroid from Example 2.

Finally, if an expert has assessed more than one set of alternatives within several groups, the information of each different situation can be combined as follows:

Definition 12. Let ϵ be a DM that has assessed the sets of alternatives $\Lambda_1, \ldots, \Lambda_m$ within the groups $G_1, \ldots, G_m$ respectively. Then:

(a) The *preciseness of* ϵ is defined as $\pi^m(\epsilon) = \frac{\sum_{l=1}^{m} \pi_{\Lambda_l}(\epsilon)}{m}$.

(b) The *dissent of* ϵ is defined as $\sigma^m(\epsilon) = \frac{\sum_{l=1}^{m} \sigma_{\Lambda_l}^{G_l}(\epsilon)}{m}$.

(c) The *precision-dissension profile of* ϵ is defined as $\Phi^m(\epsilon) = (m, \pi^m(\epsilon), \sigma^m(\epsilon))$.

With $\Phi^m(\epsilon)$ one can know the characteristics of the assessments of expert ϵ regarding precision and dissension after evaluating m different sets of alternatives within their respective groups.

5. Conclusions and future work

Based on the weak points of existing consensus measures for GDM by means of HFLTSs, two consensus measures are defined in this paper in order to capture

differences among situations in which the previous measures are not able to make a difference.

On the one hand, a consensus level is defined for the whole group on a specific alternative as a normalization of the addition of distances from a central opinion to the opinion of each expert of the group, and an analogous definition is given for a set of several alternatives instead of just one of them. On the other hand, the consensus level is defined for each expert with respect to the rest of the group based on the distance between his/her opinion and the central opinion for both one specific alternative and a set of alternatives.

Additionally, a study is carried out to compare the presented measures with the similar existing ones and concludes that the measures presented in this paper are more accurate in situations in which existing measures consider the level of agreement to be the same but where common sense suggests they should be different. Moreover, the comparison study also shows that the collective degree of consensus presented in this paper has a lower time complexity than the existing measures.

Lastly, a profile of an expert is presented to keep track of the precision and dissension in his/her assessments with a view to using this information for future experts selection processes.

Future work will focus on two main directions. From a theoretical point of view, a dynamical study will be carried out on both the consensus-reaching process and the precision-dissension profile of DMs in several GDM processes. In particular, the proposed consensus measures will be used to measure polarization in this kind of scenarios. From a practical point of view, all the introduced concepts are already being implemented in a real case example framed in the city tourism management field.

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