

The Frobenius problem: A Geometric Approach ^{*}

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Abstract

For the well known Frobenius problem, we present a new geometric approach, based on the use of the n -dimensional lattice \mathbb{Z}^n , where n is the number of generators. Within this approach we are able to study the cases of two and three generators. The main feature of our geometric representation is that we can nicely visualize the set of *gaps*, i.e., the non-representable positive integers.

In the case of two generators, we give a description of the set of gaps. Moreover, for any positive integer, m , we derive a simple expression for the denumerant $d(m; a, b)$.

We show that we can use the 2-dimensional lattice associated to the set of generators $\{a, b\}$ to study the Frobenius problem with generators $\{a, b, c\}$. In particular, we give, as for two generators, a graphical representation of the set of gaps. For a large set of possible values of c , this representation allows us to simplify the computation of the Frobenius number and compute the number of gaps.

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1 Introduction

Given a finite set of positive integers, $A = \{a_1, \dots, a_n\}$, the Frobenius problem for A asks for the maximum natural number that can not be represented as a positive combination of the elements of A . The solution of the Frobenius problem, called the Frobenius number of A and denoted by $g(a_1, \dots, a_n)$, does exist, provided that $\gcd(a_1, \dots, a_n) = 1$.

Several interesting problems are related with the Frobenius problem. Namely, the computation of the number of non-representable integers, called *gaps* in the context of numerical semigroups, and their description, and the computation of the denumerant of a representable positive integer m , i.e. the number of non-negative representations of m on A .

This problem was introduced by Frobenius in its lectures. In 1884 Sylvester set that the Frobenius number for $n = 2$ is $g(a_1, a_2) = a_1 a_2 - a_1 - a_2$ and the number of non-representable positive integers is $N(a_1, a_2) = \frac{(a_1-1)(a_2-1)}{2}$ [9]. Since then, the Frobenius problem has been widely

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studied, from very different points of view, ranging from arithmetics and algebra to algorithmics and complexity.

The Frobenius problem turns out to be a difficult problem. For fixed values of n , it is known to be polynomial, whilst it is proved to be NP-complete if n is part of the input [5]. It is shown in [2] that closed formulas doesn't exist for the Frobenius number with a fixed number of generators $n \geq 3$. We found in the litterature a variety of works giving formulas for the Frobenius number in particular cases. One of the classical ones is [1], where the authors also study the case of three generators. Other methods for computing $G(a_1, a_2, a_3)$ are given in [3, 4, 8]. From a more algebraic point of view, the set of representable integers is a numerical semigroup. The fundamental gaps of a semigroup, a concept that inspired part of our results, are explicitly described in [7].

A review of the existing work up to 2005 is the book [6]. We refer the reader to this book and the references therein for a complete overview of the state of the art.

Notation Throughout the paper we use the following notation. We say that a set of natural numbers $A = \{a_1, \dots, a_n\}$ satisfying $\gcd(a_1, \dots, a_n) = 1$ is a set of generators of dimension n . Given such a set:

- $R(A) = R(a_1, \dots, a_n) = \{m = x_1a_1 + x_2a_2 + \dots + x_na_n \mid x_i \in \mathbb{N}\}$ denotes the set of integers which are representable as a non-negative linear combination of elements of A . The expression $m = x_1a_1 + x_2a_2 + \dots + x_na_n$ is a representation of m on A .
- $\overline{R}(A) = \overline{R}(a_1, \dots, a_n) = \mathbb{N} \setminus R(A)$ denotes the set of non-representable integers, which we will also call *gaps*, as $R(A)$ is a numerical semigroup.
- $g(a_1, \dots, a_n) = \max \overline{R}(A)$ denotes the Frobenius number of A .
- $N(A) = |\overline{R}(A)|$ denotes the number of gaps.
- If m is a non-negative integer, the denumerat of m and A is the number of non-negative representations of m on A , denoted by $d(m; A)$. Hence, m is a gap if and only if $d(m; A) = 0$.

Our results For the Frobenius problem we present a new geometric approach. Within this approach, most of the known results for $n = 2$ can easily be derived and visualized. Moreover, we are able to study the case of three generators.

Our approach is based in the following idea. Given $A = \{a_1, \dots, a_n\}$, we associate to the point $(x_1, \dots, x_n) \in \mathbb{Z}^n$ the integer $m = x_1a_1 + \dots + x_na_n$. This gives a labeling of the integer n -dimensional lattice.

For $n = 2$, $A = \{a, b\}$, we identify a particular triangle in the plane containing exactly the set of non-representable positive integers. This allows us to give a complete description of this set. Moreover, we derive a simple expression for the denumerant, $d(n; a, b)$.

Let $A = \{a, b, c\}$ be a set of positive integers, with $\gcd(a, b) = 1$ and $a < b$. We use the 2-dimensional lattice and the labeling associated to $\{a, b\}$ to study the Frobenius problem for $A = \{a, b, c\}$. In particular, if c is non-representable on $\{a, b\}$ and $2c$ is, then we can easily compute $g(a, b, c)$ and $N(a, b, c)$, and also characterize the set of integers which are non-representable by $\{a, c, b\}$. The same results are proved for a particular configuration of the multiples of c in the set of gaps for $\{a, b\}$.

73	78	83	88	93	98	103	108	113	118	123	128	133	138	143	148	153	158
60	65	70	75	80	85	90	95	100	105	110	115	120	125	130	135	140	145
47	52	57	62	67	72	77	82	87	92	97	102	107	112	117	122	127	132
34	39	44	49	54	59	64	69	74	79	84	89	94	99	104	109	114	119
21	26	31	36	41	46	51	56	61	66	71	76	81	86	91	96	101	106
8	13	18	23	28	33	38	43	48	53	58	63	68	73	78	83	88	93
-5	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80
-18	-13	-8	-3	2	7	12	17	22	27	32	37	42	47	52	57	62	67
-31	-26	-21	-16	-11	-6	-1	4	9	14	19	24	29	34	39	44	49	54
-44	-39	-34	-29	-24	-19	-14	-9	-4	1	6	11	16	21	26	31	36	41
-57	-52	-47	-42	-37	-32	-27	-22	-17	-12	-7	-2	3	8	13	18	23	28
-70	-65	-60	-55	-50	-45	-40	-35	-30	-25	-20	-15	-10	-5	0	5	10	15
-83	-78	-73	-68	-63	-58	-53	-48	-43	-38	-33	-28	-23	-18	-13	-8	-3	2
-96	-91	-86	-81	-76	-71	-66	-61	-56	-51	-46	-41	-36	-31	-26	-21	-16	-11
-109	-104	-99	-94	-89	-84	-79	-74	-69	-64	-59	-54	-49	-44	-39	-34	-29	-24

Figure 1: The Frobenius lattice for $a = 5$ and $b = 13$.

2 The geometric approach

For a given set of generators, $A = \{a_1, \dots, a_n\}$, we can consider the linear map

$$\begin{aligned} \ell : \quad \mathbb{Z}^n &\rightarrow \mathbb{Z} \\ (x_1, \dots, x_n) &\mapsto x_1 a_1 + \dots + x_n a_n \end{aligned}$$

This map can be viewed as an integer labeling of \mathbb{Z}^n . If $m = x_1 a_1 + \dots + x_n a_n$, we label the point of coordinates (x_1, \dots, x_n) by the integer m . Notice that the function ℓ is linear and surjective, and its kernel is the set

$$\ker \ell = \{(x_1, \dots, x_n) \in \mathbb{Z}^n \mid x_1 a_1 + \dots + x_n a_n = 0\}$$

that is, the set of integer points lying on the hyperplane of \mathbb{R}^n of equation $x_1 a_1 + \dots + x_n a_n = 0$.

In this paper we study the first two cases, $n = 2$ and $n = 3$. (See Figure 1 for a simple example.)

2.1 The case of two generators

Let us first concentrate on the case of two generators. We define the Frobenius lattice and study its geometric properties.

Definition 2.1 (Frobenius lattice) *Given an ordered set of generators $A = \{a, b\}$, with $a < b$, the Frobenius lattice associated to A is the pair (\mathbb{Z}^2, ℓ) , where ℓ is the map from the infinite lattice \mathbb{Z}^2 to \mathbb{Z}*

$$\begin{aligned} \ell : \quad \mathbb{Z}^2 &\rightarrow \mathbb{Z} \\ (x, y) &\mapsto xa + yb \end{aligned}$$

We call $\ell(x, y)$ the label of (x, y) .

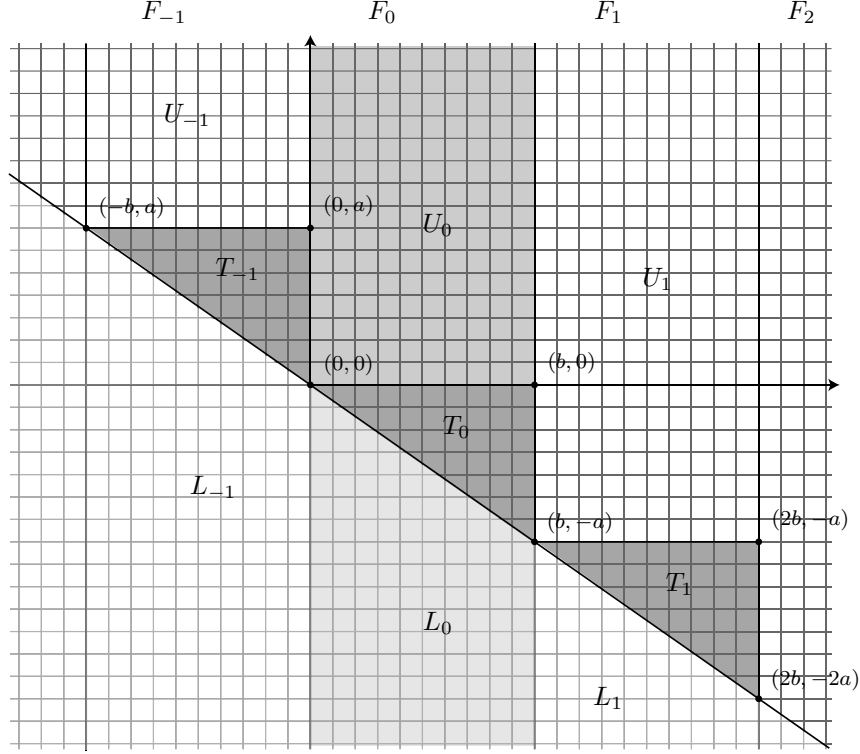


Figure 2: The Frobenius lattice and its basic features.

We next describe the geometric objects associated to the Frobenius lattice. A graphical representation is given in Figure 2.

The set of points labeled 0 are the integer points which lie on the line of equation $xa + yb = 0$

$$\ker \ell = \{(\lambda b, -\lambda a) \mid \lambda \in \mathbb{Z}\}$$

As a consequence, for every $m \in \mathbb{Z}$, the set of points labeled m are the integer points which lie on the line of equation $xa + yb = m$. If $m = ua + vb = \ell(u, v)$ then for every $\lambda \in \mathbb{Z}$

$$\ell((u, v) + (\lambda b, -\lambda a)) = m$$

The labeling ℓ is a covering of \mathbb{Z}^2 by \mathbb{Z} and, thus, induces a partition of the plane. We say that the plane is decomposed into strips, which are defined as follows.

For every $m \in \mathbb{Z}$ and $\lambda \in \mathbb{Z}$, there is exactly one point (x, y) such that $m = \ell(x, y)$ and $\lambda b \leq x < (\lambda + 1)b$. This allows us to define the strip associated to λ .

Definition 2.2 (λ -strip associated to $\{a, b\}$) For every $\lambda \in \mathbb{Z}$, we define the strip F_λ or the λ -strip associated to the set $\{a, b\}$ as the subset of the Frobenius lattice

$$F_\lambda = \{(x, y) \in \mathbb{Z}^2 \mid \lambda b \leq x < (\lambda + 1)b\} = [0, b) \times \mathbb{Z}$$

The collection of sets $\{F_\lambda\}_{\lambda \in \mathbb{Z}}$ constitutes a partition of \mathbb{Z}^2 . Indeed, they are pairwise disjoint and they cover \mathbb{Z}^2 . Moreover, for every integer λ , the restriction $\ell|_{F_\lambda} : F_\lambda \rightarrow \mathbb{Z}$ is a bijection.

For every pair of integers λ_1, λ_2 , the translation of vector $((\lambda_2 - \lambda_1)b, -(\lambda_2 - \lambda_1)a)$ is a bijection between F_{λ_1} and F_{λ_2} which applies every point of F_{λ_1} onto the point in F_{λ_2} with the same label.

For any given λ , we now give the definition of three subsets in the λ -strip whose respective sets of labels are the negative integers, the non-representable positive integers, $\overline{R}(a, b)$, and the representable integers $R(a, b)$ (see Figure 2).

- The points below the line $xa + yb = 0$ are labeled by the negative integers. We define

$$L_\lambda = \{(x, y) \in F_\lambda \mid xa + yb < 0\}$$

- The positive integers are the labels of the points satisfying $xa + yb \geq 0$. Among these points, we distinguish two sets.

$$T_\lambda = \{(x, y) \in F_\lambda \mid xa + yb > 0, y < -\lambda a\}$$

and

$$U_\lambda = \{(x, y) \in F_\lambda \mid xa + yb \geq 0, y \geq -\lambda a\}$$

When using this notation, we have to take into account that a representable integer has also non-valid representations ($m = \ell(u, v)$ is a valid representation of m if and only if $u, v \geq 0$).

In F_0 , the set of representable numbers, $R(a, b)$, is the set of labels of the points satisfying $y \geq 0$

$$\ell(U_0) = R(a, b)$$

and the set of gaps, $\overline{R}(a, b)$, is the set of labels of the triangle T_0

$$\ell(T_0) = \overline{R}(a, b)$$

Thanks to this representation we can easily deduce most of the known results on the Frobenius problem with two generators. In particular, both the Frobenius number and the number of gaps follow nicely from the Frobenius lattice and the definition of T_0 .

- The Frobenius number of A is the labeling of the up-right corner of T_0 ,

$$g(a, b) = \max\{\ell(x, y) \mid (x, y) \in T_0\} = \ell(b - 1, -1) = (b - 1)a - b = ab - a - b$$

because the labeling function is increasing in both coordinates and, in T_0 , the point of maximum first coordinate is the point of maximum second coordinate.

- The number of gaps generated by A is the number of integer points in T_0 ,

$$N(A) = |\overline{R}(A)| = \frac{(a - 1)(b - 1)}{2}$$

because no points in T_0 lie on the line $xa + yb = 0$.

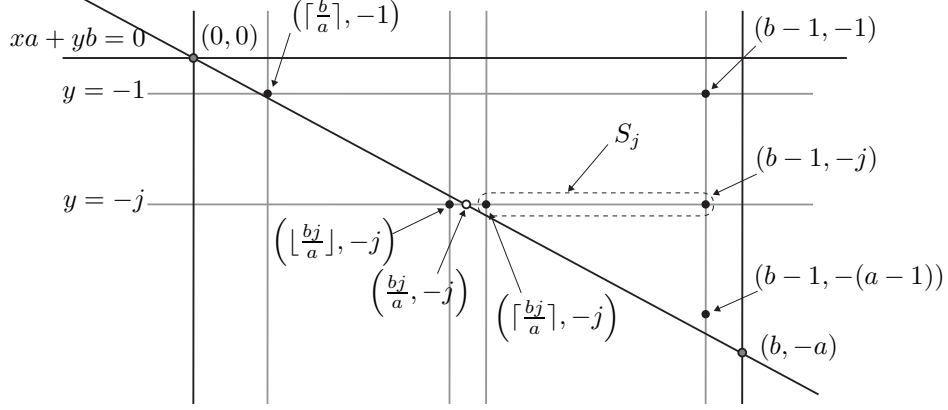


Figure 3: The set of gaps.

2.1.1 Description of the set of gaps

Let us see how the elements of $\overline{R}(a, b)$ are distributed in T_0 . For a graphical representation, see Figure 3.

Notice that, for every $u_1, u_2 \in \mathbb{Z}$, $\ell(u_1, v) \equiv \ell(u_2, v) \equiv bv \pmod{a}$. This implies that $\overline{R}(A) = \ell(T_0)$ can be partitioned into congruence classes modulo a , which correspond to horizontal segments in the geometric representation. We denote by S_j the segment $T_0 \cap \{(x, y) \mid y = -j\}$.

For every j , $1 \leq j \leq a-1$, the number of points in S_j is

$$n_j = b - \left\lceil \frac{bj}{a} \right\rceil = \left\lfloor \frac{b}{a}(a-j) \right\rfloor$$

By symmetry, $n_{a-j} = \lfloor \frac{bj}{a} \rfloor$ and $n_j + n_{a-j} = b-1$.

- The points $(i, -j) \in S_j$ are

$$\left(\left\lceil \frac{bj}{a} \right\rceil, -j \right), \left(\left\lceil \frac{bj}{a} \right\rceil + 1, -j \right), \dots, (b-1, -j)$$

and their respective labels are

$$\ell \left(\left\lceil \frac{bj}{a} \right\rceil + k, -j \right) = a \left\lceil \frac{bj}{a} \right\rceil + ak - bj, \quad 0 \leq k \leq n_j - 1$$

- In S_j , the minimum and the maximum labels are

$$\begin{aligned} \min \ell(S_j) &= \ell \left(\left\lceil \frac{bj}{a} \right\rceil, -j \right) = a \left\lceil \frac{bj}{a} \right\rceil - bj \\ \max \ell(S_j) &= \ell \left(\left\lceil \frac{bj}{a} \right\rceil + n_j, -j \right) = \ell(b-1, -j) = a(b-1) - bj \end{aligned}$$

If $r = b \pmod{a}$, then the smallest non-representable integers are

$$\{1, 2, 3, \dots, a-1\} = \{r \pmod{a}, 2r \pmod{a}, \dots, (a-1)r \pmod{a}\}$$

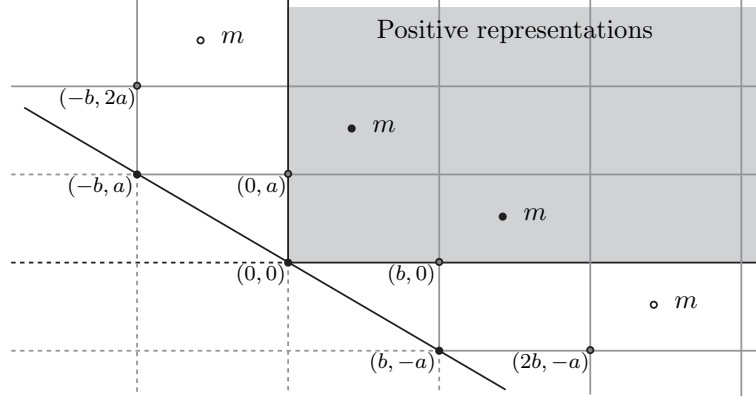


Figure 4: The denumerant of m with respect to $\{a, b\}$, $d(m; a, b)$. We represent by \bullet the points which give a positive representation of its label.

Each of these points is the minimum of one of the segments S_j .

Let us concentrate on $r = b - a \lfloor \frac{b}{a} \rfloor$. Notice that $r \in \ell(S_{a-1})$ and thus

$$r = \ell \left(\left\lceil \frac{b(a-1)}{a} \right\rceil, -(a-1) \right)$$

Moreover, $rj \in S_{a-j}$ and

$$rj = \ell \left(\left\lceil \frac{b(a-j)}{a} \right\rceil, -(a-j) \right)$$

In particular, the coordinates of 1 in T_0 can be obtained by solving the equation

$$kr \pmod{a} = 1$$

Then, $\ell(b - \lfloor \frac{kb}{a} \rfloor, -(a-k)) = 1$.

2.1.2 The denumerant

We can use the Frobenius lattice to give a simple expression of the denumerant of a given integer. This expression relates the denumerant of m with the position in the plane of any of the points labeled m . Figure 4 illustrates our result, which is given in the following Proposition.

Proposition 2.3 *Let $m \in \mathbb{N}$ and $m = \ell(u, v)$. Then,*

$$d(m; a, b) = \left\lfloor \frac{u}{b} \right\rfloor + \left\lfloor \frac{v}{a} \right\rfloor + 1$$

Proof. If $m = \ell(u, v) = \ell(u', v')$ then $u' = u + \lambda b$ and $v' = v - \lambda a$, for some integer λ . It is clear that

$$\left\lfloor \frac{u'}{b} \right\rfloor = \left\lfloor \frac{u}{b} \right\rfloor + \lambda \quad \text{and} \quad \left\lfloor \frac{v'}{a} \right\rfloor = \left\lfloor \frac{v}{a} \right\rfloor - \lambda$$

This implies that $\lfloor \frac{u}{b} \rfloor + \lfloor \frac{v}{a} \rfloor + 1$ is the same for every $(u, v) \in \ell^{-1}(m)$. Hence, we can assume w.l.o.g. that $(u, v) \in F_0$, i.e. $0 \leq u < b$. In this case, the set of non-negative representations of m is $\{(u, v) + \lambda(b, -a) \mid 0 \leq \lambda \leq \lfloor \frac{v}{a} \rfloor\}$. The number of elements in this set is $d(m; a, b) = \lfloor \frac{v}{a} \rfloor + 1$. Since in this strip $\lfloor \frac{u}{b} \rfloor = 0$, we can write $d(m; a, b) = \lfloor \frac{u}{b} \rfloor + \lfloor \frac{v}{a} \rfloor + 1$. \blacksquare

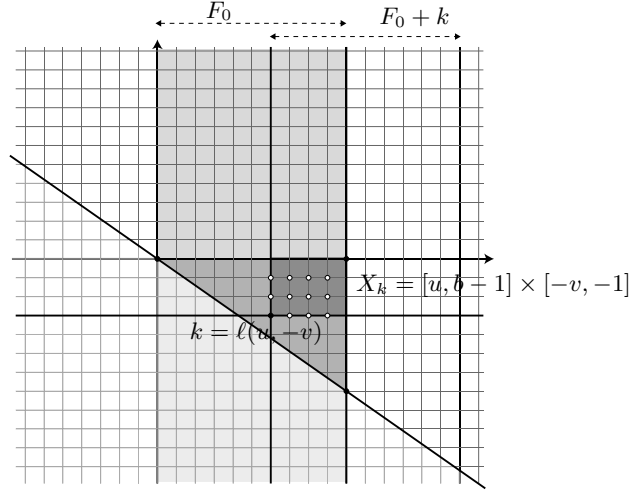


Figure 5: Adding a gap k to the set of generators $\{a, b\}$.

Remark that $d(m; a, b) = 0$ if and only if m is a gap.

3 Some results on the case of three generators

In this section we deal with the Frobenius problem for $\{a, b, c\}$ with $\gcd(a, b) = 1$, $a < b$ and $c \in \overline{R}(a, b)$. For this purpose, we will use the 2-dimensional lattice associated to $\{a, b\}$ instead of the 3-dimensional lattice associated to $\{a, b, c\}$.

First, we give the plane construction that allows us to represent $g(a, b, c)$ in \mathbb{Z}^2 . Our concern is to determine the distribution of the multiples of c in the triangle T_0 , for which we know $\ell(T_0) = \overline{R}(a, b)$.

In fact, the Frobenius problem with three generators $\{a, b, c\}$ is easy to solve if c is a gap but $2c$ is not. This case covers half of the possible values of c . We show also that, in two particular cases, the set of gaps $\overline{R}(a, b, c)$ has a nice plane representation as a subset of T_0 , and both the Frobenius number and the number of gaps are easy to compute.

3.1 The representation of $\overline{R}(a, b, c)$ in \mathbb{Z}^2

For three generators, $\{a, b, c\}$, the labeling of $(x, y, z) \in \mathbb{Z}^3$ is defined as $\ell(x, y, z) = xa + yb + zc$. This gives a labeling of the points of the three dimensional lattice.

Given $c = ua - vb \in \overline{R}(a, b)$, with $(u, -v) \in T_0$, the translation of vector $(u, -v)$ applied to F_0 gives a new strip, denoted by $F_0 + c$. We denote by $U_0 + c$ the set obtained from U_0 by the same translation. Since $(u, -v) \in T_0$, $(U_0 + c) \cap T_0$ is a rectangle containing the gaps of $\overline{R}(a, b)$ but have a non-negative representation of the form $xa + yb + c$. Notice that $xa + yb + c$ can be represented in \mathbb{Z}^3 by $(x, y, 1)$.

In this way, the set of gaps for $\{a, b, c\}$ can be obtained from T_0 , by removing the points in $U_0 + c$, $U_0 + 2c$, $U_0 + 3c$, and so on. In Figure 5 we show the representation of the 3-dimensional lattice in the plane. More formally, we have the following.

Definition 3.1 (k -representable integer) Let $\{a, b\}$ be a set of two positive integers with $a < b$,

$\gcd(a, b) = 1$. For a given non-negative integer k we define the set of k -representable integers by

$$R_k(a, b) = \{m \in \mathbb{Z} \mid m = \lambda_1 a + \lambda_2 b + k, \lambda_1 \geq 0, \lambda_2 \geq 0\} = \ell(U_0 + k)$$

In particular, if $k = 0$, $R_0(a, b) = R(a, b)$.

Definition 3.2 (k -rectangle) Let $\{a, b\}$ be a set of two positive integers with $a < b$, $\gcd(a, b) = 1$. For a given non-negative integer k we define the k -rectangle by the set of points

$$X_k = (U_0 + k) \cap T_0$$

Notice that, with this definition $\ell(X_k) = R_k(a, b) \setminus R(a, b)$. That is, if k is a gap on $\{a, b\}$, then the k -rectangle contains the points in T_0 which are gaps on $\{a, b\}$ but have a representation on $\{a, b, k\}$ of the form $xa + yb + k$.

The k -rectangle is empty if k is representable on $\{a, b\}$. If k is a gap, easy computations give the following property (see Figure 5).

Property 1 Let $\{a, b\}$ be a set of two positive integers with $a < b$, $\gcd(a, b) = 1$. If $k = ua - vb$ is the representation of k in T_0 then

$$X_k = [u, b - 1] \times [-1, -v] \quad (1)$$

As the two following properties show, in order to describe $\overline{R}(a, b, c)$, we are interested on the ic -representable integers, $\{R_{ic}(a, b)\}_{i \geq 0}$ and the ic -rectangles, $\{X_{ic}\}_{i \geq 1}$.

Property 2 The set of integers which have a non-negative representation by $\{a, b, c\}$ is

$$R(a, b, c) = \bigcup_{i=0}^{\infty} R_{ic}(a, b)$$

Property 3 The set of points in T_0 with labels in $R(a, b, c)$ is

$$\bigcup_{i=1}^{\infty} X_{ic}$$

In what follows we show that the set of gaps on $\{a, b, c\}$ depends only on a relatively small number of multiples of c .

Definition 3.3 Let $\{a, b\}$ be a set of two positive integers with $a < b$, $\gcd(a, b) = 1$. For a given $c \in \overline{R}(a, b)$, we define the first representable multiple as the positive number, denoted by $s > 1$, such that

$$c, 2c, \dots, (s-1)c \in \overline{R}(a, b)$$

and

$$sc \in R(a, b)$$

Proposition 3.4 If $c \in \overline{R}(a, b)$ and s is the first representable multiple of c on $\{a, b\}$, then

$$R(a, b, c) = \bigcup_{i=0}^{s-1} R_{ic}(a, b)$$

Proof. By Property 2, we need only to prove that

$$R(a, b, c) \subseteq \bigcup_{i=0}^{s-1} R_{ic}(a, b)$$

Since $sc \in R(a, b)$, there are two integers $m_1 \geq 0$ and $m_2 \geq 0$ such that $sc = m_1a + m_2b$. Let $m = xa + yb + zc \in R(a, b, c)$, with $x \geq 0$, $y \geq 0$ and $z \geq 0$. If $z < s$, then $m \in \bigcup_{i=0}^{s-1} R_{ic}(a, b)$. Assume that $z \geq s$, and let $z = ps + q$ with $0 \leq q \leq s - 1$ the integer division of z by s .

We can write $m = xa + yb + zc = xa + yb + (ps + q)c = xa + yb + p(m_1a + m_2b) + qc = (x + pm_1)a + (y + pm_2)b + qc$, where $0 \leq q \leq s - 1$. Thus, $m \in \bigcup_{i=0}^{s-1} R_{ic}(a, b)$. ■

Corollary 3.5 $\forall c \in \overline{R}(a, b)$, if $c, 2c, \dots, (s-1)c \in \overline{R}(a, b)$ and $sc \in R(a, b)$,

$$\overline{R}(a, b, c) = \overline{R}(a, b) \setminus \bigcup_{i=1}^{s-1} \ell(X_{ic}).$$

Proof. It follows straightforward from Proposition 3.4 and the definition of X_{ic} . ■

3.2 Solving the Frobenius problem

Although the Frobenius number for three generators is known, we have defined a nice geometric description of the set of gaps which will be used to give a more intuitive computation of $g(a, b, c)$, together with the computation of the number of gaps $N(a, b, c)$.

In fact, given $c \in \overline{R}(a, b)$, if the first representable multiple of c is small, we can give simple expressions for both the Frobenius number $g(a, b, c)$ and the number of gaps $N(a, b, c)$. We start by completely characterize the case $s = 2$, that is, c is a gap on $\{a, b\}$ and $2c$ is not. An example of this case is shown in Figure 6.

Proposition 3.6 *Let $c \in \overline{R}(a, b)$ and $c = ua - vb$ the representation of c in T_0 . Then, the first representable multiple of c is $s = 2$ if and only if $2 = \lceil \frac{b}{u} \rceil \leq \lfloor \frac{a}{v} \rfloor$.*

Under these conditions, the number of gaps and the Frobenius number corresponding to the set of generators $\{a, b, c\}$ are, respectively,

$$N(a, b, c) = N(a, b) - (b - u)v$$

and

$$g(a, b, c) = \max\{\ell(u - 1, -1), \ell(b - 1, -v - 1)\}$$

Proof. If $c = ua - vb$ is the representation of c in T_0 , then $1 \leq u < b$ and $1 \leq v < a$. The first representable multiple of c is 2 if and only if $(2u, -2v) \in U_1$. This is equivalent to

$$u < b \leq 2u \text{ and } 2v \leq a \Leftrightarrow$$

$$1 < \frac{b}{u} \leq 2 \text{ and } 2 \leq \frac{a}{v} \Leftrightarrow$$

$$2 = \lceil \frac{b}{u} \rceil \leq \lfloor \frac{a}{v} \rfloor$$

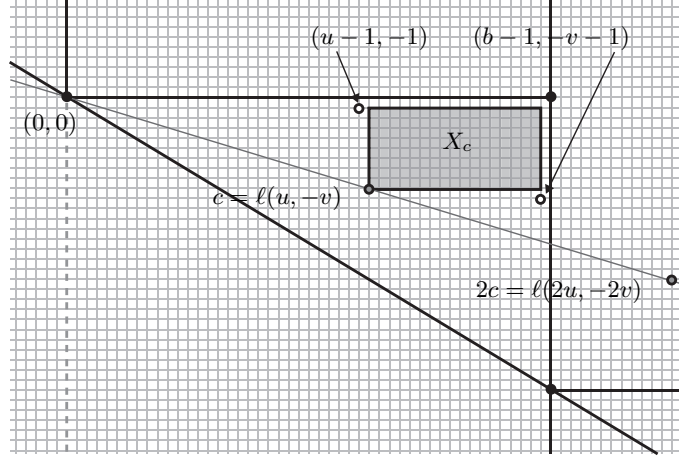


Figure 6: For $c = \ell(u, -v)$, with $2 = \lceil \frac{b}{u} \rceil \leq \lfloor \frac{a}{v} \rfloor$, the Frobenius number of $\{a, b, c\}$ is $g(a, b, c) = \max\{\ell(u-1, -1), \ell(b-1, -v-1)\}$ and the number of gaps is $N(a, b, c) = N(a, b) - (b-u)v$.

By Corollary 3.5, the set of gaps $\overline{R}(a, b, c)$ are represented in the plane by the set $T_0 \setminus X_c$. Consequently,

$$N(a, b, c) = |T_0 \setminus X_c| = N(a, b) - |X_c|$$

and

$$g(a, b, c) = \max \ell(T_0 \setminus X_c)$$

Property 1 implies $|X_c| = (b-u)v$ and, thus

$$N(a, b, c) = N(a, b) - (b-u)v$$

Let us compute the maximum label of the points in $T_0 \setminus X_c$. By the monotonicity of the labeling function ℓ , we can distinguish two local maxima (see Figure 6).

- In the set $\{(x, y) \in T_0 \setminus X_c \mid y \in [-v, -1]\}$, the maximum label is attained in the point $C_1 = (u-1, -1)$, since $C_1 + (0, 1) \in U_0$ and $C_1 + (1, 0) \in X_c$.
- In the set $\{(x, y) \in T_0 \setminus X_c \mid y \in [-a, -v-1]\}$, the maximum label is attained in the point $C_2 = (b-1, -v-1)$, since $C_2 + (0, 1) \in X_c$ and $C_2 + (1, 0) \in U_1$.

This gives

$$g(a, b, c) = \max\{\ell(C_1), \ell(C_2)\} = \max\{\ell(u-1, -1), \ell(b-1, -v-1)\}$$

which completes the proof. ■

We have shown that, if c is a gap for $\{a, b\}$, but $2c$ is not, then we can use a simple geometric reasoning to compute the number of gaps for $\{a, b, c\}$ as well as the Frobenius number. If $s \neq 2$ a similar reasoning can be applied in some particular cases. Roughly speaking, if c is represented in T_0 by $(u, -v)$, Lemma 3.7 gives a necessary and sufficient condition to ensure that $(2u, -2v)$ is in T_0 (remark that it could be in T_1) and the points $(iu, -iv)$ are also in T_0 , for $i = 2, \dots, s-1$. The

computation of the number of gaps and the Frobenius number under the conditions of Lemma 3.7 are given in Theorem 3.9.

The geometric representation of the Frobenius lattice gives an intuitive approach to our results (see Figures 6 and 7). The basic idea is that computations are easy if the points in T_0 , the labels of which are the multiples of c , are aligned. In fact, the hypothesis of Theorem 3.9 and Theorem 3.10 are symmetric, as shown in Figures 7 and 8.

Lemma 3.7 *Let $c \in \overline{R}(a, b)$ and $c = ua - vb$ the representation of c in T_0 . Then, $s = \lceil \frac{b}{u} \rceil \leq \lfloor \frac{a}{v} \rfloor$ if and only if*

$$(u, -v), (2u, -2v), \dots, ((s-1)u, -(s-1)v) \in T_0 \text{ and } (su, -sv) \in U_1 \quad (2)$$

Proof. Assume that $(u, -v) \in T_0$. This is equivalent to $0 < u < b$ and $0 < v < a$. Then,

$$\begin{aligned} s = \lceil \frac{b}{u} \rceil \leq \lfloor \frac{a}{v} \rfloor &\Leftrightarrow \\ s - 1 < \frac{b}{u} \leq s \text{ and } 0 < s \leq \frac{a}{v} &\Leftrightarrow \\ (s-1)u < b \leq su \text{ and } 0 < sv < a & \end{aligned}$$

By definition of T_0 and U_1 , this is equivalent to

$$((s-1)u, -(s-1)v) \in T_0 \text{ and } (su, -sv) \in U_1$$

And, since $(u, -v) \in T_0$, this is equivalent to (2). ■

Next we give a technical Lemma that will allow us to do easy computations in the proof of Theorem 3.9.

Lemma 3.8 *Let $c \in \overline{R}(a, b)$, $c = ua - vb$ the representation of c in T_0 , and $r > 0$ such that*

$$(u, -v), (2u, -2v), \dots, (ru, -rv) \in T_0 \quad (3)$$

For every $i = 1, \dots, r$, let $Y_{ic} = [iu, b-1] \times [-iv, -(i-1)v-1]$, then

$$\bigcup_{i=1}^r X_{ic} = \bigcup_{i=1}^r Y_{ic} \quad (4)$$

and the rectangles $\{Y_{ic}\}_{i=1..r}$ are pairwise disjoint.

Proof. Equation (4) follows from Equation (1) in Property 1. The pairwise disjointness of the collection of rectangles is clear from its definition. ■

Theorem 3.9 *Let $c \in \overline{R}(a, b)$ and $c = ua - vb$ the representation of c in T_0 . If $s = \lceil \frac{b}{u} \rceil \leq \lfloor \frac{a}{v} \rfloor$, then s is the first representable multiple of c . Moreover,*

$$N(a, b, c) = N(a, b) - (s-1)bv + \frac{s(s-1)}{2}uv$$

and

$$g(a, b, c) = \max\{\ell((s-1)u-1, -(s-2)v-1), \ell(b-1, -(s-1)v-1)\}$$

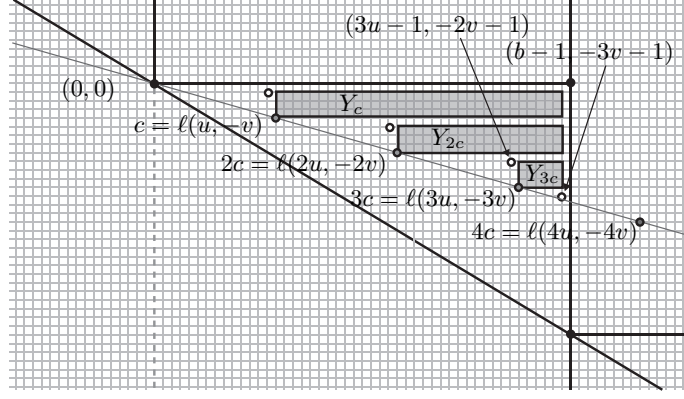


Figure 7: For $c = \ell(u, -v)$, with $4 = \lceil \frac{b}{u} \rceil \leq \lfloor \frac{a}{v} \rfloor$, the Frobenius number of $\{a, b, c\}$ is $g(a, b, c) = \max\{\ell(3u - 1, -2v - 1), \ell(b - 1, -3v - 1)\}$ and the number of gaps is $N(a, b, c) = N(a, b) - (b - u)v - (b - 2u)v - (b - 3u)v$.

Proof. Lemma 3.7 implies that s is the first representable multiple of c (on $\{a, b\}$).

To prove the Theorem, we have to compute the number of gaps and the Frobenius number, for the set of generators $\{a, b, c\}$. (See Figure 7 for a graphical explanation.)

By Corollary 3.5,

$$N(a, b, c) = N(a, b) - \left| \bigcup_{i=1}^{s-1} X_{ic} \right|$$

Now, by Lemma 3.8, with $Y_{ic} = [iu, b - 1] \times [-iv, -(i - 1)v - 1]$ as in the mentioned Lemma,

$$\left| \bigcup_{i=1}^{s-1} X_{ic} \right| = \left| \bigcup_{i=1}^{s-1} Y_{ic} \right| = \sum_{i=1}^{s-1} |Y_{ic}| = \sum_{i=1}^{s-1} (b - iu)v = (s - 1)bv - \frac{s(s - 1)}{2}uv$$

Thus,

$$N(a, b, c) = N(a, b) - (s - 1)bv + \frac{s(s - 1)}{2}uv$$

The Frobenius number is the maximum label of the points in $T_0 \setminus \bigcup_{i=1}^{s-1} X_{ic}$. By Lemma 3.8,

$$g(a, b, c) = \max \ell \left(T_0 \setminus \bigcup_{i=1}^{s-1} Y_{ic} \right)$$

A reasoning analogous to the proof of Proposition 3.6 gives that the maximum is attained in one of the local maxima

$$C_i = (iu - 1, -(i - 1)v - 1), \text{ for } i = 1, \dots, s - 1$$

and

$$C_s = (b - 1, -(s - 1)v - 1)$$

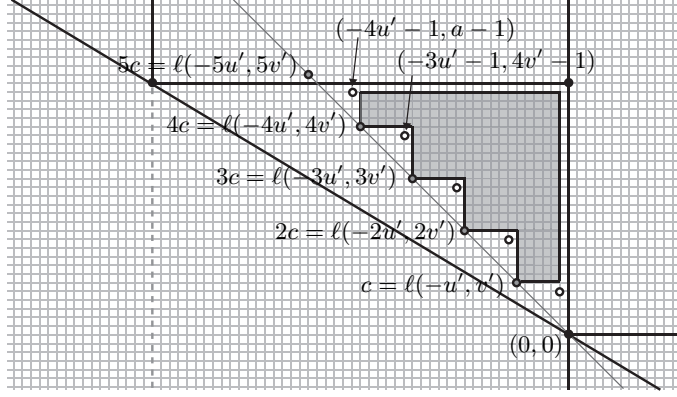


Figure 8: For $c = \ell(-u', v')$, with $5 = \lceil \frac{a}{v'} \rceil \leq \lfloor \frac{b}{u'} \rfloor$, the Frobenius number of $\{a, b, c\}$ is $g(a, b, c) = \max\{\ell(-4u' - 1, a - 1), \ell(-3u' - 1, 4v' - 1)\}$ and the number of gaps is $N(a, b, c) = N(a, b) - u'v' - 2u'v' - 3u'v' - 4u'(a - 4v')$.

By noticing that $\ell(C_i) = \ell(iu - 1, -(i - 1)v - 1) = (iu - 1)a - (i - 1)vb - b = ic - a - b + vb$, which is increasing in i , we can conclude that

$$g(a, b, c) = \max\{\ell((s - 1)u - 1, -(s - 2)v - 1), \ell(b - 1, -(s - 1)v - 1)\}$$

■

Theorem 3.10 Let $c = \ell(u, -v) \in \overline{R}(a, b)$. If $s = \lceil \frac{a}{a-v} \rceil \leq \lfloor \frac{b}{b-u} \rfloor$, then s is the first representable multiple of c . Moreover,

$$N(a, b, c) = N(a, b) - (b - u)(s - 1)a + (b - u)(a - v) \frac{s(s - 1)}{2}$$

and

$$g(a, b, c) = \max\{\ell(-(s - 2)(b - u) - 1, (s - 1)(a - v) - 1), \ell(-(s - 1)(b - u) - 1, a - 1)\}$$

Proof. Let $u' = b - u$ and $v' = a - v$. Then, $(-u', v')$ is the representation of c in the triangle of gaps T_{-1} . Analogously to Lemma 3.7, we have that $s = \lceil \frac{a}{v'} \rceil \leq \lfloor \frac{b}{u'} \rfloor$ if and only if

$$(-u', v'), (-2u', 2v'), \dots, (-(s - 1)u', (s - 1)v') \in T_{-1} \text{ and } (-su', sv') \in U_{-1}$$

Moreover, for $1 \leq i \leq s - 1$, we can replace the rectangles Y_{ic} , introduced in Lemma 3.8, by $Z_{ic} = [-iu', -(i - 1)u' - 1] \times [iv', a - 1]$ and compute the number of gaps and the Frobenius number for $\{a, b, c\}$ in T_{-1} , as we did in T_0 when proving Theorem 3.9. (See Figure 8 for a graphical explanation.)

This gives

$$N(a, b, c) = N(a, b) - \sum_{i=1}^{s-1} |Z_{ic}| = N(a, b) - \sum_{i=1}^{s-1} u'(a - iv')$$

and thus

$$N(a, b, c) = N(a, b) - u'(s-1)a + u'v' \frac{s(s-1)}{2} \quad (5)$$

On the other hand,

$$g(a, b, c) = \max \ell \left(T_{-1} \setminus \bigcup_{i=1}^{s-1} Z_{ic} \right)$$

which is attained in one of the points

$$C_i = (-(i-1)u' - 1, iv' - 1), \text{ for } i = 1, \dots, s-1$$

and

$$C_s = (-(s-1)u' - 1, a - 1)$$

By noticing that $\ell(C_i) = \ell(-(i-1)u', iv' - 1) = -(i-1)u'a - a + iv'b - b = ic - a - b + u'a$, which is increasing in i , we can conclude that

$$g(a, b, c) = \max\{\ell(-(s-2)u' - 1, (s-1)v' - 1), \ell(-(s-1)u' - 1, a - 1)\} \quad (6)$$

Replacing u' by $b - u$ and v' by $a - v$ in Equations (5) and (6), we get

$$N(a, b, c) = N(a, b) - (b - u)(s - 1)a + (b - u)(a - v) \frac{s(s - 1)}{2}$$

and

$$g(a, b, c) = \max\{\ell(-(s-2)(b-u) - 1, (s-1)(a-v) - 1), \ell(-(s-1)(b-u) - 1, a-1)\}$$

This concludes the proof. ■

4 Conclusion

For the Frobenius problem with generators $\{a_1, \dots, a_n\}$ we have proposed a geometric approach. This approach is based on a labeling of the n -dimensional integer lattice, where n is the number of generators. Our results in the cases 2 and 3 follow from an exploration of the set of gaps which can nicely be visualized thanks to our geometric approach.

For three generators $\{a, b, c\}$, with $a < b$ and $\gcd(a, b) = 1$, we propose a technique that allows to compute in a simple fashion both the Frobenius number and the number of gaps. As a final remark, notice that our computations need only $O(\log_2 a)$ operations, as they rely on the solution of the diophantine equation $au + bv = 1$. Moreover, our results are proved in a geometric flavor which gives a deeper understanding of the Frobenius problem.

A natural question that arises from Theorem 3.9 and Theorem 3.10 is: Given a pair of generators $\{a, b\}$, with $a < b$ and $\gcd(a, b) = 1$, which values of $c \in \overline{R}(a, b)$ meet the hypothesis of one of the mentioned theorems? In other words, for which values of $c \in \overline{R}(a, b)$ are the Frobenius problem, and maybe other problems related with, geometrically easy to solve?

In Figure 10 we show with a simple example an intuitive answer to this question. In fact, if $c = ua - vb$ is the representation of c in T_0 and the multiples of c are not all aligned in T_0 , we can not apply our technique.

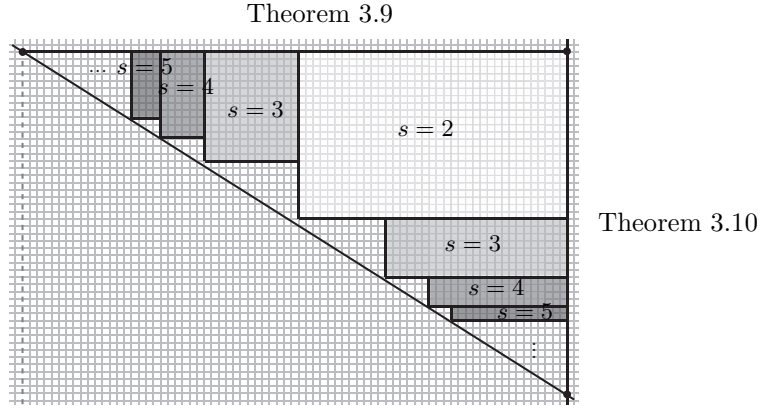


Figure 9: The portion of T_0 which is covered by our technique.

Another natural question, related with our previous results, is: Given a pair of generators $\{a, b\}$, with $a < b$ and $\gcd(a, b) = 1$, how many values of $c \in \overline{R}(a, b)$ meet the hypothesis of one of the mentioned theorems? In other words, which portion of the triangle of gaps T_0 corresponds to points for which the Frobenius problem, and maybe other problems related with, are geometrically easy to solve?

In Figure 9 we represent the answer to this question.

Related to this question, it's worth to mention that Proposition 3.6 says that in case $s = 2$ Theorem 3.9 and Theorem 3.10 are equivalently fulfilled. Indeed, it is easy to see that

$$2 = \left\lceil \frac{b}{u} \right\rceil \leq \left\lfloor \frac{a}{v} \right\rfloor \Leftrightarrow 2 = \left\lceil \frac{a}{a-v} \right\rceil \leq \left\lfloor \frac{b}{b-u} \right\rfloor$$

Moreover, a point $(u, -v) \in T_0$ satisfies this condition if and only if

$$\left\lceil \frac{b}{2} \right\rceil \leq u \leq b - 1 \text{ and } 1 \leq v \leq \left\lfloor \frac{a}{2} \right\rfloor$$

This gives a rectangle in T_0 of size

$$\left\lfloor \frac{a}{2} \right\rfloor \cdot \left\lceil \frac{b}{2} \right\rceil$$

which we know to be exactly the points corresponding to values of c satisfying $s = 2$. This rectangle contains almost half of the points in T_0 , and is the big white rectangle in Figure 9.

In fact, we can say more. By a simple geometric reasoning, it is easy to see that if $2 \leq s \leq 4$ then we are under the hypothesis of either Theorem 3.9 or Theorem 3.10.

Open problems We let some open problems, concerning the use of the technique we propose.

- We have studied the case $\{a, b, c\}$ with $a < b$ and $\gcd(a, b) = 1$. To apply the same technique to the general case needs the consideration of (at least) $d = \gcd(a, b)$ different 2-dimensional lattices.

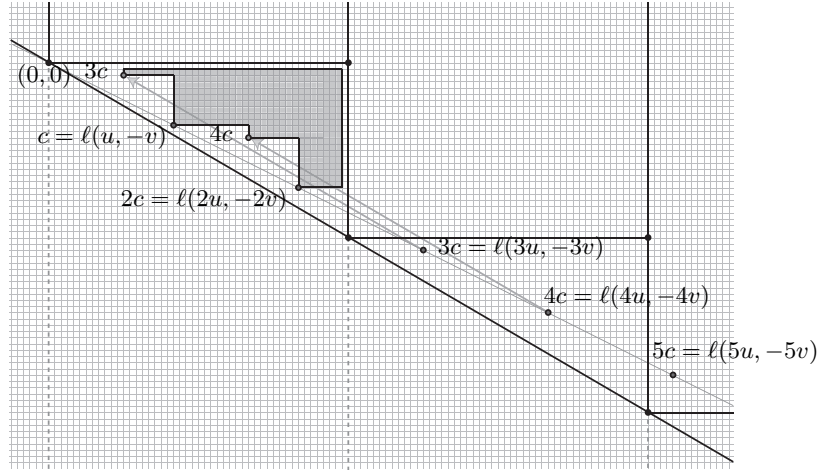


Figure 10: For $c = \ell(u, -v)$ with $s = 5$, $(u, -v), (2u, -2v) \in T_0$ and $(3u, -3v), (4u, -4v) \in T_1$, the Frobenius number and the number of gaps for $\{a, b, c\}$ is not so easy to compute. The points in T_0 corresponding to $3c$ and $4c$ are obtained from $(3u, -3v)$ and $(4u, -4v)$ by the translation of vector $(-b, a)$.

- Also, we think that some work can be done using the 3-dimensional lattice. This approach has the drawback that visualization becomes more difficult. For instance, the line $xa + yb = 0$ in the plane translates to a plane in the space, the plane $xa + yb + zc = 0$. The triangle T_0 becomes a tetrahedron. But, obviously, only a small portion of the tetrahedron correspond to the set of gaps. A possible question is: is there a nice representation of the set of gaps, as a subset of the tetrahedron determined by $xa + yb + zc = 0$?
- The generalization of our method to higher dimension could probably give some new result.

References

- [1] A. Brauer and J.E. Shockley, On a problem of Frobenius, *J. Reine Angew. Math.*, **211** (1962) pp. 399-408 (see Math. Rev. 47#127)
- [2] F. Curtis, On formulas for the Frobenius number of a numerical semigroup, *Math. Scand.*, **67** (1990) 190–192.
- [3] M. Hujter and B. Vizvari, The exact solution to the Frobenius problem with three variables, *J. Ramanujan Math. Soc.*, **55(2)** (1987), 117–143.
- [4] I.D. Kan, B.S. Stechkin, and I.V. Sharkov, Frobenius problem for three arguments, *Math. Notes*, **62(4)** (1997), 521–523.
- [5] J.L. Ramírez-Alfonsín, Complexity of the Frobenius Problem, *Combinatorica*, **16**(1996) 143–147.
- [6] J.L. Ramírez-Alfonsín, *The Diophantine Frobenius Problem*, Oxford University Press (2005).

- [7] J.C. Rosales, Fundamental gaps of numerical semigroups generated by two elements, *Linear Algebra Appl.*, **405** (2005), 200–208.
- [8] E. S. Selmer and Ö. Beyer, On the linear diophantine problem of Frobenius in three variables, *J. Reine Angew. Math.* **301** (1978), 161–170.
- [9] J.J. Sylvester, Problem 7382, *Educational Times*, **37** (1884), 26. Reprinted in: Mathematical questions with their solution, *Educational Times*, **41** (1884), 21.