Stabbers of line segments in the plane *

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Abstract

The problem of computing a representation of the stabbing lines of a set S of n line segments in the plane was solved by Edelsbrunner et al. with an $\Theta(n \log n)$ time and $O(n)$ space algorithm. We present a study of different types of stabbers such as wedges, double-wedges, 2-level trees, and zigzags; providing efficient algorithms whose time and space complexities depend on the number of combinatorially different extreme lines h_S or critical lines c_S, and the number k_S of different slopes that appear in S.

1 Introduction

Let $S = \{s_1, \ldots, s_n\}$ be a set of line segments (or segments) in the plane. For convenience, we require that if p and q are endpoints of a segment, then $p \neq q$, and consequently, lines, rays, and points are not considered to be segments. In order to avoid tedious case analysis, we assume that the endpoints of the segments are in general position. Nevertheless, the results presented in the paper can be extended to arbitrarily segment sets.

A line is a *transversal* of (or *stabs*) S if it intersects each segment of S. Edelsbrunner et al. [7] presented an $\Theta(n \log n)$ time and $O(n)$ space algorithm for solving the problem of constructing a representation of all traversal lines or stabbing lines of S. See Edelsbrunner [6] for an analysis of this problem from both a combinatorial and computational point of view. The lower bound from Edelsbrunner et al. [7] does not apply to the decision problem: *determining if there exists a line stabber for S*. Avis et al. [2] presented an $\Omega(n \log n)$ time lower bound in the fixed order algebraic decision tree model to determine the existence of a line stabber for S. For a set of n vertical segments, a stabbing line can be computed in $O(n)$ time.

A stabbing line ℓ for S classify the endpoints of the segments in two classes: endpoints above ℓ, say red points; and endpoints below ℓ, say blue points. The endpoint on ℓ is classified according to the other endpoint. Thus, we can see the problem of stabbing S as a problem of classifying the endpoints of the segments into disjoint monochromatic red and blue regions defined by the stabber, i.e., as a separability problem.

Since we want that the stabbers for S classify the endpoints of the segments in that way, we can consider the condition that there is no segment stabbed by more than one element of the stabber. We call this condition the *separability condition*. So we look for stabbers for S such that we can assign red and blue colors to the endpoints of the segments and split the plane into disjoint monochromatic regions, i.e., obtaining a red/blue classification of the endpoints of the segments. Hurtado et al. [11] classified red and blue points in the plane with separators which are similar to our stabbers.

Following this line of research, we deal with the problem of finding different kinds of "simple" stabbers when there exists no stabbing line for S. Concretely, we shall consider the structures shown in Figure 1. The goal is to design efficient algorithms for computing these stabbers for S with or without the separability condition.

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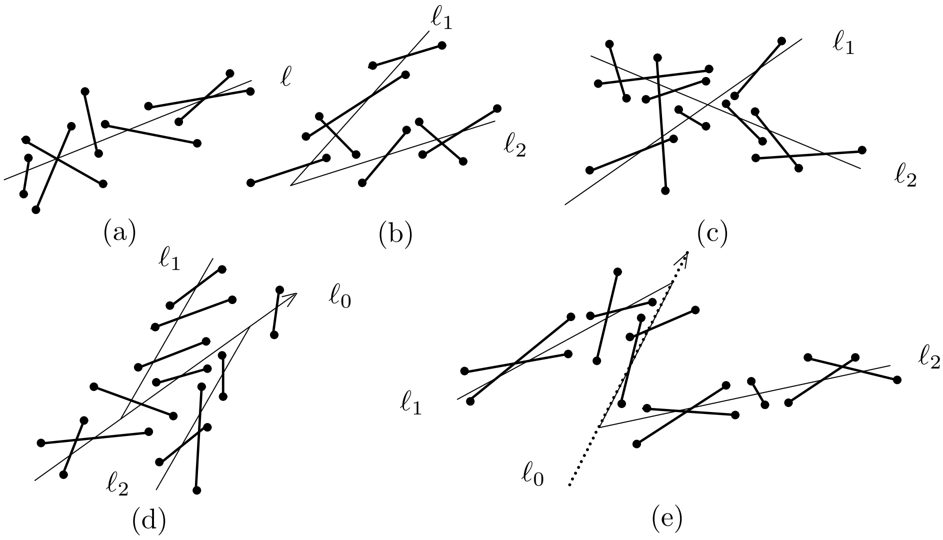

Figure 1: Stabbers from (a) to (e): line, wedge, double-wedge, 2-level tree, zigzag.

A standard geometric tool which will be used throughout this work is *duality* [7]: the geometric transform denoted by $\mathcal{D}$ which maps a point into a non-vertical line and vice versa. Given a point $p := (a, b)$ and a line $\ell := y = cx + d$, we have $\mathcal{D}(p) := y = ax + b$ and $\mathcal{D}(\ell) := (-c, d)$. A segment $s_i \in S$ is determined by its endpoints. The endpoints are transformed by D into two lines. If s_i is not vertical, $\mathcal{D}(s_i)$ is a double-wedge which does not contain a vertical line in its interior. Thus, the double-wedge is formed by two upper rays and two lower rays. If s_i is a vertical segment, $\mathcal{D}(s_i)$ is a *strip*. The set of endpoints of the segments in S is transformed by $\mathcal{D}$ into an arrangement of $2n$ lines denoted by $\mathcal{A}(S)$.

The transform $\mathcal{D}$ satisfies the following properties: (i) the transform $\mathcal{D}$ maintains the relative position (above/below) of points and lines; (ii) a line ℓ intersects a segment s_i if and only if the point $\mathcal{D}(\ell)$ lies in the double-wedge $\mathcal{D}(s_i)$; (iii) the stabbing lines of S stand in one-to-one correspondence with the intersection points of their double-wedges, i.e., $\bigcap_{s_i \in S} \mathcal{D}(s_i)$.

Related works. Claverol [5] as a part of her PhD thesis initiated the study here developed. In this paper we improve the complexities she obtained and some other stabbing problems are also considered. Atallah and Bajaj [1] presented an $O(n\alpha(n) \log n)$ algorithm for line stabbing n simple objects in the plane, where $\alpha(n)$ is the inverse of the Ackerman's function. A simple object is an object which has an $O(1)$ store description and for which common tangents and intersections can be computed in $O(1)$ time. Edelsbrunner, Guibas and Sharir [8] showed how to construct a representation of the line stabbers of convex polygons with a total of n vertices in $O(n\alpha(n) \log n)$ time. Later improved to $O(n \log n)$ using an $O(n \log n)$ time algorithm from Hershberger [10] for finding the lower envelope of a segment set in the plane. O'Rourke [14] presented an algorithm for finding (if it exists) a stabbing line of vertical line segments. Goodrich and Snoeyink [9] presented a natural variant considering another type of stabbers different from the lines by solving the problem of computing a transversal convex polygon for a set of parallel segments in $O(n \log n)$ time. Bhattacharya et al. [3] worked on the problem of computing the shortest transversal segment for a set of lines in the plane and also for a set of convex polygons. Lyons et al. [12] studied the problem of computing the minimum perimeter convex polygon which stabs a set of isothetic line segments. Rappaport [15] considered the problem of computing a simple polygon with minimum perimeter which stabs or contains a set of line segments. Bhattacharya et al. [4] and Mukhopadhyay et al. [13] considered the problem of computing the minimum area convex polygon which stabs a set of parallel line segments.

2 Ideas and tools

Some ideas and tools are shared by most of our results. In this section we present them in a unified context. The lines containing the segments of S can have different slopes. Denote by m_i the slope of (the line containing) the segment $s_i \in S$. The complexity of many of the algorithms that we present here depends on the number of different slopes of the lines containing the segments of S, written as k_S. It is due to the following fact: the endpoints of the segments fall into two classes determined by a stabbing line ℓ, endpoints above ℓ and endpoints below ℓ, say red and blue respectively. The endpoint of a segment on ℓ is classified according to its other endpoint. Thus, for a given slope of ℓ our problem of stabbing the set S can be viewed as a red-blue separability problem of classifying the endpoints of the segments into disjoint monochromatic regions of the plane determined by the stabber. In fact, it is not difficult to show that there exist as many different classifications of the endpoints of S as k_S.

A relevant property about our stabbers is that not all the lines can be candidate to define one of these structures. For instance, consider a ray ℓ which is part of a wedge that stabs S, and its extension denoted by ℓ'. We have that all the segments that are not intersected by ℓ must lie on the same half-plane defined by ℓ'. Thus, we say that a line ℓ' is an *extreme line* for S if ℓ' stabs a subset of segments $S_1 \subseteq S$, $S_1 \neq \emptyset$, and the remaining segments $S_2 = S \setminus S_1$ are in only one of the open half-planes defined by ℓ'. Otherwise, the line ℓ' is said to be a non-extreme line for S. Thus, as we have mentioned before, our interest in extreme lines comes from the following fact: *the line containing any ray of a stabbing wedge for S is extreme line for S.*

In this paper, extreme lines are studied from two different points of view: computational and combinatorial view. The reason is that our algorithms depend on the computation of the set of extreme lines for S. Thus, we say that two non-vertical lines, ℓ_1 and ℓ_2, are *combinatorially different* with respect to S if either: (1) the subset of segments $S_1 \subseteq S$ stabbed by ℓ_1 is different from the subset of segments $S_2 \subseteq S$ stabbed by ℓ_2; or (2) if $S_1 = S_2$ then either: (i) the subset of endpoints of segments of S above ℓ_1 is different from the subset of endpoints of segments of S above ℓ_2, or (ii) the subset of endpoints of segments of S below ℓ_1 is different from the subset of endpoints of segments of S below ℓ_2. If h_S is the number of combinatorially different extreme lines of S, we compute a representation of the combinatorially different extreme lines for S in $O(h_S + n \log n)$ time and $O(h_S + n)$ space. Observe that the number of combinatorially different extreme lines for S is at most $O(n^2)$, but depending on the properties of the stabbing problem, we can consider only a subset of them named the *critical extreme lines* which size c_S is at most linear.

3 Stabbing wedges

Our first aim is to study the problem of deciding whether the set S can be stabbed by a wedge, and computing this structure in case of existence. Obviously, it is assumed that the set S is not stabbed by a line. We distinguish two cases: stabbing wedges W satisfying the separability condition (described in Section 1) or those that do not satisfy such condition. In the first case, we provide an $O(h_S k_S \log n + n \log n)$ time and $O(h_S + n)$ space algorithm. The range for h_S is from $O(1)$ to $O(n^2)$, and the range for k_S is from $O(1)$ to $O(n)$. In the second case, we design an $O(c_S k_S \log n + n \log n)$ time and $O(n)$ space algorithm, with range for c_S in between $O(1)$ and $O(n)$.

We now introduce some useful notation for our purpose. Given a line ℓ and a segment s, we can classify the endpoints of s with respect to ℓ whenever ℓ and the line containing s are not parallel. It suffices to do a parallel sweep with ℓ until it crosses s, leaving one endpoint in ℓ^+, and the other one in ℓ^-. These endpoints are denoted by e^+ and e^-, respectively.

Let W be a stabbing wedge for S. The two rays that form W are denoted by ℓ_1 and ℓ_2, and W is written as $W = \{\ell_1, \ell_2\}$. The line containing ℓ_i for $i = 1, 2$ is denoted by ℓ'_i. The half-planes defined by ℓ_i are written as $\ell_i'^+$ and $\ell_i'^-$. Suppose that ℓ_1 stabs a subset of segments $S_1 \subsetneq S$, $S_1 \neq \emptyset$, and the set $S_2 = S \setminus S_1$ is stabbed by ℓ_2.

3.1 Stabbing wedges satisfying the separability condition

Since by definition we consider W as a separator structure, a segment can not be stabbed by both rays. Let S_1^+ (S_1^-) be the set of endpoints of the segments of S_1 classified as e^+ (e^-) with respect to ℓ'_1. Analogously, S_2^+ (S_2^-) is the set of endpoints of the segments of S_2 classified as e^+ (e^-) with respect to ℓ'_2. Thus, S_1^- and S_2^+ are contained inside the wedge W, and S_1^+ and S_2^- are located outside it. Notice that these assignments $\{+,-\}$ depend on the relative position of the lines ℓ'_1 and ℓ'_2, i.e., on the slope and the aperture angle of the stabbing wedge. We concentrate in a particular case but the remaining constant number of cases can be handle in a similar way.

Lemma 3.1. *If $W = \{\ell_1, \ell_2\}$ is a stabbing wedge for S, ℓ'_1 and ℓ'_2 are extreme lines for S and at least half of the segments of S are stabbed by either ℓ_1 or ℓ_2.*

The next lemma assumes the following conditions: (i) let ℓ'_1 be an extreme line for S where $S_1 \subsetneq S$, $S_1 \neq \emptyset$, is the subset of segments stabbed by ℓ'_1. Let S_1^+ and S_1^- be the classification of the endpoints of the segments of S_1 given by ℓ'_1, and let $S_2 = S \setminus S_1$. (ii) Let m be a fixed slope and let S_2^+ and S_2^- be the classification of the endpoints of the segments of S_2 by sweeping a line with slope m.

Lemma 3.2. *There exists a stabbing wedge $W = \{\ell_1, \ell_2\}$ for S with ℓ_1 contained in ℓ'_1 if and only if S_2^- is line separable from $S_1^- \cup S_2^+$. The locus of apices of the stabbing wedges for S respecting this classification of endpoints is a (possible unbounded and degenerate) convex quadrilateral Q defined by the following four lines: the interior supported lines between $CH(S_1^+)$ and $CH(S_1^- \cup S_2^+)$ and the interior supported lines between $CH(S_1^- \cup S_2^+)$ and $CH(S_2^-)$.*

Lemmas 3.1 and 3.2 are the key tools to design an algorithm that proves the following result.

Theorem 3.3. *The set of combinatorially different stabbing wedges for S with the separability condition, together with a representation of them formed by the locus of apices of the stabbing wedges can be computed in $O(h_S k_S \log n + n \log n)$ time and $O(h_S + n)$ space.*

3.2 Stabbing wedges not satisfying the separability condition

Let $W = \{\ell_1, \ell_2\}$ be a stabbing wedge for S not verifying the separability condition. A property that only holds for this type of wedges is the following: *there always exists a stabbing wedge W for S (not satisfying the separability condition) formed by two rays both anchored on fixed points of S, i.e., both rays are contained in critical extreme lines.* This property let us prove the following result.

Theorem 3.4. *The set of combinatorially different stabbing wedges for S not satisfying the separability condition can be computed in $O(c_S k_S n + n \log n)$ time and $O(n)$ space.*

Next we show an $\Omega(n \log n)$ lower bound for the problem of deciding if there exists a stabbing wedge for a set of arbitrarily segments. We reduce the decision of the stabbing wedge problem to the problem of deciding whether there exists a stabbing line for a segment set, which has an $\Omega(n \log n)$ time lower bound in the fixed order algebraic decision tree model [2]).

Theorem 3.5. *Deciding whether there exists a stabbing wedge for an arbitrary segment set requires $\Omega(n \log n)$ time in the fixed order algebraic decision tree model.*

4 Stabbing wedges for parallel segments with equal length

Let $S = \{s_1, \ldots, s_n\}$ be a set of n parallel segments in the plane *with equal length* which are not stabbed by a line. We now consider the problem of computing a stabbing wedge W for S satisfying the separability condition.

Up to symmetry with respect to either the x-axis or the y-axis, we distinguish three types of wedges according to the relative position of the rays ℓ_1 and ℓ_2 of the possible stabbing wedge $W = \{\ell_1, \ell_2\}$ for S. Let α_W be the *aperture angle* or *interval direction* defined by the rays ℓ_1 and ℓ_2 of W. The three types are the following: (a) α_W contains the vertical direction; (b) α_W contains the horizontal direction; and (c) both rays ℓ_1 and ℓ_2 of W have positive slope.

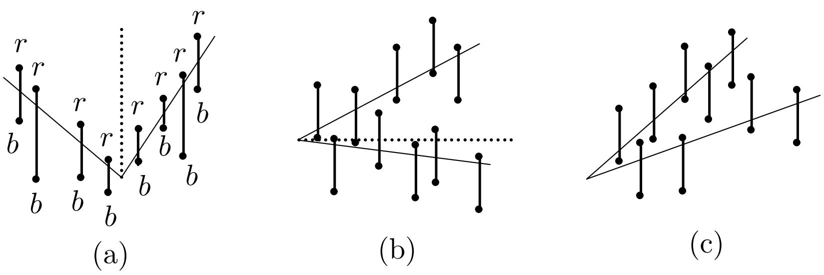

Figure 2: The three types of stabbing wedges for a parallel segment set.

4.1 Type (a)

When α_W contains the vertical direction, it is possible to design the following $O(n \log n)$ time algorithm for computing a stabbing wedge for S. It is based on an $O(n \log n)$ time and $O(n)$ space algorithm for deciding the wedge separability of a red-blue point set in the plane [11].

1. In $O(n)$ time, classify the endpoints of the segments of S as follows. For each segment s, color red the endpoint of s with bigger y-coordinate and color blue the other endpoint. Let R and B be the sets of red and blue endpoints.
2. In $O(n \log n)$ time, decide whether there exists a separating wedge for R and B, and compute it in case of existence. In the same time, we can compute the locus of apices of all the separating wedges formed by convex quadrilaterals.

Theorem 4.1. *Given the set S, a stabbing wedge of type (a) for S can be computed in $O(n \log n)$ time and $O(n)$ space.*

The assumption that the segments have equal length is not used in the algorithm. If the slopes of the rays of the wedge are known, there exists an $O(n)$ time algorithm to compute a stabbing wedge for S using the median of the x-coordinates of the endpoints of the segments.

4.2 Type (b)

We have also obtained an $O(n \log n)$ time algorithm for computing a stabbing wedge for S when α_W contains the horizontal direction. For each segment $s_i \in S$, consider its midpoint ρ_i. In $O(n \log n)$ time, sort these midpoints by decreasing y-coordinate, and let $\preceq_y$ denote this order. Denote by $S^* = \{s_1^*, \ldots, s_n^*\}$ the set of segments of S sorted by the $\preceq_y$ order of their midpoints. The key tools to design our algorithm are given by the following lemma.

Lemma 4.2. *If there exists a stabbing wedge for S, $W = \{\ell_1, \ell_2\}$, such that α_W contains the horizontal direction, then the midpoints of the segments stabbed by ℓ_1 appear before in the $\preceq_y$ order than the midpoints of the segments stabbed by ℓ_2.*

Let ℓ_1' be a line with positive slope that stabs $S_1 \subsetneq S$, $S_1 \neq \emptyset$, and let ℓ_2' be a line with negative slope that stabs $S_2 = S \setminus S_1$. Consider the classification of the endpoints of S provided by ℓ_1' and ℓ_2'. There exists a stabbing wedge $W = \{\ell_1, \ell_2\}$ for S if and only if both S_1^+ and S_2^- are line separable from $S_1^- \cup S_2^+$.

Theorem 4.3. *Given the set S, a stabbing wedge of type (b) for S can be computed in $O(n \log n)$ time and $O(n)$ space.*

4.3 Type (c)

When both rays ℓ_1 and ℓ_2 of W have positive slope, the main problem is to obtain a consistent classification of the midpoints of the segments according to the possible stabbing wedge of type (c). Denote by d the length of the segments of S, and recall that ρ_i is the midpoint of the segment $s_i \in S$.

Assume that there exists a stabbing wedge of type (c) for S, denoted by $W = \{\ell_1, \ell_2\}$. Suppose also that α_W is known. Let ℓ'_i for $i = 1, 2$, be the line containing the ray ℓ_i. Denote by ℓ''_1 the line below and parallel to ℓ'_1, such that the *vertical distance* between the two lines ℓ''_1 and ℓ'_1 is exactly $d/2$. Similarly, ℓ''_2 is the line above and parallel to ℓ'_2 such that the vertical distance between ℓ''_2 and ℓ'_2 is also $d/2$. Let ℓ be the bisector of the angle defined by ℓ'_1 and ℓ'_2 or any line with slope between the slopes of ℓ'_1 and ℓ'_2 (Figure 3).

Obviously the angle defined by ℓ''_1 and ℓ''_2 is α_W. Let ℓ be the bisector line of α_W. In fact, as ℓ we can take any line with slope within the slope interval defined by α_W. Consider the double-wedge DW formed by the lines ℓ''_1 and ℓ''_2, and the corresponding upper rays and lower rays of DW. By definition of ℓ''_1 and ℓ''_2, all the midpoints of the segments of S stabbed by ℓ_1 (ℓ_2) are above (below) or over the upper rays (lower rays) of DW. Let $\preceq_\ell$ be the order of the midpoints of the segments of S according to a sweeping by the bisector line ℓ of α_W. Thus, if we know that $\alpha_W \geq \alpha$, for some given α, we can compute a constant number $\lceil \frac{\pi}{2\alpha} \rceil = t$ of slope candidates for line ℓ and check each one in $O(n \log n)$ time and $O(n)$ space. For stabbing wedges with very small aperture angle α_W, the value t can dominate n.

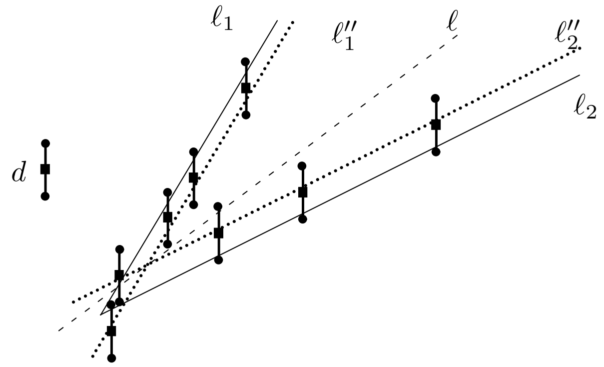

Figure 3: Both rays ℓ_1 and ℓ_2 of a stabbing wedge have positive slope.

Theorem 4.4. *Given the set S and a positive value α, a stabbing wedge of type (c) for S with aperture angle $\alpha_W \geq \alpha$ can be computed in $O(nt \log n)$ time and $O(n)$ space.*

5 Other stabbers

Table 1 summarizes the obtained results for the decisional problems of the stabbers we have considered. By (sc) and (nsc) we denote *satisfying separability condition* and *not satisfying the separability condition*, respectively.

Stabber	**Time**	**Space**
Wedge (sc)	$O(h_S k_S \log n + n \log n)$	$O(h_S + n)$
Wedge (nsc)	$O(c_S k_S n + n \log n)$	$O(n)$
Double-wedge (sc)	$\min\{O(n^4), O(n^3 k_S \log n)\}$	$O(n^2)$
Double-wedge (nsc)	$O(n^2 k_S \log n)$	$O(n^2)$
2-level tree (nsc)	$O(n^2 k_S \log n)$	$O(n^2)$
Zigzag (sc)	$O(n^2 k_S \log n)$	$O(n^2)$
Zigzag (nsc)	$O(n^3 k_S)$	$O(n^2)$

Table 1: Summary of results of decision problem.

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