Bisemivalues for bicooperative games *

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Abstract

We introduce bisemivalues for bicooperative games and we also provide an interesting characterization of this kind of values by means of weighting coefficients in a similar way as it was given for semivalues in the context of cooperative games. Moreover, the notion of induced bisemivalues on lower cardinalities also makes sense and an adaptation of Dragan's recurrence formula is obtained. For the particular case of (p,q)–bisemivalues, a computational procedure in terms of the multilinear extension of the game is given.

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1 Introduction

Cooperative games in a finite set of players are often defined in terms of a characteristic function, which specifies the worth that each coalition can achieve for itself. It can be interpreted as the maximal gain or minimal cost that the inner players of the coalition can achieve themselves against the best offensive threat by the complementary coalition. Most applications of cooperative games are found in economics and politics. In these games we are interested in what players can achieve by cooperation. In these games, each player has only two options: either to join a coalition or to stay aside. For a player who chooses the first option, he or she is supposed to cooperate in order to obtain the maximum worth of the coalition. However, there are many economical or political situations that cannot be described by using this classical model.

Let us consider the following example. Two insurance companies are always in competition in order to obtain the maximum number of clients in a region. Each one of the insurance agents has an owner clients' list. We are interested in study the benefits when some agents

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work for the first company, whereas another agents work for the second company and the remaining agents do not work for either one of them. This kind of situations can be described by using *bicooperative games*, introduced by Bilbao [2] as a generalization of classical cooperative games, where each player can participate positively to the game, negatively, or do not participate. Then, in these games ordered pairs of disjoint coalitions of players are considered. Thus, each such pair yields a partition of the set of players in three groups: (*i*) players in the first coalition are defenders of an option; (*ii*) players in the second coalition do not agree with it and they will take another option; and (*iii*) the remaining players are not in favour of adopting either option.

One may think that bicooperative games can be seen as a particular case of games with n players and r alternatives (for $r = 3$), introduced by Bolger in [8] and [9]. On the other hand, such games are also considered to be isomorphic to *multichoice games*, proposed by Hsiao and Raghavan [22]. In these games, each player has several possible levels of participation (among a finite number of possible levels that are ordered from non–participation to complete participation) to the game. The contribution of a player to a game depends on his level of participation. However, bicooperative games cannot be seen as a particular case of multichoice games because, for instance, the worth of a multichoice game if all players choose the lowest level of participation is zero, whereas the worth of a bicooperative game if all players are against is nonpositive. For an interesting discussion about these two games, we refer the reader to [20].

A central question in game theory is to define a solution concept for a game, that is, a function which assigns to every game a set of real–valued vectors, each one of them represents a payoff distribution among the players. In the context of bicooperative games this concept has also been studied and different solution concepts have been introduced. In 2008, Bilbao *et al* [4] introduced the Shapley value for bicooperative games. In [3] and [7] Bilbao *et al* introduced the *core*, the *Weber set* and the *selectope* for bicooperative games. In [5] Bilbao *et al* defined and characterized *biprobabilistic values* for bicooperative games following Weber's characterization [28] of probabilistic values on cooperative games.

In 2010 Bilbao *et al* [6] analized *ternary bicooperative games*, which are a refinement of the *ternary voting games* introduced in [13], and defined and axiomatized the Banzhaf power index for these games. Several works by Freixas [16], [17] and Freixas and Zwicker [15] have been devoted to the study of voting systems with several ordered levels of approval in the input and in the output. In their model, the abstention is a level of input approval intermediate between yes and no votes.

Other different definitions of values for bicooperative games can be found in Grabisch and Labreuche [20] and [21]. In 2012, Borkotokey and Sarmah [10] introduce the notion of a bicooperative game with fuzzy bicoalitions and an explicit form of the Shapley value as a possible solution concept to a particular class of such games is also obtained.

The aim of this paper is to introduce and to characterize bisemivalues for bicooperative games– as a particular family of biprobabilistic values– that parallels the existing statements for semivalues on cooperative games given by Dubey *et al* in [12]. Moreover, a subfamily of these values, called (p,q)–bisemivalues is also introduced and as a particular case of it, we found the *binomial bisemivalues*, that extend the concept of binomial values to bicooperative games. For more than a decade, our research group has been studying *semivalues*, a subfamily of probabilistic values introduced by Dubey et al. [12], characterized by anonymity and including the Shapley value as the only *efficient* member. From this experience, we feel that bisemivalues in general and (p,q)–bisemivalues in particular, can be also used in the study of bicooperative games because they offer a deal of flexibility greater than the classical values, and hence many more possibilities to introduce additional information when evaluating a game.

(p,q)–bisemivalues provide tools to study not only games *in abstracto* (i.e. from a merely structural viewpoint) but also the influence of players' *personality* on the issue. They are assessment techniques that do not modify the game but only the criteria by which payoffs are allocated. In the (p,q)–bisemivalue case two parameters are used to cope with different attitudes the players may hold when playing a given game, even if they are not individuals but countries, enterprises, parties, trade unions, or collectivities of any other kind. For all player, we will attach to parameter p the meaning of *generical tendency to support a player in his decision* and to parameter q *generical tendency to go against him*. We think that these bisemivalues are suited for the study of bicooperative games where players show two different tendencies to form coalitions.

Summing up, the paper tries to present bisemivalues in general and (p,q)–bisemivalues in particular, as a consistent alternative or complement to the values defined up to now. Players' tendencies can encompass a variety of situations that cannot be analyzed, without modifying the game, by means of another values, which are concerned only with the structure of the game.

The organization of the paper is as follows. In Section 2, we include a minimum of preliminaries that refers to semivalues for cooperative games and biprobabilistics values for bicooperative games. Section 3 is devoted to define bisemivalues for bicooperative games and to give the main theorem of the paper, that clearly reminds the characterization obtained by Dubey, Neyman and Weber [12] for semivalues on cooperative games. Moreover we deals with *induced bisemivalues* on lower cardinalities and an adaptation of Dragan's recurrence formula [11] is obtained. In Section 4 we introduce the (p,q)–bisemivalues and prove that their weighting coefficients lie in geometric progression, the simplest form of monotonicity. We also give a computational procedure in terms of the multilinear extension (MLE) of the game to calculate them. Finally, Section 5 contains an application of the bisemivalues to the analysis of an example.

2 Preliminaries

2.1 Cooperative games and semivalues

Let N be a finite set of *players* and 2^N be the set of its *coalitions* (subsets of N). A *cooperative game* on N is a function $v : 2^N \to \mathbb{R}$, that assigns a real number $v(S)$ to each coalition $S \subseteq N$, with $v(\emptyset) = 0$. A game v is *monotonic* if $v(S) \leq v(T)$ whenever $S \subseteq T \subseteq N$. A player $i \in N$ is a *dummy* in v iff $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \backslash \{i\}$. Endowed with the natural operations for real–valued functions, i.e. $v + v'$ and λv for all $\lambda \in \mathbb{R}$, the set of all cooperative games on N is a vector space $\mathcal{G}_N$.

By a *value* on $\mathcal{G}_N$ we will mean a map $\Psi : \mathcal{G}_N \to \mathbb{R}^N$, that assigns to every game v a vector $\Psi[v]$ with components $\Psi_i[v]$ for all $i \in N$.

According Weber's [28] axiomatic description, $\Psi : \mathcal{G}_N \to \mathbb{R}^N$ is a *semivalue* iff it satisfies the following properties:

(i) *linearity*: $\Psi[\alpha v + \beta v'] = \alpha\Psi[v] + \beta\Psi[v']$, for all $v, v' \in \mathcal{G}_N$ and α, $\beta \in \mathbb{R}$;
(ii) *anonymity*: $\Psi_{\pi i}[\pi v] = \Psi_i[v]$ for all permutation π on N, where $\pi v(\pi S) = v(S)$ and $\pi S = \{\pi i : i \in S\}$, $i \in N$, and $v \in \mathcal{G}_N$;
(iii) *positivity*: if v is monotonic, then $\Psi[v] \geq 0$;
(iv) *dummy player property*: if $i \in N$ is a dummy in game v, then $\Psi_i[v] = v(\{i\})$.

There is an interesting characterization of semivalues, by means of *weighting coefficients*, due to Dubey, Neyman and Weber [12]. Set $n = |N|$. Then: (a) for every *weighting vector* $\{p_k\}_{k=0}^{n-1}$ such that $\sum_{k=0}^{n-1} p_k \binom{n-1}{k} = 1$ and $p_k \geq 0$ for all k, the expression

$$\Psi_i[v] = \sum_{S \subseteq N \setminus \{i\}} p_s[v(S \cup \{i\}) - v(S)] \quad \text{for all } i \in N \text{ and all } v \in \mathcal{G}_N,$$

where $s = |S|$, defines a semivalue Ψ; (b) conversely, every semivalue can be obtained in this way; (c) the correspondence given by $\{p_k\}_{k=0}^{n-1} \mapsto \psi$ is bijective.

Thus, the payoff that a semivalue allocates to every player in any game is a weighted sum of his marginal contributions in the game. If p_k is interpreted as the probability that a given player i joins a coalition of size k, provided that all the coalitions of a common size have the same probability of being joined, then $\Psi_i[v]$ is the expected marginal contribution of that player to a random coalition he joins.

Well known examples of semivalues are the *Shapley value* φ (Shapley [27]), for which $p_k = 1/n\binom{n-1}{k}$, and the *Banzhaf value* β (Owen [24]), for which $p_k = 2^{1-n}$. The Shapley value φ is the only *efficient* semivalue, in the sense that $\sum_{i \in N} \varphi_i[v] = v(N)$ for every $v \in \mathcal{G}_N$.

Notice that these values are defined for each N. The same happens with the *binomial semivalues*, introduced by Puente [26] (see also Giménez [18] or Amer and Giménez [1]) as follows. Let $p \in [0,1]$ and $p_k = p^k(1-p)^{n-k-1}$ for $k = 0, 1, \ldots, n-1$. Then $\{p_k\}_{k=0}^{n-1}$ is a weighting vector and defines a semivalue that will be denoted as Ψ^p and called the *p–binomial semivalue*. Using the convention that $0^0 = 1$, the definition makes sense also for $p = 0$ and $p = 1$, where we respectively get the *dictatorial index* Ψ^0 and the *marginal index* Ψ^1, introduced by Owen [25] and such that $\Psi_i^0[v] = v(\{i\})$ and $\Psi_i^1[v] = v(N) - v(N \setminus \{i\})$ for all $i \in N$ and all $v \in \mathcal{G}_N$. Of course, $p = 1/2$ gives $\Psi^{1/2} = \beta$ —the Banzhaf value.

Finally, the *multilinear extension* [1] of a game $v \in \mathcal{G}_N$, introduced by Owen [23], is the real–valued function defined in $\mathbb{R}^n$ by

$$f(x_1, x_2, \ldots, x_n) = \sum_{S \subseteq N} \prod_{i \in S} x_i \prod_{j \in N \setminus S} (1 - x_j) v(S).$$

As is well known, both the Shapley and Banzhaf values of any cooperative game v can be obtained from its multilinear extension. Indeed, $\varphi[v]$ can be calculated by integrating the partial derivatives of the multilinear extension of the game along the main diagonal $x_1 = x_2 = \cdots = x_n$ of the cube $[0,1]^n$ [23], while the partial derivatives of that multilinear extension,

[1] The term "multilinear" means that, for each $i \in N$, the function is linear in x_i, that is, of the form $f_v(x_1, x_2, \ldots, x_n) = g_i(x_1, x_2, \ldots, \hat{x}_i, \ldots, x_n)x_i + h_i(x_1, x_2, \ldots, \hat{x}_i, \ldots, x_n)$.

evaluated at point $(1/2, 1/2, \ldots, 1/2)$, give $\beta[v]$ [24]. This latter procedure extends well to any p–binomial semivalue (see Puente [26], Freixas and Puente [14] or Amer and Giménez [1]) by evaluating the derivatives at point $(p, p, \ldots, p)$.
In 1988, Weber [28] went further, dropped anonymity, and defined the family of *probabilistic values*, each one of which requires weighting coefficients p_S^i for each player i and each coalition $S \subseteq N \backslash \{i\}$ (of course, anonymity characterizes semivalues within this new family). The payoff that a probabilistic value allocates to each player is thus, again, a weighted sum of his marginal contributions in the game. We quote from Weber [28]. As we will see in the following section, Bilbao *et al* [5] defined and characterized *biprobabilistic values for bicooperative games*.

2.2 Bicooperative games and biprobabilistic values

Let N be a finite set of *players* and $3^N = \{(S,T) : S,T \subseteq N, S \cap T = \emptyset\}$ be the set of all ordered pairs of disjoint coalitions. Grabisch and Labreuche [19] proposed a relation in 3^N given by

$$(A,B) \sqsubseteq (C,D) \Leftrightarrow A \subseteq C, B \supseteq D.$$

Following [2], a *bicooperative game* on N is a function $b : 3^N \to \mathbb{R}$, that assigns a real number $b(S,T)$ to each pair of coalitions $(S,T) \in 3^N$, with $b(\emptyset,\emptyset) = 0$. For each $(S,T) \in 3^N$, the worth $b(S,T)$ represents the maximal gain (if $b(S,T) > 0$) or the minimal loss (if $b(S,T) < 0$) that is obtained when players in S are in favor of a change in the situation, players in T are against the change and players in $N \setminus (S \cup T)$ are indifferent. Then $b(\emptyset, N)$ is the cost obtained when all players are against the change and $b(N,\emptyset)$ is the maximal gain obtained when all players want to change the initial situation. Endowed with the natural operations for real–valued functions, the set of all bicooperative games on N is a vector space $\mathcal{BG}_N$. For every $(S,T) \in 3^N$ such that $(S,T) \neq (\emptyset,\emptyset)$, the *identity game* $\delta_{(S,T)}$ is defined by

$$\delta_{(S,T)}(A,B) = \begin{cases} 1 & \text{if } (A,B) = (S,T) \\ 0 & \text{otherwise} \end{cases}$$

and it is easily checked that the set of all identity games is a basis for $\mathcal{BG}_N$, so that $\dim(\mathcal{BG}_N) = 3^n - 1$ if $n = |N|$.
A bicooperative game is *monotonic* if $b(S,T) \leq b(S',T')$ whenever $(S,T) \sqsubseteq (S',T')$. A player $i \in N$ is a *dummy* in b if $b(S \cup \{i\}, T) = b(S,T) + b(\{i\},\emptyset)$ and $b(S, T \cup \{i\} = b(S,T) + b(\emptyset, \{i\})$ for all $(S,T) \in 3^{N\backslash\{i\}}$, and *null* in b if, moreover, $b(\{i\},\emptyset) = b(\emptyset,\{i\}) = 0$.
By a *value* on $\mathcal{BG}_N$ we will mean a map $\Psi : \mathcal{BG}_N \to \mathbb{R}^N$, that assigns to every game b a vector $\Psi[b]$ with components $\Psi_i[b]$ for all $i \in N$.
In [5] Bilbao *et al* defined and characterized *biprobabilistic values* for bicooperative games as follows.

Definition 2.1 A value ϕ for player i on $\mathcal{BG}_N$ is a biprobabilistic value if there exist two collections of real numbers $\{p^i_{(S,T)} : (S,T) \in 3^{N\backslash\{i\}}\}$ and $\{q^i_{(S,T)} : (S,T) \in 3^{N\backslash\{i\}}\}$ satisfying

$p^i_{(S,T)} \geq 0$, $q^i_{(S,T)} \geq 0$, $\sum_{(S,T)\in 3^{N\setminus\{i\}}} p^i_{(S,T)} = 1$ and $\sum_{(S,T)\in 3^{N\setminus\{i\}}} q^i_{(S,T)} = 1$ such that,

$$\phi_i[b] = \sum_{(S,T)\in 3^{N\setminus\{i\}}} \left[p^i_{(S,T)}(b(S\cup\{i\},T) - b(S,T)) + q^i_{(S,T)}(b(S,T) - b(S,T\cup\{i\})) \right]$$

for every game $b \in \mathcal{BG}_N$.

Notice that $\phi_i[b]$ is a weighted sum of his marginal contributions $b(S\cup\{i\},T) - b(S,T)$, whenever i joins coalition $S \subseteq N\setminus\{i\}$ and his marginal contributions $b(S,T) - b(S,T\cup\{i\})$ whenever i leaves coalition $T\cup\{i\}$, where $p^i_{(S,T)}$ is the probability that player i joins S and $q^i_{(S,T)}$ is the probability that player i leaves $T\cup\{i\}$.
Following the axiomatic description given by Bilbao *et al* [5], a value ϕ on $\mathcal{BG}_N$ is a probabilistic value if and only if it satisfies the following properties:
(i) *linearity*: $\phi[\alpha b + \beta b'] = \alpha\phi[b] + \beta\phi[b']$, for all $b, b' \in \mathcal{GB}_N$ and α, $\beta \in \mathbb{R}$;
(ii) *positivity*: if b is monotonic, then $\phi[b] \geq 0$;
(iii) *dummy player property*: if $i \in N$ is a dummy in game b, then $\phi_i[b] = b(\{i\},\emptyset) - b(\emptyset,\{i\})$.
Among biprobabilistic values the Shapley value [4], denoted here by φ, for which

$$p^i_{s,t} = \frac{(n+s-t)!\,(n+t-s-1)!}{(2n)!}\,2^{n-s-t} \text{ and } q^i_{s,t} = \frac{(n+t-s)!\,(n+s-t-1)!}{(2n)!}\,2^{n-s-t},$$

for all $i \in N$ and for all $(S,T) \in 3^{N\setminus i}$ with $s = |S|$ and $t = |T|$, was characterized by Bilbao *et al* [4] as the only efficient bisemivalue –in the sense that its total power for every $b \in \mathcal{BG}_N$ is $\sum_{i\in N} \varphi_i[b] = b(N,\emptyset) - b(\emptyset,N)$– satisfying the *structural axiom*.
From now on we will denote $S\cup\{i\}$ by $S\cup i$, $S\setminus\{i\}$ by $S\setminus i$ and $|S| = s$ for all $S \subseteq N$.

3 Bisemivalues for bicooperative games

In this section we introduce and study bisemivalues for bicooperative games. This includes, besides the axiomatic description, a characterization of them by means of weighting coefficients that parallels the existing characterization of semivalues given by Dubey, Neyman and Weber [12] in the context of cooperative games.
In a similar way as the cooperative case, for the comparison of roles in a game to be meaningful, the evaluation of a particular position should depend on the structure of the game but not on the labels of the players.
In order to define this family we need a new axiom, introduced by Bilbao *et al* in [4].

Definition 3.1 *Anonymity axiom.* $\phi_{\pi i}[\pi b] = \phi_i[b]$ for all permutation π over N, $i \in N$, and $b \in \mathcal{BG}_N$, where $\pi b(\pi S, \pi T) = b(S,T)$ and $\pi S = \{\pi i : i \in S\}$.

Now we are ready to introduce bisemivalues on bicooperative games following Weber's axiomatic description of semivalues on cooperative games.

Definition 3.2 A bisemivalue on $\mathcal{BG}_N$ is a map $\psi : \mathcal{BG}_N \to \mathbb{R}^N$ that satisfies linearity, anonymity, positivity and dummy player property.

As we will see, anonymity characterizes bisemivalues within the family of biprobabilistic values.

Theorem 3.3 *A value ψ on $\mathcal{BG}_N$ is a bisemivalue if and only if there exist two collections of real numbers $\{p_{s,t}\}$ and $\{q_{s,t}\}$ satisfying:*

$$\begin{aligned} &p_{s,t} \geq 0, q_{s,t} \geq 0, \\ &\sum_{s=0}^{n-1} \binom{n-1}{s} \left[\sum_{t=0}^{n-s-1} \binom{n-s-1}{t} p_{s,t} \right] = 1, \\ &\sum_{t=0}^{n-1} \binom{n-1}{t} \left[\sum_{s=0}^{n-t-1} \binom{n-t-1}{s} q_{s,t} \right] = 1, \end{aligned} \tag{1}$$

such that

$$\psi_i[b] = \sum_{(S,T)\in 3^{N\setminus i}} [p_{s,t}(b(S\cup i,T) - b(S,T)) + q_{s,t}(b(S,T) - b(S,T\cup i))]$$

for all $i \in N$ and all $b \in \mathcal{BG}_N$, where $s = |S|$ and $t = |T|$.

Proof ($\Leftarrow$) Taking into account that bisemivalues are a particular case of biprobabilistic values, linearity, dummy and positivity are proved in [5]. Anonimity follows from the fact that the weighting coefficients only depend of the cardinality of S and T.
($\Rightarrow$) Following [4], it is easy to prove that if a biprobabilistic value satisfies the anonymity axiom then $p^i_{(S,T)} = p_{s,t}$ and $q^i_{(S,T)} = q_{s,t}$ for all $(S,T) \in 3^{N\setminus i}$ with $s = |S|$ and $t = |T|$, for all $i \in N$. $\square$

Remark 3.4 (a) The payoff that a bisemivalue allocates to every player in any game is a weighted sum of his marginal contributions $b(S\cup i,T) - b(S,T)$ whenever i joins coalition $S \subseteq N\setminus i$ and his marginal contributions $b(S,T) - b(S,T\cup i)$ whenever i leaves coalition $T \cup i$, where $p_{s,t}$ is the probability that player i joins S in presence of the players in T and $q_{s,t}$ is the probability that player i leaves $T \cup i$ in presence of the players in S, provided that all the coalitions of a common size have the same probability of being joined and lived. Notice that among biprobabilistic values, bisemivalues are characterized by the fact that all coalitions of a given size share common weights with regard to all players.
(b) Among bisemivalues, we found the Shapley value [4] – whose weighting coefficients defined in Section 2 are independent of player i and only depend on cardinalities of S and T – and the Banzhaf value [6] denoted here by β, for which $p_{s,t} = q_{s,t} = \left(\frac{1}{3}\right)^{n-1}$ is the only bisemivalue with constant weighting coefficients, that is, weighting coefficients do not depend on the size of the coalitions S and T.
(c) As it is well known, semivalues for cooperative games are defined on cardinalities rather than on specific player sets: this means that a weighting vector $\{p_k\}_{k=0}^{n-1}$ defines a semivalue

Ψ on all N such that $n = |N|$. When necessary, we shall write $\Psi^{(n)}$ for a semivalue on cardinality n and p^n_k for its weighting coefficients. A semivalue $\Psi^{(n)}$ induces semivalues $\Psi^{(t)}$ for all cardinalities $t < n$, recurrently defined by the Pascal triangle (inverse) formula given by Dragan [11]:

$$p^t_k = p^{t+1}_k + p^{t+1}_{k+1} \quad \text{for} \quad 0 \leq k < t, \tag{2}$$

A series $\Psi = \{\Psi^{(n)}\}_{n=1}^{\infty}$ of semivalues, one for each cardinality, satisfies Dragan's recurrence formula. and we will say that Ψ is a *multisemivalue*. Particularly, the Shapley, the Banzhaf values and all binomial semivalues are multisemivalues.
As we will see, things are very similar to bisemivalues on bicooperative games.

Following Theorem 3.3, analogously to the cooperative case, bisemivalues are also defined on cardinalities rather than on specific player set: that is, two weighting vectors $p_{s,t}$ and $q_{s,t}$ define a bisemivalue ψ on all N such that $n = |N|$. When necessary, we shall write $\psi^{(n)}$ for a bisemivalue on cardinality n, $p^n_{s,t}$ and $q^n_{s,t}$ for its weighting coefficients.

Proposition 3.5 *Given a bisemivalue $\psi^{(n)}$ on $\mathcal{BG}_N$ with weighting coefficients $p^n_{s,t}$ and $q^n_{s,t}$, the recursively obtained numbers*

$$\begin{aligned} p^{m-1}_{s,t} &= p^m_{s+1,t} + p^m_{s,t} + p^m_{s,t+1}, \\ q^{m-1}_{s,t} &= q^m_{s+1,t} + q^m_{s,t} + q^m_{s,t+1} \end{aligned} \tag{3}$$

for $0 \leq s,t < m \leq n$, define a induced bisemivalue $\psi^{(m)}$ on the space of bicooperative games with m players.

Proof Let $\psi^{(n)}$ be a bisemivalue with weighting coefficients $p^n_{s,t}$ and $q^n_{s,t}$.
It suffices to prove that if $\psi^{(n)}$ is a bisemivalue on $\mathcal{BG}_N$ then the induced weighting coefficients $p^{n-1}_{s,t}$ and $q^{n-1}_{s,t}$ obtained from (3) define a bisemivalue $\psi^{(n-1)}$ on bicooperative games with $n-1$ players.
We have to check that the induced weighting coefficients satisfy (1). It is straightforward to verify $p^{n-1}_{s,t} \geq 0$ and $q^{n-1}_{s,t} \geq 0$. The remaining condition for the weighting coefficients $p^{n-1}_{s,t}$ follows from the fact that:

$$\begin{aligned} &\sum_{s=0}^{n-2} \binom{n-2}{s} \left[\sum_{t=0}^{n-s-2} \binom{n-s-2}{t} p^{n-1}_{s,t} \right] \\ &\quad = \sum_{s=0}^{n-2} \binom{n-2}{s} \left[\sum_{t=0}^{n-s-2} \binom{n-s-2}{t} (p^n_{s+1,t} + p^n_{s,t} + p^n_{s,t+1}) \right] \\ &\quad = \sum_{s=0}^{n-2} \binom{n-2}{s} \left[\sum_{t=0}^{n-s-1} \binom{n-s-1}{t} p^n_{s,t} + \sum_{t=0}^{n-s-2} \binom{n-s-2}{t} p^n_{s+1,t} \right] \\ &\quad = \sum_{s=0}^{n-1} \binom{n-1}{s} \left[\sum_{t=0}^{n-s-1} \binom{n-s-1}{t} p^n_{s,t} \right] = 1. \end{aligned}$$

And analogously for the weighting coefficients $q^{n-1}_{s,t}$. □

Definition 3.6 A series $\psi = \{\psi^{(n)}\}_{n=1}^{\infty}$ of bisemivalues, one for each cardinality, is a *multi-bisemivalue* if and only if it satisfies (3).

Proposition 3.7 *The expression of the weighting coefficients of any induced bisemivalue $\psi^{(m)}$ in terms of the coefficients of the original bisemivalue $\psi^{(n)}$, are*

$$p^m_{s,t} = \sum_{i=0}^{n-m} \binom{n-m}{i} \sum_{j=0}^{n-m-i} \binom{n-m-i}{j} p^n_{s+i,t+j},$$
$$q^m_{s,t} = \sum_{i=0}^{n-m} \binom{n-m}{i} \sum_{j=0}^{n-m-i} \binom{n-m-i}{j} p^n_{s+i,t+j} \tag{4}$$

for $\quad 0 \leq s,t < m < n.$

Proof It follows by applying (3) repeatedly. □

4 $(\mathbf{p},\mathbf{q})$–bisemivalues

In this section we introduce a subfamily of bisemivalues, called (p,q)–bisemivalues. As we will see, for each one of them, the weighting coefficients depend on two parameters $p, q \in [0,1]$. These bisemivalues are suited for the study of bicooperative games where players show two different tendencies to form coalitions. These tendencies are defined for all players by *parameters* p and q. From now on we assume that p is the probability to support a player in his decision and q is the probability to go against him.

Proposition 4.1 *Let $p, q \in [0,1]$ with $p+q \leq 1$, then the coefficients $p_{s,t} = p^s q^t (1-p-q)^{n-s-t-1}$ and $q_{s,t} = p^t q^s (1-p-q)^{n-s-t-1}$ define a bisemivalue for the bicooperative games.*

Proof We have to prove that the weighting coefficients satisfy (1). It is straightforward to verify that $p_{s,t} \geq 0$ and $q_{s,t} \geq 0$. The remaining condition for the weighting coefficients $p_{s,t}$ follows from the fact that:

$$\sum_{s=0}^{n-1} \binom{n-1}{s} \left[\sum_{t=0}^{n-s-1} \binom{n-s-1}{t} p_{s,t} \right] = \sum_{s=0}^{n-1} \binom{n-1}{s} \left[\sum_{t=0}^{n-s-1} \binom{n-s-1}{t} p^s q^t (1-p-q)^{n-s-t-1} \right]$$

$$= \sum_{s=0}^{n-1} \binom{n-1}{s} p^s \left[\sum_{t=0}^{n-s-1} \binom{n-s-1}{t} q^t (1-p-q)^{n-s-t-1} \right] = \sum_{s=0}^{n-1} \binom{n-1}{s} p^s (1-p)^{n-s-1} = 1$$

The case of $q_{s,t}$ follows similarly. □

Definition 4.2 *(i)* The family of *dictatorial indices* D, for player $i \in N$ is given by

$$D_i[b] = \sum_{(S,T)\in 3^{N\setminus i}} p^i_T \left[b(i,T) - b(\emptyset,T) \right] + q^i_S \left[b(S,\emptyset) - b(S,i) \right], \text{ for all } b \in \mathcal{BG}_N,$$

where $p^i_T \geq 0$, $\sum_{T\subseteq N\setminus i} p^i_T = 1$ and $q^i_S \geq 0$, $\sum_{S\subseteq N\setminus i} q^i_S = 1$.

Particularly, the *super-dictatorial index SD* for player $i \in N$ is given by

$$SD_i[b] = b(i, N \setminus i) - b(\emptyset, N \setminus i) + b(N \setminus i, \emptyset) - b(N \setminus i, i) \text{ for all } b \in \mathcal{BG}_N.$$

(ii) The *marginal index M* for player $i \in N$ is given by

$$M_i[b] = b(N, \emptyset) - b(N \setminus i, \emptyset) + b(\emptyset, N \setminus i) - b(\emptyset, N) \text{ for all } b \in \mathcal{BG}_N.$$

Definition 4.3 Let $p, q \in [0,1]$ with $p+q \leq 1$. The (p,q)–bisemivalue ψ^{pq} on $\mathcal{BG}_N$ is defined by the coefficients $p_{s,t} = p^s q^t (1-p-q)^{n-s-t-1}$ and $q_{s,t} = p^t q^s (1-p-q)^{n-s-t-1}$.

Using the convention $0^0 = 1$, in case of $p = 0$ we obtain a subfamily of dictatorial indices D^q, $q \in [0,1]$, given by

$$D_i^q[b] = \sum_{(S,T) \in 3^{N \setminus i}} q^t (1-q)^{n-t-1} \left[b(i,T) - b(\emptyset,T)\right] + q^s (1-q)^{n-s-1} \left[b(S,\emptyset) - b(S,i)\right], \text{ for all } b \in \mathcal{BG}_N.$$

If moreover $q = 1$, we obtain the super-dictatorial index *SD*. Finally, for $p = 1$ and $q = 0$ the marginal index *M* is obtained.

Remark 4.4 Notice that for all $p, q \in (0,1)$ with $p+q < 1$, the weighting coefficients of ψ^{pq} are in geometric progression $\frac{p_{s+1,t}}{p_{s,t}} = \frac{q_{s,t+1}}{q_{s,t}} = \frac{p}{1-p-q}$. That is, technically they are given by the (simplest form of) monotonicity of the weighting coefficients.

In 2000, Puente [26] (see also Giménez [18] or Amer and Giménez [1]) defined a special family of semivalues on cooperative games, *binomial semivalues*: for each one of them, the weighting coefficients depend on a unique parameter $p \in [0,1]$—the Banzhaf value corresponds to $p = 1/2$. These semivalues are especially suited for the study of cooperative games where the players show some (common) tendency to form coalitions. This tendency is defined by *parameter p*.
Which is reason for letting p range from 0 to 1? Notice that a reasonable regularity assumption on players' behavior is that the probability to form coalitions follows a monotonic (increasing or decreasing) behavior. Then, the only semivalues such that $p_{k+1} = \lambda p_k$ for all k are precisely the p–binomial semivalues, in which case $\lambda = p/(1-p)$ for each $p \in [0,1]$.
Following this idea, we introduce in the following definition a subfamily of (p,q)–bisemivalues, obtained when $p = q$, and called *binomial bisemivalues*. As we will see, they "extend" the concept of binomial semivalues to bicooperative games.

Definition 4.5 Let $p \in [0,1/2]$. The p–binomial bisemivalue ψ^p on $\mathcal{BG}_N$ is defined by the coefficients $p_{s,t} = q_{s,t} = p^{s+t}(1-2p)^{n-s-t-1}$. Of course, $p = 1/3$ gives the Banzhaf bisemivalue.

Remark 4.6 The only (p,q)–bisemivalues satisfying $p_{s,t} = q_{s,t}$, that is, which weights in a same way the marginal contributions of players in favor or against the change are for: *(i)* $p = q$, corresponding to the binomial bisemivalues; *(ii)* $p = 0$, corresponding to the family of dictatorial indices; *(iii)* $q = 0$ and *(iv)* $p+q = 1$. The Marginal index is obtained when the two last cases are given simultaneously.

4.1 Computational procedure

The MLE technique has been a useful tool for the calculus of values on cooperative games: it applies to e.g. the Shapley value (Owen [23]), the Banzhaf value (Owen [24]) and all binomial semivalues (Puente [26]). In this section first we introduce the *multilinear extension* of a bicooperative game that parallels the existing *multilinear extension* of a cooperative game given by Owen in [23] and then, we provide a method to compute (p,q)–bisemivalues by means of the multilinear extension of the game.
We identify each $(S,T) \in 3^N$ by vectors (X,Y) of $\mathbb{R}^{2n}$ such that $X = (x_1,\ldots,x_n)$, $Y = (y_1,\ldots,y_n)$, and

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad y_i = \begin{cases} 1 & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases}$$

For instead, if $N = \{1,2,3\}$ the coalitions $(\{1,3\},\{2\})$ and $(\{1,2\},\emptyset)$ are identified by $(X,Y) = (1,0,1,0,1,0)$ and $(X,Y) = (1,1,0,0,0,0)$ respectively.

Definition 4.7 The *multilinear extension* of a game $b \in \mathcal{BG}_N$ is the real-valued function defined on $\mathbb{R}^{2n}$ by

$$f(X,Y) = \sum_{(S,T)\in 3^N} \left[\prod_{i\in S} x_i \prod_{j\in T} y_j \prod_{k\in N\setminus(S\cup T)} (1 - x_k - y_k) \right] b(S,T). \tag{5}$$

It is easy to prove that f coincides with b where b is defined.

Proposition 4.8 *If ψ^{pq} is a (p,q)–bisemivalue and f is the multilinear extension of a game $b \in \mathcal{BG}_N$ then*

$$\psi_i^{pq}[b] = \frac{\partial f}{\partial x_i}\Big(P,Q\Big) - \frac{\partial f}{\partial y_i}\Big(Q,P\Big)$$

for all $i \in N$, where $P = (p,\overset{n}{\ldots},p)$ and $Q = (q,\overset{n}{\ldots},q)$.

Proof From Definition 4.7 the partial derivatives of f with respect to x_i and y_i are:

$$\frac{\partial f}{\partial x_i}(X,Y) = \sum_{(S,T)\in 3^{N\setminus i}} \left[\prod_{j\in S} x_j \prod_{k\in T} y_k \prod_{l\in N\setminus(S\cup T\cup i)} (1 - x_l - y_l) \right] [b(S\cup i,T) - b(S,T)], \tag{6}$$

$$\frac{\partial f}{\partial y_i}(X,Y) = \sum_{(S,T)\in 3^{N\setminus i}} \left[\prod_{j\in S} x_j \prod_{k\in T} y_k \prod_{l\in N\setminus(S\cup T\cup i)} (1 - x_l - y_l) \right] [b(S,T\cup i) - b(S,T)]. \tag{7}$$

Finally, valuating (6) at point (P,Q) and (7) at point (Q,P) and by subtracting these two results, we obtain the (p,q)–bisemivalue

$$\psi_i^{pq}[b] = \sum_{(S,T)\in 3^{N\setminus i}} p^s q^t (1-p-q)^{n-s-t-1}(b(S\cup i,T) - b(S,T))$$
$$+ p^t q^s (1-p-q)^{n-s-t-1}(b(S,T) - b(S,T\cup i)). \ \square$$

Corollary 4.9 *Let $p \in [0,1/2]$. If ψ^p is the p–binomial bisemivalue and f is the multilinear extension of a game $b \in \mathcal{BG}_N$ then*

$$\psi_i^p[b] = \frac{\partial f}{\partial x_i}(P,P) - \frac{\partial f}{\partial y_i}(P,P).$$

Notice that this result extends well the result obtained by Puente [26] for binomial semivalues on cooperative games.

5 An example

In this section we present an example of bicooperative game. The allocations obtained by the players will be analyzed by using (p,q)–bisemivalues and we will compute them by using the MLE technique given in Proposition 4.8.

Example 5.1 Two insurance companies, A_1 and A_2, are always in competition in order to obtain the maximum number of clients in a region. If N is the set of insurance agents, each one of them with an owner clients' list, we can define the bicooperative game $b(S,T)$ as A_1's benefits when players in S work for A_1, players in T work for A_2 and players in $N \setminus (S \cup T)$ do not work for A_1 neither A_2.
Consider $N = \{1,2,3\}$ the number of insurance agents and assume that players 1 and 3 are the agents with the biggest and the smallest clients' list respectively. If an agent leaves company A_1, he can go to A_2 and take part or the whole list of his clients or, on the contrary, go to another type of company unrelated to insurances, to be retired, ... In the first case company A_1 is more damaged than in the second one.
In this situation, let b be the bicooperative game defined by

$$\begin{array}{lll}
b(\{1,2,3\},\emptyset) = 100, & b(\emptyset,\emptyset) = 0, & b(\emptyset,\{1,2,3\}) = -60, \\
b(\{1,3\},\emptyset) = 85, & b(\{2,3\},\emptyset) = 75, & b(\{1,2\},\emptyset) = 90, \\
b(\{1,3\},\{2\}) = 50, & b(\{2,3\},\{1\}) = 20, & b(\{1,2\},\{3\}) = 60, \\
b(\{3\},\emptyset) = 65, & b(\{2\},\emptyset) = 70, & b(\{1\},\emptyset) = 80, \\
b(\{3\},\{1\}) = 5, & b(\{3\},\{2\}) = 15, & b(\{2\},\{1\}) = 10, \\
b(\{2\},\{3\}) = 35, & b(\{1\},\{2\}) = 40, & b(\{1\},\{3\}) = 50, \\
b(\{3\},\{1,2\}) = -25, & b(\{2\},\{1,3\}) = -20, & b(\{1\},\{2,3\}) = 5, \\
b(\emptyset,\{1\}) = -30, & b(\emptyset,\{2\}) = -15, & b(\emptyset,\{3\}) = -10, \\
b(\emptyset,\{2,3\}) = -30, & b(\emptyset,\{1,3\}) = -40, & b(\emptyset,\{1,2\}) = -50.
\end{array}$$

From Definition 4.7 the MLE of b is

$$\begin{aligned}
f(X,Y) = {} & 80x_1 + 70x_2 + 65x_3 - 30y_1 - 15y_2 - 10y_3 - 60x_1x_2 - 60x_1x_3 - 25x_1y_2 - 20x_1y_3 \\
& - 60x_2x_3 - 30x_2y_1 - 25x_2y_3 - 30x_3y_1 - 35x_3y_2 - 5y_1y_2 - 5y_2y_3 + 65x_1x_2x_3 \\
& + 25x_1x_2y_3 + 40x_1x_3y_2 + 35x_2x_3y_1 + 5x_2y_1y_3 + 25x_3y_1y_2 + 5y_1y_2y_3
\end{aligned}$$

We compute (p,q)–bisemivalues by using the MLE technique for each player i:

$$\psi_1^{pq}[b] = 60p^2 + 35pq - 35q^2 - 115p + 15q + 110$$
$$\psi_2^{pq}[b] = 60p^2 + 35pq - 35q^2 - 110p + 5q + 85$$
$$\psi_3^{pq}[b] = 60p^2 + 70pq - 115p - 20q + 75$$

Table 1 shows the (p,q)–bisemivalues for each player i and for several values of p and q.

$\psi_i^{pq}[b]$	$(p,q) = (0.1, 0.7)$	$(p,q) = (0.2, 0.6)$	$(p,q) = (0.6, 0.2)$	$(p,q) = (0.7, 0.1)$
$i = 1$	94.9 (44.49%)	90.0 (44.82%)	68.4 (47.24%)	62.5 (48.34%)
$i = 2$	63.4 (29.72%)	60.0 (29.88%)	44.4 (30.66%)	40.0 (30.93%)
$i = 3$	55.0 (25.79%)	50.8 (25.30%)	32.0 (22.10%)	26.8 (20.73%)

Table 1: (p,q)–bisemivalues for each player i and for several values of p and q

If $q = p$ we obtain the p–binomial bisemivalues. Figure 1 shows the p–binomial bisemivalues for each player i and Table 2 shows the p–binomial bisemivalues for each player i and for several values of p.

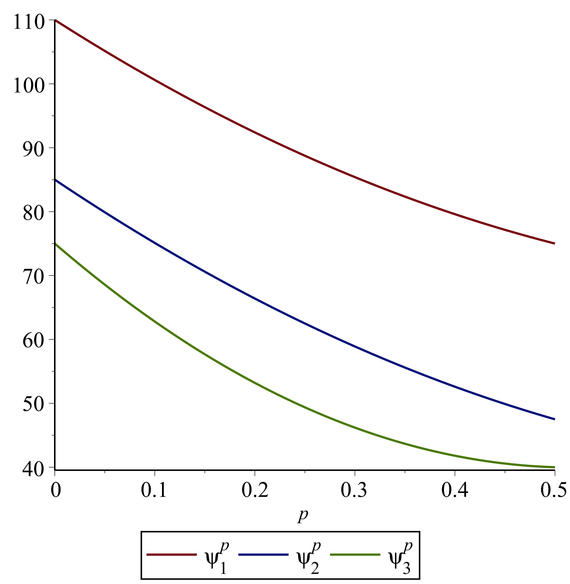

Figure 1: p–binomial bisemivalues for each player i

i	$\psi_i^p[b]$	$\psi_i^{0.1}[b]$	$\psi_i^{1/3}[b]$ (Banzhaf)	$\psi_i^{0.4}[b]$
1	$60p^2 - 100p + 110$	100.6 (42.18 %)	83.3333 (45.18 %)	79.6 (45.75 %)
2	$60p^2 - 105p + 85$	75.1 (31.49 %)	56.6667 (30.72 %)	52.6 (30.23 %)
3	$130p^2 - 135p + 75$	62.8 (26.33 %)	44.4445 (24.10 %)	41.8 (24.02 %)

Table 2: p–bisemivalues for each player i and for several values of p

From Figure 1, it follows that $\psi_1^p[v] \geq \psi_2^p[v] \geq \psi_3^p[v]$ for all $p \in [0, 1/2]$ and the three players' maximum and minimum allocations, $\psi_i^p[v]$, $i = 1, 2, 3$, are obtained when $p = 0$ and $p = 1/2$, respectively. The ratio among players' allocations varies in a significant way, as it is showed in Table 2 for values $p = 0.1$, $1/3$ and 0.4.

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