Characterization of symmetric M-matrices as resistive inverses

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Abstract

We aim here at characterizing those nonnegative matrices whose inverse is an irreducible Stieltjes matrix. Specifically, we prove that any irreducible Stieltjes matrix is a resistive inverse. To do that we consider the network defined by the off-diagonal entries of the matrix and we identify the matrix with a positive definite Schrödinger operator which ground state is determined by the its lowest eigenvalue and the corresponding positive eigenvector. We also analyze the case in which the operator is positive semidefinite which corresponds to the study of singular irreducible symmetric M-matrices. The key tool is the definition of the effective resistance with respect to a nonnegative value and a weight. We prove that these effective resistances verify similar properties to those satisfied by the standard effective resistances which leads us to carry out an exhaustive analysis of the generalized inverses of singular irreducible symmetric M-matrices. Moreover we pay special attention on those generalized inverses identified with Green operators, which in particular includes the analysis of the Moore-Penrose inverse.

Keywords: M-matrices, Schrödinger operators, Green kernel, generalized inverse, Moore-Penrose Inverse, effective resistance, Kirchhoff index.

1 Introduction

In view of their numerous applications, for instance in numerical methods, probability and economics, M-matrices have deserved a great attention and many of their properties has been studied. An important problem related with M-matrices is the so-called *inverse M-matrix problem*, that consists in characterizing all nonnegative matrices whose inverses are M-matrices. This is a longstanding and difficult problem that has generated a big amount of literature and has been partially solved. M. Martínez *et al.* in their celebrated paper [16] proved that the inverse of any strictly ultrametric matrix is a diagonally dominant Stieltjes matrix and C. Dellacherie *et al.* in [10] extend this result by proving that the inverse of any nonsingular ultrametric matrix is a weakly diagonally dominant Stieltjes matrix. Two years later, M. Fiedler characterized in [11] this type of matrices as *resistive inverses* associated with networks. Specifically, if M is an irreducible weakly diagonally dominant Stieltjes matrix of order n, then there exists a connected network with $n+1$ vertices such that $\mathsf{M}^{-1} = (g_{ij})$ where $g_{ij} = \frac{1}{2}\left(R_{i,n+1} + R_{j,n+1} - R_{ij}\right)$ and (R_{ij}) is the effective resistance matrix of the network.

In this work we extend the above characterization to the case of irreducible Stieltjes matrices, making unnecessary the diagonally dominance hypothesis. The key idea is to identify any irreducible Stieltjes matrix with a positive definite Schrödinger operator on a suitable connected network and take advantage of the previous work developed by the authors, [6, 7]. In this context we need to define the concept of effective resistance with respect to a nonnegative value and a weight on the network. We prove that such effective resistances verify properties that are analogues to those verified by the standard effective resistance. In particular, they determine a distance on the network and hence they are of potential application specially in Chemistry, [18]. Moreover, we give a formula for the inverse of the resistance matrix that generalized the known formula for the usual resistances, see [2]. In this context, the generalized inverses and their relation with the effective resistances are of interest and have been widely studied, [1, 2, 12, 13, 15, 17]. Therefore, we are also concern with the generalized inverses of positive semidefinite Schrödinger operators paying special attention to the associated Green operators and mainly to the one identified with the Moore-Penrose inverse of a irreducible symmetric M-matrix.

2 Preliminaries

Given a finite set V, the set of real valued functions on V is denoted by $\mathcal{C}(V)$. In particular, for any $x \in V$, $\varepsilon_x \in \mathcal{C}(V)$ stands for the Dirac function at x, whereas a function $\omega \in \mathcal{C}(V)$ is called a *weight* if it verifies that $\omega(x) > 0$ for any $x \in V$. The standard inner product on $\mathcal{C}(V)$ is denoted by $\langle \cdot, \cdot \rangle$ and hence if $u, v \in \mathcal{C}(V)$ then $\langle u, v \rangle = \sum\limits_{x \in V} u(x)\, v(x)$.

If $\mathcal{K}$ is an endomorphism of $\mathcal{C}(V)$ its *adjoint* is denoted by $\mathcal{K}^*$. The endomorphism $\mathcal{K}$ is called *self-adjoint* when $\mathcal{K}^* = \mathcal{K}$, *positive semi-definite* when $\langle \mathcal{K}(u), u \rangle \geqslant 0$ for any $u \in \mathcal{C}(V)$ and *positive definite* when $\langle \mathcal{K}(u), u \rangle > 0$ for any non-null $u \in \mathcal{C}(V)$. Clearly any positive definite endomorphism of $\mathcal{C}(V)$ is invertible. We will say that an endomorphism $\mathcal{K}$ is *conditionally positive definite with respect to a weight* ω iff $\langle \mathcal{K}(u), u \rangle > 0$ for any non-null $u \in \mathcal{C}(V)$ such that $\langle u, \omega \rangle = 0$.

A function $K\colon V \times V \longrightarrow \mathbb{R}$ is generically called *a kernel on* V and determines an endomorphism of $\mathcal{C}(V)$ by assigning to any $u \in \mathcal{C}(V)$ the function $\mathcal{K}(u) = \sum\limits_{y \in V} K(\cdot, y)\, u(y)$. Conversely, the *Kernel Theorem*, see [6], establishes that each endomorphism of $\mathcal{C}(V)$ is determined by the kernel given by $K(x,y) = \mathcal{K}(\varepsilon_y)(x)$ for any $x, y \in V$. Therefore, the kernel of $\mathcal{K}^*$ is given by $K^*(x,y) = K(y,x)$ for any $x, y \in V$. In particular, an endomorphism $\mathcal{K}$ is self-adjoint iff its kernel is a symmetric function. If $\mathcal{K}$ is an endomorphism of $\mathcal{C}(V)$ whose associated kernel is K, then the value $\mathsf{tr}\, \mathcal{K} = \sum\limits_{x \in V} K(x,x)$ is called the *trace of* $\mathcal{K}$. We will say that an endomorphism $\mathcal{K}$ is *zero axial* iff $K(x,x) = 0$ for any $x \in V$. It is well-known that if ω is a weight, then any self-adjoint zero axial endomorphism conditionally positive definite with respect to ω is invertible, see [4, Lemma 4.3.5].

Given $\tau \in \mathcal{C}(V)$, we denote by $\mathcal{D}_\tau$ the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function $\mathcal{D}_\tau(u) = \tau\, u$ and hence its kernel is $D(x,x) = \tau(x)$ and $D(x,y) = 0$ when $y \neq x$.

Given $\omega, \tau \in \mathcal{C}(V)$, we denote by $\mathcal{P}_{\omega,\tau}$ the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function $\mathcal{P}_{\omega,\tau}(u) = \langle \tau, u\rangle\, \omega$ and hence, its kernel is $\omega \otimes \tau$. In particular, when $\omega \neq 0$ and $\tau = \omega\, \langle \omega, \omega\rangle^{-1}$, the above endomorphism is denoted simply by $\mathcal{P}_\omega$. Clearly, $\mathcal{P}_\omega(\omega) = \omega$ and hence, $\mathcal{P}_\omega \circ \mathcal{P}_{\omega,\tau} = \mathcal{P}_{\omega,\tau}$ for any $\tau \in \mathcal{C}(V)$. In addition, $\mathcal{P}^*_{\omega,\tau} = \mathcal{P}_{\tau,\omega}$ and hence $\mathcal{P}_{\omega,\tau}$ is self-adjoint iff $\mathcal{P}_{\omega,\tau} = a\mathcal{P}_\omega$ with $a \in \mathrm{I\!R}$. In this case, $a\mathcal{P}_\omega$ is positive semi-definite iff $a \geqslant 0$, since $\langle \mathcal{P}_\omega(u), u\rangle = \dfrac{\langle \omega, u\rangle^2}{\langle \omega, \omega\rangle}$, for any $u \in \mathcal{C}(V)$. Moreover, if $\mathcal{K}$ is an endomorphism of $\mathcal{C}(V)$ then $\mathcal{K} \circ \mathcal{P}_{\omega,\tau} = \mathcal{P}_{\mathcal{K}(\omega),\tau}$ and $\mathcal{P}_{\omega,\tau} \circ \mathcal{K} = \mathcal{P}_{\omega,\mathcal{K}^*(\tau)}$.

If we consider an arbitrary order of the elements of V, then kernels, and hence its associated endomorphisms of $\mathcal{C}(V)$ can be identified with matrices of order $|V|$, whereas functions on V can be alternatively identified with (column) vectors of $\mathrm{I\!R}^{|V|}$ or diagonal matrices. Therefore, for any $\omega, \tau \in \mathcal{C}(V)$, the matrix identified with $\mathcal{P}_{\omega,\tau}$ is $\omega\tau^*$. On the other hand, an endomorphism $\mathcal{K}$ of $\mathcal{C}(V)$ is identified with a symmetric *Z-matrix* iff its kernel, K, is symmetric and verifies that $K(x,y) \leqslant 0$ for any $x, y \in V$ with $x \neq y$. Moreover $\mathcal{K}$ is identified with a symmetric *M-matrix*, respectively an *Stieltjes matrix*, iff in addition it is positive semidefinite, respectively positive definite.

Keeping in mind the above identifications, along the paper we mainly make use of the terminology of endomorphisms of $\mathcal{C}(V)$ and their kernels. We have preferred to do that because then we do not need to choose any order for the elements of V and also because our methodology appears as the discrete counterpart of the standard treatment of resolvent operators in Riemannian manifolds. On the other hand, we also take into account that given an arbitrary symmetric Z-matrix of order n with null diagonal entries and a the kernel identified with it, then $c = -a$ can be seen as the conductance function of a network whose vertex subset is V, see below for definitions, and moreover the matrix is irreducible iff the network is connected. So, to prove any question about a given irreducible Z-matrix, we always refer to the associated network.

In the sequel $\Gamma = (V, E, c)$ denotes a finite network; that is, a finite connected graph without loops nor multiple edges, with vertex set V and edge set E, in which each edge $\{x, y\}$ has been assigned a *conductance* $c(x,y) > 0$. So, the conductance can be considered as a symmetric function $c\colon V \times V \longrightarrow [0, +\infty)$ such that $c(x,x) = 0$ for any $x \in V$ and moreover, vertex x is adjacent to vertex y iff $c(x,y) > 0$. Definitely, a finite network is entirely characterized by its vertex set and its conductance function and hence in the sequel it will be represented as $\Gamma = (V, c)$. Moreover, n represents the cardinality of V and for any $x \in V$, the value $k(x) = \sum\limits_{y\in V} c(x,y)$ is called *total conductance at x* or *degree of x*. Given $x, y, z \in V$, we say that *z separates x and y* iff the set $V \setminus \{z\}$ is not connected and x and y belong to different connected components.

The *combinatorial Laplacian* or simply the *Laplacian* of the network Γ, that we denote by $\mathcal{L}$, is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function

$$\mathcal{L}(u)(x) = \sum_{y \in V} c(x,y)\Big(u(x) - u(y)\Big), \quad x \in V. \tag{1}$$

It is well-known, that the Laplacian is a self-adjoint positive semidefinite operator an moreover $\mathcal{L}(u) = 0$ iff u is a constant function.

Given $q \in \mathcal{C}(V)$, the *Schrödinger operator* on Γ with *ground state* q is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function $\mathcal{L}_q(u) = \mathcal{L}(u) + qu$, see for instance [6, 7].

Clearly, the positive semidefiniteness of $\mathcal{L}$ implies that any Schrödinger operator with non-negative ground state is positive semidefinite and positive definite when in addition the ground state is non-null.

The properties of the matrices identified with Schrödinger operators are described in the following result, whose proof is straightforward.

Proposition 2.1 *Given any order on the vertices of* Γ*, the set of irreducible symmetric* Z*-matrices whose off-diagonal elements are described by* $-c$ *is identified with the set of the Schrödinger operators on* Γ*. Moreover one of these matrices is a* M*-matrix, respectively an Stieltjes matrix, iff the corresponding Schrödinger operator is positive semidefinite, respectively positive definite. In addition, such a matrix is a weakly diagonally dominant* M*-matrix iff the ground state of its corresponding Schrödinger operator is nonnegative.*

If ω is a weight, we call *ground state determined by* ω the function

$$q_\omega = -\frac{1}{\omega}\,\mathcal{L}(\omega) = -k + \frac{1}{\omega}\sum_{y \in V} c(\cdot, y)\omega(y).$$

If μ is another weight, then $q_\mu = q_\omega$ iff $\mu = a\omega$ for some $a > 0$, see [6]. In particular, $q_\omega = 0$ iff ω is a positive constant. More generally, $\langle \omega, q_\omega \rangle = 0$, which implies that q_ω takes positive and negative values, except when ω is constant. Moreover, for any proper subset $F \subset V$ it is possible to choose a weight ω such that $q_\omega(x) < 0$ for any $x \in F$, see [6].

As we have seen, the ground state q_ω univocally determines ω up to multiplicative positive constant. Although this lack of uniqueness is not important for most of the results of this work, we normalize the weights and hence, each ground state of the form q_ω characterizes its corresponding weight. So, in the sequel the word weight will refer to a function $\omega \in \mathcal{C}(V)$ such that $\omega(x) > 0$ for any $x \in V$ and moreover $\langle \omega, \omega \rangle = n$. The set of weights on V that verify the above property is denoted by $\Omega(V)$. Observe that the unique constant function in $\Omega(V)$ is given by $\omega(x) = 1$ for any $x \in V$. In the sequel we denote this constant weight by 1.

We remark that the notion of Schrödinger operator includes different operators built from the combinatorial Laplacian of the network and that have been extensively studied in the literature. In particular, in the context of the paper it is worthy to mention the so-called *normalized Laplacian*, see [9], and the *q-Laplacian*, see [3].

3 The generalized Poisson equation

In this section we develop a variational treatment of the discrete analogue of integro-differential equations that allows us to obtain existence and uniqueness results that will be useful to the study of generalized inverses. Moreover, we prove a monotonicity property that leads to the non-negativeness of the corresponding inverses. That equations are built from positive semidefinite Schrödinger operators.

Fixed the network Γ, for any ground state $q \in \mathcal{C}(V)$ we consider the following problem, known as the *Poisson equation for* $\mathcal{L}_q$ on Γ: given $f \in \mathcal{C}(V)$ find $u \in \mathcal{C}(V)$ such that $\mathcal{L}_q(u) = f$. It is well-known that if the ground state is nonnegative and non null, then the Poisson equation has a unique solution for any data $f \in \mathcal{C}(V)$, whereas when $q = 0$ the Poisson equation has a solution for data $f \in \mathcal{C}(V)$ iff it verifies that $\langle f, 1\rangle = 0$ and moreover the solution is unique up to additive constant. In [6] some of the authors extended the above results to Schrödinger operators with more general ground states. Specifically, we characterized when $\mathcal{L}_q$ is a positive semidefinite operator and gave a variational treatment of the corresponding Poisson equation. The key result is the following, see [6, Proposition 3.3].

Proposition 3.1 *The Schrödinger operator $\mathcal{L}_q$ is positive semidefinite iff there exist $\omega \in \Omega(V)$ and $\lambda \geqslant 0$ such that $q = q_\omega + \lambda$. Moreover, ω and λ are univocally determined. In addition, $\mathcal{L}_q$ is not positive definite iff $\lambda = 0$, in which case $\langle \mathcal{L}_{q_\omega}(v), v\rangle = 0$ iff $v = a\omega$, $a \in \mathbb{R}$.*

Observe that Proposition 3.1 says that the Schrödinger operator $\mathcal{L}_q$ is positive semidefinite iff there exists a weight ω such that $q - q_\omega$ is a nonnegative constant. In addition, if $q \neq q_\omega$, then $\mathcal{L}_q$ is invertible, whereas if $q = q_\omega$ then $\mathcal{L}_q$ is singular. In both cases if $\lambda = q - q_\omega$, then λ is the lowest eigenvalue of $\mathcal{L}_q$ and its associated eigenfunctions are multiple of ω.

Given any order of the vertices of Γ, the interpretation of Schrödinger operators on Γ as Z-matrices whose off-diagonal entries are given by $-c$ together with the above result allows us to characterize the sets of irreducible symmetric M-matrices and Stieltjes matrices whose off-diagonal entries are given by $-c$, that we represent by $\mathcal{M}(c)$ and by $\mathcal{S}(c)$, respectively. Moreover, from now on for any $\omega \in \Omega$, we will consider $\mathsf{L}(\omega)$ the matrix whose off-diagonal elements are given by $-c$ and whose diagonal entries are given by the function $\dfrac{1}{\omega}\sum_{y\in V} c(\cdot, y)\omega(y)$ and $\mathsf{P}(\omega) = \frac{1}{n}\omega\omega^*$ the matrix associated with the endomorphism $\mathcal{P}_\omega$.

Corollary 3.2 *Given any order of the vertices of Γ, for any $\omega \in \Omega(V)$, $\mathsf{L}(\omega)$ is the matrix identified with $\mathcal{L}_{q_\omega}$ and, in addition,*

$$\mathcal{M}(c) = \{\mathsf{L}(\omega) + \lambda\mathsf{I} : \lambda \geqslant 0,\ \omega \in \Omega(V)\} \quad \textit{and} \quad \mathcal{S}(c) = \{\mathsf{L}(\omega) + \lambda\mathsf{I} : \lambda > 0,\ \omega \in \Omega(V)\},$$

where I is the identity matrix. Moreover, for any $\lambda \geqslant 0$ and $\omega \in \Omega(V)$, $\mathsf{L}(\omega) + \lambda\mathsf{I}$ is the unique M-matrix whose off-diagonal elements are given by $-c$ such that λ is its lowest eigenvalue with ω as corresponding eigenvector.

When the Schrödinger operator $\mathcal{L}_q$ is positive semidefinite, we get the following result that generalizes the variational treatment of the Poisson equation given in [6, 7].

Proposition 3.3 *Let $\lambda \geqslant 0$, $\omega \in \Omega(V)$, $q = q_\omega + \lambda$, $f \in \mathcal{C}(V)$ and for any $\alpha \leqslant \lambda$, consider the quadratic functional $\mathfrak{J}\colon \mathcal{C}(V) \longrightarrow \mathbb{R}$ determined by the expression*

$$\mathfrak{J}(u) = \langle \mathcal{L}_q(u), u\rangle - \alpha \langle \mathcal{P}_\omega(u), u\rangle - 2\langle f, u\rangle, \qquad \textit{for any } u \in \mathcal{C}(V).$$

Then, $u \in \mathcal{C}(V)$ minimizes $\mathfrak{J}$ iff it verifies

$$\mathcal{L}_q(u) - \alpha\mathcal{P}_\omega(u) = f,$$

which implies that $\mathfrak{J}(u) = -\langle f, u\rangle = \alpha \langle \mathcal{P}_\omega(u), u\rangle - \langle \mathcal{L}_q(u), u\rangle$. Moreover if $\alpha < \lambda$, then $\mathfrak{J}$ has a unique minimum, whereas when $\alpha = \lambda$, $\mathfrak{J}$ has a minimum iff $\mathcal{P}_\omega(f) = 0$ in which case there exists a unique minimum up to a multiple of ω.

Proof. Observe first that $\mathcal{Q}(u) = \langle \mathcal{L}_q(u), u\rangle - \dfrac{\alpha}{n} \langle \omega, u\rangle^2$ is the quadratic form associated with the bilinear form $\mathcal{B}(u, v) = \langle \mathcal{L}_q(u), v\rangle - \alpha \langle \mathcal{P}_\omega(u), v\rangle$. Therefore, $\mathfrak{J}$ is a convex functional, respectively an strictly convex functional, iff $\mathcal{Q}(u) \geqslant 0$ for any $u \in \mathcal{C}(V)$, respectively $\mathcal{Q}(u) > 0$ for any non-null $u \in \mathcal{C}(V)$. In addition, the standard theory of minimization of convex quadratic functionals establishes that $u \in \mathcal{C}(V)$ is a minimum of $\mathfrak{J}$ iff it verifies the so-called *Euler-Lagrange Identity*; that is, $\mathcal{L}_q(u) - \alpha\mathcal{P}_\omega(u) = f$, and also that it has a minimum iff $\langle f, v\rangle = 0$ for any $v \in \mathcal{C}(V)$ such that $\mathcal{Q}(v) = 0$. Moreover, when this condition holds, if u is a minimum, then v is another minimum of $\mathfrak{J}$, iff $\mathcal{Q}(u - v) = 0$. Therefore, when $\mathfrak{J}$ is strictly convex it has a unique minimum whereas when $\mathfrak{J}$ is convex but not strictly convex, then it has a unique minimum up to functions v such that of $\mathcal{Q}(v) = 0$. In addition, if $\mathcal{L}_q(u) - \alpha\mathcal{P}_\omega(u) = f$, then $\langle f, u\rangle = \langle \mathcal{L}_q(u), u\rangle - \alpha \langle \mathcal{P}_\omega(u), u\rangle$ and the expression for $\mathfrak{J}(u)$ follows.

Given $u \in \mathcal{C}(V)$, from Proposition 3.1, $\langle \mathcal{L}_q(u), u\rangle \geqslant \lambda \langle u, u\rangle$ with equality iff u is a multiple of ω, whereas $\langle \omega, u\rangle^2 \leqslant n \langle u, u\rangle$ with equality iff u is a multiple of ω, by applying the Cauchy-Schwartz inequality. Therefore, we obtain that $\mathcal{Q}(u) \geqslant (\lambda - \alpha) \langle u, u\rangle \geqslant 0$ for any $u \in \mathcal{C}(V)$, with equality iff u is a multiple of ω. In conclusion, if u is non-null then $\mathcal{Q}(u) > 0$, except when $\alpha = \lambda$ in which case $\mathcal{Q}(u) = 0$ iff $u = a\omega$, with $a \in \mathbb{R}$. Moreover when this occurs it is clear that condition $\langle f, v\rangle = 0$ for any $v \in \mathcal{C}(V)$ such that $\mathcal{Q}(v) = 0$ is equivalent to $\mathcal{P}_\omega(f) = 0$. ∎

The proof of the above Proposition assures that given $\lambda \geqslant 0$, $\omega \in \Omega(V)$ and $q = q_\omega + \lambda$, then for any $\alpha \leqslant \lambda$ the operator $\mathcal{L}_q - \alpha\mathcal{P}_\omega$ is self-adjoint and positive semidefinite, in fact positive definite when $\alpha < \lambda$. Moreover, $\lambda - \alpha$ is the lowest eigenvalue whose eigenfunctions are all multiple of ω, which implies that it is a simple eigenvalue. Therefore, when $\alpha < \lambda$, the operator $\mathcal{L}_q - \alpha\mathcal{P}_\omega$ is invertible, which in particular implies the invertibility of the Schrödinger operator $\mathcal{L}_q$ when $\lambda > 0$, whereas when $\alpha = \lambda$ it is singular. We call *generalized Schrödinger operator with respect to λ and ω*, $\mathcal{L}_q - \lambda\mathcal{P}_\omega$ and *generalized Poisson equation on Γ with respect to λ and ω*, $\mathcal{L}_q(u) - \lambda\mathcal{P}_\omega(u) = f$ where $f \in \mathcal{C}(V)$. Observe that when $\lambda = 0$ the generalized Poisson equation is nothing else than the standard Poisson equation on Γ relative to the positive semidefinite Schrödinger operator $\mathcal{L}_{q_\omega}$, whereas when $\lambda > 0$ it should be interpreted as a discrete version of an integro-differential equation. Proposition 3.3 establishes that the generalized Poisson equation has solution iff the data verifies that $\mathcal{P}_\omega(f) = 0$ and then the solution is unique up to a multiple of ω. This loss of uniqueness can be avoided if we demand suitable additional properties, as we can see in the following result.

Proposition 3.4 *Given $\lambda \geqslant 0$, $\omega \in \Omega(V)$, $q = q_\omega + \lambda$ and $f \in \mathcal{C}(V)$ such that $\mathcal{P}_\omega(f) = 0$, then the generalized Poisson equation Γ with respect to λ and ω, $\mathcal{L}_q(u) - \lambda\mathcal{P}_\omega(u) = f$, has a unique solution $u \in \mathcal{C}(V)$ such that $\mathcal{P}_\omega(u) = 0$.*

Proof. From Proposition 3.3 we know the generalized Poisson equation has solution and that if $v \in \mathcal{C}(V)$ is a solution then the set $\{v + a\omega : a \in \mathbb{R}\}$ describes all the solutions. On the other hand, if $u = v + a\omega$, then $\mathcal{P}_\omega(u) = 0$ iff $a = -\frac{1}{n}\langle\omega, v\rangle$ and hence iff $u = v - \mathcal{P}_\omega(v)$. ∎

Corollary 3.5 *Given $\lambda \geqslant 0$, $\omega \in \Omega(V)$, $q = q_\omega + \lambda$, $f \in \mathcal{C}(V)$, consider $\hat{u} \in \mathcal{C}(V)$ the unique solution of the generalized Poisson equation $\mathcal{L}_q(\hat{u}) - \lambda\mathcal{P}_\omega(\hat{u}) = f - \mathcal{P}_\omega(f)$ such that $\mathcal{P}_\omega(\hat{u}) = 0$. Then, for any $\alpha < \lambda$ the function $u = \hat{u} + \frac{1}{\lambda-\alpha}\mathcal{P}_\omega(f)$ is the unique solution of the equation $\mathcal{L}_q(u) - \alpha\mathcal{P}_\omega(u) = f$.*

Proof. If $\hat{f} = f - \mathcal{P}_\omega(f)$, then $\mathcal{P}_\omega(\hat{f}) = 0$ and applying the above proposition the generalized Poisson equation has a unique solution $\hat{u}$ such that $\mathcal{P}_\omega(\hat{u}) = 0$. On the other hand, Proposition 3.3 says that when $\alpha < \lambda$ the equation $\mathcal{L}_q(u) - \alpha\mathcal{P}_\omega(u) = f$ has a unique solution. Moreover, the self-adjointness of $\mathcal{L}_q - \alpha\mathcal{P}_\omega$ implies that

$$\langle f, \omega\rangle = \langle\mathcal{L}_q(u) - \alpha\mathcal{P}_\omega(u), \omega\rangle = \langle\mathcal{L}_q(\omega) - \alpha\mathcal{P}_\omega(\omega), u\rangle = (\lambda - \alpha)\langle\omega, u\rangle,$$

or equivalently $\mathcal{P}_\omega(f) = (\lambda - \alpha)\mathcal{P}_\omega(u)$. Therefore,

$$\mathcal{L}_q(u) - \lambda\mathcal{P}_\omega(u) = \mathcal{L}_q(u) - \alpha\mathcal{P}_\omega(u) - (\lambda - \alpha)\mathcal{P}_\omega(u) = f - \mathcal{P}_\omega(f) = \hat{f},$$

which implies that $\hat{u} = u - \mathcal{P}_\omega(u) = u - \frac{1}{\lambda-\alpha}\mathcal{P}_\omega(f)$. ∎

We can also avoid the lack of uniqueness of solutions for the generalized Poisson equation by demanding a local characteristic instead of demanding a global property, as was stated in Proposition 3.4.

Proposition 3.6 *Given $\lambda \geqslant 0$, $\omega \in \Omega(V)$, $q = q_\omega + \lambda$ and $f \in \mathcal{C}(V)$ such that $\mathcal{P}_\omega(f) = 0$, for any $z \in V$ the equation*

$$\mathcal{L}_q(u) - \lambda\mathcal{P}_\omega(u) = f$$

has a unique solution $u_z \in \mathcal{C}(V)$ such that $u_z(z) = 0$. Moreover u_z is the unique minimum of the functional

$$\mathfrak{J}(u) = \langle \mathcal{L}_q(u), u\rangle - \lambda\langle \mathcal{P}_\omega(u), u\rangle - 2\langle f, u\rangle$$

on the set $\{u \in \mathcal{C}(V) : u(z) = 0\}$. In addition, the function $u = \dfrac{1}{n}\sum_{z\in V} u_z\omega^2(z)$ is the unique solution of the generalized Poisson equation $\mathcal{L}_q(u) - \lambda\mathcal{P}_\omega(u) = f$ such that $\mathcal{P}_\omega(u) = 0$.

Proof. If v is a solution of the given generalized Poisson equation, then $u_z = v - \dfrac{v(z)}{\omega(z)}\,\omega$ is the unique function verifying the prescribed property. The second claim is an straightforward consequence of the variational characterization of the generalized Poisson equation. Finally, if $u \in \mathcal{C}(V)$ is the unique solution of the given equation such that $\mathcal{P}_\omega(u) = 0$, then for any $z \in V$ we get that $u_z = u - \dfrac{u(z)}{\omega(z)}\,\omega$. Therefore, $\sum\limits_{z\in V} u_z\omega^2(z) = n\,u - \omega\langle\omega, u\rangle = n\,(u - \mathcal{P}_\omega(u)) = n\,u$ and the last conclusion follows. ∎

The following results are directed at showing a monotonicity property verified by the operators of the form $\mathcal{L}_q - \alpha\mathcal{P}_\omega$ when $0 \leqslant \alpha \leqslant \lambda$ that generalizes the well-known monotonicity property for positive semidefinite Schrödinger operators, see [6, Proposition 4.10]. For this, it is useful to introduce a suitable network, that in some sense extends Γ and in which operator $\mathcal{L}_q - \alpha\mathcal{P}_\omega$ appears as the restriction to $\mathcal{C}(V)$ of a positive semidefinite Schrödinger operator. Specifically, fixed $\lambda \geqslant 0$ and $\omega \in \Omega(V)$, consider a new vertex $\hat{x} \notin V$ and for any $0 \leqslant \alpha \leqslant \lambda$ the network $\Gamma_\alpha = (V \cup \{\hat{x}\}, c_\alpha)$ where $c_\alpha(x, y) = c(x, y) + \dfrac{\alpha}{n}\,\omega(x)\omega(y)$ when $x, y \in V$ and $c_\alpha(\hat{x}, x) = (\lambda - \alpha)\,\omega(x)$ for any $x \in V$. So, when $0 \leqslant \alpha < \lambda$ the network Γ_α is connected, when $\alpha = \lambda$, then $\hat{x}$ is an isolated vertex and finally, when $0 < \alpha < \lambda$ the underlying graph to Γ_α is the complete one.

We also consider $\sigma \in \Omega(V \cup \{\hat{x}\})$ defined as $\sigma(x) = \omega(x)$ when $x \in V$ and as $\sigma(\hat{x}) = 1$. In addition, we denote by $\mathcal{L}^\alpha$ the combinatorial Laplacian of Γ_α.

Proposition 3.7 *If $q = q_\omega + \lambda$ and $0 \leqslant \alpha \leqslant \lambda$, then it is verified that*

$$q_\sigma = q - \Big(\lambda - \alpha + \frac{\alpha}{n}\,\langle\omega, 1\rangle\Big)\,\omega \quad \text{on } V \qquad \text{and} \qquad q_\sigma(\hat{x}) = (\lambda - \alpha)\Big(n - \langle\omega, 1\rangle\Big).$$

Moreover, for any $u \in \mathcal{C}(V \cup \{\hat{x}\})$ we get that $\mathcal{L}^\alpha_{q_\sigma}(u)(\hat{x}) = (\lambda - \alpha)\Big(n\,u(\hat{x}) - \langle\omega, u_{|_V}\rangle\Big)$ and

$$\mathcal{L}^\alpha_{q_\sigma}(u) = \mathcal{L}_q(u_{|_V}) - \alpha\,\mathcal{P}_\omega(u_{|_V}) - (\lambda - \alpha)\,\omega\,u(\hat{x}) = \mathcal{L}_q(u_{|_V}) - \lambda\,\mathcal{P}_\omega(u_{|_V}) - \frac{\omega}{n}\,\mathcal{L}^\alpha_{q_\sigma}(u)(\hat{x}) \quad \text{on } V.$$

Proof. Given $u \in \mathcal{C}(V \cup \{\hat{x}\})$, then for any $x \in V$ we get that

$$\mathcal{L}^\alpha(u)(x) = \mathcal{L}(u_{|_V})(x) - \alpha\,\mathcal{P}_\omega(u_{|_V})(x) + \Big(\lambda - \alpha + \frac{\alpha}{n}\,\langle\omega, 1\rangle\Big)u(x)\omega(x) - (\lambda - \alpha)\omega(x)u(\hat{x}).$$

In particular, tacking $u=\sigma$ it is verified that

$$\mathcal{L}^{\alpha}(\sigma)(x)=\mathcal{L}(\omega)(x)-\lambda\omega(x)+\Big(\lambda-\alpha+\frac{\alpha}{n}\,\langle\omega,1\rangle\Big)\,\omega^2(x),$$

which implies that

$$-q_\sigma(x)=-q(x)+\Big(\lambda-\alpha+\frac{\alpha}{n}\,\langle\omega,1\rangle\Big)\,\omega(x)$$

and hence from the expression of $\mathcal{L}^{\alpha}(u)(x)$ we obtain that

$$\mathcal{L}^{\alpha}_{q_\sigma}(u)(x)=\mathcal{L}_q(u_{|_V})(x)-\alpha\,\mathcal{P}_\omega(u_{|_V})(x)-(\lambda-\alpha)\,\omega(x)\,u(\hat{x}).$$

On the other hand, $\mathcal{L}^{\alpha}(u)(\hat{x})=(\lambda-\alpha)\Big(u(\hat{x})\langle\omega,1\rangle-\langle\omega,u_{|_V}\rangle\Big)$, which in particular implies that $-q_\sigma(\hat{x})=\mathcal{L}^{\alpha}(\sigma)(\hat{x})=(\lambda-\alpha)\Big(\langle\omega,1\rangle-n\Big)$. Therefore, for any $u\in\mathcal{C}(V\cup\{\hat{x}\})$ we get that $\mathcal{L}^{\alpha}_{q_\sigma}(u)(\hat{x})=(\lambda-\alpha)\,\Big(n\,u(\hat{x})-\langle\omega,u_{|_V}\rangle\Big)$, which in particular implies that $(\lambda-\alpha)\,\omega\,u(\hat{x})=\dfrac{\omega}{n}\,\mathcal{L}^{\alpha}_{q_\sigma}(u)(\hat{x})+(\lambda-\alpha)\mathcal{P}_\omega(u_{|_V})$ and the second identity for $\mathcal{L}^{\alpha}_{q_\sigma}(u)$ follows. ∎

Corollary 3.8 *Let $f\in\mathcal{C}(V)$ and define $f(\hat{x})=-\langle\omega,f\rangle$. If $0\leqslant\alpha<\lambda$ consider $u\in\mathcal{C}(V\cup\{\hat{x}\})$ the unique solution of the Poisson equation $\mathcal{L}^{\alpha}_{q_\sigma}(u)=f$ on $V\cup\{\hat{x}\}$ such that $u(\hat{x})=0$, then $u_{|_V}$ is the unique solution of the equation $\mathcal{L}_q(u)-\alpha\mathcal{P}_\omega(u)=f$ on V.*

Proof. First, we observe that $\langle f,\sigma\rangle=f(\hat{x})+\langle f,\omega\rangle=0$ which implies that $\mathcal{P}_\sigma(f)=0$. Therefore, applying Proposition 3.6 to the network Γ_α and the Schrödinger operator $\mathcal{L}^{\alpha}_{q_\sigma}$ we obtain that the Poisson equation for data f has a unique solution $u\in\mathcal{C}(V\cup\{\hat{x}\})$ such that $u(\hat{x})=0$. Therefore applying the above theorem it results that

$$f=\mathcal{L}^{\alpha}_{q_\sigma}(u_{|_V})=\mathcal{L}_q(u_{|_V})-\alpha\mathcal{P}_\omega(u_{|_V})-(\lambda-\alpha)\omega\,u(\hat{x})=\mathcal{L}_q(u_{|_V})-\alpha\mathcal{P}_\omega(u_{|_V})\ \text{ on } V$$

and the claims follow. ∎

Now we can obtain the claimed monotonicity property of generalized Schrödinger operators.

Theorem 3.9 *Consider $\lambda\geqslant 0$, $\omega\in\Omega(V)$, $q=q_\omega+\lambda$ and $0\leqslant\alpha\leqslant\lambda$. Let $F\subset V$ be a non empty subset and suppose that it is not simultaneously true that $F=V$ and $\alpha=\lambda$. If $u\in\mathcal{C}(V)$ verifies that $\mathcal{L}_q(u)\geqslant\alpha\mathcal{P}_\omega(u)$ on F and $u(x)\geqslant 0$ for any $x\in V\setminus F$, then $u(x)\geqslant 0$ for any $x\in V$. Moreover when $\alpha>0$, then either $u=0$ on F or $u(x)>0$ for any $x\in F$, whereas when $\alpha=0$, then if $H\subset F$ is a connected component of F either $u=0$ on H or $u(x)>0$ for any $x\in H$.*

Proof. We can suppose that $u \in \mathcal{C}(V \cup \{\hat{x}\})$ by defining $u(\hat{x}) = 0$. Then applying the above theorem we get that $\mathcal{L}^{\alpha}_{q_\sigma}(u) \geqslant 0$ on F and $u(x) \geqslant 0$ for any $x \in V \cup \{\hat{x}\} \setminus F$. Therefore, the conclusion is a consequence of the monotonicity of the positive semidefinite Schrödinger operator $\mathcal{L}^{\alpha}_{q_\sigma}$, taking into account that if $\alpha > 0$ then any subset of V is connected in Γ_α. ∎

4 Green operators of the network

In this section we are concern with the resolvent operators that help us to solve the generalized Poisson equation. This study includes the analysis of Green operators and more generally of generalized inverses. In particular, we prove that any generalized inverse can be obtained throughout a Green kernel plus some projection operators related with the weight. The matrix version of that result tell us that any generalized inverse of an irreducible symmetric M-matrix is the Moore-Penrose inverse plus the matrix associated with the projection operator.

Along this section we consider fixed the network $\Gamma = (V, c)$ and for any $\lambda \geqslant 0$ and $\omega \in \Omega(V)$ we also consider the ground state $q = q_\omega + \lambda$ and the corresponding Schrödinger operator $\mathcal{L}_q$. The results of the above section imply that 0 is the lowest eigenvalue of the operator $\mathcal{L}_q - \lambda\mathcal{P}_\omega$ whose eigenfunctions are all multiple of ω. Moreover, given $f \in \mathcal{C}(V)$, then the generalized Poisson equation

$$\mathcal{L}_q(u) - \lambda\mathcal{P}_\omega(u) = f$$

has solution iff $\mathcal{P}_\omega(f) = 0$. In particular, these properties imply that

$$(\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{P}_\omega = \mathcal{P}_\omega \circ (\mathcal{L}_q - \lambda\mathcal{P}_\omega) = 0,$$

or equivalently

$$\mathcal{L}_q \circ \mathcal{P}_\omega = \mathcal{P}_\omega \circ \mathcal{L}_q = \lambda\mathcal{P}_\omega. \tag{2}$$

Next we show that the above identities characterize the endomorphism $\mathcal{P}_\omega$.

Lemma 4.1 *If $\mathcal{K}$ is an endomorphism of $\mathcal{C}(V)$, then it is verified that $(\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{K} = 0$, respectively $\mathcal{K} \circ (\mathcal{L}_q - \lambda\mathcal{P}_\omega) = 0$, iff $\mathcal{K} = \mathcal{P}_{\omega,\tau}$, respectively $\mathcal{K} = \mathcal{P}_{\tau,\omega}$, with $\tau \in \mathcal{C}(V)$. In particular, $(\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{K} = \mathcal{K} \circ (\mathcal{L}_q - \lambda\mathcal{P}_\omega) = 0$, iff $\mathcal{K} = a\mathcal{P}_\omega$, where $a \in \mathbb{R}$.*

Proof. Clearly $(\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{P}_{\omega,\tau} = 0$, since for any $u \in \mathcal{C}(V)$ we get that $\mathcal{P}_{\omega,\tau}(u)$ is a multiple of ω. Conversely, if $(\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{K} = 0$, for any $y \in V$, there exist $\tau(y) \in \mathbb{R}$ such that $\mathcal{K}(\varepsilon_y) = \tau(y)\omega$. Therefore, the kernel associated with $\mathcal{K}$ is $\omega \otimes \tau$, or equivalently $\mathcal{K} = \mathcal{P}_{\omega,\tau}$. On the other hand, $\mathcal{K} \circ (\mathcal{L}_q - \lambda\mathcal{P}_\omega) = 0$ iff $(\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{K}^* = 0$ and hence iff $\mathcal{K}^* = \mathcal{P}_{\omega,\tau}$. To prove the last conclusion it is enough to observe that $\mathcal{P}_{\omega,\tau} = \mathcal{P}_{\hat{\tau},\omega}$ iff $\tau = \hat{\tau} = a\omega$. ∎

The main problem in this context is to find a solution of the generalized Poisson equation for any data f verifying the compatibility condition. So, we call *generalized inverse of $\mathcal{L}_q - \lambda\mathcal{P}_\omega$*

any endomorphism of $\mathcal{C}(V)$, $\mathcal{K}$, that assign to any f such that $\mathcal{P}_\omega(f) = 0$ a solution of the generalized Poisson equation. It is well-known that an endomorphism $\mathcal{K}$ is a generalized inverse iff it satisfies the identity

$$(\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{K} \circ (\mathcal{L}_q - \lambda\mathcal{P}_\omega) = \mathcal{L}_q - \lambda\mathcal{P}_\omega, \tag{3}$$

see [5, Theorem 2.2]. We focus in this section on a special class of generalized inverses. For that, note first that given $f \in \mathcal{C}(V)$, then $\hat{f} = f - \mathcal{P}_\omega(f)$ satisfies that $\mathcal{P}_\omega(\hat{f}) = 0$ and hence the generalized Poisson equation

$$\mathcal{L}_q(u) - \lambda\mathcal{P}_\omega(u) = \hat{f}$$

has solution. We call *Green operator for* Γ *with respect to* λ *and* ω, any endomorphism of $\mathcal{C}(V)$ that assigns to any $f \in \mathcal{C}(V)$ a solution of the above generalized Poisson equation; that is, any endomorphism, say $\mathcal{G}$, of $\mathcal{C}(V)$ such that

$$(\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{G} = \mathcal{I} - \mathcal{P}_\omega, \tag{4}$$

where $\mathcal{I}$ denotes de Identity on $\mathcal{C}(V)$. In addition, the kernel of any Green operator is called *Green function of the network* Γ *with respect to* λ *and* ω.

Clearly any Green operator with respect to λ and ω is a generalized inverse of $\mathcal{L}_q - \lambda\mathcal{P}_\omega$. The converse is not true, but any generalized inverse can be easily obtained from any Green operator as we will see later.

From the definition of the Green operator we obtain that $(\mathcal{L}_q - \lambda\mathcal{P}_\omega)(\mathcal{G}(\omega)) = 0$ and hence $\mathcal{G}(\omega) = a\omega$, where $a \in \mathbb{R}$. Therefore, if we consider $\lambda_{\mathcal{G}} = \frac{1}{n}\langle \mathcal{G}(\omega), \omega\rangle$, then $\lambda_{\mathcal{G}}$ is an eigenvalue of $\mathcal{G}$ that has ω as associated eigenfunction. In addition, $\lambda_{\mathcal{G}} = 0$ iff $\mathcal{G} \circ \mathcal{P}_\omega = 0$; that is, iff $\mathcal{G} = \mathcal{G} \circ (\mathcal{I} - \mathcal{P}_\omega)$ or in other words, iff for any $f \in \mathcal{C}(V)$, $\mathcal{G}$ assigns the same function to f and to $f - \mathcal{P}_\omega(f)$. In this case, it is also verified that $\mathcal{G} \circ (\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{G} = \mathcal{G}$; *i.e.*, $\mathcal{G} \circ \mathcal{L}_q \circ \mathcal{G} = \mathcal{G}$.

The lack of uniqueness of solution for generalized Poisson equations implies that there exist infinite Green operators and Green functions for Γ, as it is shown in the following result.

Proposition 4.2 *A kernel* $G\colon V \times V \longrightarrow \mathbb{R}$ *is a Green function of the network* Γ *with respect to* λ *and* ω*, iff for any* $y \in V$*, the function* $G(\cdot, y)$ *is a solution of the generalized Poisson equation* $\mathcal{L}_q(u) - \lambda\mathcal{P}_\omega(u) = \varepsilon_y - \dfrac{1}{n}\omega(y)\,\omega$ *and hence* $\sum\limits_{x\in V} G(x,y)\omega(x)\omega(y) < n\,G(y,y)$*, for any* $y \in V$.

Proof. If G is a Green function and $\mathcal{G}$ is its corresponding Green operator, then for any $f \in \mathcal{C}(V)$, $\mathcal{G}(f)$ is a solution of the generalized Poisson equation $\mathcal{L}_q(u) - \lambda\mathcal{P}_\omega(u) = f - \mathcal{P}_\omega(f)$. Therefore, fixed $y \in V$, $u_y = G(\cdot, y) = \mathcal{G}(\varepsilon_y)$ is a solution of the Poisson equation

$$\mathcal{L}_q(u) - \lambda\mathcal{P}_\omega(u) = \varepsilon_y - \frac{1}{n}\omega(y)\,\omega.$$

In particular, applying the positive semidefiniteness of $\mathcal{L}_q - \lambda\mathcal{P}_\omega$ and that $(\mathcal{L}_q - \lambda\mathcal{P}_\omega)(u_y) \neq 0$, we get

$$0 < \langle \mathcal{L}_q(u_y) - \lambda\mathcal{P}_\omega(u_y), u_y\rangle = \langle \varepsilon_y - \frac{1}{n}\omega(y)\,\omega, u_y\rangle = G(y,y) - \frac{1}{n}\sum_{x\in V} G(x,y)\omega(x)\omega(y).$$

Conversely, if for any $y \in V$ we consider u_y a solution of the above generalized Poisson equation and the linear operator, $\mathcal{G}$, whose kernel is the function given by $G(x,y) = u_y(x)$, then for any $f \in \mathcal{C}(V)$ the function $u = \sum\limits_{y\in V} u_y f(y)$ satisfies that

$$(\mathcal{L}_q - \lambda\mathcal{P}_\omega)(u) = \sum_{y\in V}(\mathcal{L}_q - \lambda\mathcal{P}_\omega)(u_y)f(y) = \sum_{y\in V}\Big(\varepsilon_y - \frac{1}{n}\,\omega(y)\,\omega\Big)\,f(y) = f - \mathcal{P}_\omega(f)$$

and hence $\mathcal{G}$ is a Green operator for Γ with respect to λ and ω. ∎

Proposition 4.3 *There exist infinite Green operators for Γ with respect to λ and ω. If $\mathcal{G}$ is one of them, then $\hat{\mathcal{G}}$ is a Green operator with respect to λ and ω iff there exists a unique $\tau \in \mathcal{C}(V)\}$ such that $\hat{\mathcal{G}} = \mathcal{G} + \mathcal{P}_{\omega,\tau}$. In addition, $\mathcal{K}$ is a generalized inverse of $\mathcal{L}_q - \lambda\mathcal{P}_\omega$ iff there exist $\tau, \hat{\tau} \in \mathcal{C}(V)$ such that $\mathcal{K} = \mathcal{G} + \mathcal{P}_{\omega,\tau} + \mathcal{P}_{\hat{\tau},\omega}$ and $\mathcal{K}$ is a Green operator iff ω is an eigenfunction of $\mathcal{K}$. Moreover, τ and $\hat{\tau}$ are univocally determined up to a multiple of ω.*

Proof. Clearly any operator of the form $\mathcal{G} + \mathcal{P}_{\omega,\tau}$ is a Green operator and any operator of the form $\mathcal{G} + \mathcal{P}_{\omega,\tau} + \mathcal{P}_{\hat{\tau},\omega}$ is a generalized inverse of $\mathcal{L}_q - \lambda\mathcal{P}_\omega$. Conversely, if $\mathcal{G}$ and $\hat{\mathcal{G}}$ are Green operators, then $(\mathcal{L}_q - \lambda\mathcal{P}_\omega)\circ(\hat{\mathcal{G}} - \mathcal{G}) = 0$ and applying Lemma 4.1 we get that $\hat{\mathcal{G}} = \mathcal{G} + \mathcal{P}_{\omega,\tau}$, for unique $\tau \in \mathcal{C}(V)$. Moreover, if $\mathcal{K}$ is a generalized inverse then, $(\mathcal{L}_q - \lambda\mathcal{P}_\omega)\circ\Big(\mathcal{K}\circ(\mathcal{L}_q - \lambda\mathcal{P}_\omega) - \mathcal{I}\Big) = 0$ and hence $\mathcal{K}\circ(\mathcal{L}_q - \lambda\mathcal{P}_\omega) = \mathcal{I} + \mathcal{P}_{\omega,\rho}$, for some $\rho \in \mathcal{C}(V)$. Therefore,

$$\mathcal{K}\circ(\mathcal{L}_q - \lambda\mathcal{P}_\omega)\circ\mathcal{G} = \mathcal{K}\circ(\mathcal{I} - \mathcal{P}_\omega) = \mathcal{G} + \mathcal{P}_{\omega,\rho}\circ\mathcal{G}$$

and hence $\mathcal{K} = \mathcal{G} + \mathcal{P}_{\omega,\rho}\circ\mathcal{G} + \mathcal{K}\circ\mathcal{P}_\omega = \mathcal{G} + \mathcal{P}_{\omega,\tau} + \mathcal{P}_{\hat{\tau},\omega}$, where $\tau = \mathcal{G}^*(\rho)$ and $\hat{\tau} = \frac{1}{n}\,\mathcal{K}(\omega)$. If we suppose that $\mathcal{K} = \mathcal{G} + \mathcal{P}_{\omega,\tau} + \mathcal{P}_{\hat{\tau},\omega} = \mathcal{G} + \mathcal{P}_{\omega,\mu} + \mathcal{P}_{\hat{\mu},\omega}$, then $\mathcal{P}_{\omega,(\tau-\mu)} = \mathcal{P}_{(\hat{\mu}-\hat{\tau}),\omega}$ and therefore $\tau - \mu = \hat{\mu} - \hat{\tau} = a\omega$. Finally, if $\mathcal{K}$ is a Green operator, then ω is an eigenfunction associated with the eigenvalue $\frac{1}{n}\langle\mathcal{K}(\omega),\omega\rangle$. Conversely, if $\mathcal{K} = \mathcal{G} + \mathcal{P}_{\omega,\tau} + \mathcal{P}_{\hat{\tau},\omega}$, then $\mathcal{K}(\omega) = (\lambda_{\mathcal{G}} + \langle\tau,\omega\rangle)\,\omega + n\hat{\tau}$. Therefore, ω is an eigenfunction for $\mathcal{K}$ iff $\hat{\tau} = a\omega$ and hence $\mathcal{K} = \mathcal{G} + \mathcal{P}_{\omega,\rho}$, where $\rho = \tau + a\omega$. ∎

The existence of multiple Green operators for Γ with respect to λ and ω, due to the existence of multiple solutions for each generalized Poisson equation, can be avoided if we demand suitable additional properties. In fact, applying Proposition 3.4, we know that for any $y \in V$ there exists a unique solution of the generalized Poisson equation $\mathcal{L}_q(u) - \lambda\mathcal{P}_\omega(u) = \varepsilon_y - \dfrac{1}{n}\,\omega(y)\,\omega$ such that $\mathcal{P}_\omega(u) = 0$. Therefore, for any $\lambda \geqslant 0$ and any $\omega \in \Omega(V)$ there exists a unique Green operator for Γ with respect to λ and ω, that we denote by $\mathcal{G}_{\lambda,\omega}$, such that $\mathcal{P}_\omega\circ\mathcal{G}_{\lambda,\omega} = 0$. Clearly, given $f \in \mathcal{C}(V)$, the function $u = \mathcal{G}_{\lambda,\omega}(f)$ is the unique solution of the generalized Poisson equation $\mathcal{L}_q(v) - \lambda\mathcal{P}_\omega(v) = f$ such that $\mathcal{P}_\omega(u) = 0$. In particular, we obtain that $\lambda_{\mathcal{G}_{\lambda,\omega}} = 0$ and hence

$$\mathcal{G}_{\lambda,\omega}\circ\mathcal{P}_\omega = \mathcal{P}_\omega\circ\mathcal{G}_{\lambda,\omega} = 0. \tag{5}$$

The Green function associated with $\mathcal{G}_{\lambda,\omega}$ is denoted by $G_{\lambda,\omega}$. Moreover the *Kirchhoff index of the network* Γ *with respect to* λ *and* ω, is defined as $\mathsf{k}(\lambda,\omega) = \operatorname{tr} \mathcal{G}_{\lambda,\omega}$. The Kirchhoff index have been introduced in the context of Organic Chemistry when $\lambda = 0$ and ω is constant. Note that we have chosen as definition for the Kirchhoff index that for many authors is a characterization, see for instance [9, 18]. Then, we will obtain as a characterization of the Kirchhoff index what they use as definition.

Proposition 4.4 *Given* $\lambda \geqslant 0$ *and* $\omega \in \Omega(V)$, $\mathcal{G}_{\lambda,\omega}$ *is a self-adjoint and positive semidefinite operator verifying that* $\mathcal{L}_q \circ \mathcal{G}_{\lambda,\omega} = \mathcal{G}_{\lambda,\omega} \circ \mathcal{L}_q = \mathcal{I} - \mathcal{P}_\omega$ *and*

$$(\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{G}_{\lambda,\omega} \circ (\mathcal{L}_q - \lambda\mathcal{P}_\omega) = \mathcal{L}_q - \lambda\mathcal{P}_\omega \quad \text{and} \quad \mathcal{G}_{\lambda,\omega} \circ (\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{G}_{\lambda,\omega} = \mathcal{G}_{\lambda,\omega}.$$

Moreover, $G_{\lambda,\omega}(x,x) > 0$, *for any* $x \in V$ *and if* $\mathcal{G}$ *is any Green operator for* Γ *with respect to* λ *and* ω, *then* $\mathcal{G}_{\lambda,\omega} = (\mathcal{I} - \mathcal{P}_\omega) \circ \mathcal{G}$ *and hence* $\mathsf{k}(\lambda,\omega) = \operatorname{tr} \mathcal{G} - \lambda_{\mathcal{G}}$.

Proof. Identity (4) implies that $(\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{G}_{\lambda,\omega} = \mathcal{I} - \mathcal{P}_\omega$; that is, $\mathcal{L}_q \circ \mathcal{G}_{\lambda,\omega} = \mathcal{I} - \mathcal{P}_\omega$, since $\mathcal{P}_\omega \circ \mathcal{G}_{\lambda,\omega} = 0$. Moreover, given $u \in \mathcal{C}(V)$ and $f = (\mathcal{L}_q - \lambda\mathcal{P}_\omega)(u)$, then $\mathcal{G}_{\lambda,\omega}(f) = u - \mathcal{P}_\omega(u)$ and hence, $\mathcal{G}_{\lambda,\omega} \circ (\mathcal{L}_q - \lambda\mathcal{P}_\omega) = \mathcal{I} - \mathcal{P}_\omega$, or equivalently $\mathcal{G}_{\lambda,\omega} \circ \mathcal{L}_q = \mathcal{I} - \mathcal{P}_\omega$, since $\mathcal{G}_{\lambda,\omega} \circ \mathcal{P}_\omega = 0$.

Now, if $f \in \mathcal{C}(V)$ and $u = \mathcal{G}_{\lambda,\omega}(f)$, then $(\mathcal{L}_q - \lambda\mathcal{P}_\omega)(u) = f - \mathcal{P}_\omega(f)$ and hence

$$\langle \mathcal{G}_{\lambda,\omega}(f), f\rangle = \langle (\mathcal{I} - \mathcal{P}_\omega)(f), \mathcal{G}_{\lambda,\omega}(f)\rangle = \langle (\mathcal{L}_q - \lambda\mathcal{P}_\omega)(u), u\rangle \geqslant 0,$$

since $\mathcal{L}_q - \lambda\mathcal{P}_\omega$ is positive semidefinite. Therefore, $\mathcal{G}_{\lambda,\omega}$ is also positive semidefinite.

On the other hand, we know that an endomorphism of $\mathcal{C}(V)$ is self-adjoint iff its kernel is a symmetric function. If given $x, y \in V$, consider the functions $u = G_{\lambda,\omega}(\cdot, x)$ and $v = G_{\lambda,\omega}(\cdot, y)$, then $(\mathcal{L}_q - \lambda\mathcal{P}_\omega)(u) = \varepsilon_x - \frac{1}{n}\,\omega(x)\,\omega$, $(\mathcal{L}_q - \lambda\mathcal{P}_\omega)(v) = \varepsilon_y - \frac{1}{n}\,\omega(y)\,\omega$ and applying the self-adjointness of the operator $\mathcal{L}_q - \lambda\mathcal{P}_\omega$, we get that

$$G_{\lambda,\omega}(y,x) = u(y) = \langle u, (\mathcal{L}_q - \lambda\mathcal{P}_\omega)(v)\rangle = \langle v, (\mathcal{L}_q - \lambda\mathcal{P}_\omega)(u)\rangle = v(x) = G_{\lambda,\omega}(x,y),$$

since $\langle u, \omega\rangle = \langle v, \omega\rangle = 0$. In addition, identity $(\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{G}_{\lambda,\omega} \circ (\mathcal{L}_q - \lambda\mathcal{P}_\omega) = \mathcal{L}_q - \lambda\mathcal{P}_\omega$ is consequence of being $\mathcal{G}_{\lambda,\omega}$ a Green operator, whereas identity $\mathcal{G}_{\lambda,\omega} \circ (\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{G}_{\lambda,\omega} = \mathcal{G}_{\lambda,\omega}$ is consequence of being $\mathcal{G}_{\lambda,\omega}(\omega) = 0$.

The positiveness of $G_{\lambda,\omega}(x,x)$ is a direct consequence of Proposition 4.3 and the identity $\mathcal{P}_\omega \circ \mathcal{G}_{\lambda,\omega} = 0$. Finally, if $\mathcal{G}$ is a Green operator for Γ with respect to λ and ω, then $\mathcal{G}_{\lambda,\omega} = \mathcal{G} + \mathcal{P}_{\omega,\tau}$, $\tau \in \mathcal{C}(V)$. Therefore, $\mathcal{P}_\omega \circ \mathcal{G}_{\lambda,\omega} = \mathcal{P}_\omega \circ \mathcal{G} + \mathcal{P}_{\omega,\tau}$, which implies that $\mathcal{P}_{\omega,\tau} = -\mathcal{P}_\omega \circ \mathcal{G}$ and hence that $\mathcal{G}_{\lambda,\omega} = (\mathcal{I} - \mathcal{P}_\omega) \circ \mathcal{G}$. The last conclusion is a consequence of the fact that if $\mathcal{K}$ is an endomorphism of $\mathcal{C}(V)$, then $\operatorname{tr}(\mathcal{P}_\omega \circ \mathcal{K}) = \frac{1}{n}\langle \mathcal{K}(\omega), \omega\rangle$. ∎

Note that the above Proposition says, in particular, that $\lambda_{\mathcal{G}_{\lambda,\omega}} = 0$ is the lowest eigenvalue of $\mathcal{G}_{\lambda,\omega}$ and its associated eigenfunctions are all multiple of ω. In particular, this implies that $\mathcal{G}_{\lambda,\omega}$ is conditionally definite positive with respect to ω.

Corollary 4.5 *If $\mathcal{K}$ is a generalized inverse of $\mathcal{L}_q-\lambda\mathcal{P}_\omega$, then it is conditionally definite positive with respect to ω. Moreover, it is a self-adjoint generalized inverse of $\mathcal{L}_q-\lambda\mathcal{P}_\omega$ iff there exists $\tau\in\mathcal{C}(V)$ such that $\mathcal{K}=\mathcal{G}_{\lambda,\omega}+\mathcal{P}_{\omega,\tau}+\mathcal{P}_{\tau,\omega}$.*

Proof. From Proposition 4.3, $\mathcal{K}=\mathcal{G}_{\lambda,\omega}+\mathcal{P}_{\omega,\sigma}+\mathcal{P}_{\hat\sigma,\omega}$ and hence it is conditionally definite positive with respect to ω since $\mathcal{G}_{\lambda,\omega}$ is. Moreover, from Proposition 4.4, $\mathcal{K}$ is self-adjoint iff $\mathcal{P}_{\omega,\sigma}+\mathcal{P}_{\hat\sigma,\omega}$ is self-adjoint and hence iff $\hat\sigma-\sigma=2a\omega$, $a\in\mathbb{R}$. Finally, it is enough to choose $\tau=\sigma+an\omega$. ∎

Corollary 4.6 *If $\mathcal{G}$ is an endomorphism of $\mathcal{C}(V)$, then it is a self-adjoint Green operator with respect to $\lambda\geqslant 0$ and $\omega\in\Omega(V)$ iff $\mathcal{G}=\mathcal{G}_{\lambda,\omega}+\lambda_{\mathcal{G}}\mathcal{P}_\omega$ and hence*

$$\mathcal{G}\circ\mathcal{P}_\omega=\mathcal{P}_\omega\circ\mathcal{G}=\lambda_{\mathcal{G}}\mathcal{P}_\omega \quad \textit{and} \quad (\mathcal{L}_q-\lambda\mathcal{P}_\omega)\circ\mathcal{G}=\mathcal{G}\circ(\mathcal{L}_q-\lambda\mathcal{P}_\omega)=\mathcal{I}-\mathcal{P}_\omega.$$

In particular, $\mathcal{G}$ is a positive definite self-adjoint Green operator with respect to λ and ω iff $\lambda_{\mathcal{G}}>0$ and then it is verified that $\mathcal{G}=(\mathcal{L}_q-\alpha\mathcal{P}_\omega)^{-1}$, where $\alpha=\lambda-\frac{1}{\lambda_{\mathcal{G}}}$. Therefore, $\mathcal{G}_{\lambda,\omega}$ is the unique self-adjoint, positive semidefinite and non positive definite Green operator for Γ with respect to λ and ω.

Proof. From Proposition 4.3 we know that $\mathcal{G}$ is a Green operator with respect to λ and ω iff $\mathcal{G}=\mathcal{G}_{\lambda,\omega}+\mathcal{P}_{\omega,\tau}$, $\tau\in\mathcal{C}(V)$. As $\mathcal{G}_{\lambda,\omega}$ is self-adjoint, $\mathcal{G}$ is self-adjoint iff $\mathcal{P}_{\omega,\tau}$ is and hence iff τ is a multiple of ω. Therefore, $\mathcal{G}=\mathcal{G}_{\lambda,\omega}+\lambda_{\mathcal{G}}\mathcal{P}_\omega$ and moreover, the commuting properties are a consequence of the commuting properties for $\mathcal{G}_{\lambda,\omega}$ established in the above proposition, together with Identities (2) and (5).

On the other hand, if $\mathcal{G}$ is a positive semidefinite operator necessarily $\lambda_{\mathcal{G}}\geqslant 0$. Conversely if $\lambda_{\mathcal{G}}\geqslant 0$, then for any $u\in\mathcal{C}(V)$ we get that

$$\langle\mathcal{G}(u),u\rangle=\langle\mathcal{G}_{\lambda,\omega}(u),u\rangle+\lambda_{\mathcal{G}}\langle\mathcal{P}_\omega(u),u\rangle\geqslant 0,$$

which implies that $\mathcal{G}$ is positive semidefinite and positive definite when $\lambda_{\mathcal{G}}>0$. In particular, $\mathcal{G}_{\lambda,\omega}$ is the unique self-adjoint positive semidefinite and non positive definite Green operator for Γ with respect to λ and ω. Finally, $\lambda_{\mathcal{G}}>0$ iff $\lambda_{\mathcal{G}}=\frac{1}{\lambda-\alpha}$ where $\alpha<\lambda$ and the last conclusion is a straightforward consequence of Corollary 3.5. ∎

We remark that the identities in the preceding results imply that if we consider any order of the vertices of V, then for any $\lambda\geqslant 0$ and $\omega\in\Omega(V)$, the matrix identified with the operator $\mathcal{G}_{\lambda,\omega}$ is the Moore-Penrose inverse of $\mathsf{L}(\omega)+\lambda(\mathsf{I}-\mathsf{P}(\omega))$; *i.e.*, the singular M-matrix identified with $\mathcal{L}_q-\lambda\mathcal{P}_\omega$, where $q=q_\omega+\lambda$. Recall that $\mathsf{M}^\dagger$ stands for the Moore-Penrose inverse of M.

Corollary 4.7 *Given any order of the vertices of Γ, consider $\lambda\geqslant 0$ and $\omega\in\Omega(V)$. Then, K is a generalized inverse of $\mathsf{L}(\omega)+\lambda(\mathsf{I}-\mathsf{P}(\omega))$ iff there exist $\tau,\hat\tau\in\mathbb{R}^n$ such that*

$$\mathsf{K}=\Big(\mathsf{L}(\omega)+\lambda(\mathsf{I}-\mathsf{P}(\omega))\Big)^\dagger+\omega\tau^*+\hat\tau\omega^*.$$

In particular, K *is symmetric iff* $\mathsf{K} = \Big(\mathsf{L}(\omega) + \lambda(\mathsf{I} - \mathsf{P}(\omega))\Big)^{\dagger} + \omega\tau^* + \tau\omega^*$, *for some* $\tau \in \mathbb{R}^n$. *Moreover, for any* $\alpha < \lambda$, *the matrix* $\mathsf{L}(\omega) + \lambda\mathsf{I} - \alpha\,\mathsf{P}(\omega)$ *is invertible and*

$$\Big(\mathsf{L}(\omega) + \lambda\mathsf{I} - \alpha\,\mathsf{P}(\omega)\Big)^{-1} - \frac{1}{\lambda - \alpha}\,\mathsf{P}(\omega) = \Big(\mathsf{L}(\omega) + \lambda(\mathsf{I} - \mathsf{P}(\omega))\Big)^{\dagger}.$$

In particular, when $\omega = 1$, then $n\mathsf{P}(1) = \mathsf{J}$, the matrix whose entries are all equal to 1, and hence if $\mathsf{L} = \mathsf{L}(1)$ the last Corollary can be rewritten for $\lambda = 0$ and $\alpha = -1$ as

$$\mathsf{K} = \mathsf{L}^{\dagger} + 1\tau^* + \tau 1^* \quad \text{and} \quad \mathsf{L}^{\dagger} = \Big(\mathsf{L} + \frac{1}{n}\,\mathsf{J}\Big)^{-1} - \frac{1}{n}\,\mathsf{J},$$

that are well-known identities, see for instance [2, 12, 13]. In addition, when $\lambda = \varepsilon^{-1} > 0$ and $\alpha = 0$, we get

$$(\varepsilon\mathsf{L} + \mathsf{I})^{-1} - \frac{1}{n}\,\mathsf{J} = \Big(\varepsilon\mathsf{L} + \mathsf{I} - \frac{1}{n}\mathsf{J}\Big)^{\dagger}.$$

We remark that in [8], the matrix $(\varepsilon\mathsf{L} + \mathsf{I})^{-1}$ is called *relative forest accessibility matrix* since its entries are related with the number of spanning rooted forests on the network Γ.

Clearly, for $0 \leqslant \alpha < \lambda$, $\mathsf{L}(\omega) + \lambda\mathsf{I} - \alpha\,\mathsf{P}(\omega)$ is an Stieltjes matrix and then the entries of its inverse are nonnegative. As an application of the monotonicity property, we prove in the following result that these entries are in fact positive.

Proposition 4.8 *Consider* $\mathcal{G}$ *a positive definite self-adjoint Green operator with respect to* $\lambda > 0$ *and* $\omega \in \Omega(V)$ *and such that* $\lambda_{\mathcal{G}} \geqslant \dfrac{1}{\lambda}$. *If* $q = q_\omega + \lambda$ *and* G *is the Green function associated with* $\mathcal{G}$, *then* $0 < G(x,y)\omega(y) \leqslant G(y,y)\omega(x)$ *for any* $x, y \in V$ *and the second inequality is an equality iff* $x = y$. *In addition,* $G(x,x) > \dfrac{\lambda_{\mathcal{G}}}{n}\,\omega^2(x)$ *for any* $x \in V$ *and*

$$-\frac{1}{n\lambda}\,\omega(x)\omega(y) < G_{\lambda,\omega}(x,y) < G_{\lambda,\omega}(y,y)\frac{\omega(x)}{\omega(y)} \quad \textit{for any } x, y \in V \textit{ with } x \neq y.$$

Proof. If we consider $\alpha = \lambda - \dfrac{1}{\lambda_{\mathcal{G}}}$, then $0 \leqslant \alpha < \lambda$ and applying Corollary 4.6, it results that $\mathcal{G} = (\mathcal{L}_q - \alpha\mathcal{P}_\omega)^{-1}$. Therefore, given $y \in V$, then $u = G(\cdot, y) = \mathcal{G}(\varepsilon_y)$ is the unique solution of the equation $\mathcal{L}_q(u) - \alpha\mathcal{P}_\omega(u) = \varepsilon_y$. Moreover, from Theorem 3.9 we obtain that $u > 0$ on V, since $\varepsilon_y \geqslant 0$ and V is connected. In addition, if we consider the function $v = \dfrac{\omega}{\omega(y)} - \dfrac{u}{u(y)}$, then $v(y) = 0$ and $\mathcal{L}_q(v) - \alpha\mathcal{P}_\omega(v) = \dfrac{(\lambda - \alpha)}{\omega(y)}\,\omega - \dfrac{1}{u(y)}\varepsilon_y \geqslant 0$ on $V \setminus \{y\}$ and hence, Theorem 3.9 newly implies that $v(x) > 0$ for any $x \in V \setminus \{y\}$. On the other hand, as $\mathcal{G} = \mathcal{G}_{\lambda,\omega} + \lambda_{\mathcal{G}}\mathcal{P}_\omega$, we get that $G(x,x) > \dfrac{\lambda_{\mathcal{G}}}{n}\,\omega^2(x)$ for any $x \in V$ since $G_{\lambda,\omega}(x,x) > 0$. Moreover, the inequalities just proved for G imply that $-\dfrac{\lambda_{\mathcal{G}}}{n}\,\omega(x)\omega(y) < G_{\lambda,\omega}(x,y) < G_{\lambda,\omega}(y,y)\dfrac{\omega(x)}{\omega(y)}$ for any $x, y \in V$ with $x \neq y$. ∎

If we take into account Proposition 3.6, then given $\lambda \geqslant 0$, $\omega \in \Omega(V)$ and fixed $z \in V$, for any $y \in V$ there exists a unique solution $G^z_{\lambda,\omega}(\cdot, y)$ of the generalized Poisson equation $(\mathcal{L}_q - \lambda\mathcal{P}_\omega)(u) = \varepsilon_y - \frac{\omega(y)}{\omega(z)}\,\varepsilon_z$ verifying that $G^z_{\lambda,\omega}(z, y) = 0$. The kernel $G^z_{\lambda,\omega}$ and its associated endomorphism $\mathcal{G}^z_{\lambda,\omega}$ are called respectively *Green function and Green operator of $V \setminus \{z\}$ with respect to λ and ω.*

Proposition 4.9 *Given $\lambda \geqslant 0$ and $\omega \in \Omega(V)$ for any $z \in V$, $\mathcal{G}^z_{\lambda,\omega}$ is a self-adjoint and positive definite generalized inverse of $\mathcal{L}_q - \lambda\mathcal{P}_\omega$. Moreover, for any $f \in \mathcal{C}(V)$ the function $u = \mathcal{G}^z_{\lambda,\omega}(f)$ is the unique element in $\mathcal{C}(V)$ that verifies $\mathcal{L}_q(u) - \lambda\mathcal{P}_\omega(u) = f$ on $V \setminus \{z\}$ and $u(z) = 0$. In addition, $G^z_{\lambda,\omega}(\cdot, z) = 0$, $0 \leqslant G^z_{\lambda,\omega}(x, y)\omega(y) \leqslant G^z_{\lambda,\omega}(y, y)\omega(x)$ for any $x, y \in V \setminus \{z\}$ and the first inequality is an equality iff $\lambda = 0$ and z separates x and y, whereas the second one is an equality iff $\lambda = 0$ and y separates x and z.*

Proof. If $u = \mathcal{G}^z_{\lambda,\omega}(f)$, then $u(x) = \sum\limits_{y\in V} G^z_{\lambda,\omega}(x, y)\, f(y)$ for any $x \in V$ and hence $u(z) = 0$, since $G^z_{\lambda,\omega}(z, y) = 0$ for any $y \in V$. Moreover,

$$(\mathcal{L}_q - \lambda\mathcal{P}_\omega)(u) = \sum_{y\in V} f(y)(\mathcal{L}_q - \lambda\mathcal{P}_\omega)(G^z_{\lambda,\omega}(\cdot, y)) = f - \frac{\langle f, \omega\rangle}{\omega(z)}\,\varepsilon_z,$$

which in particular implies that $(\mathcal{L}_q - \lambda\mathcal{P}_\omega)(u) = f$ on $V \setminus \{z\}$ and that $(\mathcal{L}_q - \lambda\mathcal{P}_\omega)(u) = f$ on V, when $\mathcal{P}_\omega(f) = 0$; i.e., $\mathcal{G}^z_{\lambda,\omega}$ is a generalized inverse of $\mathcal{L}_q - \lambda\mathcal{P}_\omega$.

If we suppose now that $\tilde{u} \in \mathcal{C}(V)$ verifies that $(\mathcal{L}_q - \lambda\mathcal{P}_\omega)(\tilde{u}) = f$ on $V \setminus \{z\}$, then we get that $\omega(z)(\mathcal{L}_q - \lambda\mathcal{P}_\omega)(\tilde{u})(z) = f(z)\omega(z) - \langle f, \omega\rangle$, since $\mathcal{P}_\omega \circ (\mathcal{L}_q - \lambda\mathcal{P}_\omega) = 0$. Therefore, $(\mathcal{L}_q - \lambda\mathcal{P}_\omega)(\tilde{u}) = (\mathcal{L}_q - \lambda\mathcal{P}_\omega)(u)$, which implies that $\tilde{u} = u + a\omega$, $a \in \mathbb{R}$, and $\tilde{u} = u$ if we impose that $\tilde{u}(z) = 0$.

Consider $x, y \in V$ and the functions $v_x, v_y \in V$ defined respectively as $v_x = G^z_{\lambda,\omega}(\cdot, x)$ and $v_y = G^z_{\lambda,\omega}(\cdot, y)$. Then, $v_x(z) = v_y(z) = 0$ and applying the self-adjointness of the operator $\mathcal{L}_q - \lambda\mathcal{P}_\omega$, we have that

$$G^z_{\lambda,\omega}(y, x) = v_x(y) = \langle v_x, (\mathcal{L}_q - \lambda\mathcal{P}_\omega)(v_y)\rangle = \langle v_y, (\mathcal{L}_q - \lambda\mathcal{P}_\omega)(v_x)\rangle = v_y(x) = G^z_{\lambda,\omega}(x, y),$$

which implies the self-adjointness of $\mathcal{G}^z_{\lambda,\omega}$. Moreover, given $f \in \mathcal{C}(V)$ and $u = \mathcal{G}^z_{\lambda,\omega}(f)$, then $\mathcal{L}_q(u) - \lambda\mathcal{P}_\omega(u) = f$ on $V \setminus \{z\}$, $u(z) = 0$ and applying now the positive definiteness of the operator $\mathcal{L}_q - \lambda\mathcal{P}_\omega$, we have that

$$\langle \mathcal{G}^z_{\lambda,\omega}(f), f\rangle = \langle u, (\mathcal{L}_q - \lambda\mathcal{P}_\omega)(u)\rangle \geqslant 0.$$

Moreover, the last inequality is an equality iff $u = a\omega$ where $a \in \mathbb{R}$ and hence $u = 0$, since $u(z) = 0$ implies that $a = 0$. In conclusion, the operator $\mathcal{G}^z_{\lambda,\omega}$ is positive definite.

In addition, if $F = V \setminus \{z\}$, then $\mathcal{L}_q(v_y) = \varepsilon_y + \lambda\mathcal{P}_\omega(v_y) \geqslant \lambda\mathcal{P}_\omega(v_y)$ on F. As $v_y(z) = 0$, applying Theorem 3.9 we obtain that $v_y \geqslant 0$ on V. Moreover, if F_y is the connected component of F that contains y, then $v_y(x) > 0$ for any $x \in F_y$, since $\mathcal{L}_q(v_y)(y) = 1 + \lambda \mathcal{P}_\omega(v_y)(y) \geqslant 1$. This implies that if $v_y(x) = G^z_{\lambda,\omega}(x, y) = 0$, necessarily $\lambda = 0$ and $x \notin F_y$; *i.e.*, z separates x and y.

Finally, if $u = G^y_{\lambda,\omega}(\cdot, z)$, then $(\mathcal{L}_q - \lambda \mathcal{P}_\omega)(\omega(z)\, v_y + \omega(y)\, u) = 0$ and hence, there exists $a \in \mathbb{R}$ such that $G^z_{\lambda,\omega}(x, y)\omega(z) + G^y_{\lambda,\omega}(x, z)\omega(y) = a\omega(x)$, which implies that $a\omega(y) = G^z_{\lambda,\omega}(y, y)\omega(z)$ and then,

$$G^z_{\lambda,\omega}(x, y) + G^y_{\lambda,\omega}(x, z)\frac{\omega(y)}{\omega(z)} = \frac{\omega(x)}{\omega(y)}\, G^z_{\lambda,\omega}(y, y).$$

Therefore, $G^z_{\lambda,\omega}(x, y)\omega(y) \leqslant G^z_{\lambda,\omega}(y, y)\omega(x)$, since $G^y_{\lambda,\omega}(x, z) \geqslant 0$ and moreover the inequality is an equality iff $G^y_{\lambda,\omega}(x, z) = 0$ and hence iff $\lambda = 0$ and y separates x and z. ∎

Corollary 4.10 *Fixed any order on V, for any $z \in V$ let L^z and G^z the matrices of order $n-1$ obtained by deleting the row and column corresponding to z from the matrices associated to operators $\mathcal{L}_q - \lambda \mathcal{P}_\omega$ and $\mathcal{G}^z_{\lambda,\omega}$, respectively. Then L^z and G^z are inverses of each other.*

The relation between kernels $G^z_{\lambda,\omega}$ when z run over the vertices of V and the Green functions for the Poisson equation is given in the following result.

Proposition 4.11 *Given $\lambda \geqslant 0$ and $\omega \in \Omega(V)$, consider $\mathcal{K}$ a generalized inverse of $\mathcal{L}_q - \lambda \mathcal{P}_\omega$ and K its associated kernel. Then, for any $x, y, z \in V$ it is verified that*

$$G^z_{\lambda,\omega}(x, y) = K(x, y) - \frac{1}{\omega(z)}\Big[K(x, z)\omega(y) + K(z, y)\omega(x)\Big] + \frac{\omega(x)\omega(y)}{\omega^2(z)}\, K(z, z).$$

Conversely, $\mathcal{G} = \mathcal{G}_{\lambda,\omega} + \mathsf{k}(\lambda, \omega)\mathcal{P}_\omega$ is a self-adjoint and positive definite Green operator for Γ with respect to λ and ω, whose kernel is given by

$$G(x, y) = \frac{1}{n}\sum_{z \in V} G^z_{\lambda,\omega}(x, y)\omega^2(z), \qquad \textit{for any } x, y \in V.$$

Proof. If we consider $f = \varepsilon_y - \dfrac{\omega(y)}{\omega(z)}\, \varepsilon_z$, then $\mathcal{P}_\omega(f) = 0$ and hence

$$G^z_{\lambda,\omega}(\cdot, y) = \sum_{t \in V} K(\cdot, t)f(t) + a(y, z)\omega = K(\cdot, y) - \frac{\omega(y)}{\omega(z)}\, K(\cdot, z) + a(y, z)\omega,$$

where $a(y, z) = \dfrac{\omega(y)}{\omega^2(z)}\, K(z, z) - \dfrac{1}{\omega(z)}\, K(z, y)$, since $G^z_{\lambda,\omega}(z, y) = 0$. Therefore, the first claim follows. In particular, taking $K = G_{\lambda,\omega}$ we get that

$$G^z_{\lambda,\omega}(x, y) = G_{\lambda,\omega}(x, y) - \frac{1}{\omega(z)}\Big[G_{\lambda,\omega}(x, z)\omega(y) + G_{\lambda,\omega}(z, y)\omega(x)\Big] + \frac{\omega(x)\omega(y)}{\omega^2(z)}\, G_{\lambda,\omega}(z, z)$$

and hence multiplying by $\omega^2(z)$ both sides of the above equality, summing in z and tacking into account that $\mathcal{G}_{\lambda,\omega}(\omega) = 0$, we obtain that

$$\sum_{z\in V} G^z_{\lambda,\omega}(x,y)\omega^2(z) = nG_{\lambda,\omega}(x,y) + \omega(x)\omega(y)\mathsf{k}(\lambda,\omega)$$

and the last claim follows from Corollary 4.6. ∎

Corollary 4.12 *Given $\lambda \geqslant 0$ and $\omega \in \Omega(V)$, for any $z \in V$, $\mathcal{G}^z$ is the unique self-adjoint generalized inverse of $\mathcal{L}_q - \lambda\mathcal{P}_\omega$ that assigns to ε_z the null function. Moreover,*

$$\mathcal{G}^z_{\lambda,\omega} = \mathcal{G}_{\lambda,\omega} + \mathcal{P}_{\omega,\tau} + \mathcal{P}_{\tau,\omega},$$

where $\tau = -\dfrac{1}{\omega(z)}\Big(G_{\lambda,\omega}(\cdot,z) - \dfrac{G_{\lambda,\omega}(z,z)}{2\omega(z)}\,\omega\Big)$. *Conversely,*

$$\mathcal{G}_{\lambda,\omega} = \frac{1}{n}\langle \mathcal{G}^z_{\lambda,\omega}(\omega),\omega\rangle\,\mathcal{P}_\omega + \mathcal{G}^z_{\lambda,\omega} - \mathcal{P}_\omega\circ\mathcal{G}^z_{\lambda,\omega} - \mathcal{G}^z_{\lambda,\omega}\circ\mathcal{P}_\omega.$$

Proof. The first identity is a straightforward consequence of the above proposition. In addition, $\mathcal{G}^z_{\lambda,\omega}(\omega) = \langle\tau,\omega\rangle\omega + n\tau = \dfrac{nG_{\lambda,\omega}(z,z)}{2\omega^2(z)}\,\omega + n\tau$, which implies that $\langle\mathcal{G}^z_{\lambda,\omega}(\omega),\omega\rangle = \dfrac{n^2G_{\lambda,\omega}(z,z)}{\omega^2(z)}$. On the other hand, the expression for $\mathcal{G}_{\lambda,\omega}$ follows from the equalities

$$\mathcal{P}_\omega\circ\mathcal{G}^z_{\lambda,\omega} = \frac{1}{n}\,\mathcal{P}_{\omega,\mathcal{G}^z_{\lambda,\omega}(\omega)} = \mathcal{P}_{\omega,\tau} + \frac{nG_{\lambda,\omega}(z,z)}{2\omega^2(z)}\,\mathcal{P}_\omega \quad\text{and}\quad \mathcal{G}^z_{\lambda,\omega}\circ\mathcal{P}_\omega = \mathcal{P}_{\tau,\omega} + \frac{nG_{\lambda,\omega}(z,z)}{2\omega^2(z)}\,\mathcal{P}_\omega. \quad ∎$$

The following result that constitutes the matrix version of the above one tell us how to obtain the Moore-Penrose inverse of $\mathsf{L}(\omega) + \lambda(\mathsf{I} - \mathsf{P}(\omega))$ in terms of the inverse of a $(n-1)$-matrix.

Corollary 4.13 *Given any order of the vertices of Γ, consider $\lambda \geqslant 0$, $\omega \in \Omega(V)$ and $z = x_n$. Then, if $\mathsf{P}^z(\omega)$ and ω^z denote the matrix and vector obtained respectively from $\mathsf{P}(\omega)$ and from ω by deleting the row and column and the entry corresponding to z, then*

$$\Big(\mathsf{L}(\omega) + \lambda(\mathsf{I} - \mathsf{P}(\omega))\Big)^\dagger = \frac{(\omega^z)^*\mathsf{G}^z\omega^z}{n}\,\mathsf{P}(\omega) + \begin{bmatrix} \mathsf{G}^z - \mathsf{G}^z\mathsf{P}^z(\omega) - \mathsf{P}^z(\omega)\mathsf{G}^z & -\frac{1}{n}\mathsf{G}^z\omega^z \\ -\frac{1}{n}(\omega^z)^*\mathsf{G}^z & 0 \end{bmatrix}.$$

Observe that when $\lambda = 0$ and $\omega = 1$ the above expression becomes

$$\mathsf{L}^\dagger = \frac{1^*\mathsf{G}^z 1}{n^2}\,\mathsf{J} + \begin{bmatrix} \mathsf{G}^z - \frac{1}{n}\mathsf{G}^z\mathsf{J} - \frac{1}{n}\mathsf{J}\mathsf{G}^z & -\frac{1}{n}\mathsf{G}^z 1 \\ -\frac{1}{n}1^*\mathsf{G}^z & 0 \end{bmatrix},$$

that was obtained in [14], where the matrix G^z was called the *bottleneck matrix of* L *based at* z.

5 The effective resistances of a network

In the standard setting, the effective resistance of the network Γ between vertices x and y is defined throughout the solution of the Poisson equation $\mathcal{L}(u) = f$ when the data is the dipole with poles at x and y; that is, $f = \varepsilon_x - \varepsilon_y$. Important properties of electrical networks can be deduced from the knowledge of the effective resistance, see for instance [12]. One of them establishes that the Green function of the complementary of any vertex can be obtained in terms of the effective resistances, see for instance [11, 13]. As a by-product of Proposition 4.11, this also occurs for any Green function. Moreover, Corollary 4.10 and the relation between M-matrices and Schrödinger operators given in Lemma 2.1 implies in particular, that any irreducible and weakly diagonal dominant Stieltjes matrix is the resistive inverse associated with a suitable network Γ, which is precisely the main result in [11].

Along this section we generalize the above mentioned facts in several ways. First we use the definition of the dipole with respect to a weight introduced in [6] to define the concept of the effective resistance between two points of a network with respect to a value $\lambda \geqslant 0$ and a weight $\omega \in \Omega(V)$. So, we obtain that the Green function of the complementary of any vertex, associated with a positive semidefinite Schrödinger operator, singular or not, can be expressed in terms of the effective resistances with respect to a non-negative value and a weight. As a by-product, we obtain a new version of Fiedler's result and moreover we can eliminate the hypothesis of diagonally dominance to obtain that any irreducible Stieltjes matrix is a resistive inverse. Moreover we introduce here the concept of total resistance of a vertex with respect to a positive value and a weight, that in some sense generalized the notion of *status of a vertex* introduced in [14], and that together with effective resistances allows us to obtain a expression for any Green function. The matrix version of these results allows us to obtain the expression of the Moore-Penrose inverse of any irreducible symmetric M-matrix in terms of the matrix of effective resistance.

In the sequel we consider fixed the network $\Gamma = (V, c)$, the value $\lambda \geqslant 0$, the weight $\omega \in \Omega(V)$ and $\mathcal{L}_q$ the Schrödinger operator with ground state $q = q_\omega + \lambda$. Given $x, y \in V$, the *ω-dipole between x and y* is the function $f_{xy} = \dfrac{1}{\omega}(\varepsilon_x - \varepsilon_y)$. Observe that $f_{xx} = 0$ for any $x \in V$, whereas when ω is constant the ω-dipole between x and y is simply the standard dipole. Clearly, for any $x, y \in V$ it is verified that $\mathcal{P}_\omega(f_{xy}) = 0$ and then the hypotheses of Proposition 3.3 are in force. Consequently, given $x, y \in V$ the functional $\mathfrak{J}_{x,y}\colon \mathcal{C}(V) \longrightarrow \mathbb{R}$ determined for any $u \in \mathcal{C}(V)$ by the expression

$$\mathfrak{J}_{x,y}(u) = 2\left[\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)}\right] - \langle \mathcal{L}_q(u), u \rangle \tag{6}$$

attains a maximum value. In addition, $v \in \mathcal{C}(V)$ is a maximum of $\mathfrak{J}_{x,y}$ iff it satisfies the Poisson equation $\mathcal{L}_q(v) = f_{xy}$.

In view of the above result for any $x, y \in V$, we define the *Effective Resistance between x and y with respect to λ and ω,* as the value

$$R_{\lambda,\omega}(x, y) = \max_{u \in \mathcal{C}(V)} \{\mathfrak{J}_{x,y}(u)\}.$$

The kernel $R_{\lambda,\omega}: V \times V \longrightarrow \mathbb{R}$ is called the *Effective Resistance of the network* Γ *with respect to* λ *and* ω, whereas its associated endomorphism, $\mathcal{R}_{\lambda,\omega}$, is called *Effective Resistance operator of the network* Γ *with respect to* λ *and* ω.

Observe that when $x = y$, the functional $\mathfrak{J}_{x,y}$ attains its maximum value at $v = a\omega$, where $a = 0$ when $\lambda > 0$ and $a \in \mathbb{R}$, otherwise. In any case, $R_{\lambda,\omega}(x, x) = 0$.

When $\lambda = 0$ we usually omit the subindex λ in the above expressions and we refer to R_ω and $\mathcal{R}_\omega$ as the effective resistance and the effective resistance operator of Γ with respect to ω. If, in addition, $\omega = 1$ we also omit the subindex ω and we refer to R and $\mathcal{R}$ simply as the effective resistance and the effective resistance operator of Γ. Therefore, R is nothing else than the standard effective resistance of the network.

Next, we study the basic properties of the effective resistance with respect to λ and ω.

Proposition 5.1 *Given* $x, y \in V$*, then*

$$R_{\lambda,\omega}(x, y) = \langle \mathcal{L}_q(v), v\rangle = \frac{v(x)}{\omega(x)} - \frac{v(y)}{\omega(y)},$$

where $v \in \mathcal{C}(V)$ *is any solution of the Poisson equation* $\mathcal{L}_q(v) = f_{xy}$*. In addition,* $R_{\lambda,\omega}$ *is symmetric non-negative and* $R_{\lambda,\omega}(x, y) = 0$ *iff* $x = y$*.*

Proof. The expression for the effective resistance is newly a consequence of the Proposition 3.3, specifically of the Euler-Lagrange Identity. On the other hand, given $u \in \mathcal{C}(V)$ we have that $\mathfrak{J}_{x,y}(u) = \mathfrak{J}_{y,x}(-u)$ and hence $R_{\lambda,\omega}(x, y) = R_{\lambda,\omega}(y, x)$ for any $x, y \in V$. Moreover, we know that $R_{\lambda,\omega}(x, x) = 0$ for any $x \in V$ and also that $R_{\lambda,\omega}(x, y) = 0$ iff $\langle \mathcal{L}_q(v), v\rangle = 0$ for any solution of the Poisson equation $\mathcal{L}_q(v) = f_{xy}$. So, $v = a\omega$ where $a = 0$ when $\lambda > 0$, that in any case implies that $\mathcal{L}_q(v) = 0$ and hence $f_{xy} = 0$ or equivalently, $x = y$. ∎

The relation between the effective resistance and the Poisson equation whose data is the ω-dipole between x and y, leads to the following relations between the effective resistance and the generalized inverses.

Proposition 5.2 *Given* K *the kernel of a generalized inverse of* $\mathcal{L}_q - \lambda\mathcal{P}_\omega$*, then*

$$R_{\lambda,\omega}(x, y)\omega^2(x)\omega^2(y) = K(x, x)\omega^2(y) + K(y, y)\omega^2(x) - [K(x, y) + K(y, x)]\,\omega(x)\omega(y),$$

for any $x, y \in V$ *and hence* $\mathsf{k}(\lambda, \omega) = \dfrac{1}{2n}\langle \mathcal{R}_{\lambda,\omega}(\omega^2), \omega^2\rangle$.

Proof. It suffices to proof the first claim for $G_{\lambda,\omega}$, since from Proposition 4.3 we know that $K = G_{\lambda,\omega} + \omega \otimes \tau + \hat{\tau} \otimes \omega$ for some $\tau, \hat{\tau} \in \mathcal{C}(V)$.

Given $x, y \in \mathbb{R}$, if $u = \mathcal{G}_{\lambda,\omega}(f_{xy})$ then, applying Corollary 3.5 we know that $\mathcal{L}_q(u) = f_{xy}$, since $\mathcal{P}_\omega(f_{xy}) = 0$. Therefore, $u = \dfrac{G_{\lambda,\omega}(\cdot,x)}{\omega(x)} - \dfrac{G_{\lambda,\omega}(\cdot,y)}{\omega(y)}$ and hence taking into account the symmetry of $G_{\lambda,\omega}$, we obtain that

$$R_{\lambda,\omega}(x,y) = \frac{G_{\lambda,\omega}(x,x)}{\omega^2(x)} + \frac{G_{\lambda,\omega}(y,y)}{\omega^2(y)} - \frac{2G_{\lambda,\omega}(x,y)}{\omega(x)\omega(y)}.$$

The last conclusion is a direct consequence of the above equality taking into account that $\mathcal{G}_{\lambda,\omega}(\omega) = 0$. ∎

The above result suggests that it will be useful to define the *total resistance at $x \in V$ with respect to λ and ω*, as the value $r_{\lambda,\omega}(x) = \dfrac{G_{\lambda,\omega}(x,x)}{\omega^2(x)}$. Moreover, the function $r_{\lambda,\omega} \in \mathcal{C}(V)$ is called *the total resistance of the network Γ with respect to λ and ω* and it has properties very similar to those verified by the effective resistance, as we show in the next result.

Proposition 5.3 *For any $x \in V$, if we consider the functional $\mathfrak{J}_x\colon \mathcal{C}(V) \longrightarrow \mathbb{R}$ given for any $u \in \mathcal{C}(V)$ by*

$$\mathfrak{J}_x(u) = 2\left[\frac{u(x)}{\omega(x)} - \frac{1}{n}\langle u, \omega\rangle\right] - \langle \mathcal{L}_q(u), u\rangle,$$

then $r_{\lambda,\omega}(x) = \max\limits_{u \in \mathcal{C}(V)}\{\mathfrak{J}_x(u)\} = \langle \mathcal{L}_q(v), v\rangle = \dfrac{v(x)}{\omega(x)} - \dfrac{1}{n}\langle v, \omega\rangle$ where v is any solution of the Poisson equation $\mathcal{L}_q(v) = \dfrac{1}{\omega}\varepsilon_x - \dfrac{\omega}{n}$. Moreover, $\langle r_{\lambda,\omega}, \omega^2\rangle = \mathsf{k}(\lambda,\omega)$ and for any $x \in V$

$$0 < \frac{1}{n^2}\langle \mathcal{G}^x_{\lambda,\omega}(\omega), \omega\rangle = r_{\lambda,\omega}(x) = \frac{1}{n}\left[\mathcal{R}_{\lambda,\omega}(\omega^2)(x) - \mathsf{k}(\lambda,\omega)\right].$$

In addition, if $\mathcal{G}$ is any self-adjoint Green operator for Γ with respect to λ and ω and G is its associated Green function, then for any $x \in V$, $\dfrac{G(x,x)}{\omega^2(x)} = r_{\lambda,\omega}(x) + \dfrac{\lambda_{\mathcal{G}}}{n}$.

Proof. If $f_x = \dfrac{1}{\omega}\varepsilon_x - \dfrac{\omega}{n} = \dfrac{1}{\omega(x)}(\varepsilon_x - \dfrac{1}{n}\omega(x)\omega)$, then $u = \dfrac{G_{\lambda,\omega}(\cdot,x)}{\omega(x)}$ is a solution of the generalized Poisson equation $\mathcal{L}_q(u) - \lambda\mathcal{P}_\omega(u) = f_x$ and hence a solution of the Poisson equation $\mathcal{L}_q(u) = f_x$, since $\mathcal{P}_\omega(u) = 0$. Therefore, by applying Proposition 3.3 u maximizes $\mathfrak{J}_x$ and then, the Euler-Lagrange Identity implies that

$$\mathfrak{J}_x(u) = \langle \mathcal{L}_q(u), u\rangle = \frac{u(x)}{\omega(x)} - \frac{1}{n}\langle u, \omega\rangle = \frac{u(x)}{\omega(x)} = r_{\lambda,\omega}(x),$$

since $\mathcal{P}_\omega \circ \mathcal{G}_{\lambda,\omega} = 0$ and the same properties are valid for any solution of the Poisson equation.

On the other hand, from the definition of $r_{\lambda,\omega}$, it is clear that $\langle r_{\lambda,\omega}, \omega^2\rangle = \mathsf{k}(\lambda,\omega)$; whereas the equality $r_{\lambda,\omega}(x) = \dfrac{1}{n}\left[\mathcal{R}_{\lambda,\omega}(\omega^2)(x) - \mathsf{k}(\lambda,\omega)\right]$, follows from Proposition 5.2, since $\mathcal{G}_{\lambda,\omega}(\omega) = 0$. Moreover, from Corollary 4.12 we get that $0 < \langle \mathcal{G}^x_{\lambda,\omega}(\omega), \omega\rangle = n^2 r_{\lambda,\omega}(x)$, since $\mathcal{G}^x_{\lambda,\omega}$ is positive definite.

Finally, if G is the Green function associated with $\mathcal{G}$, then $\dfrac{G(x,x)}{\omega^2(x)} = r_{\lambda,\omega}(x) + \dfrac{\lambda_{\mathcal{G}}}{n}$, since $\mathcal{G} = \mathcal{G}_{\lambda,\omega} + \lambda_{\mathcal{G}}\mathcal{P}_\omega$. ∎

The first equality in the following result is a generalization of the well-known characterization of the so-called *Campbell-Youla inverse*, see [17], whereas the second identity agrees with the formula for the inverse of the resistive matrix obtained in [2, 3] for $\lambda = 0$ and $\omega = 1$.

Proposition 5.4 *For any $\lambda \geqslant 0$ and $\omega \in \Omega(V)$, it is verified that*

$$\mathcal{R}_{\lambda,\omega} = \mathcal{P}_{r_{\lambda,\omega},1} + \mathcal{P}_{1,r_{\lambda,\omega}} - 2\mathcal{D}_\omega^{-1} \circ \mathcal{G}_{\lambda,\omega} \circ \mathcal{D}_\omega^{-1},$$

and hence the endomorphism $-\frac{1}{2}\mathcal{D}_\omega \circ \mathcal{R}_{\lambda,\omega} \circ \mathcal{D}_\omega$ is the unique self-adjoint zero-axial generalized inverse of $\mathcal{L}_q - \lambda\mathcal{P}_\omega$. Moreover, $\mathcal{R}_{\lambda,\omega}$ is non singular and

$$\mathcal{R}^{-1}_{\lambda,\omega} = -\frac{1}{2}\,\mathcal{D}_\omega \circ (\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{D}_\omega + \frac{\langle \nu,\nu\rangle}{\langle \mathcal{R}_{\lambda,\omega}(\nu),\nu\rangle}\,\mathcal{P}_\nu,$$

where $\nu = \omega(\mathcal{L}_q - \lambda\mathcal{P}_\omega)(\omega r_{\lambda,\omega}) + \frac{2}{n}\,\omega^2$.

Proof. The first identity is a straightforward consequence of Proposition 5.2. Moreover, if $\mathcal{K} = -\frac{1}{2}\mathcal{D}_\omega \circ \mathcal{R}_{\lambda,\omega} \circ \mathcal{D}_\omega$, then $\mathcal{K}$ is self-adjoint and zero axial since $\mathcal{R}_{\lambda,\omega}$ is. In addition, from the first equality we get that $\mathcal{K} = \mathcal{G}_{\lambda,\omega} + \mathcal{P}_{\omega,\tau} + \mathcal{P}_{\tau,\omega}$, where $\tau = -\frac{1}{2}\,\omega\, r_{\lambda,\omega}$ and hence $\mathcal{K}$ is a generalized inverse of $\mathcal{L}_q - \lambda\mathcal{P}_\omega$. So, $\mathcal{K}$ is a zero-axial and conditionally positive definite with respect to ω, which implies that it is invertible and hence that $\mathcal{R}_{\lambda,\omega}$ is also non singular. Moreover,

$$-\frac{1}{2}\,\mathcal{D}_\omega \circ (\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{D}_\omega \circ \mathcal{R}_{\lambda,\omega} = -\frac{1}{2}\mathcal{P}_{\nu,1} + \frac{1}{n}\mathcal{P}_{\omega^2,1} + \mathcal{D}_\omega \circ (\mathcal{I} - \mathcal{P}_\omega)\mathcal{D}_\omega^{-1} = -\frac{1}{2}\mathcal{P}_{\nu,1} + \mathcal{I}$$

since $(\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{G}_{\lambda,\omega} = \mathcal{I} - \mathcal{P}_\omega$ and $(\mathcal{L}_q - \lambda\mathcal{P}_\omega)(\omega) = 0$. On the other hand,

$$\langle \nu, 1\rangle = \langle (\mathcal{L}_q - \lambda\mathcal{P}_\omega)(\omega r_{\lambda,\omega}), \omega\rangle + \frac{2}{n}\langle \omega,\omega\rangle = \langle \omega r_{\lambda,\omega}, (\mathcal{L}_q - \lambda\mathcal{P}_\omega)(\omega)\rangle + 2 = 2.$$

Therefore, $\mathcal{P}_{\nu,1}(\nu) = 2\nu$ and hence $-\frac{1}{2}\,\mathcal{D}_\omega \circ (\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{D}_\omega \circ \mathcal{R}_{\lambda,\omega}(\nu) = -\frac{1}{2}\mathcal{P}_{\nu,1}(\nu) + \nu = 0$. This equality implies that $\omega\mathcal{R}_{\lambda,\omega}(\nu) = a\omega$, that is, $\mathcal{R}_{\lambda,\omega}(\nu) = a \cdot 1$, where $a \neq 0$ since $\nu \neq 0$ and $\mathcal{R}_{\lambda,\omega}$ is non-singular. Moreover, $a = \frac{1}{2}\,\langle \mathcal{R}_{\lambda,\omega}(\nu),\nu\rangle$ and

$$-\frac{1}{2}\mathcal{P}_{\nu,1} = -\frac{1}{\langle \mathcal{R}_{\lambda,\omega}(\nu),\nu\rangle}\,\mathcal{P}_{\nu,\mathcal{R}_{\lambda,\omega}(\nu)} = -\frac{\langle \nu,\nu\rangle}{\langle \mathcal{R}_{\lambda,\omega}(\nu),\nu\rangle}\,\mathcal{P}_\nu \circ \mathcal{R}_{\lambda,\omega}.$$

Therefore,

$$-\frac{1}{2}\,\mathcal{D}_\omega \circ (\mathcal{L}_q - \lambda \mathcal{P}_\omega) \circ \mathcal{D}_\omega \circ \mathcal{R}_{\lambda,\omega} = -\frac{\langle \nu,\nu\rangle}{\langle \mathcal{R}_{\lambda,\omega}(\nu),\nu\rangle}\,\mathcal{P}_\nu \circ \mathcal{R}_{\lambda,\omega} + \mathcal{I}$$

and the last claim follows. ∎

Proposition 5.2 allows us to obtain the effective resistances throughout generalized inverses. Of course, we can act in the opposite way so that the knowledge of effective resistances of the network permit us to obtain the generalized inverses and in particular the Green operators. In fact, after Proposition 4.3 it suffices to obtain $\mathcal{G}_{\lambda,\omega}$ from the effective resistance of Γ.

Corollary 5.5 *For any $\lambda \geqslant 0$ and $\omega \in \Omega(V)$, it is verified that*

$$\mathcal{G}_{\lambda,\omega} = \frac{1}{2}\,\mathcal{D}_\omega \circ (\mathcal{P}_{r_{\lambda,\omega},1} + \mathcal{P}_{1,r_{\lambda,\omega}} - \mathcal{R}_{\lambda,\omega}) \circ \mathcal{D}_\omega = -\frac{1}{2}\mathcal{D}_\omega \circ \Big(\mathcal{I} - \frac{1}{n}\mathcal{P}_{1,\omega^2}\Big) \circ \mathcal{R}_{\lambda,\omega} \circ \Big(\mathcal{I} - \frac{1}{n}\mathcal{P}_{\omega^2,1}\Big) \circ \mathcal{D}_\omega,$$

or equivalently, for any $x,y \in V$ we get that

$$\begin{aligned} G_{\lambda,\omega}(x,y) &= \frac{1}{2}\,\omega(x)\omega(y)\,\Big(r_{\lambda,\omega}(x) + r_{\lambda,\omega}(y) - R_{\lambda,\omega}(x,y)\Big) \\ &= \frac{1}{2n}\,\omega(x)\omega(y) \sum_{z\in V} \Big(R_{\lambda,\omega}(x,z) + R_{\lambda,\omega}(y,z) - R_{\lambda,\omega}(x,y)\Big)\omega^2(z) - \frac{1}{n}\,\omega(x)\omega(y)\,\mathsf{k}(\lambda,\omega). \end{aligned}$$

In particular, for any $x,y \in V$ we get that $\Big|r_{\lambda,\omega}(x) - r_{\lambda,\omega}(y)\Big| \leqslant R_{\lambda,\omega}(x,y)$ with equality iff $x = y$ and moreover, when $\lambda > 0$ it is also verified $R_{\lambda,\omega}(x,y) < r_{\lambda,\omega}(x) + r_{\lambda,\omega}(y) + \frac{2}{n\lambda}$.

Proof. The identities for the kernels are consequence of Proposition 5.2 and of the equality $r_{\lambda,\omega}(x) = \dfrac{1}{n}\,[\mathcal{R}_{\lambda,\omega}(\omega^2)(x) - \mathsf{k}(\lambda,\omega)]$, for any $x \in V$ and hence the equalities for the operators follow. On the other hand, the last inequalities are consequence of the last ones in Proposition 4.8. ∎

The matrix counterpart of the above result is given in the following corollary.

Corollary 5.6 *Given any order of the vertices of Γ, for any $\lambda \geqslant 0$ and $\omega \in \Omega(V)$, let R, D and S the matrices identified with $\mathcal{R}_{\lambda,\omega}$, $\mathcal{D}_\omega$ and $\frac{1}{n}\mathcal{P}_{\omega^2,1}$, respectively. Then,*

$$\Big(\mathsf{L}(\omega) + \lambda(\mathsf{I} - \mathsf{P}(\omega))\Big)^\dagger = -\frac{1}{2}\,\mathsf{D}(\mathsf{I} - \mathsf{S}^*)\mathsf{R}(\mathsf{I} - \mathsf{S})\,\mathsf{D}.$$

In particular, when $\lambda = 0$ and ω is constant, then $\mathsf{D} = \mathsf{I}$ and $\mathsf{S} = \frac{1}{n}\,\mathsf{J}$ and hence the above corollary becomes the well-known identity

$$\mathsf{L}^\dagger = -\frac{1}{2}\Big[\mathsf{R} - \frac{1}{n}\,[\mathsf{J}\,\mathsf{R} + \mathsf{R}\,\mathsf{J}] + \frac{1}{n^2}\,\mathsf{J}\,\mathsf{R}\,\mathsf{J}\Big],$$

see [13, Theorem 7] and [14, Theorem 3.7], that was obtained there in a different way. A similar identity can be found in [1] in the context of spherical Euclidean distance matrices.

Observe that the above results allow us to characterize when the Moore-Penrose inverse of $\mathsf{L}(\omega)+\lambda(\mathsf{I}-\mathsf{P}(\omega))$ is a M-matrix. Specifically, $\Big(\mathsf{L}(\omega)+\lambda(\mathsf{I}-\mathsf{P}(\omega))\Big)^{\dagger}$ is an M-matrix iff for any $x\neq y$ it is verified that $r_{\lambda,\omega}(x)+r_{\lambda,\omega}(y)\leqslant R_{\lambda,\omega}(x,y)$ or, in an equivalent manner, iff

$$\sum_{z\in V}\Big(R_{\lambda,\omega}(x,z)+R_{\lambda,\omega}(y,z)\Big)\omega^2(z)\leqslant n\,R_{\lambda,\omega}(x,y)+\frac{1}{n}\sum_{t,z\in V}R_{\lambda,\omega}(t,z)\,\omega^2(t)\omega^2(z).$$

The above bound is tight since for the complete graph and in the case of $\lambda=0$ and $\omega=1$ $R(x,y)=\frac{2}{n}$ for all $x,y\in V$ and hence the equality happens. In fact, $\mathsf{L}=n\mathsf{I}-\mathsf{J}$ and $\mathsf{L}^{\dagger}=\frac{1}{n^2}\,\mathsf{L}$. Moreover, when Γ is a weighted tree, $\lambda=0$ and $\omega=1$, it is enough that the above condition is verified for adjacent vertices, as was proved in [14]. In fact, in [15] it was proved that for $n\geqslant 5$ this occurs iff Γ is a weighted star.

The following proposition represents a generalization of the well-known result that establishes that the Green function for the complementary of a vertex can be expressed in terms of the effective resistance and also shows that the generalized resistance is a distance, see for instance [9, 18] for the standard case and [6, 7] for the case $\lambda=0$.

Proposition 5.7 *Given $\lambda\geqslant 0$ and $\omega\in\Omega(V)$, then for any $x,y,z\in V$ it is verified that*

$$G^z_{\lambda,\omega}(x,y)=\frac{1}{2}\,\omega(x)\omega(y)\Big(R_{\lambda,\omega}(x,z)+R_{\lambda,\omega}(y,z)-R_{\lambda,\omega}(x,y)\Big).$$

In particular, $R_{\lambda,\omega}$ defines a distance on Γ and moreover $R_{\lambda,\omega}(x,y)=R_{\lambda,\omega}(x,z)+R_{\lambda,\omega}(y,z)$ iff $\lambda=0$ and z separates x and y.

Proof. Since $\mathcal{G}^z_{\lambda,\omega}$ is a generalized inverse from Proposition 5.2 we get

$$R_{\lambda,\omega}(x,y)\omega^2(x)\omega^2(y)=G^z_{\lambda,\omega}(x,x)\omega^2(y)+G^z_{\lambda,\omega}(y,y)\omega^2(x)-2G^z_{\lambda,\omega}(x,y)\omega(x)\omega(y).$$

In particular, $G^z_{\lambda,\omega}(x,x)=R_{\lambda,\omega}(x,z)\omega^2(x)$, for any $x\in V$, since $G^z_{\lambda,\omega}(x,z)=0$ and hence we obtain the first claim. The rest of the results are a direct consequence of the second part of Proposition 4.9. ∎

Corollary 5.8 *Under the hypotheses of Corollary 4.10, fixed $z\in V$, let J_z the matrix whose entries are null except those corresponding to the row z that are equal to 1. Then, G^z is the matrix obtained from $\frac{1}{2}\,\mathsf{D}\Big(\mathsf{J}_z\mathsf{R}+\mathsf{R}\mathsf{J}_z^*-\mathsf{R}\Big)\mathsf{D}$ by deleting the row and column corresponding to z.*

It is possible to obtain an analogue expression to the one obtained in Proposition 5.7 for some self-adjoint definite positive Green operators. To do that, we need to consider the network defined at the end of Section 3. Specifically, suppose that $\lambda > 0$ and let $\mathcal{G}$ be a self-adjoint and positive definite Green operator for Γ with respect to λ and ω such that $\lambda_{\mathcal{G}} \geqslant \dfrac{1}{\lambda}$. If we take $\alpha = \lambda - \dfrac{1}{\lambda_{\mathcal{G}}}$, then $0 \leqslant \alpha < \lambda$ and hence we can consider $\Gamma_\alpha = (V \cup \{\hat{x}\}, c_\alpha)$ and $\mathcal{L}^\alpha$ its Laplacian operator. Recall that $\sigma \in \Omega(V \cup \{\hat{x}\})$ is the weight given by $\sigma(\hat{x}) = 1$ and $\sigma(x) = \omega(x)$, when $x \in V$. Then, Proposition 3.7 says that if $u \in \mathcal{C}(V \cup \{\hat{x}\})$ then $\mathcal{L}^\alpha_{q_\sigma}(u)(\hat{x}) = \dfrac{1}{\lambda_{\mathcal{G}}}\Big(n\,u(\hat{x}) - \langle \omega, u_{|_V}\rangle\Big)$ and also that

$$\mathcal{L}^\alpha_{q_\sigma}(u) = \mathcal{L}_q(u_{|_V}) - \lambda\,\mathcal{P}_\omega(u_{|_V}) - \frac{\omega}{n}\,\mathcal{L}^\alpha_{q_\sigma}(u)(\hat{x}) \ \text{ on } V.$$

Moreover, we denote by $\mathcal{G}_\sigma$ the unique self-adjoint positive semidefinite and non positive definite Green function for Γ_α with respect to 0 and σ and by G_σ its associated Green function. The following result establishes the relation between G_σ and the Green function associated with $\mathcal{G}$.

Proposition 5.9 *It is verified that* $G_\sigma(\cdot,\hat{x}) = \dfrac{\lambda_{\mathcal{G}}}{(n+1)^2}\Big((n+1)\,\varepsilon_{\hat{x}} - \sigma\Big)$. *Moreover, if G is the Green function associated with $\mathcal{G}$, then* $G_{\sigma|_{V\times V}} = G - \dfrac{\lambda_{\mathcal{G}}(n+2)}{(n+1)^2}\,\omega \otimes \omega$.

Proof. If $u = G_\sigma(\cdot,\hat{x})$, then $\mathcal{L}^\alpha_{q_\sigma}(u) = \varepsilon_{\hat{x}} - \dfrac{\sigma}{n+1}$, which implies that $\mathcal{L}^\alpha_{q_\sigma}(u)(\hat{x}) = \dfrac{n}{n+1}$ and hence that $\mathcal{L}_q(u_{|_V}) - \lambda\,\mathcal{P}_\omega(u_{|_V}) = 0$ on V. Therefore, $u_{|_V} = a\omega$ where $a \in \mathbb{R}$. Moreover, $u(\hat{x}) = \dfrac{\lambda_{\mathcal{G}}}{n}\,\mathcal{L}^\alpha_{q_\sigma}(u)(\hat{x}) + \dfrac{1}{n}\langle \omega, u_{|_V}\rangle = \dfrac{\lambda_{\mathcal{G}}}{n+1} + a$ and hence $u = a\sigma + \dfrac{\lambda_{\mathcal{G}}}{n+1}\,\varepsilon_{\hat{x}}$. Finally, the condition $\langle \sigma, u\rangle = 0$ implies that $a = -\dfrac{\lambda_{\mathcal{G}}}{(n+1)^2}$ and the first identity follows.

On the other hand, let $y \in V$ and consider $v = G_\sigma(\cdot,y)$. Then, $\mathcal{L}^\alpha_{q_\sigma}(v) = \varepsilon_y - \dfrac{\sigma\,\omega(y)}{n+1}$, which implies that $\mathcal{L}^\alpha_{q_\sigma}(v)(\hat{x}) = -\dfrac{\omega(y)}{n+1}$ and hence that $\mathcal{L}_q(v_{|_V}) - \lambda\,\mathcal{P}_\omega(v_{|_V}) = \varepsilon_y - \dfrac{1}{n}\,\omega\,\omega(y)$ on V. Therefore, $v_{|_V} = G_{\lambda,\omega}(\cdot,y) + a\omega$ where $an = \langle \omega, v_{|_V}\rangle$, since $\langle G_{\lambda,\omega}(\cdot,y), \omega\rangle = 0$. The same reason implies that $0 = \langle \sigma, v\rangle = v(\hat{x}) + \langle \omega, v_{|_V}\rangle$ and hence that $\langle \omega, v_{|_V}\rangle = \dfrac{\lambda_{\mathcal{G}}}{(n+1)^2}\,\omega(y)$. In conclusion, we obtain that $v_{|_V} = G_{\lambda,\omega}(\cdot,y) + \dfrac{\lambda_{\mathcal{G}}}{n(n+1)^2}\,\omega\,\omega(y)$, or equivalently that $v_{|_V} = G(\cdot,y) - \dfrac{\lambda_{\mathcal{G}}(n+2)}{(n+1)^2}\,\omega\,\omega(y)$. ∎

As a by-product of the above proposition we can obtain the expression of the effective resistance, R^α_σ, and the total resistance, r^α_σ, of the network Γ_α with respect to σ in terms of $R_{\lambda,\omega}$ and $r_{\lambda,\omega}$ the effective resistance and the total resistance of Γ with respect to λ and ω.

Corollary 5.10 *For any $x, y \in V$ it is verified that*

$$r^\alpha_\sigma(\hat{x}) = \frac{n\lambda_{\mathcal{G}}}{(n+1)^2}, \qquad r^\alpha_\sigma(x) = r_{\lambda,\omega}(x) + \frac{\lambda_{\mathcal{G}}}{n(n+1)^2},$$

$$R^\alpha_\sigma(x,\hat{x}) = r_{\lambda,\omega}(x) + \frac{\lambda_{\mathcal{G}}}{n}, \qquad R^\alpha_\sigma(x,y) = R_{\lambda,\omega}(x,y).$$

Moreover, the Kirchhoff index of Γ_α with respect to σ is given by $\mathsf{k}(\lambda,\omega) + \dfrac{\lambda_{\mathcal{G}}}{(n+1)}$.

Proof. We know that $r^\alpha_\sigma(x) = \dfrac{G_\sigma(x,x)}{\sigma^2(x)}$, for any $x \in V \cup \{\hat{x}\}$. Therefore, by using the expression for G_σ obtained in the above proposition we get the value for $r^\alpha_\sigma(\hat{x})$ and also that for any $x \in V$,

$$r^\alpha_\sigma(x) = \frac{G(x,x)}{\omega^2(x)} - \frac{\lambda_{\mathcal{G}}(n+2)}{(n+1)^2} = r_{\lambda,\omega}(x) + \frac{\lambda_{\mathcal{G}}}{n} - \frac{\lambda_{\mathcal{G}}(n+2)}{(n+1)^2} = r_{\lambda,\omega}(x) + \frac{\lambda_{\mathcal{G}}}{n(n+1)^2}.$$

On the other hand, Proposition 5.2 establishes that

$$R^\alpha_\sigma(x,y) = \frac{G_\sigma(x,x)}{\sigma^2(x)} + \frac{G_\sigma(y,y)}{\sigma^2(y)} - \frac{2G_\sigma(x,y)}{\sigma(x)\sigma(y)} \quad \text{for any } x, y \in V \cup \{\hat{x}\},$$

which implies that

$$R^\alpha_\sigma(x,\hat{x}) = \frac{G(x,x)}{\omega^2(x)} - \frac{\lambda_{\mathcal{G}}(n+2)}{(n+1)^2} + \frac{n\lambda_{\mathcal{G}}}{(n+1)^2} + \frac{2\lambda_{\mathcal{G}}}{(n+1)^2} = r_{\lambda,\omega}(x) + \frac{\lambda_{\mathcal{G}}}{n} \quad \text{for any } x \in V,$$

and also that

$$R^\alpha_\sigma(x,y) = \frac{G(x,x)}{\omega^2(x)} + \frac{G(y,y)}{\omega^2(y)} - \frac{2G(x,y)}{\omega(x)\omega(y)} = R_{\lambda,\omega}(x,y) \quad \text{for any } x, y \in V.$$

Finally, we know that the Kirchhoff index of Γ_α with respect to σ is given by

$$\langle r^\alpha_\sigma, \sigma^2 \rangle = r^\alpha_\sigma(\hat{x}) + \langle r_{\lambda,\omega}, \omega^2 \rangle + \frac{\lambda_{\mathcal{G}}}{(n+1)^2} = \mathsf{k}(\lambda,\omega) + \frac{\lambda_{\mathcal{G}}}{n+1}. \qquad \blacksquare$$

Theorem 5.11 *Let $\mathcal{G}$ be a self-adjoint and positive definite Green operator for Γ with respect to $\lambda > 0$ and $\omega \in \Omega(V)$ and G its corresponding Green function. If $\lambda_{\mathcal{G}} \geqslant \dfrac{1}{\lambda}$ and we consider $\alpha = \lambda - \dfrac{1}{\lambda_{\mathcal{G}}}$, then $G = G^{\hat{x}}_{\sigma|_{V\times V}}$ and hence for any $x, y \in V$ it is verified that*

$$G(x,y) = \frac{1}{2}\,\omega(x)\,\omega(y)\left(R^\alpha_\sigma(x,\hat{x}) + R^\alpha_\sigma(y,\hat{x}) - R^\alpha_\sigma(x,y)\right).$$

Proof. Proposition 4.11 together with Proposition 5.9 imply that for any $x, y \in V$

$$G_\sigma^{\hat{x}}(x,y) = G_\sigma(x,y) - G_\sigma(x,\hat{x})\omega(y) - G_\sigma(y,\hat{x})\omega(x) + G_\sigma(\hat{x},\hat{x})\,\omega(x)\omega(y)$$
$$= G_\sigma(x,y) + \frac{\lambda_{\mathcal{G}}(n+2)}{(n+1)^2}\,\omega(x)\omega(y) = G(x,y).$$

Now the last identity is a straightforward consequence of Proposition 5.7. ∎

We end the paper with the characterization of the inverse of any irreducible symmetric M-matrix, singular or not, in terms of the effective resistances of suitable networks, or equivalently, we prove that any irreducible symmetric M-matrix is a resistive inverse. These characterizations follow from the main results in this section.

Proposition 5.12 *Let* M *be a singular irreducible and symmetric* M*-matrix of order* n *and* $\mathsf{M}^\dagger = (g_{ij})$ *its Moore-Penrose inverse. Then there exist a network* $\Gamma = (V,c)$ *with* $|V| = n$ *and* $\omega \in \Omega(V)$ *such that* $\mathsf{M} = \mathsf{L}(\omega)$*. Moreover, if* R_{ij}*,* $i,j = 1,\dots,n$ *are the effective resistances of* Γ *with respect to* ω*, then*

$$g_{ij} = -\frac{\omega_i\omega_j}{2}\left(R_{ij} - \frac{1}{n}\sum_{k=1}^{n}(R_{ik}+R_{jk})\omega_k^2 + \frac{1}{n^2}\sum_{k,l=1}^{n} R_{kl}\omega_k^2\omega_l^2\right).$$

Proposition 5.13 *Let* M *be an irreducible Stieltjes matrix of order* n *and* $\mathsf{M}^{-1} = (g_{ij})$ *its inverse. Then there exist a network* $\Gamma = (V,c)$ *with* $|V| = n$*,* $\lambda > 0$ *and* $\omega \in \Omega(V)$ *such that* $\mathsf{M} = \mathsf{L}(\omega) + \lambda\mathsf{I}$*. Moreover, if we consider the extended network* $\hat{\Gamma} = (V \cup \{x_{n+1}\}, \hat{c})$ *and* R_{ij}*,* $i,j = 1,\dots,n+1$ *are the effective resistances of* $\hat{\Gamma}$ *with respect to* $\sigma = (\omega, 1)$*, then*

$$\hat{\mathsf{L}}(\sigma) = \begin{bmatrix} \mathsf{M} & -\mathsf{M}\omega \\ -\omega^*\mathsf{M} & \omega^*\mathsf{M}\omega \end{bmatrix} = \begin{bmatrix} \mathsf{M} & -\lambda\omega \\ -\lambda\omega^* & n\lambda \end{bmatrix}$$

and

$$g_{ij} = \frac{\omega_i\omega_j}{2}\left(R_{in+1} + R_{jn+1} - R_{ij}\right), \quad \text{for any } i,j = 1,\dots,n.$$

Observe that $(-c_{ij}) = \mathsf{M} = \mathsf{L}(\omega) + \lambda\mathsf{I}$ is weakly diagonally dominant iff

$$\lambda \geqslant \frac{1}{\omega_i}\sum_{j=1}^{n} c_{ij}(\omega_i - \omega_j), \quad i = 1,\dots,n$$

with strict inequality for at least one index. So, the result of the above proposition generalizes the main result obtained by M. Fiedler in [11] where the inverses of weakly diagonal dominant Stieltjes matrices are characterized.

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