Classification of b^m-Nambu structures of top degree

Eva Miranda [a] Arnau Planas [b],

[a] *Laboratory of Geometry and Dynamical Systems, Department of Mathematics-UPC and BGSMath in Barcelona and CEREMADE (Université de Paris Dauphine)-IMCCE (Observatoire de Paris)- IMJ (Université de Paris Diderot) in Paris, Postal address: Observatoire de Paris, 77 Avenue Denfert Rochereau, 75014, Paris, France.*

[b] *Laboratory of Geometry and Dynamical Systems, Department of Mathematics-UPC, Barcelona*

Received *****; accepted after revision +++++

Presented by

Abstract

In this paper we classify b^m-Nambu structures via b^m-cohomology. The complex of b^m-forms is an extension of De Rham complex which allows to consider *singular* forms. b^m-Cohomology is well-understood thanks to Scott [12] and it can be expressed in terms of De Rham cohomology of the manifold and the critical hypersurface using a Mazzeo-Melrose-type formula. Each of the terms in b^m-Mazzeo-Melrose formula acquires a geometrical interpretation in this classification. We also give equivariant versions of this classification scheme. *To cite this article: A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

Résumé

Classification de structures b^m-Nambu de degré maximal

On classifie les structures b^m-Nambu de degré maximal en utilisant la b^m-cohomologie. Le complexe des b^m-formes est une extension du complexe de De Rham et permet considérer des formes *singulières.* La b^m-cohomologie est bien comprise grâce à Scott [12] et elle peut être exprimée en termes de la cohomologie de De Rham de la variété et de l'hypersurface critique en utilisant une formule de type Mazzeo-Melrose. Chacun des termes dans la formule de b^m-Mazzeo-Melrose acquiert une interpretation géométrique dans cette classification. On donne aussi des versions équivariantes des théorèmes de classification.

Pour citer cet article : A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).

Email addresses: `eva.miranda@upc.edu, Eva.Miranda@obspm.fr` (Eva Miranda), `arnau.planas@upc.edu` (Arnau Planas).

1 . Eva Miranda is supported by the Catalan Institution for Research and Advanced Studies via an ICREA Academia 2016 Prize, by a *Chaire d'Excellence* de la Fondation Sciences Mathématiques de Paris, and partially supported by the Ministerio de Economía y Competitividad project with reference MTM2015-69135-P/FEDER and by the Generalitat de Catalunya project with reference 2014SGR634. This work is supported by a public grant overseen by the French National Research Agency (ANR) as part of the *"Investissements d'Avenir"* program (reference : ANR-10-LABX-0098). Arnau Planas is partially supported by the project MTM2015-69135-P/FEDER and 2014SGR634.

1. Introduction

In this article we focus our attention on b^m-Nambu structures. Nambu structures were introduced by Nambu [11] and Takhtajan [13] as a generalization of Poisson structures. Unlike the domain of Poisson Geometry, Nambu geometry is not so well-explored. In this short note we give a classification theorem for a class of Nambu structures using a generalization of De Rham cohomology called b^m-cohomology. Our result generalizes a former classification theorem by Martínez-Torres for generic Nambu structures of top degree [8].

Recently a class of Poisson structures called in the literature b-Poisson structures (see for instance, [3],[4],[6] and [2]) has been widely studied. A b-Poisson manifold is an even dimensional Poisson manifold (M^{2n}, Π) where the Poisson structure Π satisfies the following transversality condition: Π^n cuts the zero section of the bundle $\Lambda^{2n}T(M^{2n})$ transversally. As a consequence the vanishing set of Π^n is a smooth submanifold of codimension 1 which is called *critical hypersurface*.

The transversality condition can be relaxed in a way the critical hypersurface is still a smooth submanifold. This is the case of b^m-Poisson manifolds introduced by Scott [12]. In this paper we generalize this setting to the Nambu world and classify these structures. This class of singular Nambu structures was already considered by Arnold in [1]. The classification theorem we prove here is an extension of Moser's classification theorem [10] for volume forms on a manifold. As an outcome of this classification scheme a geometrical interpretation is given to the Mazzeo-Melrose decomposition theorem (see section 2.16 in [9] for m=1 and [12] for general m) which expresses the b^m-cohomology in terms of the classical De Rham cohomology groups of the manifold and the critical hypersurface.

2. Constructions and classification of b^m-Nambu structures

Nambu structures of b^m-type can be described using forms which are singular along a smooth hypersurface. These forms, called b^m-forms, were studied by Scott [12] in his thesis. We start introducing the language of b^m-forms: We follow [12] for these definitions and main properties. The set-up in Scott [12] allows to consider smooth hypersurfaces without a globally defining function. For the sake of simplicity in this paper we will consider Z a smooth hypersurfaces (not necessarily connected) and attach to it a defining function f.

Take a local set of coordinates $(x, \ldots x_{n-1})$ in a neighborhood of a point p in the critical set, the b^m-tangent bundle can be defined as the bundle whose sections are locally generated by:

$$\{x^m \frac{\partial}{\partial x}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}\}, \tag{1}$$

with x such that $|x| = \lambda$, and λ is the distance function to z. For globally defining functions $f = x$.

As done in the case $m = 1$ in [3] we can define the dual bundle, the b^m-cotangent bundle ${}^{b^m}T^*(M)$. Sections of powers of these bundles are called b^m-forms.

A **Laurent Series** of a closed b^m-form ω is a decomposition of ω in a tubular neighborhood U of the critical set Z of the form

$$\omega = \frac{dx}{x^m} \wedge (\sum_{i=0}^{m-1} \pi^*(\alpha_i)x^i) + \beta \tag{2}$$

with $\pi : U \to Z$ the projection of the tubular neighborhood onto Z, α_i a closed smooth De Rham form on Z and β a De Rham form on M.

In [12] it is proved that in a neighborhood of Z, every closed b^m-form ω can be written in a Laurent form of type (2) once a defining function has been fixed.

The complex of b^m-forms endowed with a natural extension of De Rham differential defines b^m-Cohomology. The follow theorem tells us that b^m-cohomology can be read off from de Rham cohomology thus generalizing the classical Mazzeo-Melrose decomposition theorem in Section 2.16 in [9]:

Theorem 2.1 (b^m-Mazzeo-Melrose, [12]) *The b^m-cohomology groups can be determined from De Rham cohomology groups as follows:*

$$^{b^m}H^p(M) \cong H^p(M) \oplus (H^{p-1}(Z))^m. \tag{3}$$

We now introduce b^m-Nambu structures of top degree,

Definition 2.2 *A b^m-Nambu structure of top degree on a pair (M^n, Z) with Z a smooth hypersurface is given by a smooth n-multivector field Λ such that there exists a local system of coordinates for which*

$$\Lambda = x_1^m \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n} \tag{4}$$

and Z is defined by $x_1 = 0$ in a neighborhood of Z.

Dualizing the local expression of the Nambu structure we obtain the form

$$\Theta = \frac{1}{x_1^m} dx_1 \wedge \ldots \wedge dx_n \tag{5}$$

(which is not a smooth de Rham form), but it is a b^m-form of degree n defined on a b^m-manifold. As it is done in [3], we can check that this dual form is non-degenerate. So we may define a b^m-Nambu form as follows.

Mimicking the same condition as for b^m-symplectic forms we can talk about non-degenerate b^m-forms of top degree. This means that seen as a section of $\Lambda^n(^bT^*M)$ the form does not vanish.

Notation: We will denote by Λ the Nambu multivectorfield and by Θ its dual.

Definition 2.3 *A b^m-Nambu form is a non-degenerate b^m-form of top degree.*

We first include a collection of motivating examples, and then prove an equivariant classification theorem.

2.1. *Examples*

(i) **b^m-symplectic surfaces:** Any b^m-symplectic surface is a b^m-Nambu manifold with Nambu structure of top degree.

(ii) **b^m-symplectic manifolds as b^m-Nambu manifolds:** Let (M^{2n}, ω) be a b^m-symplectic manifold, then $(M^{2n}, \underbrace{\omega \wedge \ldots \wedge \omega}_{n})$ is automatically b^m-Nambu.

(iii) **Orientable manifolds:** Let (M^n, Ω) be any orientable manifold (with Ω a volume form) and let f be a defining function for Z, then $(1/f^m)\Omega$ defines a b^m-Nambu structure of top degree having Z as critical set.

Any Nambu structure can be written in this way if the hypersurface can be globally described as the vanishing set of a smooth function.

(iv) **Spheres:** In [8], it was given special importance to the example $(S^n, \sqcup_i S_i^{(n-1)})$ because of the Schoenflies theorem [2], which imposes the associated graph to be a tree. The nice feature of this ex-

2. The nature of this theorem is purely topological in dimension equal or greater than four, and so is its construction.

ample is that $O(n)$ acts on the b^m-manifold $(S^n, S^{(n-1)})$, and it makes sense to consider its classification under these symmetries. This also works for other homogeneous spaces of type $(G_1/G_2, G_2/G_3)$ with G_2 and G_3 with codimension 1 in G_1 and G_2 respectively.

2.2. *b^m-Nambu structures of top degree and orientability*

We start proving:

Theorem 2.4 *A compact n-dimensional manifold M admitting a b^{2k}-Nambu structure is orientable.*

Proof: Consider a collar of charts for the b^{2k}-Nambu structure such that in local coordinates the Nambu structure can be written as $x_1^{2k}\frac{\partial}{\partial x_1}\wedge\ldots\wedge\frac{\partial}{\partial x_n}$ with compatible orientations in a neighborhood of each connected component of Z.

Consider a 2:1 orientable covering $(\tilde{M}, \tilde{Z})$ of the manifold and denote by $\rho : \mathbb{Z}/2\mathbb{Z}\times\tilde{M}\to\tilde{M}$ the deck transformation. For each point $p\in\tilde{Z}$ take a neighborhood U_p which does not contain other points identified by ρ thus $U_p\cong\pi(U_p) =: V_p$, and $\Theta = \frac{1}{x^{2k}}dx_1\wedge\ldots\wedge dx_n$. This form defines an orientation on $V_p\setminus\pi(Z)$. Take a symmetric covering of such neighborhoods to define a collar of Z with compatible orientations, and compatible with the covering. The compatible orientations and the symmetric coverings descend to (M, Z), thus defining an orientation in (M, Z). Thus, we have an orientation in $V\setminus Z$. By perturbing Θ in V we obtain a volume form on V, $\tilde{\omega}$, and thus an orientation in V. These can be glued to define an orientation via the volume form $\tilde{\Theta}$ on the whole M proving that M is oriented.

2.3. *Classification of b^m-Nambu structures of top degree and b^m-cohomology*

We present the definitions contained in [8] of modular period attached to the connected component of an orientable Nambu structure using the language of b^m-forms.

Let Θ be the dual to the multivectorfield Λ defining a Nambu structure. From the general decomposition of b^m-forms as it was set in Equation 2 we may write:

$$\Theta = \Theta_0\wedge\frac{df}{f^m}$$

with $\Theta_0\in\Omega^{n-1}(M)$.

This decomposition is valid in a neighborhood of Z whenever the defining function is well-defined. For non-orientable manifolds a similar decomposition can be proved by replacing the defining function f by an adapted distance (see [7]).

With this language in mind, the the **modular $(n-1)$-vector field** in [8] of Θ along Z is the dual of the form Θ_0 in the decomposition above which is indeed the **modular $(n-1)$-form** along Z in [8].

Recall from [8] in our language:

Definition 2.5 *The **modular period** T_Λ^Z of the component Z of the zero locus of Λ is*

$$T_\Lambda^Z := \int_Z \Theta_0 > 0.$$

In fact, this positive number determines the Nambu structure in a neighborhood of Z up to isotopy as it was proved in [8].

The following theorem gives a classification of b^m-Nambu structures.

Theorem 2.6 *Let Θ_0 and Θ_1 be two b^m-Nambu forms of degree n on a compact orientable manifold M^n. If $[\Theta_0] = [\Theta_1]$ in b^m-cohomology then there exists a diffeomorphism ϕ such that $\phi^*\Theta_1 = \Theta_0$.*

Proof: We will apply the techniques of [10] with the only difference that we work with b^m-volume forms instead of volume forms.

Since Θ_0 and Θ_1 are non-degenerate b^m-forms both of them are a multiple of a volume form and thus the linear path $\Theta_t = (1-t)\Theta_0 + t\Theta_1$ is a path of non-degenerate b^m-forms.

Because Θ_0 and Θ_1 determine the same cohomology class:

$$\Theta_1 - \Theta_0 = d\beta$$

with d the b^m-De Rham differential and β a b^m-form of degree $n-1$.

Now consider the Moser equation:

$$\iota_{X_t}\Theta_t = -\beta. \tag{6}$$

Observe that since β is a b^m-form and Θ_t is non-degenerate. The vector field X_t is a b^m-vector field. Let ϕ_t be the t-dependent flow integrating X_t.

The ϕ_t gives the desired diffeomorphism $\phi_t : M \to M$, leaving Z invariant (since X_t is tangent to Z) and $\phi_t^*\Theta_t = \Theta_0$.

In particular we recover the classification of b-Nambu structures of top degree in [8]:

Theorem 2.7 (Classification of b-Nambu structures of top degree, [8]) *A generic b-Nambu structure Θ is determined, up to orientation preserving diffeomorphism, by the following three invariants: the diffeomorphism type of the oriented pair (M, Z), the modular periods and the regularized Liouville volume.*

By Theorem 2.1,

$$^bH^n(M) \cong H^n(M) \oplus H^{n-1}(Z).$$

The first term on the right hand side is the Liouville volume image by the De Rham theorem, as it was done in [4] for b-symplectic forms. The second term collects the periods of the modular vector field. So if the three invariants coincide then they determine the same b-cohomology class.

In other words, the statement in [8] is equivalent to the following theorem in the language of b-cohomology.

Theorem 2.8 *Let Θ_1 and Θ_2 be two b-Nambu forms on an orientable manifold M. If $[\Theta_1] = [\Theta_2]$ in b-cohomology then there exists a diffeomorphism ϕ such that $\phi^*\Theta_1 = \Theta_2$.*

This global Moser theorem for b^m-Nambu structures admits an equivariant version,

Theorem 2.9 *Let Θ_0 and Θ_1 be two b^m-Nambu forms of degree n on a compact orientable manifold M^n and let $\rho : G \times M \longrightarrow M$ be a compact Lie group action preserving both b^m-forms. If $[\Theta_0] = [\Theta_1]$ in b^m-cohomology then there exists an equivariant diffeomorphism ϕ such that $\phi^*\Theta_1 = \Theta_0$.*

Proof: As in the former proof, write

$$\Theta_1 - \Theta_0 = d\beta$$

with d the b^m-De Rham differential and β a b^m-form of degree $n-1$. Observe that the path $\Theta_t = (1-t)\Theta_0 + t\Theta_1$ is a path of invariant b^m-forms.

Now consider Moser's equation:

$$\iota_{X_t}\Theta_t = -\beta. \tag{7}$$

Since Θ_t is invariant we can find an invariant $\tilde{\beta}$. For instance take $\tilde{\beta} = \int_G \rho_g^*(\beta)d\mu$ with μ a de Haar measure on G and ρ_g the induced diffeomorphism $\rho_g(x) := \rho(g, x)$.

Now replace β by $\tilde{\beta}$ to obtain,

$$\iota_{X_t^G}\Theta_t = -\tilde{\beta} \tag{8}$$

with $X_t^G = \int_G \rho_{g_*} X_t d\mu$. The vector field X_t^G is an invariant b-vector field. Its flow ϕ_t^G preserves the action and $\phi_t^{G*}\Theta_t = \Theta_0$.

Playing the equivariant b^m-Moser trick using the 2:1 cover of a non-orientable manifold and taking as G the group of deck transformations we obtain,

Corollary 2.10 *Let Θ_0 and Θ_1 be two b^m-Nambu forms of degree n on a manifold M^n (not necessarily oriented). If $[\Theta_0] = [\Theta_1]$ in b^m-cohomology then there exists a diffeomorphism ϕ such that $\phi^*\Theta_1 = \Theta_0$.*

References

[1] Arnold, V. I. *Remarks on Poisson structures on a plane and on other powers of volume elements.* (Russian) Trudy Sem. Petrovsk. No. 12 (1987), 37–46, 242; translation in J. Soviet Math. 47 (1989), no. 3, 2509–2516.

[2] A. Delshams, A. Kiesenhofer and E. Miranda, *Examples of integrable and non-integrable systems on singular symplectic manifolds*, Journal of Geometry and Physics (2016), in press, doi:10.1016/j.geomphys.2016.06.011.

[3] V. Guillemin, E. Miranda, A.R. Pires, *Symplectic and Poisson geometry on b-manifolds*, *Adv. Math.* **264**, pp. 864-896 (2014).

[4] V. Guillemin, E. Miranda, A. R. Pires and G. Scott, *Toric actions on b-symplectic manifolds*, Int Math Res Notices (2015) 2015 (14): 5818-5848. doi: 10.1093/imrn/rnu108 .

[5] V. Guillemin, E. Miranda and J. Weitsman, *Desingularizing b^m-symplectic structures*, International Mathematics Research Notices, rnx126, https://doi.org/10.1093/imrn/rnx126. .

[6] A. Kiesenhofer and E. Miranda, *Cotangent Models for Integrable Systems.* Comm. Math. Phys. 350 (2017), no. 3, 1123–1145.

[7] I. Marcut and B. Osorno, *Deformations of log-symplectic structures* , J. Lond. Math. Soc. (2) 90 (2014), no. 1, 197–212.

[8] D. Martinez-Torres, *Global classification of generic multi-vector fields of top degree*, Journal of the London Mathematical Society, 69:3, 2004, pp. 751–766.

[9] R. Melrose, *Atiyah-Patodi-Singer Index Theorem* (book), Research Notices in Mathematics, A.K. Peters, Wellesley, 1993.

[10] J. Moser, *On the volume elements on a manifold.* Trans. Amer. Math. Soc. 120 1965 286–294.

[11] Y. Nambu, *Generalized Hamiltonian Dynamics* , Physical Review D, 7 8, 2405-2412 (1973).

[12] G. Scott, *The Geometry of b^k Manifolds*, J. Symplectic Geom. 14 (2016), no. 1, 71–95.

[13] L. Takhtajan, *On Foundation of the Generalized Nambu Mechanics* , Commun. Math Phys, 160 , 295–315 (1994).