Classification of $b^m$-Nambu structures of top degree

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Abstract

In this paper we classify $b^m$-Nambu structures via $b^m$-cohomology. The complex of $b^m$-forms is an extension of De Rham complex which allows to consider singular forms. $b^m$-Cohomology is well-understood thanks to Scott [12] and it can be expressed in terms of De Rham cohomology of the manifold and the critical hypersurface using a Mazzeo-Melrose-type formula. Each of the terms in $b^m$-Mazzeo-Melrose formula acquires a geometrical interpretation in this classification. We also give equivariant versions of this classification scheme. To cite this article: A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).

Résumé

Classification de structures $b^m$-Nambu de degré maximal

On classe les structures $b^m$-Nambu de degré maximal en utilisant la $b^m$-cohomologie. Le complexe des $b^m$-formes est une extension du complexe de De Rham et permet considérer des formes singulières. La $b^m$-cohomologie est bien comprise grâce à Scott [12] et elle peut être exprimée en termes de la cohomologie de De Rham de la variété et de l'hypersurface critique en utilisant une formule de type Mazzeo-Melrose. Chacun des termes dans la formule de $b^m$-Mazzeo-Melrose acquiert une interprétation géométrique dans cette classification. On donne aussi des versions équivariantes des théorèmes de classification.

1. Introduction

In this article we focus our attention on \( b^m \)-Nambu structures. Nambu structures were introduced by Nambu [11] and Takhtajan [13] as a generalization of Poisson structures. Unlike the domain of Poisson Geometry, Nambu geometry is not so well-explored. In this short note we give a classification theorem for a class of Nambu structures using a generalization of De Rham cohomology called \( b^m \)-cohomology. Our result generalizes a former classification theorem by Martínez-Torres for generic Nambu structures of top degree [8].

Recently a class of Poisson structures called in the literature \( b \)-Poisson structures (see for instance, [3],[4],[6] and [2]) has been widely studied. A \( b \)-Poisson manifold is an even dimensional Poisson manifold \((M^{2n}, \Pi)\) where the Poisson structure \( \Pi \) satisfies the following transversality condition: \( \Pi \) cuts the zero section of the bundle \( \Lambda^{2n}T(M^{2n}) \) transversally. As a consequence the vanishing set of \( \Pi \) is a smooth submanifold of codimension 1 which is called critical hypersurface.

The transversality condition can be relaxed in a way the critical hypersurface is still a smooth submanifold. This is the case of \( b^m \)-Poisson manifolds introduced by Scott [12]. In this paper we generalize this setting to the Nambu world and classify these structures. This class of singular Nambu structures was already considered by Arnold in [1]. The classification theorem we prove here is an extension of Moser’s classification theorem [10] for volume forms on a manifold. As an outcome of this classification scheme a geometrical interpretation is given to the Mazzeo-Melrose decomposition theorem (see section 2.16 in [9] for \( m=1 \) and [12] for general \( m \)) which expresses the \( b^m \)-cohomology in terms of the classical De Rham cohomology groups of the manifold and the critical hypersurface.

2. Constructions and classification of \( b^m \)-Nambu structures

Nambu structures of \( b^m \)-type can be described using forms which are singular along a smooth hypersurface. These forms, called \( b^m \)-forms, were studied by Scott [12] in his thesis. We start introducing the language of \( b^m \)-forms: We follow [12] for these definitions and main properties. The set-up in Scott [12] allows to consider smooth hypersurfaces without a globally defining function. For the sake of simplicity in this paper we will consider \( Z \) a smooth hypersurface (not necessarily connected) and attach to it a defining function \( f \).

Take a local set of coordinates \((x, \ldots x_{n-1})\) in a neighborhood of a point \( p \) in the critical set, the \( b^m \)-tangent bundle can be defined as the bundle whose sections are locally generated by:

\[
\{x^m \frac{\partial}{\partial x}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}\},
\]

with \( x \) such that \( |x| = \lambda \) and \( \lambda \) is the distance function to \( z \). For globally defining functions \( f = x \).

As done in the case \( m = 1 \) in [3] we can define the dual bundle, the \( b^m \)-cotangent bundle \( b^m T^\ast(M) \). Sections of powers of these bundles are called \( b^m \)-forms.

A Laurent Series of a closed \( b^m \)-form \( \omega \) is a decomposition of \( \omega \) in a tubular neighborhood \( U \) of the critical set \( Z \) of the form

\[
\omega = \frac{dx}{x^m} \wedge \left( \sum_{i=0}^{m-1} \pi^\ast(\alpha_i)x^i \right) + \beta
\]

with \( \pi : U \to Z \) the projection of the tubular neighborhood onto \( Z \), \( \alpha_i \) a closed smooth De Rham form on \( Z \) and \( \beta \) a De Rham form on \( M \).
In [12] it is proved that in a neighborhood of $Z$, every closed $b^m$-form $\omega$ can be written in a Laurent form of type (2) once a defining function has been fixed.

The complex of $b^m$-forms endowed with a natural extension of De Rham differential defines $b^m$-cohomology. The follow theorem tells us that $b^m$-cohomology can be read off from de Rham cohomology thus generalizing the classical Mazzeo-Melrose decomposition theorem in Section 2.16 in [9]:

**Theorem 2.1 (b$^m$-Mazzeo-Melrose, [12])** The $b^m$-cohomology groups can be determined from De Rham cohomology groups as follows:

$$b^m H^p(M) \cong H^p(M) \oplus (H^{p-1}(Z))^m.$$ (3)

We now introduce $b^m$-Nambu structures of top degree.

**Definition 2.2** A $b^m$-Nambu structure of top degree on a pair $(M^n, Z)$ with $Z$ a smooth hypersurface is given by a smooth $n$-multivector field $\Lambda$ such that there exists a local system of coordinates for which

$$\Lambda = x_1^m \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n}$$ (4)

and $Z$ is defined by $x_1 = 0$ in a neighborhood of $Z$.

Dualizing the local expression of the Nambu structure we obtain the form

$$\Theta = \frac{1}{x_1^m} dx_1 \wedge \ldots \wedge dx_n$$ (5)

(which is not a smooth de Rham form), but it is a $b^m$-form of degree $n$ defined on a $b^m$-manifold. As it is done in [3], we can check that this dual form is non-degenerate. So we may define a $b^m$-Nambu form as follows.

**Notation:** We will denote by $\Lambda$ the Nambu multivectorfield and by $\Theta$ its dual.

**Definition 2.3** A $b^m$-Nambu form is a non-degenerate $b^m$-form of top degree.

We first include a collection of motivating examples, and then prove an equivariant classification theorem.

### 2.1. Examples

(i) **$b^m$-symplectic surfaces:** Any $b^m$-symplectic surface is a $b^m$-Nambu manifold with Nambu structure of top degree.

(ii) **$b^m$-symplectic manifolds as $b^m$-Nambu manifolds:** Let $(M^n, \omega)$ be a $b^m$-symplectic manifold, then $(M^n, \omega \wedge \ldots \wedge \omega)$ is automatically $b^m$-Nambu.

(iii) **Orientable manifolds:** Let $(M^n, \Omega)$ be any orientable manifold (with $\Omega$ a volume form) and let $f$ be a defining function for $Z$, then $(1/f^m)\Omega$ defines a $b^m$-Nambu structure of top degree having $Z$ as critical set.

Any Nambu structure can be written in this way if the hypersurface can be globally described as the vanishing set of a smooth function.

(iv) **Spheres:** In [8], it was given special importance to the example $(S^n, \cup_i S^{(n-1)}_i)$ because of the Schoenflies theorem $^2$, which imposes the associated graph to be a tree. The nice feature of this ex-

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$^2.$ The nature of this theorem is purely topological in dimension equal or greater than four, and so is its construction.

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ample is that $O(n)$ acts on the $b^m$-manifold $(S^n, S^{(n-1)})$, and it makes sense to consider its classification under these symmetries. This also works for other homogeneous spaces of type $(G_1/G_2, G_2/G_3)$ with $G_2$ and $G_3$ with codimension 1 in $G_1$ and $G_2$ respectively.

2.2. $b^m$-Nambu structures of top degree and orientability

We start proving:

**Theorem 2.4** A compact $n$-dimensional manifold $M$ admitting a $b^{2k}$-Nambu structure is orientable.

**Proof:** Consider a collar of charts for the $b^{2k}$-Nambu structure such that in local coordinates the Nambu structure can be written as $x_1^{2k} \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n}$ with compatible orientations in a neighborhood of each connected component of $Z$.

Consider a 2:1 orientable covering $(\tilde{M}, \tilde{Z})$ of the manifold and denote by $\rho: \tilde{Z}/\mathbb{Z} \times \tilde{M} \to \tilde{M}$ the deck transformation. For each point $p \in \tilde{Z}$ take a neighborhood $U_p$ which does not contain other points identified by $\rho$ thus $U_p \cong \pi(U_p) =: V_p$, and $\Theta = \frac{1}{fx} dx_1 \wedge \ldots \wedge dx_n$. This form defines an orientation on $V_p \setminus \pi(Z)$. Take a symmetric covering of such neighborhoods to define a collar of $Z$ with compatible orientations, and compatible with the covering. The compatible orientations and the symmetric coverings descend to $(M, Z)$, thus defining an orientation in $(M, Z)$. Thus, we have an orientation in $V \setminus Z$. By perturbing $\Theta$ in $V$ we obtain a volume form on $V$, $\tilde{\omega}$, and thus an orientation in $V$. These can be glued to define an orientation via the volume form $\Theta$ on the whole $M$ proving that $M$ is oriented.

2.3. Classification of $b^m$-Nambu structures of top degree and $b^m$-cohomology

We present the definitions contained in [8] of modular period attached to the connected component of an orientable Nambu structure using the language of $b^m$-forms.

Let $\Theta$ be the dual to the multivectorfield $\Lambda$ defining a Nambu structure. From the general decomposition of $b^m$-forms as it was set in Equation 2 we may write:

$$\Theta = \Theta_0 \wedge \frac{df}{f^m}$$

with $\Theta_0 \in \Omega^{n-1}(M)$.

This decomposition is valid in a neighborhood of $Z$ whenever the defining function is well-defined. For non-orientable manifolds a similar decomposition can be proved by replacing the defining function $f$ by an adapted distance (see [7]).

With this language in mind, the the **modular** $(n-1)$-vector field in [8] of $\Theta$ along $Z$ is the dual of the form $\Theta_0$ in the decomposition above which is indeed the **modular** $(n-1)$-form along $Z$ in [8].

Recall from [8] in our language:

**Definition 2.5** The **modular period** $T^Z_{\Lambda}$ of the component $Z$ of the zero locus of $\Lambda$ is

$$T^Z_{\Lambda} := \int_Z \Theta_0 > 0.$$ 

In fact, this positive number determines the Nambu structure in a neighborhood of $Z$ up to isotopy as it was proved in [8].

The following theorem gives a classification of $b^m$-Nambu structures.

**Theorem 2.6** Let $\Theta_0$ and $\Theta_1$ be two $b^m$-Nambu forms of degree $n$ on a compact orientable manifold $M^n$. If $[\Theta_0] = [\Theta_1]$ in $b^m$-cohomology then there exists a diffeomorphism $\phi$ such that $\phi^* \Theta_1 = \Theta_0$. 


**Proof:** We will apply the techniques of [10] with the only difference that we work with $b^m$-volume forms instead of volume forms.

Since $\Theta_0$ and $\Theta_1$ are non-degenerate $b^m$-forms both of them are a multiple of a volume form and thus the linear path $\Theta_t = (1-t)\Theta_0 + t\Theta_1$ is a path of non-degenerate $b^m$-forms.

Because $\Theta_0$ and $\Theta_1$ determine the same cohomology class:

$$\Theta_1 - \Theta_0 = d\beta$$

with $d$ the $b^m$-De Rham differential and $\beta$ a $b^m$-form of degree $n - 1$.

Now consider the Moser equation:

$$\iota_{X_t} \Theta_t = -\beta. \quad (6)$$

Observe that since $\beta$ is a $b^m$-form and $\Theta_t$ is non-degenerate. The vector field $X_t$ is a $b^m$-vector field.

Let $\phi_t$ be the $t$-dependent flow integrating $X_t$. The $\phi_t$ gives the desired diffeomorphism $\phi_t : M \to M$, leaving $Z$ invariant (since $X_t$ is tangent to $Z$) and $\phi_t^* \Theta_t = \Theta_0$.

In particular we recover the classification of $b$-Nambu structures of top degree in [8]:

**Theorem 2.7 (Classification of $b$-Nambu structures of top degree, [8]):** A generic $b$-Nambu structure $\Theta$ is determined, up to orientation preserving diffeomorphism, by the following three invariants: the diffeomorphism type of the oriented pair $(\Theta, t)$, the modular periods and the regularized Liouville volume. By Theorem 2.1,

$$H^n(M) \cong H^n(M) \oplus H^{n-1}(Z).$$

The first term on the right hand side is the Liouville volume image by the De Rham theorem, as it was done in [4] for $b$-symplectic forms. The second term collects the periods of the modular vector field. So if the three invariants coincide then they determine the same $b$-cohomology class.

In other words, the statement in [8] is equivalent to the following theorem in the language of $b$-cohomology.

**Theorem 2.8** Let $\Theta_1$ and $\Theta_2$ be two $b$-Nambu forms on an orientable manifold $M$. If $[\Theta_1] = [\Theta_2]$ in $b$-cohomology then there exists a diffeomorphism $\phi$ such that $\phi^* \Theta_1 = \Theta_2$.

This global Moser theorem for $b^m$-Nambu structures admits an equivariant version,

**Theorem 2.9** Let $\Theta_0$ and $\Theta_1$ be two $b^m$-Nambu forms of degree $n$ on a compact orientable manifold $M^n$ and let $\rho : G \times M \to M$ be a compact Lie group action preserving both $b^m$-forms. If $[\Theta_0] = [\Theta_1]$ in $b^m$-cohomology then there exists an equivariant diffeomorphism $\phi$ such that $\phi^* \Theta_1 = \Theta_0$.

**Proof:** As in the former proof, write

$$\Theta_1 - \Theta_0 = d\beta$$

with $d$ the $b^m$-De Rham differential and $\beta$ a $b^m$-form of degree $n - 1$. Observe that the path $\Theta_t = (1-t)\Theta_0 + t\Theta_1$ is a path of invariant $b^m$-forms.

Now consider Moser’s equation:

$$\iota_{X_t^G} \Theta_t = -\beta. \quad (7)$$

Since $\Theta_t$ is invariant we can find an invariant $\tilde{\beta}$. For instance take $\tilde{\beta} = \int_G \rho^*_g(\beta)d\mu$ with $\mu$ a de Haar measure on $G$ and $\rho_g$ the induced diffeomorphism $\rho_g(x) := \rho(g, x)$.

Now replace $\beta$ by $\tilde{\beta}$ to obtain,

$$\iota_{X_t^G} \Theta_t = -\tilde{\beta} \quad (8)$$

with $X_t^G = \int_G \rho^*_g X_t d\mu$. The vector field $X_t^G$ is an invariant $b$-vector field. Its flow $\phi_t^G$ preserves the action and $\phi_t^G \Theta_t = \Theta_0$. 

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Playing the equivariant $b^n$-Moser trick using the 2:1 cover of a non-orientable manifold and taking as $G$ the group of deck transformations we obtain,

**Corollary 2.10** Let $\Theta_0$ and $\Theta_1$ be two $b^n$-Nambu forms of degree $n$ on a manifold $M^n$ (not necessarily oriented). If $[\Theta_0] = [\Theta_1]$ in $b^m$-cohomology then there exists a diffeomorphism $\phi$ such that $\phi^*\Theta_1 = \Theta_0$.

**References**


