Abstract

Let \( E \) be an elliptic curve over \( \mathbb{Q} \) and let \( \varrho \) be an odd, irreducible two-dimensional Artin representation. This article proves the Birch and Swinnerton-Dyer conjecture in analytic rank zero for the Hasse-Weil-Artin \( L \)-series \( L(E, \varrho, s) \), namely, the implication

\[
L(E, \varrho, 1) \neq 0 \implies (E(H) \otimes \varrho)^{\text{Gal}(H/\mathbb{Q})} = 0,
\]

where \( H \) is the finite extension of \( \mathbb{Q} \) cut out by \( \varrho \). The proof relies on \( p \)-adic families of global Galois cohomology classes arising from Beilinson-Flach elements in a tower of products of modular curves.

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Introduction

This paper completes the project undertaken in [BDR12] of proving the Birch and Swinnerton-Dyer conjecture in analytic rank 0 for the \( \varrho \)-isotypic components of Mordell-Weil groups of elliptic curves, when \( \varrho \) is an odd, irreducible two-dimensional Artin representation.

In the early 90’s, Kato unveiled a new approach to the Birch–Swinnerton-Dyer conjecture and to the cyclotomic Iwasawa main conjecture for (modular) elliptic curves, resting on Beilinson elements in the second \( K \)-groups of modular curves and on their \( p \)-adic deformations. This approach led to a proof of the implication

\[ L(E, \chi, 1) \neq 0 \Rightarrow \text{Hom}_{G \mathbb{Q}}(\chi, E(\bar{\mathbb{Q}}) \otimes \mathbb{C}) = 0, \]

for all elliptic curves \( E \) over \( \mathbb{Q} \) and all Dirichlet characters \( \chi \) (viewed as Galois characters via class field theory). The article [BCDDPR] places Kato’s strategy in the broader framework of “Euler systems of Garrett-Rankin-Selberg type”. Roughly speaking, the global objects exploited by Kato are distinguished elements in the higher Chow group \( \text{CH}^2(X_1(Np^s), 2) \) attached to a pair of modular units, whose logarithmic derivatives are Eisenstein series of weight two. Replacing either one, or both, of these Eisenstein series by cuspidal eigenforms leads to the study of Beilinson-Flach elements in \( \text{CH}^2(X_1(Np^s)^2, 1) \) and of Gross-Kudla-Schoen diagonal cycles in the Chow group \( \text{CH}^2(X_1(Np^s)^3) \) of codimension two cycles on the triple product of \( X_1(Np^s) \). Chapters 2.2 and 2.3. of [BCDDPR] explain how the images of these elements under \( p \)-adic étale regulators and Abel-Jacobi maps, when made to vary in \( p \)-adic families and specialised at suitable classical weight one points, might be parlayed into a proof of the implications

\[
\begin{align*}
(0.1) \quad L(E, \varrho, 1) \neq 0 & \Rightarrow \text{Hom}_{G \mathbb{Q}}(\varrho, E(\bar{\mathbb{Q}}) \otimes \mathbb{C}) = 0, \\
(0.2) \quad L(E, \varrho_1 \otimes \varrho_2, 1) \neq 0 & \Rightarrow \text{Hom}_{G \mathbb{Q}}(\varrho_1 \otimes \varrho_2, E(\bar{\mathbb{Q}}) \otimes \mathbb{C}) = 0,
\end{align*}
\]

where \( \varrho, \varrho_1 \), and \( \varrho_2 \) are odd, irreducible two-dimensional (complex) Artin representations for which \( \varrho_1 \otimes \varrho_2 \) is self-dual. The details of this program are carried out in [DR2] to obtain the proof of (0.2). The goal of the present work is to deliver on the other promise made in [BCDDPR] by proving the following result:

**Theorem A.** Let \( E \) be an elliptic curve over \( \mathbb{Q} \) and let \( \varrho \) be an odd, irreducible, two-dimensional Artin representation. Assume that the conductors of \( E \) and \( \varrho \) are prime to each other. Then (0.1) holds.
Prior to Theorem A, the only irreducible two-dimensional Artin representations for which (0.1) had been proved were those induced from ring class characters of

(i) imaginary quadratic fields, by a series of works building on the methods of Kolyvagin [Ko89];
(ii) real quadratic fields, under a mild analytic non-vanishing hypothesis, by Corollary A1 of [DR2].

In addition to Artin representations with projective image isomorphic to $A_4$, $S_4$ and $A_5$, Theorem A also applies to a large class of two-dimensional representations induced from general ray class characters of quadratic fields:

**Theorem B.** Let $E$ be an elliptic curve over $\mathbb{Q}$ of conductor $N$, let $K$ be a quadratic field of discriminant $D$, and let $\psi : \text{Gal}(H/K) \rightarrow \mathbb{C}^\times$ be a ray class character of conductor $f \triangleleft \mathcal{O}_K$. Assume that $\gcd(N, D \cdot \text{norm}(f)) = 1$, and that $\psi$ is of mixed signature if $K$ is real quadratic. Then

$$L(E/K, \psi, 1) \neq 0 \implies E(H)\psi = 0,$$

where

$$E(H)\psi := \{ P \in E(H) \otimes \mathbb{C} \text{ s.t. } \sigma P = \psi(\sigma)P, \ \forall \sigma \in \text{Gal}(H/K) \}.$$ 

Even for imaginary quadratic fields, Theorem B goes well beyond what can be obtained by the methods of Kolyvagin which apply only to ring class characters. For real quadratic $K$, there is simply no overlap between Theorem B and [DR2, Cor. A1], since ring class characters are either totally even or totally odd.

It is worth pointing out that the representation $\varrho$ arising in Theorem A is typically not self-dual. Because of this, the root number arising in the functional equation of $L(E, \varrho, s)$ does not control the parity of its order of vanishing at $s = 1$, and it is expected that $L(E, \varrho, 1)$ vanishes only in rare, sporadic instances, so that the non-vanishing hypothesis in Theorem A is almost always satisfied. Any quantitative non-vanishing statement along these lines would find immediate, unconditional applications to the arithmetic of elliptic curves via Theorems A and B. It should also be noted that $L(E, \varrho, 1) \neq 0$ if and only if $L(E, \varrho^\vee, 1) \neq 0$, where $\varrho^\vee$ is the contragredient representation of $\varrho$. Likewise, the triviality of the complex vector space

$$\text{Hom}_{\mathbb{G}_m}(\varrho, E(\bar{\mathbb{Q}}) \otimes \mathbb{C}) = (E(\bar{\mathbb{Q}}) \otimes \varrho^\vee)^{\mathbb{G}_m}$$

is equivalent to that of $\text{Hom}_{\mathbb{G}_m}(\varrho^\vee, E(\bar{\mathbb{Q}}) \otimes \mathbb{C})$.

The proof of Theorem A is based on the eponymous Beilinson-Flach elements of this article whose definition is recalled in §2 below, and on their
variation in $p$-adic families. For the purpose of the introduction, suffice it to say that Beilinson-Flach elements are distinguished elements in the higher Chow group $\text{CH}^2(X_1(N)^2, 1)$, which can also be interpreted as the motivic cohomology group $H^3_{\text{mot}}(X_1(N)^2, \mathbb{Q}(2))$. They were first introduced in the 1980’s by Beilinson [Bei, Ch. 2, §6], who related their image under his complex regulator to the value at $s = 2$ of the Rankin $L$-series $L(f \otimes g, s)$ attached to the convolution of weight two newforms $f$ and $g$ on $\Gamma_1(N)$.

In the early 1990’s, Flach [Fl] exploited (a slight variant of) these elements to prove the finiteness of the Selmer group of the symmetric square of an elliptic curve. Shortly thereafter, these finiteness results found a spectacular application in Wiles’ epoch-making proof of the Shimura-Taniyama conjecture and Fermat’s Last Theorem. Flach’s approach to bounding the Selmer group of the symmetric square featured prominently in Wiles’ original strategy, but was later obviated by the simpler and more flexible approach based on “Taylor-Wiles systems” introduced in [TW].

The power and versatility of Taylor-Wiles systems for questions surrounding the deformation theory of Galois representations may explain why, with a few exceptions ([MWe], [We02], ...), Beilinson-Flach elements garnered relatively less attention in the intervening decades. Yet their arithmetic relevance is not confined to the symmetric square motive of a modular form: guided by an analogy between Beilinson-Flach elements and the study of Gross-Kudla-Schoen diagonal cycles undertaken in [DR1] and [DR2], the article [BDR12] established a $p$-adic analogue of Beilinson’s formula for Beilinson-Flach elements, pointing out a number of arithmetic applications of this formula based on the “Euler system philosophy” propounded by Kato and Perrin-Riou, and lying ostensibly outside the scope of the Taylor-Wiles method. (Cf. the introduction of loc.cit., and [BCDDPR, Sec. 2.2.].)

The recent work of Kings, Lei, Loeffler and Zerbes makes significant strides in furthering the program initiated in [BDR12], applying the ideas of the present work to more general settings and introducing key technical improvements along the way. Thus, [LLZ1] constructs a full cyclotomic Euler system of Beilinson-Flach elements and applies it to the study of the Selmer group of the tensor product of two motives attached to cusp forms, while [LLZ2] applies similar techniques to study the Iwasawa theory of elliptic curves over the full two-variable $\mathbb{Z}_p$-extension of an imaginary quadratic field. The recent preprint [KLZ] exploits an interpolation of the Beilinson-Flach elements in all weights at once based on the notion of “Rankin-Iwasawa classes” arising in [Ki13], to establish a general explicit reciprocity law relating these families to Hida’s three variable $p$-adic Rankin $L$-function, with important applications to the associated three variable Main conjecture. It is worth pointing out that
the approach to Theorem A in the present work rests solely on $p$-adic families of Beilinson-Flach elements of minimal level and on their local behaviour at $p$; it can therefore be compared to the strategy followed by Coates-Wiles [CoWi] to prove the finiteness of Mordell-Weil groups of CM elliptic curves. By exploiting the full collection of Beilinson-Flach elements and their Euler system properties, notably their “tame deformations” at auxiliary primes $\ell \neq p$, [KLZ] also establishes (just as in Thaine and Rubin’s strengthening of the Coates-Wiles method) the finiteness of certain $p$-parts of the relevant Selmer and Shafarevich-Tate groups in the setting of Theorem A.

Shortly after [BDR12], Dasgupta [Da] had the idea of exploiting Beilinson-Flach elements to factor Hida’s $p$-adic Rankin $L$-function of the tensor square $f \otimes f$ into the Kubota-Leopoldt $p$-adic zeta-function and the Coates-Schmidt $p$-adic $L$-function attached to the symmetric square of $f$. The main result of [Da] is a deep manifestation of the Artin formalism for Hida’s $p$-adic $L$-series and rests on a comparison between circular units and a new construction of units in number fields arising from Beilinson-Flach elements. Dasgupta’s approach, which is inspired by a theorem of Gross [Gr] for the Katz $p$-adic $L$-function of an imaginary quadratic field and builds on the ideas of the present work, also makes essential use of the improvements described in [KLZ].

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1. Strategy of proof

Not surprisingly, modularity plays a key part in the proof of Theorem A. The elliptic curve $E$ is associated to a weight 2 newform $f = f_E \in S_2(N_f)$ thanks to [BCDT], while the Artin representation $\varrho$ is associated to a weight 1 newform $g = g_\varrho \in S_1(N_g, \chi)$ thanks to [KW09]. The Hasse-Weil-Artin $L$-series $L(E, \varrho, s)$ can therefore be identified with the convolution $L$-series $L(f \otimes g, s)$, whose analytic continuation follows from Rankin’s method.

Let $L \subset \mathbb{Q}_{ab}$ denote a field of coefficients over which $\varrho$ can be described, and let $H$ be the finite extension of $\mathbb{Q}$ cut out by $\varrho$ (i.e., the fixed field of the kernel of $\varrho$). Write

$$E(H)^\varrho_L := \text{hom}_{G_\varrho}(V_\varrho, E(H) \otimes L)$$
for the \( p \)-isotypic part of the Mordell-Weil group of \( E \). Theorem A asserts the triviality of this vector space when \( L(E, \varrho, 1) \neq 0 \).

The standard approach to bounding \( E(H) \) consists in embedding it in an appropriate Selmer group attached to the choice of a rational prime \( p \), which throughout the paper is assumed to be prime to \( N_f \) and \( N_g \). Throughout the paper, \( p \) is ordinary for \( f \), i.e., \( p \) does not divide \( a_p(f) \). To define this Selmer group, let

\[
V_f := V_p(E) := H^1(E_{\overline{Q}}, \mathbb{Q}_p)(1) = (\lim_{\leftarrow, n} E[p^n]) \otimes \mathbb{Q}_p
\]

denote Deligne’s \( p \)-adic representation attached to \( f \), which occurs as a direct factor of the \( \acute{e} \)tale cohomology \( H^1_{\acute{e}}(X_1(N)_{\overline{Q}}, \mathbb{Q}_p)(1) \), and let \( V_g = V_{g, \sigma} \) denote the Artin representation attached to \( g \), viewed as a \( p \)-adic representation by fixing some embedding \( \sigma \) of its coefficient field \( L \) into \( \overline{L} \subset \overline{\mathbb{Q}}_p \). The four dimensional \( L_p \)-vector space

\[
V_p(E) \otimes V_g = V_f \otimes V_g,
\]

where the tensor product is taken over \( \mathbb{Q}_p \), is a \( p \)-adic representation of \( G_{\mathbb{Q}} \). The connecting homomorphism

\[
E(H)_{\varrho}^0 := E(H)_{\varrho}^0 \otimes_L L_p \longrightarrow H^1(H, V_p(E) \otimes V_g^{\vee})^{\text{Gal}(H/\mathbb{Q})}
\]

of Kummer theory arising from the \( p \)-power descent exact sequence on \( E(H) \), composed with the inverse

\[
\text{res}_H^{-1} : H^1(H, V_p(E) \otimes V_g^{\vee})^{\text{Gal}(H/\mathbb{Q})} \longrightarrow H^1(Q, V_f \otimes V_g^{\vee})
\]

of the restriction map from \( \mathbb{Q} \) to \( H \) in Galois cohomology, gives a linear injection

\[
\delta : E(H)_{\varrho}^0 \longrightarrow H^1(Q, V_f \otimes V_g^{\vee})
\]

of \( L_p \)-vector spaces, where \( V_g^{\vee} := V_g \otimes \det(V_g)^{-1} \) is the \( L_p \)-linear dual of the representation \( V_g \). For each place \( v \) of \( \mathbb{Q} \) the corresponding map on local points yields an injection

\[
\delta_v : E(H_v)_{\varrho}^0 \longrightarrow H^1(Q_v, V_f \otimes V_g^{\vee}),
\]

where \( \text{Gal}(H/\mathbb{Q}) \) acts in the natural way on the local points

\[
E(H_v) := \oplus_{w|v} E(H_w).
\]

The image of \( \delta_v \) in \( H^1(Q_v, V_f \otimes V_g^{\vee}) \) is called the finite part of the local cohomology at \( v \), and is denoted \( H^1_{\text{fin}}(Q_v, V_f \otimes V_g^{\vee}) \). Likewise, \( H^1_{\text{fin}}(Q_v, V_f \otimes V_g) \) is defined to be the image of \( E(H_v)_{\varrho}^{\varphi} \) by the appropriate local coboundary map.
When \( v = p \), the finite part of the local cohomology admits an alternate description in terms of the general local conditions attached by Bloch and Kato [BK, §3] to any \( p \)-adic representation \( V \) of \( G_{\mathbb{Q}_p} \):

\[
H^1_{\text{exp}}(\mathbb{Q}_p, V) := \ker \left( H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, V \otimes B_{\text{cris}}^{\phi=1}) \right),
\]

\[
(1.1) H^1_{\text{fin}}(\mathbb{Q}_p, V) := \ker (H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, V \otimes B_{\text{cris}})) = \text{Ext}^1_{\text{cris}}(\mathbb{Q}_p, V),
\]

\[
H^1_{\text{geom}}(\mathbb{Q}_p, V) := \ker (H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, V \otimes B_{\text{dR}})) = \text{Ext}^1_{\text{dR}}(\mathbb{Q}_p, V),
\]

where \( B_{\text{cris}} \subset B_{\text{dR}} \) are Fontaine’s rings of periods for cristalline and de Rham representations, respectively, and \( \phi \) denotes the cristalline frobenius. The subscripts exp, fin, and geom stand for “exponential”, “finite”, and “geometric” respectively, and one has the obvious inclusions

\[
H^1_{\text{exp}}(\mathbb{Q}_p, V) \subset H^1_{\text{fin}}(\mathbb{Q}_p, V) \subset H^1_{\text{geom}}(\mathbb{Q}_p, V).
\]

When \( V = V_f \otimes V_g \) or \( V_f \otimes V_g^\vee \), the condition \( p \nmid N_f N_g \) shows, as in the proof of equation (2.15), that the Dieudonné module of \( V \) has no \( \phi \)-invariant vectors (for reasons of weight), and thus

\[
\delta_p(E(H_p)^{\phi=1}) = H^1_{\text{exp}}(\mathbb{Q}_p, V_f \otimes V_g^\vee) = H^1_{\text{fin}}(\mathbb{Q}_p, V_f \otimes V_g^\vee) = H^1_{\text{geom}}(\mathbb{Q}_p, V_f \otimes V_g^\vee).
\]

The cup product in Galois cohomology combined with the Weil pairing \( V_f \times V_f \to \mathbb{Q}_p(1) \) gives rise to the perfect local Tate pairing

\[
(\_ , \_ ) : H^1(\mathbb{Q}_p, V_f \otimes V_g^\vee) \times H^1(\mathbb{Q}_p, V_f \otimes V_g) \to H^2(\mathbb{Q}_p, \mathbb{Q}_p(1)) = \mathbb{Q}_p.
\]

For all \( v \), the finite subspaces \( H^1_{\text{fin}}(\mathbb{Q}_v, V_f \otimes V_g) \) and \( H^1_{\text{fin}}(\mathbb{Q}_v, V_f \otimes V_g^\vee) \) are orthogonal complements of each other under the local Tate pairing. Let

\[
H^1_{\text{sing}}(\mathbb{Q}_v, V_f \otimes V_g) := \frac{H^1(\mathbb{Q}_v, V_f \otimes V_g)}{H^1_{\text{fin}}(\mathbb{Q}_v, V_f \otimes V_g)}
\]

denote the singular quotient of the local cohomology at \( v \), and write

\[
\partial_v : H^1(\mathbb{Q}, V_f \otimes V_g) \to H^1_{\text{sing}}(\mathbb{Q}_v, V_f \otimes V_g)
\]

for the natural projection, which is referred to as the residue map at \( v \). Both the finite part and the singular quotient of \( H^1(\mathbb{Q}_p, V_f \otimes V_g) \) are two-dimensional over \( L_p \). By local Tate duality, similar statements hold for the finite part and the singular quotient of \( H^1(\mathbb{Q}_p, V_f \otimes V_g^\vee) \).

The Selmer group attached to \((E, V_g^\vee)\) is the subgroup of \( H^1(\mathbb{Q}, V_f \otimes V_g^\vee) \), denoted \( H^1_{\text{fin}}(\mathbb{Q}, V_f \otimes V_g^\vee) \), which fits into the cartesian square

\[
\begin{array}{c}
H^1_{\text{fin}}(\mathbb{Q}, V_f \otimes V_g^\vee) \to H^1(\mathbb{Q}, V_f \otimes V_g^\vee) \\
\downarrow \quad \downarrow \\
\prod_v H^1_{\text{fin}}(\mathbb{Q}_v, V_f \otimes V_g^\vee) \to \prod_v H^1(\mathbb{Q}_v, V_f \otimes V_g^\vee).
\end{array}
\]
The Selmer group attached to \((E, V_g)\) is defined likewise, by replacing \(V_g^\vee\) by \(V_g\).

The standard approach for bounding \(E(H)_L^\varnothing\) rests on the following statement whose proof is explained in Prop. 6.3 of [DR2].

**Lemma 1.1.** Assume that the residue map \(\partial_p\) attached to the \(p\)-adic representation \(\nu_p \otimes V_g,\sigma\) is a surjective map of \(L_p\)-vector spaces, for all embeddings \(\sigma : L \rightarrow L_p\). Then \(E(H)_L^\varnothing = 0\).

The proof of theorem A has thus been reduced to the problem of constructing two independent classes in the global cohomology \(H_1(Q, V_f \otimes V_g)\) whose images generate the singular quotient \(H_{\text{sing}}^1(Q_p, V_f \otimes V_g)\). Let \(\alpha_g\) and \(\beta_g\) be the two (not necessarily distinct) roots of the Hecke polynomial

\[x^2 - a_p(g)x + \chi(p) = (x - \alpha_g)(x - \beta_g)\]

attached to \(g\) at \(p\), and let \(g_\alpha\) and \(g_\beta\) denote the associated \(p\)-stabilisations of \(g\) satisfying

\[U_pg_\alpha = \alpha_g g_\alpha, \quad U_pg_\beta = \beta_g g_\beta.\]

When \(\alpha_g \neq \beta_g\), these stabilisations are distinct, and the unramified representation \(V_g\) decomposes as a direct sum

\[V_g = V_g^\alpha \oplus V_g^\beta\]

of one-dimensional eigenspaces for the arithmetic frobenius element at \(p\) attached to these two eigenvalues. When \(\alpha_g = \beta_g\), set \(V_g^\alpha := V_g\). Theorem A will now follow from the following result.

**Theorem 1.2.** There exists a global cohomology class

\[\kappa(f, g_\alpha) \in H^1(Q, V_f \otimes V_g)\]

attached to to the \(p\)-stabilised form \(g_\alpha\), satisfying:

1. The residue \(\partial_p(\kappa(f, g_\alpha))\) belongs to \(H_{\text{sing}}^1(Q_p, V_f \otimes V_g^\alpha)\);
2. it is non-zero if and only if \(L(f \otimes g, 1) \neq 0\).

To see why Theorem 1.2 implies Theorem A, choose the descent prime \(p\) in such a way that the frobenius element \(\sigma_p\) acts on \(V_g\) with distinct eigenvalues \(\alpha_g\) and \(\beta_g\). Such a choice is always possible, in view of the Chebotarev density theorem and the fact that the two-dimensional \(V_g\) is irreducible and hence the image of \(\sigma\) contains a non-scalar matrix. When \(L(E, 1) \neq 0\), Theorem 1.2 implies that the residues of the global classes \(\kappa(f, g_\alpha)\) and \(\kappa(f, g_\beta)\) generate the two-dimensional \(L_p\)-vector space \(H_{\text{sing}}^1(Q_p, V_f \otimes V_g)\). Since this reasoning applies to the representation \(V_{\sigma, g}\) for any embedding \(\sigma\) of \(L\) into \(L_p\), Theorem A follows from Lemma 1.1.

The “singular” class \(\kappa(f, g_\alpha)\) of Theorem 1.2 is not expected to arise from the images of algebraic cycles or elements in \(K\)-theory under an appropriate
étale Abel-Jacobi or regulator map, since such images are typically cristalline, at least in scenarios of good reduction of the ambient variety. It is constructed instead as a $p$-adic limit of classes $\kappa(f, g_x)$ attached to (sufficiently many) weight two specialisations $g_x$ of a Hida family specialising to $g_\alpha$ in weight one. These “weight two classes” are obtained via a geometric construction, as the images of Beilinson-Flach elements under the $p$-adic étale regulator map to Galois cohomology.

Section 2 describes the construction of a geometric class $\kappa(f, g)$ attached to a classical eigenform $g$ of weight two and $p$-power nebentypus character, and summarises its key properties, including the relation between its $p$-adic logarithm and the values of Hida’s $p$-adic Rankin $L$-function attached to $f$ and $g$. This relation extends the main result of [BDR12] to settings where the modular curve has bad reduction at the prime $p$, and relies crucially on the work of Besser, Loeffler and Zerbes [BLZ] on syntomic regulators partially extending the earlier work of Besser to semistable varieties.

Section 3 explains how the weight two classes $\kappa(f, g_x)$, as $g$ ranges over all the classical specialisations of weight two and $p$-power nebentypus character of a Hida family $g$, can be interpolated into a $\Lambda$-adic cohomology class $\kappa(f, g)$. Since these ordinary weight two points are dense in the rigid analytic topology on the relevant components of the Coleman-Mazur eigencurve, it follows that $\kappa(f, g)$ is uniquely determined by its weight two specialisations.

The class $\kappa(f, g_\alpha)$ of Theorem 1.2 is obtained by letting $g$ be a Hida family specialising to $g_\alpha$ in weight one, and specialising the resulting $\kappa(f, g)$ to $g_\alpha$. The $p$-adic Beilinson formula of Section 2 is then parlayed into the explicit reciprocity law relating the Bloch-Kato dual exponential of $\kappa(f, g_\alpha)$ to the special value of the same $p$-adic $L$-function at the point attached to $g_\alpha$. Since this latter point lies in the range of classical interpolation defining the $p$-adic $L$-function, it is directly related to a classical $L$-value, and this is what leads to the proof of Theorem 1.2.

2. A $p$-adic Beilinson formula

2.1. Siegel units and Eisenstein series. Fix a modulus $M = Np^s \geq 3$ with $p \nmid N$ and $s \geq 0$. Let

$$Y_s := Y_1(Np^s) \subset X_s := X_1(Np^s)$$

be canonical models over $\mathbb{Q}$ of the (affine and projective, respectively) modular curves classifying triples $(A, i_N, i_p)$ where $A$ is a (generalised) elliptic curve and $i_N : \mu_N \rightarrow A$ and $i_p : \mu_{p^s} \rightarrow A$ are embeddings of finite group schemes.
The standard notations \( Y_0(Np) \) and \( X_0(Np)/\mathbb{Q} \) are adopted for the affine and projective modular curves classifying pairs \((A, C_{Np})\) where \( C_{Np} \subset A \) is a cyclic subgroup scheme of \( A \) of order \( Np \). We will denote by \( w_M : X_1(M) \rightarrow X_1(M) \) the standard Atkin-Lehner involution, and will simply denote it as \( w \) when the level \( M \) is clear from the context.

For \( a \in (\mathbb{Z}/M\mathbb{Z})^\times \), let \( g_{a;M} \) be the Siegel unit denoted \( g_{0,a/M} \) in [Ka98], whose \( q \)-expansion is given by

\[
(2.1) \quad g_{a;M} := (1 - \zeta_M^a q^{1/12}) \prod_{n=1}^{\infty} (1 - \zeta_M^n q^n)(1 - \zeta_M^{-n} q^n),
\]

where \( \zeta_M \) is a fixed primitive \( M \)-th root of unity. The unit \( g_{a;M} \) can naturally be viewed as belonging to \( \mathcal{O}_{Y_1(M)}^\times \otimes \mathbb{Q} \), and its \( q \)-expansion is defined over \( \mathbb{Q}(\mu_M) \), while the unit

\[
(2.2) \quad g_{w;M} := w g_{a;M}
\]

is defined over \( \mathbb{Q} \). If \( \chi \) is a primitive even Dirichlet character of conductor \( M \), taking values in a field denoted \( \mathbb{Q}(\chi) \), let

\[
G(\chi) = \sum_{a=1}^{M} \chi(a) \zeta_M^a
\]

denote the Gauss sum attached to \( \chi \) and the choice of \( \zeta_M \), and set

\[
(2.3) \quad g_{\chi} = \frac{1}{\varphi(M)} \cdot \sum_{a=1}^{M} \chi^{-1}(a) \otimes g_{a;M} \in \mathcal{O}_{Y_1(M)}^\times \otimes \mathbb{Q}(\chi), \quad g_{w} := w g_{\chi}.
\]

Given a pair \((\chi_1, \chi_2)\) of Dirichlet characters and an integer \( k \geq 2 \) such that \( \chi_1 \chi_2(-1) = (-1)^k \), define also the Eisenstein series

\[
(2.4) \quad E_k(\chi_1, \chi_2) = c_k(\chi_1, \chi_2) + \sum_{n=1}^{\infty} \left( \sum_{d|n} \chi_1(n/d) \chi_2(d) d^{k-1} \right) q^n,
\]

where

\[
c_k(\chi_1, \chi_2) := \begin{cases} \frac{1}{2} \cdot L(\chi_2, 1 - k) & \text{if } \chi_1 = 1 \\ 0 & \text{otherwise}. \end{cases}
\]

A direct calculation using (2.1), (2.2) and (2.3) shows that the logarithmic derivative of \( g_\chi \) is

\[
(2.4) \quad \frac{d}{dq} \log g_{\chi} = \frac{2G(\chi^{-1})}{\varphi(M)} \cdot E_2(\chi, 1) \frac{dq}{q} \in \Omega_{Y_1(M)}^1.
\]
2.2. Beilinson-Flach elements. The Beilinson-Flach elements of this section are elements in the higher Chow group over $\mathbb{Q}$ of the surface

$$S_s := X_0(Np) \times X_s.$$ 

For any field $F$, let $C^2(S_s,1)_F$ denote the group of finite formal linear combinations

$$D = \sum_i a_i(Z_i, u_i)$$

of pairs $(Z_i, u_i)$ with coefficients $a_i \in F$, where $Z_i$ are irreducible curves embedded in $S_s$ and $u_i$ are rational functions on $Z_i$. Write $Z^2(S_s,1)_F \subset C^2(S_s,1)_F$ for the subspace of elements $D$ satisfying

$$\text{div}(D) := \sum a_i \text{div}(u_i) = 0.$$ 

Any pair $\{\varphi, \psi\}$ of non-zero rational functions on $S_s$ gives rise to the element

$$\sum_i (Z_i, u_i)$$

in $Z^2(S_s,1)_F$, where $\{Z_i\}$ is the set of irreducible components in $S_s$ of the divisors of $\varphi$ or $\psi$, and, after setting $a_i = \text{ord}_{Z_i}(\varphi)$ and $b_i = \text{ord}_{Z_i}(\psi)$, the rational function $u_i$ on $Z_i$ is given by the tame symbol

$$u_i = \prod (-1)^{a_i b_i} \left( \frac{\varphi^{a_i}}{\psi^{b_i}} \right) |_{Z_i}.$$ 

Writing $B^2(S_s,1)_F$ for the subspace of $Z^2(S_s,1)_F$ formed by such elements, Bloch’s higher Chow group of the surface $S_s$ with coefficients in $F$ is defined as

$$\text{CH}^2(S_s,1)_F = Z^2(S_s,1)_F / B^2(S_s,1)_F.$$ 

Letting

$$\tilde{\pi}_s : X_s \longrightarrow X_0(Np) \tag{2.5}$$

denote the natural forgetful projection of modular curves which is compatible with the $U_p$ correspondence acting on both curves, define

$$\iota_s = (\tilde{\pi}_s, \text{Id}) : X_s \hookrightarrow S_s, \tag{2.6}$$

to be the closed embedding given by $\iota_s(x) = (\tilde{\pi}_s(x), x)$. Let

$$\Delta_s := \iota_s(X_s) \subset S_s$$

denote the resulting embedded curve in $S_s$.

**Lemma 2.1.** For any $u \in O_{Y_s}^\times$, there exists a negligible element $\theta_s \in C^2(S_s,1)_F$ satisfying

$$\text{div} \theta_s = \text{div}(\Delta_s, u). \tag{2.7}$$

**Proof.** The proof is given in Lemma 3.1. of [BDR12], and relies on the fact the the group of degree zero divisors supported at the cusps is torsion in the jacobian of $X_s$ by the Manin-Drinfeld theorem. \hfill $\square$

Thanks to Lemma 2.1, the class $(\Delta_s, u) - \theta_s$ lies in $Z^2(S_s,1)_\mathbb{Q}$, and its image

$$\text{BF}(u) := [(\Delta_s, u) - \theta_s] \quad \text{in} \quad \text{CH}^2_{\text{reg}}(S_s,1)_\mathbb{Q}(\mathbb{Q})$$

is independent of the particular choice of $\theta_s$ satisfying (2.7). It is called the *Beilinson-Flach element* attached to the modular unit $u \in O_{Y_s}^\times$. 
Definition 2.2. The elements of CH$_{neg}^2(S, 1)_Q(Q)$ given by
$$BF(a; Np^s) := BF(g^w_{a; Np^s}), \quad BF_s := BF(1; Np^s)$$
are called the Beilinson-Flach elements of level $Np^s$.

2.3. Etale and syntomic regulators. Let $\varphi = \sum a_n(\varphi)q^n \in S_k(N_\varphi, \chi)$ be a normalized newform of weight $k$, level $N_\varphi$ and character $\chi$, and let $\omega_\varphi := \varphi(q)^{\frac{d_\phi}{q}}$ be the associated regular differential form on $X_1(N_\varphi)$. Let $K_\varphi$ be the finite extension of $Q_p$ generated by the fourier coefficients of $\varphi$, let $O_\varphi$ be its valuation ring, and let $V_\varphi$ denote the two-dimensional Galois representation over $O_\varphi$ associated by Deligne to $\varphi$. This Galois representation is pure of weight $-1$ and is characterized up to isomorphism by the property that for any prime $\ell | N\varphi$, the characteristic polynomial of an arithmetic Frobenius element $Fr_\ell$ is equal to the Hecke polynomial $x^2 - a_\ell(\varphi)x + \chi(\ell)\ell^{k-1}$.

Write $\alpha_\varphi$ and $\beta_\varphi$ be the roots of this Hecke polynomial at $\ell = p$, ordered so that $\ord_p(\alpha_\varphi) \leq \ord_p(\beta_\varphi)$.

If $N$ is a multiple of $N_\varphi$ and $T_N$ is the Hecke algebra generated by the good Hecke operators $T_\ell$ with $\ell \nmid N$, the $\varphi$-isotypic component of a $T_N$-module $S$ is defined to be

$$S[\varphi] := \{v \in S \otimes O_\varphi : T_\ell(v) = a_\ell(\varphi)v \text{ for all } \ell \nmid N\}.$$

Generalising the setting of §1, and modifying slightly the notations therein, let $f \in S_2(Np^s)[f_0]$ now denote a normalised eigenform with trivial Nebentypus character arising from a newform $f_0$ on $\Gamma_0(N_f)$ for some $N_f | N$. In the context of §1, $f_0$ is the eigenform associated to the elliptic curve $E$. More specifically, we take

$$f(q) = \sum_{d | \frac{Np^s}{N}} \lambda_d f_0(q^d) \in S_2(Np)[f_0]$$

to be an eigenvector for all Hecke operators $U_\ell$, $\ell | Np$, and assume $f$ is ordinary, i.e., that $\alpha_f = \alpha_{f_0}$ is a $p$-adic unit.

In addition, for the rest of this section let $g \in S_2(Np^s, \chi \cdot \chi_p)$ be an eigenform of weight 2, level $Np^s$ and nebentype character $\chi \chi_p$, that we have factored in such a way that

$$\cond(\chi) \mid N, \quad \cond(\chi_p) \mid p^s.$$ 

Assume $g$ arises from a newform $g_0$ of level $N_qp^s$ for some $N_q \mid N$ and that the conductor of $\chi_p$ is equal to $p^s$. In the terminology introduced by Mazur-Wiles [MWi84], $g$ is said to be primitive at $p$. Assume also that $g$ is ordinary at $p$, i.e., that $\alpha_g = a_p(g)$ is a $p$-adic unit.
For any variety $X$ over $\mathbb{Q}$, the notation $\bar{X} := X \times \bar{\mathbb{Q}}$ will be used to designate the base change of $X$ to $\bar{\mathbb{Q}}$. Let

$V_0(Np) = H^1_{\text{et}}(\bar{X}_0(Np), \mathbb{Z}_p)(1)$ and $V_s = H^1_{\text{et}}(\bar{X}_s, \mathbb{Z}_p)(1)$ for $s \geq 1$

denote the Galois representation afforded by the Tate module of $X_0(Np)$ and $X_s$, respectively.

Let

$pr_{1,1} : H^2_{\text{et}}(\bar{S}_s, \mathbb{Z}_p)(2) \rightarrow V_0(Np) \otimes V_s$

denote the natural projection of $G_{\mathbb{Q}}$-modules induced by the K"unneth decomposition, and let

$\text{reg}_{\text{et}} : \text{CH}^2(S_s, 1)(\mathbb{Q}) \rightarrow H^1(\mathbb{Q}, H^2_{\text{et}}(\bar{S}_s, \mathbb{Z}_p)(2)) \xrightarrow{pr_{1,1}} H^1(\mathbb{Q}, V_0(Np) \otimes V_s)$

denote the composition of the $p$-adic etale regulator map (as defined for instance in chapter 2 of [Fl]) with the map induced by $pr_{1,1}$ in Galois cohomology.

The image under $\text{reg}_{\text{et}}$ of the subgroup generated by negligible element vanishes, for the cohomology class of such elements take values in the K"unneth component

$H^2_{\text{et}}(\bar{X}_0(Np), \mathbb{Z}_p) \otimes H^0_{\text{et}}(\bar{X}_s, \mathbb{Z}_p)(2) \oplus H^0_{\text{et}}(\bar{X}_0(Np), \mathbb{Z}_p) \otimes H^2_{\text{et}}(\bar{X}_s, \mathbb{Z}_p)(2)$.

Hence the etale regulator descends to a map which we continue to denote with the same symbol

$\text{reg}_{\text{et}} : \text{CH}^2_{\text{neg}}(S_s, 1)(\mathbb{Q}) \rightarrow H^1(\mathbb{Q}, V_0(Np) \otimes V_s)$.

**Definition 2.3.** The Beilinson-Flach cohomology classes of level $Np^s$ are defined to be

$\kappa(a; Np^s) := \text{reg}_{\text{et}}(\text{BF}(a; Np^s)) \in H^1(\mathbb{Q}, V_0(Np) \otimes V_s)$,

$\kappa_s := \kappa(1; Np^s)$.

We now turn to providing an analytic description of the restriction to $G_{\mathbb{Q}_p}$ of the classes $\kappa_s$, which will pave the way for establishing its relation to the values of the Hida-Rankin $p$-adic $L$-function associated to Hida families passing through $f$ and $g$. A first step towards this task is the description of $H^1_{\text{fin}}(\mathbb{Q}_p, V_{fg})$ in terms of the de Rham cohomology of the relevant modular curves.

Recall Fontaine’s ring $\mathcal{B}_{\text{cris}}$ of cristalline periods, and let

$D_f := (\mathcal{B}_{\text{cris}} \otimes_{\mathbb{Q}_p} V_f)^{G_{\mathbb{Q}_p}}$,

$D_g := (\mathcal{B}_{\text{cris}} \otimes_{\mathbb{Q}_p} V_g)^{G_{\mathbb{Q}_p(\mu_{p^r})}}$,

$D_{fg} := (\mathcal{B}_{\text{cris}} \otimes_{\mathbb{Q}_p} V_{fg})^{G_{\mathbb{Q}_p(\mu_{p^r})}}$.
be the (potentially crystalline) Dieudonné modules associated to $V_f$, $V_g$ and $V_{fg}$, respectively. Note that $D_{fg} \simeq D_f \otimes D_g$ in the category of filtered frobenius vector spaces over the field $K$. Let

$$g^* = \bar{g} = \sum a_n(g)q^n \in S_2(Np^s, \chi^{-1}\chi_p^{-1}), \quad g^w := wg$$

be the modular forms which both belong to the automorphic representation attached to the contragredient of $g$; these modular forms are eigenvectors for the good Hecke operators, and

$$V_{g^*} = V_g \otimes \chi^{-1}\chi_p^{-1} = V^\vee_g(1).$$

The Dieudonné module of the Kummer dual $V_{fg}^\vee(1) = V_{fg}^\vee(-1)$ of $V_{fg}$, denoted $D_{fg}(-1)$, is canonically identified with $D_{fg^*}$ but with a shift in the Hodge filtration. The dimensions of the relevant pieces of the Hodge filtration on $D_{fg}$ and $D_{fg^*}(-1)$ are summarised in the table below:

<table>
<thead>
<tr>
<th></th>
<th>Fil^{-2}</th>
<th>Fil^{-1}</th>
<th>Fil^{0}</th>
<th>Fil^{1}</th>
<th>Fil^{2}</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_{fg}$</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$D_{fg^*}(-1)$</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

The Bloch-Kato logarithm $\log_p$ of \([BK]\) associated to the Galois representation $V_{fg}$ gives an isomorphism

$$\log_p : H^1_{\exp}(\mathbb{Q}_p(\mu_{p^r}), V_{fg}) \rightarrow \frac{D_{fg}}{\Fil^0D_{fg} + D_{fg}^\phi=1},$$

where $H^1_{\exp}(\mathbb{Q}_p(\mu_{p^r}), V_{fg})$ is as defined in (1.1), with $\mathbb{Q}_p$ replaced by $\mathbb{Q}_p(\mu_{p^r})$. The crystalline frobenius $\phi$ acts on $D_{fg}$ with eigenvalues

$$\alpha_f \alpha_g, \quad \alpha_f \beta_g, \quad \beta_f \alpha_g, \quad \beta_f \beta_g.$$

These eigenvalues are algebraic numbers of absolute value $p$ (relative to any complex embedding of $\mathbb{Q}$) and therefore $D_{fg}^\phi=1 = 0$. It follows from \([BK, Cor. 3.8.4]\) that

$$H^1_{\exp}(\mathbb{Q}_p(\mu_{p^r}), V_{fg}) = H^1_{\fin}(\mathbb{Q}_p(\mu_{p^r}), V_{fg}).$$

The Bloch-Kato logarithm can therefore be recast in the current context as a map

$$\log_p : H^1_{\fin}(\mathbb{Q}_p(\mu_{p^r}), V_{fg}) \rightarrow \frac{D_{fg}}{\Fil^0(D_{fg})} = \Fil^0(D_{fg^*}(-1))^\vee,$$

where the last identification arises from the natural duality between $D_{fg}$ and $D_{fg^*}(-1)$, relative to which $\Fil^0(D_{fg})$ and $\Fil^0(D_{fg^*}(-1))$ are orthogonal complements of each other.

Assume furthermore that none of the eigenvalues in (2.14) are equal to $p$. This assumption is satisfied with at most finitely many exceptions when $f$ is
fixed and $g$ runs over the weight two specialisations of a Hida family, hence will suffice for the applications in this paper. It implies, by another application of [BK, cor. 3.8.4], that $H^1_{\text{exp}}(\mathbb{Q}_p(\mu_{p^s}), V_{fg}^*(-1)) = H^1_{\text{fin}}(\mathbb{Q}_p(\mu_{p^s}), V_{fg}^*(-1))$, and hence, by duality, that

$$(2.17) \quad H^1_{\text{fin}}(\mathbb{Q}_p(\mu_{p^s}), V_{fg}^*) = H^1_{\text{geom}}(\mathbb{Q}_p(\mu_{p^s}), V_{fg}^*).$$

The argument used in the proof of Lemma 2.8 of [Fl] shows that the image of the étale regulator $\text{reg}_{\text{et}}^{fg}$ consists of extensions of Galois representations occurring in the étale cohomology of varieties with potentially semistable reduction at $p$, which hence are de Rham at $p$. This image is therefore contained in $H^1_{\text{geom}}(\mathbb{Q}_p(\mu_{p^s}), V_{fg})$, and thus by (2.15) in the finite subspace of the local cohomology.

Let

$$(2.18) \quad D_0(Np) := (B_{\text{dr}} \otimes V_0(Np))^{G_{Q_p}}, \quad D_s := (B_{\text{dr}} \otimes V_s)^{G_{Q_p}(\mu_{p^s})},$$

be the de Rham Dieudonné filtered modules attached to the Galois representations $V_0(Np)$ and $V_s$. Note that these $p$-adic Galois representations are $B_{\text{dr}}$ but not $B_{\text{cris}}$-admissible, but that their $f_0$ and $g_0$-isotypic parts, denoted $V_0(Np)[f] \simeq V_f^*$ and $V_s[g] \simeq V_g^*$, are crystalline over $\mathbb{Q}_p$ and $\mathbb{Q}_p(\mu_{p^s})$ respectively.

The étale regulator map (2.11) on $\text{CH}^2(S_s, 1)_{neg}$ and the Bloch-Kato logarithm associated to the Galois representation $V_0(Np) \otimes V_s$ give rise to a diagram of maps

$$(2.19) \quad \text{CH}^2(S_s, 1)_{neg}(\mathbb{Q}_p(\mu_{p^s})) \xrightarrow{\text{reg}_{\text{et}}} H^1_{\text{geom}}(\mathbb{Q}_p(\mu_{p^s}), V_0(Np) \otimes V_s) \xrightarrow{H^1_{\text{exp}}(\mathbb{Q}_p(\mu_{p^s}), V_0(Np) \otimes V_s)} H^1_{\text{exp}}(\mathbb{Q}_p(\mu_{p^s}), V_0(Np) \otimes V_s) \xrightarrow{\log_p} \text{Fil}^0(D_0(Np) \otimes D_s(-1))^\vee.$$

Let

$$(2.20) \quad \text{reg}_{fg}: \text{CH}^2(S_s, 1)(\mathbb{Q}_p(\mu_{p^s}))[fg] \longrightarrow \text{Fil}^0(D_0(Np)[f] \otimes D_s(-1)[g^*])^\vee$$

be the map obtained by applying the $(f, g)$-isotypic projection to (2.19), invoking (2.15) and (2.17) to show that the vertical inclusion becomes an isomorphism after restricting to these $(f, g)$-isotypic subspaces.
The main goal of the next sections is proving an analytic formula for the value of \( \text{reg}^{f_p}(\mathbf{BF}_s) \) at a suitable vector in \( \text{Fil}^0(D_0(Np)[f] \otimes D_0(-1)[g^*]) \), which we now proceed to describe.

By the comparison theorem between étale and de Rham cohomology, there are isomorphisms of two-dimensional filtered Frobenius modules

\[
\begin{aligned}
D_f \simeq H^1_{dR}(X_0(Nf)/\mathbb{Q}_p)(1)[f_0], & \quad D_g \simeq H^1_{dR}(X_1(Ng^{p^s})/\mathbb{Q}_p(\mu_{p^s}))(1)[g_0] \\
\end{aligned}
\]

such that

\[
\text{Fil}^0D_f \simeq \Omega^1(X_0(Nf))[f_0] = \langle \omega_{f_0} \rangle, \quad \text{Fil}^0D_g \simeq \Omega^1(X_1(Ng^{p^s}))[g_0] = \langle \omega_{g_0} \rangle.
\]

For any projective curve \( X/F \), write

\[
\langle \cdot, \cdot \rangle : H^1_{dR}(X/F) \times H^1_{dR}(X/F) \longrightarrow F
\]

for the canonical Poincaré pairing on \( X \). Let \( \eta_{f_0} \) be an element in the one-dimensional unit-root subspace of \( D_f \), normalised so that

\[
\langle \omega_{f_0}, \eta_{f_0} \rangle_{X_0(Nf)} = 1.
\]

Fix a finite extension \( K/\mathbb{Q}_p \) containing \( K_f \) and \( K_g \), and let \( \mathcal{O} \) denote its valuation ring. For each positive divisor \( d \) of \( Np/Nf \), write \( \pi_d : X_1(Np) \longrightarrow X_1(Nf) \) for the standard degeneracy map induced by multiplication by \( d \) on the upper half plane. Let

\[
\pi_{d*} : V_0(Np)[f_0] \longrightarrow V_f, \quad V_f := V_0(Nf)[f_0] = H^1_{dR}(\tilde{X}_0(Nf), \mathcal{O})(1)[f_0]
\]

be the map in étale cohomology induced by \( \pi_d \) by covariant functoriality. Our choice of \( f \) in \( S_2(Np)[f_0] \) stated in equation (2.9) determines the projection

\[
\omega_f := \sum \lambda_d \pi_{d*} : V_0(Np) \otimes \mathcal{O} \longrightarrow V_f.
\]

Committing a slight abuse of notation, denote by \( \omega_f \) the map induced by (2.22) by functoriality on Dieudonné modules and the first isomorphism in (2.21), and write \( \omega_f^* \) for the dual map arising from Poincaré duality:

\[
\omega_f : D_0(Np) \longrightarrow D_f, \quad \omega_f^* : D_f \longrightarrow D_0(Np).
\]

The maps \( \omega_g : D_s \longrightarrow D_g \) and \( \omega_g^* : D_g \longrightarrow D_s \) are defined similarly, and we let

\[
\omega_f^* \otimes \omega_g^* : D_{fg} \longrightarrow D_0(Np) \otimes D_s
\]

denote the map induced by \( \omega_f^* \) and \( \omega_g^* \) and the Künneth decomposition.

Define \( \eta_f := \omega_f^*(\eta_{f_0}) \in D_0(Np)[f] \), and likewise set

\[
\omega_g := \omega_g^*(\omega_{g_0}) \in D_s[g], \quad \omega_g^w := \omega \omega_g \in D_s[g^*].
\]

Note that \( \eta_f \) is ordinary, by the compatibility of \( \omega_f^* \) with the \( U_p \)-operator. Since \( \eta_f \) belongs to \( \text{Fil}^{-1}(D_f) = D_f \) and \( \omega_g^w \) belongs to \( \text{Fil}^0(D_{g^*}) \subset \Omega^1(X_s) \),
the tensor product $\eta_f \otimes \omega_g$ belongs to $\operatorname{Fil}^{-1}(D_0(Np)[f] \otimes D_s[g^*])$. Because of the assumption on $g$ leading to (2.17), the expression
\[
\operatorname{reg}_{fp}^g(\mathbf{BF}_s)(\eta_f \otimes \omega_g)
\]
is defined. The main result of §2.5 is a formula for its value. As a preparation for that, we first recall in §2.4 a few needed facts on the rigid geometry of the modular curve $X_1(Np^s)$.

### 2.4. The rigid cohomology of modular curves.

This section briefly summarises the description of the de Rham cohomology of $X_s$ viewed as a curve over a $p$-adic field, following the treatment in [DR2, §3.1], with suitable modifications aimed at giving a similar description of the cohomology of the affine curve $Y_s$. We adopt the same notations as in loc.cit., which the reader is encouraged to consult for a more detailed description of the objects that are used here.

More specifically, let $F$ be any extension of $\mathbb{Q}_p(\mu_{p^r})$ over which $X_s$ admits a semistable model, and over which the closed points of $\Sigma := X_s - Y_s$ are defined.

The étale cohomology groups of $X_s$ and $Y_s$ are related by the Gysin exact sequence of semistable representations of $G_F = \operatorname{Gal}(\bar{F}/F)$:
\[
0 \to H^1_{\text{et}}(\bar{X}_s, \mathbb{Q}_p) \to H^1_{\text{et}}(\bar{Y}_s, \mathbb{Q}_p) \to \mathbb{Q}_p(-1)^t \to 0,
\]
where $t = |\Sigma| - 1$.

Let $D_{st}$ be Fontaine's comparison functor attached to the ring of semistable periods $\mathbf{B}_{st}$. By applying $D_{st}$ to the sequence (2.24), and invoking the comparison theorem between étale and de Rham cohomology, one obtains a corresponding sequence of filtered Frobenius monodromy (FFM) modules
\[
0 \to H^1_{\text{dR}}(X_s/F) \to H^1_{\text{dR}}(Y_s/F) \to F(-1)^t \to 0.
\]
(See for example [CI10], Proposition 7.6 for details.) Note that the above FFM modules are isomorphic as filtered Frobenius modules to those obtained by applying the construction of the potentially crystalline Dieudonné module of Section 2.3, in which $\mathbf{B}_{st}$ is replaced by $\mathbf{B}_{\text{cris}}$. The use of $\mathbf{B}_{st}$ equips these modules with the additional structure of a monodromy operator.

In particular, it follows from (2.25) that the Frobenius operator $\Phi$ associated to $Y_s$ acts on the subspace of $H^1_{\text{dR}}(Y_s/F)$ generated by classes of Eisenstein series with eigenvalues of absolute value $p$.

In order to make (2.25) more explicit, we recall, following [CI10], the Maier-Vietoris exact sequences describing the de Rham cohomology groups of $X_s$ and $Y_s$. Fix a proper semistable model $\mathcal{X}_s$ of $X_s$ over the ring of integers $\mathcal{O}_F$ of $F$, whose special fiber $C$ is a union of smooth geometrically connected components $C_1, \ldots, C_n$ intersecting at ordinary double points defined over
the residue field $F$ of $F$. Since $s \geq 1$, there are at least two components, i.e.,
$n \geq 2$. Write $\mathcal{G}$ for the dual graph of $C$, and $\mathcal{V}$, resp. $\mathcal{E}$ for its set of vertices,
resp. oriented edges. Recall that
\[ \mathcal{V} = \{ C_1, \ldots, C_n \}, \quad \mathcal{E} = \{ (x, C_i, C_j) : x \in C_i \cap C_j \}. \]
Given $e = (x, C_i, C_j)$ in $\mathcal{E}$, we say that $s(e) := C_i$ is the source of $e$ and
$t(e) := C_j$ is the target of $e$. Write
\[ \text{red} : X_s(C_p) \rightarrow C(\overline{F}) \]
for the reduction map, and identify $X_s(C_p)$ with the rigid-analytic space over $F$ attached to $X_s$. The wide-open space associated to $v \in \mathcal{V}$ is defined as $\mathcal{W}_v := \text{red}^{-1}(v)$. Its underlying connected affinoid space $\mathcal{A}_v$ is obtained as the inverse image by reduction of the component $v$ with the singular points removed. The wide-open annulus associated to $e = (x, v_i, v_j) \in \mathcal{E}$ is defined as $\mathcal{W}_e := \text{red}^{-1}(x)$ (so that $\mathcal{W}_e = \mathcal{W}_{v_i} \cap \mathcal{W}_{v_j}$). The collection $\{ \mathcal{W}_v \}_{v \in \mathcal{V}}$ forms an admissible covering of $X_s(C_p)$ by wide open spaces. Likewise, an admissible covering of $Y_s(C_p)$ is given by $\{ \mathcal{W}_v' \}_{v \in \mathcal{V}}$, where $\mathcal{W}_v' := \mathcal{W}_v - \Sigma$. Note that
\[ \mathcal{W}_{v_i} \cap \mathcal{W}_{v_j} = \mathcal{W}_{v_i} \cap \mathcal{W}_{v_j} = \mathcal{W}_e, \]
with $e = (x, v_i, v_j) \in \mathcal{E}$. Since $H_{\text{dR}}^1(X_s/F)$ and $H_{\text{dR}}^1(Y_s/F)$ are identified with the Cech hypercohomology groups of their associated admissible coverings, they arise as the middle terms in the following commutative diagram originating from the Maier-Victoris sequences attached to these coverings:

\[
\begin{array}{cccccc}
0 & \xrightarrow{0} & H_{\text{dR}}^0(\mathcal{W}_v) & \xrightarrow{i} & H_{\text{dR}}^1(X_s/F) & \xrightarrow{\iota} & (\oplus_{v \in \mathcal{V}} H_{\text{dR}}^1(\mathcal{W}_v)) \downarrow 0 \\
\delta \oplus & \downarrow & \oplus & \downarrow & \oplus & \downarrow & \\
0 & \xrightarrow{0} & H_{\text{dR}}^0(\mathcal{W}_v) & \xrightarrow{\iota'} & H_{\text{dR}}^1(Y_s/F') & \xrightarrow{\iota'} & (\oplus_{v \in \mathcal{V}} H_{\text{dR}}^1(\mathcal{W}_v)) \downarrow 0,
\end{array}
\]

where
\[
\begin{align*}
(\oplus_{v \in \mathcal{V}} H_{\text{dR}}^1(\mathcal{W}_v))_0 & := \ker (\oplus_{v \in \mathcal{V}} H_{\text{dR}}^1(\mathcal{W}_v) \xrightarrow{\delta} \oplus_{v \in \mathcal{E}} H_{\text{dR}}^1(\mathcal{W}_v)), \\
(\oplus_{v \in \mathcal{V}} H_{\text{dR}}^1(\mathcal{W}_v'))_0 & := \ker (\oplus_{v \in \mathcal{V}} H_{\text{dR}}^1(\mathcal{W}_v') \xrightarrow{\delta} \oplus_{v \in \mathcal{E}} H_{\text{dR}}^1(\mathcal{W}_v')).
\end{align*}
\]

In (2.26) and (2.27), the various coboundary maps $\delta$ are defined by the rule
\[ \delta(\alpha_v)(e) := \alpha_t(e)|_{\mathcal{W}_v} - \alpha_s(e)|_{\mathcal{W}_v}, \]
and we have used the fact that $H_{\text{dR}}^0(\mathcal{W}_v) = H_{\text{dR}}^0(\mathcal{W}_v')$, since $\Sigma$ is finite. Furthermore, the map $\iota$ sends the class of a collection $\{ \lambda_x \}$ of scalars with $\lambda_x \in \mathbb{C}_p$ to the class of the hypercocycle $(\{ 0_v \}, \{ \lambda_v \})$, while the map $\iota'$ sends the class of the hypercocycle $(\{ \omega_v \}, \{ f_e \})$ to the class of $\{ \omega_v \}$. Likewise for $\iota'$ and $\iota'$. 
2.5. Analytic description of the syntomic regulator. We now give an analytic description of the image under the \( p \)-adic syntomic regulator of the Beilinson-Flach element \( BF_s \) attached to the Siegel unit \( g_{1,Np^s} \). Recall the modular forms \( f \in S_2(Np) \) and the primitive form \( g \in S_2(Np^s, \chi \chi_p) \) considered in \( \S 2.3 \), arising from newforms \( f_0 \) and \( g_0 \) of level \( N_f \) and \( N_g p^s \), respectively.

Lemma 2.4. We have the equality
\[
\text{reg}_{p^s}(BF_s)(\eta_f \otimes \omega_{g^w}) = \text{reg}_{p^s}(BF(g^w)(\chi^{-1} - 1))(\eta_f \otimes \omega_{g^w}).
\]

Proof. This follows by noting that
\[
BF_s = \sum BF(g^w \psi),
\]
where the sum is being taken over the even characters \( \psi \) of modulus \( Np^s \). It follows from the equivariance of \( \text{reg}_{p^s} \) under the group of diamond operators acting functorially on both source and target that \( \text{reg}_{p^s}(BF(g^w \psi))(\eta_f \otimes \omega_{g^w}) \) is zero for all \( \psi \neq \chi^{-1} \chi_p^{-1} \).

In light of this lemma, let
\[
\omega_E := \mathcal{E}(q) \frac{dq}{q} = \frac{\varphi(Np^s)}{2} \cdot \text{dlog} \chi^{-1} \chi_p^{-1} = \mathcal{G}(\chi \chi_p) E_2(1) \frac{dq}{q},
\]
be the differential of the third kind on \( X_1(Np^s) \) associated to the Eisenstein series
\[
\mathcal{E} \in M_2(Np^s, \chi^{-1} \chi_p^{-1})
\]
introduced in equation (2.4) of \( \S 2.1 \).

Choose a polynomial \( P(x) \in F[x] \) such that the following conditions hold:

1. \( P(\Phi \times \Phi) \) annihilates the class of \( \omega_{g^w} \otimes \omega_{\mathcal{E}^w} \) in \( H^2_{\text{dR}}(Y^2_{s/F}) \),
2. \( P(\Phi) \) acts invertibly on \( H^1_{\text{dR}}(Y_{s/F}) \).

The polynomial \( P \) exists because \( \Phi \times \Phi \) acts with eigenvalues of complex absolute value \( p^{3/2} \) on the subspace of \( H^2_{\text{dR}}(Y^2_{s/F}) \) generated by the Frobenius translates of classes of the form \( \omega_{g^w} \otimes \omega_{\mathcal{E}^w} \), while \( \Phi \) acts on \( H^1_{\text{dR}}(Y_{s/F}) \) with eigenvalues of absolute value \( 1, \sqrt{p} \) and \( p \) (as is deduced for example by analysing the Mayer-Vietoris sequence (2.26)).

Just as in [DR2, \S 3.2], condition (1) in the choice of the polynomial \( P \) means that \( P(\Phi \times \Phi)(\omega_{g^w} \otimes \omega_{\mathcal{E}^w}) \) is exact and hence there exists a system of rigid one-forms
\[
\rho_p(g^w, \mathcal{E}^w) := \{ \rho_{v_1,v_2}^{v_1,v_2}(g^w, \mathcal{E}^w) \in \Omega_{\text{rig}}^1(W_{v_1} \times W_{v_2}' \times W_{v_2}') \}
\]
indexed by pairs \( (v_1, v_2) \in V \times V \), satisfying
\[
d\rho_p^{v_1,v_2}(g^w, \mathcal{E}^w) = P(\Phi \times \Phi)\left( \omega_{g^w} \otimes \omega_{\mathcal{E}^w} \mid_{W_{v_1} \times W_{v_2}'} \right).
\]
Note that the choice of the system of one-forms \( \rho_{v_1,v_2}^w(g^w, E^w) \) is not unique, as it is well-defined only up to systems of closed rigid one-forms on the products \( \mathcal{W}_{v_1} \times \mathcal{W}_{v_2} \). Nevertheless, we shall use this system to construct a canonical class in \( H^1_{\text{dR}}(X_s/F) \) associated to \( g^w \) and \( E^w \), independent of all the choices made.

For any \( a \in (\mathbb{Z}/Np^s\mathbb{Z})^\times \), let \( \langle a \rangle \in \text{Aut}(Y_s) \) denote the diamond operator associated to \( a \) on \( Y_s \), and for any pair \( (a,b) \) of such elements let \( \langle a,b \rangle \) denote the automorphism of \( Y_s \times Y_s \) given by the product of \( \langle a \rangle \) and \( \langle b \rangle \) acting on the first and second variable, respectively.

Fix an \( a \in (\mathbb{Z}/Np^s\mathbb{Z})^\times \) such that \( \chi_\rho(a) \neq 1 \) for the rest of the section, and define the linear homomorphism

\[
(2.30) \quad \delta_a^*: \oplus_{v_1,v_2} \Omega^1_{\text{rig}}(\mathcal{W}_{v_1}' \times \mathcal{W}_{v_2}') \longrightarrow \oplus_{v} \Omega^1_{\text{rig}}(\mathcal{W}_v')
\]

by

\[
\delta_a^* = \text{diag}^* \circ (\text{Id} - \langle a,1 \rangle^*) \circ (\text{Id} - \langle 1,a \rangle^*),
\]

where diag* denotes the map induced by pull-back under the diagonal embedding \( \mathcal{W}_v' \longrightarrow \mathcal{W}_v' \times \mathcal{W}_v' \).

**Lemma 2.5.** The map \( \delta_a^* \) sends collections of closed 1-forms to collections of exact 1-forms.

**Proof.** By the Künneth formula, there is a decomposition

\[
H^1_{\text{dR}}(\mathcal{W}_v') \simeq (H^0 \otimes H^1) \oplus (H^1 \otimes H^0),
\]

where we use the shorthands \( H^0 = H^0_{\text{dR}}(\mathcal{W}_v') \) and \( H^1 = H^1_{\text{dR}}(\mathcal{W}_v') \). Since the endomorphism \( \text{Id} - \langle a,1 \rangle^* \) vanishes on the first factor, and \( \text{Id} - \langle 1,a \rangle^* \) vanishes on the second, the claim follows. \( \Box \)

**Corollary 2.6.** The image \( \Xi_{\rho_a}(g^w, E^w) \) of the system of closed rigid 1-forms \( \delta_a^*(\rho_a(g^w, E^w)) \) in \( \oplus_v H^1_{\text{dR}}(\mathcal{W}_v') \) does not depend on the choice of solution of the differential equation \( (2.29) \).

As the notation indicates, the class \( \Xi_{\rho_a}(g^w, E^w) \) does depend on both our choices of \( P \) and \( a \). In order to eliminate this ambiguity, define the class

\[
(2.31) \quad \Xi_a(g^w, E^w) := (1 - \chi_\rho(a))^{-1}(1 - \chi^{-1}_P(a))^{-1} \Xi_{\rho_a}(g^w, E^w)
\]

in \( \oplus_v H^1_{\text{dR}}(\mathcal{W}_v') \). This class still depends on \( P \), but is independent of the choice of \( a \) as above. Indeed, this follows from the very definition of \( \Xi_a(g^w, E^w) \) and the fact that \( \langle a \rangle \) acts on the \( g^w \)-isotypic (resp. \( E^w \)-isotypic) component of \( H^1_{\text{dR}}(Y_{s/F}) \) as multiplication by \( \chi^{-1}_P(a) \) (resp. \( \chi_\rho(a) \)).

Following [DR2, Def.3.1], define the subspace \( H^1_{\text{dR}}(X_{s/F})^{(1)} \) of pure classes of weight one of \( H^1_{\text{dR}}(X_{s/F}) \) by the condition that \( \Phi \) acts via eigenvalues of complex absolute value \( \sqrt{p} \) on its elements. Likewise, define the subspace \( \oplus_v H^1_{\text{dR}}(\mathcal{W}_v')^{(1)} \) of \( \oplus_v H^1_{\text{dR}}(\mathcal{W}_v') \) by the same condition. Weight considerations show that \( \oplus_v H^1_{\text{dR}}(\mathcal{W}_v')^{(1)} \) is contained in the kernel of the map \( \delta \) appearing in
the commutative diagram (2.26), and that the map \( r' \circ \iota \) defined in the same diagram induces an isomorphism between the above spaces of pure weight one classes. Denote by

\[ \text{spl}_X : \oplus_v H^1_{\text{dR}}(W_v^e(1)) \longrightarrow H^1_{\text{dR}}(X_{s/F}(1)) \]

the (Frobenius-equivariant) inverse isomorphism. In light of condition (2) in our choice of \( P \), define the class

\[ \Xi(\varphi) = P(\Phi)^{-1} \text{spl}_X(\Xi_P(\varphi)) \in H^1_{\text{dR}}(X_{s/F}(1)). \]

It is immediate to check that \( \Xi(\varphi, \mathcal{E}^w) \) not depend on our choice of \( P \).

Let \( \pi_s \) and \( \varpi_s \) be the natural degeneracy maps arising in the diagram

\[ \tilde{\pi}_s : X_s \xrightarrow{\pi_s} X_0(Np^s) \xrightarrow{\varpi_s} X_0(Np), \]

where \( \tilde{\pi}_s \) is the morphism appearing in equation (2.5). The degeneracy map \( \varpi_s \) that we choose is the unique one which is compatible with the \( U_p \) operators in level \( Np^s \) and \( Np \). Let

\[ \eta_{f,s} := \varpi_s^* \eta_f, \quad \tilde{\eta}_{f,s} := \pi_s^* \eta_{f,s}. \]

The class \( \eta_{f,s} \) is ordinary, by the ordinariness of \( \eta_f \) combined with the compatibility of \( \varpi_s \) with \( U_p \).

**Proposition 2.7.** The equality

\[ \text{reg}^f_p(BF_s)(\eta_f \otimes \omega_{g^w}) = \frac{2}{\varphi(Np^s)} \langle \tilde{\eta}_{f,s}, \Xi(\varphi, \mathcal{E}^w) \rangle_{X_s} \]

holds.

**Proof.** This formula follows by applying the same argument as in the proof of [BDR12, Proposition 3.3], with the following difference. While in loc. cit. we work with the modular curve \( X_1(N) \) which admits a smooth model over \( \mathbb{Z}_p \), here the modular curve \( X_s \) has bad reduction at \( p \) and we are bound to work with a semistable model of this curve. Besser’s theory of finite polynomial cohomology has been extended recently to the setting of semistable varieties by Besser, Loeffler and Zerbes in [BLZ], building on previous work of Nekovar and Niziol [NeNi]. The proof of [BDR12, Proposition 3.3] applies verbatim after replacing Besser’s [Bes, Prop. 6.3] with [BLZ, Proposition 3.12], taking into account that

\[ \omega_{\mathcal{E}^w} = \frac{\varphi(Np^s)}{2} \cdot d \log(g_{\mathcal{X}_{g^w}}^{w-1}). \]

The fact that the nebentype characters of \( g^w \) and \( \mathcal{E}^w \) are inverse one of another implies that the class \( \Xi(g^w, \mathcal{E}^w) \) arises as the pull-back of a class – denoted with the same symbol by abuse of notation – in the cohomology of \( X_0(Np^s) \) under \( \pi_s \).
Proposition 2.8. For all $s \geq 1$,

\[(2.34) \quad \reg_p^g(BF_s)(\eta_f \otimes \omega_{g^w}) = \langle \eta_{f,s}, \Xi(g^w, \mathcal{E}^w) \rangle_{X_0(Np^s)}. \]

Proof. Because the degree of $\pi_s$ is $\frac{1}{2} \varphi(Np^s)$, and the pullback and pushforward maps are adjoint to each other relative to the Poincaré pairing,

\[(2.35) \quad \langle \eta_{f,s}, \Xi(g^w, \mathcal{E}^w) \rangle_{X_0(Np^s)} = \frac{1}{2} \varphi(Np^s) \langle \eta_{f,s}, \Xi(g^w, \mathcal{E}^w) \rangle_{X_0(Np^s)}. \]

The result follows. \hfill \Box

It will be convenient to consider the anti-ordinary counterparts of the classes $\eta_f$ and $\eta_{f,s}$:

\[
\eta_f^w := w\eta_f, \quad \eta_{f,s}^w := w\eta_{f,s},
\]

where we recall that $w$ refers in the above equations to the Atkin-Lehner involutions $w_{NP}$ and $w_{NP'}$ of levels $NP$ and $NP'$ respectively.

Let $\{W_\infty, W_0\}$ denote the standard admissible covering of $X_0(Np)$ by wide-open neighbourhoods of the Hasse domain of $X_0(N)$, as described by Coleman in [Co94]. By construction, arguing as in [DR2, Lemma 4.6], the ordinary unit root class $\eta_f \in H^1_{dR}(X_0(Np))$ is supported on $W_0$, and has vanishing residues at the supersingular annuli, while the anti-ordinary unit root class $\eta_f^w$ is supported on $W_\infty$.

Let $U_p^w = wU_p w$ be the adjoint operator of $U_p$ with respect to the Poincaré pairing on $X_0(Np^s)$, and let

\[
eqord = \lim U_p^{n!} \quad \text{and} \quad \ord^w = \lim(U_p^w)^{n!} = w\ord w
\]

denote Hida’s ordinary and anti-ordinary projectors, respectively. Pull-back under $\omega_s$ gives rise to an isomorphism between the ordinary subspaces of $H^1_{dR}(X_0(Np^s))$ and $H^1_{dR}(X_0(Np))$. Hence $\ord \Xi(g, \mathcal{E})$ arises as the pull-back under $\omega_s$ of some ordinary class in $H^1_{dR}(X_0(Np))$, that we again denote with the same symbol. In the context of the next theorem, $\ord \Xi(g, \mathcal{E})$ is to be understood in the latter sense. Since $\ord \Xi(g, \mathcal{E})$ belongs to the subspace of the cohomology which is pure of weight one, its restriction to the wide open $W_\infty$ has vanishing residues along the supersingular annuli. Therefore, it also makes sense to pair it with the (restriction of) the anti-ordinary class $\eta_f^w$.

Theorem 2.9. For all $s \geq 1$,

\[(2.36) \quad \reg_p^g(BF_s)(\eta_f \otimes \omega_{g^w}) = \alpha_f^{s-1} \cdot \langle \eta_{f,s}^w, \ord \Xi(g, \mathcal{E}) \rangle_{W_\infty}. \]

Proof. Since the class $\eta_{f,s}$ is ordinary, it is fixed by $\ord$ and thus

\[
\langle \eta_{f,s}, \Xi(g^w, \mathcal{E}^w) \rangle_{X_0(Np^s)} = \langle \ord \eta_{f,s}, \Xi(g^w, \mathcal{E}^w) \rangle_{X_0(Np^s)} = \langle \eta_{f,s}^w, \ord \Xi(g^w, \mathcal{E}^w) \rangle_{X_0(Np^s)} = \langle \eta_{f,s}^w, \ord \Xi(g, \mathcal{E}) \rangle_{X_0(Np^s)},
\]
where the self-adjointness of the involution $w$ has been exploited to derive the last equation.

But since $e_{\text{ord}} \Xi(g, \mathcal{E})$ arises as a pullback under $\varpi_s$, we can write

$$
\langle \eta \, w, e_{\text{ord}} \Xi(g, \mathcal{E}) \rangle_{X_0(Np)} = \langle \varpi_s \, \varpi_s^* \, (\eta \, w), e_{\text{ord}} \Xi(g, \mathcal{E}) \rangle_{X_0(Np)} = \langle w \circ U_{p-1}^s (\eta \, w), e_{\text{ord}} \Xi(g, \mathcal{E}) \rangle_{X_0(Np)} = \alpha_{f-1}^s \langle \eta \, w, e_{\text{ord}} \Xi(g, \mathcal{E}) \rangle_{W_\infty},
$$

where the last identity follows from the fact that the anti-ordinary unit root class $\eta \, w$ is supported on $W_\infty$. The theorem now follows from Proposition 2.8 combined with (2.36) and (2.37).

We are now in a position to state the main result of this section, namely, the evaluation of the $p$-adic syntomic regulator attached to $BF_s$ in terms of the $q$-expansions of concrete (overconvergent) $p$-adic modular forms. For a $p$-adic modular form $\varphi = \sum a_n(\varphi)q^n$, let

$$
\varphi^{[p]} = \sum_{p \nmid n} a_n(\varphi)q^n
$$

denote its “$p$-deprived” counterpart.

The Atkin-Serre operator $d = q \frac{d}{dq}$ sends $p$-adic overconvergent modular forms of weight 0 to overconvergent modular forms of weight two. The image of the $p$-depleted classical weight two Eisenstein series $E_2^{[p]}(\chi_p^{-1}, 1)$ under $d^{-1}$ is the $p$-adic overconvergent Eisenstein series of weight 0:

$$
d^{-1} E_2^{[p]}(\chi_p^{-1}, 1) = E_0^{[p]}(1, \chi_p^{-1}).
$$

The product $g^{[p]} \times E_0^{[p]}(1, \chi_p^{-1})$ is therefore an overconvergent cusp form of weight two with trivial nebentypus character, and hence its image under the ordinary projection $e_{\text{ord}}$ corresponds to a classical weight two cusp form on $\Gamma_0(Np)$, which can be viewed as an element of the first de Rham cohomology of $X_0(Np)$. The inner product $\langle \ , \ \rangle_{X_0(N)}$ in the theorem below denotes the usual Poincaré duality on this de Rham cohomology group.

**Theorem 2.10.** For all $s \geq 1$,

$$
\text{reg}_{g^{[p]}}(BF_s)(\eta \otimes \omega_g^w) = \mathcal{G}(\chi_p) \cdot \frac{\alpha_{f-1}^s}{1 - \chi_p^{-1}(p) \alpha_f^{-1} \alpha_g} \times 
\langle \eta \, w, e_{\text{ord}}(g^{[p]} \cdot E_0^{[p]}(1, \chi_p^{-1})) \rangle_{X_0(Np)}.
$$

**Proof.** The restriction to $W_\infty$ of $e_{\text{ord}} \Xi(g, \mathcal{E})$ is the cohomology class of a specific overconvergent weight two $p$-adic modular form representing a rigid analytic differential form on $W_\infty$ with vanishing annular residues. The class
\(e_{\text{ord}} \Xi(g, E)\) is defined by exactly the same analytic recipe as the class denoted \(e_{\text{ord}} \xi(\omega_g, \omega_h)\) in equation (85) of [DR2], after replacing \(\tilde{g}\) by \(g\) and setting
\[
\tilde{h} = w_p E\quad (= G(\chi)E_2(\chi^{-1}, \chi_p)).
\]
In [DR2], the form \(\tilde{h}\) was assumed to be cuspidal while \(w_p E\) is an Eisenstein series, but the calculations of loc.cit. remain valid in the latter setting. In particular, the expression
\[
w_p \tilde{h} = G(\chi_p^{-1})\alpha_h^{-s} \omega_h,
\]
that appears in the statement of Prop. 4.13 of [DR2] can be replaced by \(\omega_E\), yielding
\[
\begin{align*}
e_{\text{ord}} \Xi_P(g, E) &= e_{\text{ord}} \xi_P(\omega_g, \omega_{w_P} E) = -2e_{\text{ord}} (g \times d^{-1} E^{|p|}) \\
&= -2G(\chi P) e_{\text{ord}} (g \times E_0^{|p|}(1, \chi^{-1} \chi_p^{-1})),
\end{align*}
\]
where \(P(x)\) is the linear polynomial \(2(1 - p^{-1} \chi^{-1}(p) \alpha_g x)\), and \(\Xi_P(g, E)\) is defined as in (2.31). Combining equation (2.38) with Theorem 2.9, we obtain
\[
\begin{align*}
\text{reg}_p^{g}(\mathbf{BF}_s)(\eta_{f} \otimes \omega_{g^w}) &= -2G(\chi P) \alpha_f^{-1} \times \langle \eta_{f}^w, P(\Phi)^{-1} e_{\text{ord}} (g \cdot E_0^{|p|}(1, \chi^{-1} \chi_p^{-1}))) \rangle_{W_\infty}.
\end{align*}
\]
Note that \(g\) can be replaced by \(g^{|p|}\) in equation (2.40), as a direct calculation shows that
\[
\langle g^{|p|} - g \rangle \cdot E_0^{|p|}(1, \chi^{-1} \chi_p^{-1})
\]
belongs to the kernel of \(U_p\), and a fortiori of \(e_{\text{ord}}\). (See for instance Lemma 2.17 of [DR1].) Finally, for any class \(\Xi\) in \(H_{dR}^1(W_\infty)\), we have
\[
\langle \eta_{f}^w, \Phi \Xi \rangle_{W_\infty} = \alpha_f^{-1} \langle \Phi \eta_{f}^w, \Phi \Xi \rangle_{W_\infty} = P(\beta_f)^{-1} \langle \eta_{f}^w, \Xi \rangle_{W_\infty} = \beta_f \langle \eta_{f}^w, \Xi \rangle_{W_\infty},
\]
which in turn implies
\[
\langle \eta_{f}^w, P(\Phi)^{-1} \Xi \rangle_{W_\infty} = P(\beta_f)^{-1} \langle \eta_{f}^w, \Xi \rangle_{W_\infty}.
\]
In light of the explicit description of \(P(x)\) given above, combined with equations (2.40) and (2.41),
\[
\begin{align*}
\text{reg}_p^{g}(\mathbf{BF}_s)(\eta_{f} \otimes \omega_{g^w}) &= G(\chi P) \cdot \frac{\alpha_f^{s-1}}{1 - \chi^{-1}(p) \alpha_f^{-1} \alpha_g} \times \langle \eta_{f}^w, e_{\text{ord}} (g^{|p|} \cdot E_0^{|p|}(1, \chi^{-1} \chi_p^{-1}))) \rangle_{W_\infty}.
\end{align*}
\]
Proceeding just as in the proof of Theorem 2.9, Theorem 2.10 follows from the compatibility between the de Rham pairings on \(W_\infty \subset X_0(Np)\) and on \(X_0(Np)\), using the fact that \(\eta_{f}^w\) is anti-ordinary and supported on \(W_\infty\). \(\square\)
2.6. Hida’s \( p \)-adic L-function. A point of the rigid analytic space
\[
\Omega := \text{Hom}_{cts}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)
\]
is said to be \textit{arithmetic} if it is of the form
\[
\nu_{k,\epsilon}(x) := x^{k-1}\epsilon(x), \quad \text{(for all } x \in \mathbb{Z}_p^\times),
\]
where \( k \geq 2 \) is an integer, and \( \epsilon \) is a finite order character factoring through \((\mathbb{Z}/p\mathbb{Z})^\times\). The space \( \Omega \) is referred to as the \textit{weight space} and plays a basic role in the theory of \( p \)-adic families of modular forms.

A \textit{Hida family} of tame character \( \chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}_p^\times \) is a pair \((\Omega_\varphi, \varphi)\),

where

1. \( \Omega_\varphi \) is a rigid analytic space equipped with a finite morphism \( \text{wt} : \Omega_\varphi \longrightarrow \Omega \),

2. \( \varphi = \sum_{n=1}^{\infty} a_n(\varphi)q^n \) is a formal \( q \)-series with coefficients in the ring \( \Lambda_\varphi := \mathcal{A}(\Omega_\varphi) \) of rigid analytic functions on \( \Omega_\varphi \),

such that, for all points \( x \in \Omega_\varphi \) of arithmetic weight \( \text{wt}(x) = \nu_{k,\epsilon} \), the specialisation
\[
\varphi_x := \sum_{n=1}^{\infty} a_n(\varphi)(x)q^n
\]
is the \( q \)-series of a normalised eigenform of weight \( k \), level \( Np^s \) and character \( \chi_{\varphi_x} = \chi\epsilon\omega^{1-k} \), where \( \omega \) is the Teichmüller character of conductor \( p \).

Let \( g \) be the weight one modular form considered in Section 1, and let \( g_\alpha \) be one of its ordinary \( p \)-stabilisations. A theorem of Hida ensures the existence of a Hida family, denoted \( g_\varphi \), of tame character \( \chi \) which specialises to \( g_\alpha \) at a suitable point \( y_0 \in \Omega_\varphi \) lying above \( \nu_{1,1} \). For any arithmetic point \( y \in \Omega_\varphi \) above \( \nu_{k,\epsilon} \), for some \( k \geq 1 \) and some character \( \epsilon \) of \( p \)-power conductor, denote by \( g_y \in S_{1}(Np^s, \chi\epsilon\omega^{1-k}) \) the specialisation of \( g \) at \( y \).

Write \( E(1, \chi^{-1}) \) for the ordinary Hida family of Eisenstein series, whose specialisation at the point \( \nu_{k,\epsilon} \) in weight space is equal to the classical Eisenstein series
\[
E(1, \chi^{-1})_{\nu_{k,\epsilon}} = E_\ell(1, \chi^{-1}\epsilon\omega^{1-k})
\]
of weight \( \ell \) and character \( \chi^{-1}\epsilon\omega^{1-k} \). The Iwasawa algebra admits an involution inducing the map \( \nu_{k,\epsilon} \mapsto \nu_{2-k,\epsilon} \) on weight space. Applying the change of scalars to \( E(1, \chi^{-1}) \) via this involution yields a \( \Lambda \)-adic modular form, denoted \( E^*(1, \chi^{-1}) \), whose specialisation at the arithmetic point \( \nu_{k,\epsilon} \) is the overconvergent modular form
\[
E^*(1, \chi^{-1})_{\nu_{k,\epsilon}} = E_{2-\ell}(1, \chi^{-1}\epsilon^{-1}\omega^{1-k}).
\]
The expression $g_y[p] \cdot E_0[p] \chi^{-1} \chi_p^{-1}$ appearing in Theorem 2.10 is the specialisation at the point $y$ of weight $\nu_{2,\chi_p \omega}$ of the $\Lambda$-adic modular form

$$g_{E^\ast[p]}(1, \chi^{-1}),$$

whose specialisation at a point $y$ of weight $\nu_{\ell,\epsilon}$ is given by

$$(2.42) \quad \left\{ g_y[p] \cdot E_{E^\ast[p]}(1, \chi^{-1}) \right\}_x = g_{y}[p] \cdot E_{E^\ast[p]}(1, \chi^{-1} \epsilon^{-1} \omega^{-1}).$$

Note that for all arithmetic $y$, this specialisation is an overconvergent modular form of weight 2 and trivial character, whose ordinary projection is therefore classical of level $Np$; hence,

$$c_{\text{ord}}(g_y[p] \cdot E_{E^\ast[p]}(1, \chi^{-1})) \text{ belongs to } S_2(\Gamma_0(Np), \mathbb{Q}_p) \otimes \Lambda_2.$$

The reader is referred to Section 4.5 of [DR2] for details on dual families of modular forms, which motivate the definition of the $\Lambda$-adic modular form of equation (2.42).

Let $L_p(f,g)(x,y,j)$ be the 3-variable Rankin $p$-adic $L$-function on $\Omega_f \times \Omega_g \times \Omega$ introduced in [BDR12, §2.2], where $f$ is the Hida family passing through $f$. It interpolates the algebraic part of the critical values $L(f_x \otimes g_y, j)$, where $f_x$ and $g_y$ are classical specialisations of $f$ and $g$ whose weights $k$ and $\ell$ satisfy

$$(2.43) \quad k > \ell, \quad \frac{k + \ell - 1}{2} \leq j \leq k - 1.$$ 

Consider the restriction of $L_p(f,g)(x,y,j)$ to the set of points $(x_0, y, \nu_{\ell,\epsilon})$, where $x_0$ is such that $f_{x_0} = f$, and $y \in \Omega_2$ is a point lying above $\nu_{\ell,\epsilon}$. In the notations of [BDR12, §2.2.1], this amounts to considering only the values of $L_p(f,g)(k, \ell, j)$ when $k = 2, j = \ell, t = 1 - \ell$ and $m = \ell$. In light of equations (27) and (24) in loc.cit., combined with the equality

$$d_1^{1-\ell} E_{E^\ast[p]}(\chi^{-1} \epsilon^{-1} \omega^{-1}, 1) = E_{E^\ast[p]}(1, \chi^{-1} \epsilon^{-1} \omega^{-1}),$$

this restriction agrees, up to a non-zero fudge factor which depends only on the fixed form $f$, with the $(p$-adic analytic) function described for all such $y$ by

$$(2.44) \quad L_p(f,g)(y) := (\eta_f^w, c_{\text{ord}}(g_y[p] \cdot E_{E^\ast[p]}(1, \chi^{-1} \epsilon^{-1} \omega^{-1})))_{x_0(Np)}.$$

Note that the points of the form $(x_0, y, \nu_{\ell,\epsilon})$ lie outside the range of classical interpolation for $L_p(f,g)$, unless $g_y$ is classical and of weight one. The next result relates the values of $L_p(f,g)$ at arithmetic points $y$ of weight $\nu_{2,\epsilon}$ to the $p$-adic regulators of Beilinson-Flach elements:
Theorem 2.11. Let $\epsilon$ be a character of conductor $p^s$, let $y \in \Omega_2$ be a point over $\nu_{2,\epsilon}$ and let $g = g_y \in S_2(Np^s, \chi \omega^{-1})$ denote the specialization of $g$ at $y$. Then

$$\text{reg}_{p}(\text{BF}_s)(\eta_f \otimes \omega_gw) = \mathcal{G}(\chi \omega^{-1}) \cdot \frac{\alpha_f^{-1}}{1 - \chi(p)^{-1}\alpha_f^{-1}} \cdot L_p(f,g)(y).$$

Proof. This follows directly by combining equation (2.44) with Theorem 2.10. □

Another important property of $L_p(f,g)$ pertains to its specialisations at classical weight one points attached to the Hida family $g$.

Theorem 2.12. Let $y \in \Omega_2$ be a point over $\nu_{1,1}$, corresponding to a classical $p$-stabilised weight one form $g = g_y \in S_1(Np, \chi)$. Then

$$L_p(f,g)(y) \neq 0 \iff L(f \otimes g, 1) \neq 0.$$

Proof. By equation (2.44),

$$L_p(f,g)(y) = \langle \eta_f^p, e_{\text{ord}}\left(g[^p]_y \cdot E[^p]_1(1, \chi^{-1})\right)_{X_0(Np)} \rangle$$

(2.45)

$$= \langle \eta_f^p, g[^p]_y \cdot E[^p]_1(1, \chi^{-1}) \rangle_{X_0(Np^2)} \pmod{\bar{Q}^\times},$$

where the fact that $g[^p]_y E[^p]_1(1, \chi^{-1})$ is a classical modular form of level $Np^2$ has been used to deduce the second equality. The expression in (2.45) is an explicit non-zero multiple of the value of the classical complex Rankin-Selberg $L$-function $L(f \otimes g, s)$ of the convolution of $f$ and $g$ at $s = 1$. (See equation (27) and the one immediately following it in [BDR12, §2.2.2].) The theorem follows. □

3. $\Lambda$-adic classes

3.1. Norm-compatible systems of elements. For all $s \geq 0$, the natural $U_p$-compatible projections

(3.1) $\pi_{s+1,s} : X_{s+1} \rightarrow X_s$

of modular curves give rise to maps

(3.2) $\pi_{s+1,s} : S_{s+1} \rightarrow S_s$

on the associated surfaces, that we continue to denote with the same symbol by a slight abuse of notation. Write also $\pi_{s+1,s} : \mathcal{O}_{X_{s+1}}^\times \rightarrow \mathcal{O}_{X_s}^\times$ for the norm maps on units induced by $\pi_{s+1,s}$.

Proposition 3.1. For all $s \geq 1$,

$$\pi_{s+1,s}(\mathfrak{g}^w_{\mathfrak{d};Np^{s+1}}) = \mathfrak{g}^w_{\mathfrak{d};Np^{s}}.$$
Proof. Lemma 2.12 of [Ka98] implies that the units $g_{a,Np^{s+1}}$ are compatible relative to the pushforward via the maps $\pi_{s+1,s}^*: X_{s+1} \to X_s$ that are compatible with the operator $U'_p$. This compatibility of the units $g_{a,Np^s}$ relative to the $\pi_{s+1,s}^*: w_{p^{s+1}}s, Np^s \to Np^s$ implies the $\pi_{s+1,s}$-compatibility of the units $g_{a,Np^s}$. See also §2.11 and §2.13 of [Ka98], or Thm. 2.24 of [LLZ1], for a convenient summary of these and further calculations in the more general setting treated in [Ka98]. (Recall that the unit $g_{a,Np^s}$ is denoted $g_{0,\pi_{p^s}}$ in [Ka98] and [LLZ1].) □

Write $\pi_{s+1,s}: CH^2(S_{s+1}, 1) \to CH^2(S, 1)$ for the norm maps on higher Chow groups induced by push-forward under the maps $\pi_{s+1,s}$ of (3.2). Note that $\pi_{s+1,s}$ preserves the subspaces of negligible classes, and hence descends to a well-defined map

$$\pi_{s+1,s}: CH^2_{neg}(S_{s+1}, 1) \to CH^2_{neg}(S, 1).$$

The norm compatibilities of the units $g_{a,Np^s}$ are inherited by the associated Beilinson-Flach elements:

**Proposition 3.2.** For all $s \geq 1$, 

$$\pi_{s+1,s}(BF(a; Np^{s+1})) = BF(a; Np^s), \quad \pi_{s+1,s}(BF_{s+1}) = BF_s$$

in $CH^2_{neg}(S, 1)(\mathbb{Q})$.

**Proof.** This follows directly from Proposition 3.1 in light of the fact that the map $\pi_{s+1,s}$ of (3.2) maps the diagonally embedded $X_{s+1} \simeq \Delta_{s+1} \subset S_{s+1}$ to $\Delta_s$, and is identified with the projection $\pi_{s+1,s}$ of (3.1). □

**Corollary 3.3.** The Beilinson-Flach cohomology classes $\kappa_s$ of Definition 2.3 are compatible under the norm maps, i.e.,

$$\pi_{s+1,s}(\kappa_{s+1}) = \kappa_s.$$

In particular, the classes $\kappa_s$ can be packaged together into the inverse limit class

$$\kappa_{\infty} := (\kappa_s)_{s \geq 1} \in H^1(\mathbb{Q}, V_0(Np) \otimes V_{\infty}),$$

where

$$V_{\infty} := \lim_{\leftarrow, s} V_s.$$

**3.2. The $\Lambda$-adic Beilinson-Flach cohomology class.** The module $V_{\infty}$ of the previous section becomes more manageable when replaced by its image under the ordinary projection. Let

$$V^ord_s := e_{ord}V_s, \quad V^ord_{\infty} := e_{ord}V_{\infty} = \lim_{\leftarrow, s} e_{ord}V_s.$$

Note that the action of the group $D_s$ of diamond operators on $X_s$ endows the $G_{\mathbb{Q}}$-module $V^ord_{\infty}$ with a natural structure of module over the Iwasawa algebra.
\[ \tilde{\Lambda} = \mathbb{Z}_p[[D_\infty]] := \lim \mathbb{Z}_p[D_s]. \] A theorem of Hida asserts that this module is finitely generated and locally free over this algebra. Its Hecke eigenspaces realise the \( \Lambda \)-adic representations attached to ordinary families of eigenforms.

The \( \Lambda \)-adic Beilinson-Flach cohomology class is defined by setting

\[ \kappa_{s,\text{ord}} := e_{s,\text{ord}} \in H^1(\mathbb{Q}, V_0(Np) \otimes V_{s,\text{ord}}), \]

\[ \kappa_{\infty,\text{ord}} := e_{\infty,\text{ord}} = (\kappa_{s,\text{ord}})_{s \geq 1} \in H^1(\mathbb{Q}, V_0(Np) \otimes \mathbb{V}_\infty). \]

Recall the cusp form \( f = q + \sum_{n \geq 2} a_n(f)q^n \in S_2(Np) \) introduced in §2.3, on which \( U_p \) acts with eigenvalue \( \alpha_f \). Define

\[ \psi_f : G_{\mathbb{Q}_p} \longrightarrow \mathcal{O}_\infty \]

to be the unramified character of \( G_{\mathbb{Q}_p} \) such that \( \psi_f(Fr_p) = \alpha_f \) and let

\[ \epsilon_{\text{cyc}} : G_{\mathbb{Q}_p} \longrightarrow \mathbb{Z}_p^\times \]

denote the cyclotomic character. There is an \( \mathcal{O}[G_{\mathbb{Q}_p}] \)-module exact sequence

\[ 0 \longrightarrow V_f^+ \longrightarrow V_f \longrightarrow V_f^- \longrightarrow 0, \]

with

\[ V_f^+ \simeq \mathcal{O}(\psi_f^{-1} \epsilon_{\text{cyc}}), \quad V_f^- \simeq \mathcal{O}(\psi_f) \]

as \( \mathcal{O}[G_{\mathbb{Q}_p}] \)-modules.

Let

\[ \xi_{\text{cyc}} : G_{\mathbb{Q}_p} \longrightarrow \Lambda^\times \]

denote the \( \Lambda \)-adic cyclotomic character satisfying the interpolation property

\[ \nu_{\ell, \epsilon} \circ \xi_{\text{cyc}} = \epsilon \xi_{\text{cyc}}^{-1} \omega^{1-\ell} \]

for any \( \ell \geq 1 \) and for any Dirichlet character \( \epsilon \) of \( p \)-power conductor.

Let \( g = \sum_{n \geq 1} a_n(g)q^n \) be a \( \Lambda \)-adic cuspidal eigenform of tame level \( N \) and tame character \( \chi \), arising from a \( \Lambda \)-adic newform of level \( N_g \), and let \( \Lambda_g \) denote the finite flat extension of the Iwasawa algebra \( \Lambda := \mathbb{Z}_p[[1 + p\mathbb{Z}_p]] \) generated by the coefficients \( a_n(g) \). Hida and Wiles associated to \( g \) a two-dimensional Galois representation \( \nabla_g \) over \( \Lambda_g \), again characterized up to isomorphism by the property that the characteristic polynomial of \( Fr_\ell \) is \( T^2 - a_\ell(g)T + \xi_{\text{cyc}}(\ell) \) for primes \( \ell \nmid Np \).

Wiles [Wi88] proved a \( \Lambda \)-adic analogue of (3.5) (cf. also [Oh00]),

\[ 0 \longrightarrow V_{g}^+ \longrightarrow V_g \longrightarrow V_{g}^- \longrightarrow 0, \]

where

\[ V_{g}^+ \simeq \Lambda_g(\psi_g^{-1} \chi_{\text{cyc}}), \quad V_{g}^- \simeq \Lambda_g(\psi_g) \]

as \( \Lambda_g[G_{\mathbb{Q}_p}] \)-modules. Here

\[ \psi_g : G_{\mathbb{Q}_p} \longrightarrow \Lambda_g^\times \]
is the unramified character sending \( \text{Fr}_p \) to \( a_p(g) \).

Fix once and for all a choice of the isomorphisms in (3.6). Similarly to the previous sections, the \( \Lambda \)-adic form \( g \) gives rise to an epimorphism

\[ \varpi_g : V_\infty^{\text{ord}} \to V_g \]

of \( \Lambda[G] \)-modules. Set

\[ V_{f,g} := V_f \otimes V_g, \quad \varpi_{f,g} := \varpi_f \otimes \varpi_g : V_0(Np) \otimes V_\infty^{\text{ord}} \to V_{f,g}. \]

**Definition 3.4.** The \( \Lambda \)-adic Beilinson-Flach cohomology class associated to the pair \((f, g)\) is

\[ \kappa(f, g) = \varpi_{f,g}(\kappa_\infty) \in H^1(\mathbb{Q}, V_{f,g}). \]

Let \( \ell \geq 1 \) be an integer and \( \epsilon \) be a Dirichlet character of conductor \( p^s \) for some \( s \geq 1 \). Let \( y \) be a point in \( \Omega_g = \text{Spf}(\Lambda_g) \) such that \( \text{wt}(y) = \nu_{\ell, \epsilon} \), and assume that the specialization \( g_y \) of \( g \) at \( y \) is a classical eigenform (which is always the case if \( \ell \geq 2 \)). Then \( g_y \) belongs to \( M_\ell(Np^s, \chi\epsilon\omega^{1-\ell}) \), and is actually a cuspidal if \( \ell \geq 2 \). Define

(3.7)

\[ \kappa(f, g)_y \in H^1(\mathbb{Q}, V_{f,g}_y) \]

to be the specialization at \( y \) of the \( \Lambda \)-adic class \( \kappa(f, g) \). Let

\[ \kappa_p(f, g) \in H^1(\mathbb{Q}_p, V_{f,g}), \quad \kappa_p(f, g)_y \in H^1(\mathbb{Q}_p, V_{f,g}_y) \]

denote the restriction to \( G_{\mathbb{Q}_p} \) of the \( \Lambda \)-adic class \( \kappa(f, g) \) and of its specialization \( \kappa(f, g)_y \).

Since the \( p \)-adic syntomic regulator map is the composition of the étale regulator with Bloch-Kato’s logarithm map, Theorem 2.11 translates into the following statement:

**Theorem 3.5.** Let \( y \in \Omega_g \) be an arithmetic point of weight-character \( \nu_{2, \epsilon} \) for some character \( \epsilon \) of conductor \( p^s \). Then

\[ \log_p \kappa_p(f, g)_y(\eta_f \otimes \omega_{g_y}) = G(\chi\epsilon\omega^{-1}) \cdot \frac{\alpha_f^{s-1}}{1 - \chi(p)^{-1}} \cdot \alpha_{g_y}^{-1} \cdot L_p(f, g)(y). \]

**3.3. The explicit reciprocity law.** The \( \Lambda_g \)-module \( V_{f,g} \) admits a natural \( G_{\mathbb{Q}_p} \)-stable filtration, given by

\[ V_{f,g}^+ := V_f^+ \otimes V_g^+ \subset V_{f,g}^+ := V_f \otimes V_g^+ + V_f^+ \otimes V_g \subset V_{f,g}. \]

Note that \( V_{f,g}^+ \) and \( V_{f,g}^{-} \) have rank 1 and 3 over \( \Lambda_g \), respectively.

**Lemma 3.6.**

1. There is an isomorphism \( V_{f,g}^- / V_{f,g}^+ \cong \Lambda_g(\psi_f\psi_g) \) of \( \Lambda_g[G_{\mathbb{Q}_p}] \)-modules.
(ii) The quotient $\mathcal{V}^+_f/\mathcal{V}^+_{f^g}$ decomposes as a $\Lambda_2[G_{Q_p}]$-module as
\[ \mathcal{V}^+_f/\mathcal{V}^+_{f^g} \cong \mathcal{V}^f_{f^g} \oplus \mathcal{V}^g_{f^g}, \]
where
\[ \mathcal{V}^f_{f^g} = \Lambda_2(\psi_f^{-1} \chi \cdot \epsilon_{\text{cyc}}) \quad \mathcal{V}^g_{f^g} = \Lambda_2(\psi_1^{-1} \psi_g \epsilon_{\text{cyc}}). \]

Proof. The lemma follows directly from the definitions of $\mathcal{V}^+_{f^g}$ and $\mathcal{V}^+_{f^g}$, and the formulae described in (3.5) and (3.6). □

The natural inclusion $\mathcal{V}^+_{f^g} \hookrightarrow \mathcal{V}_{f^g}$ induces a homomorphism
\[ H^1(Q_p, \mathcal{V}^+_{f^g}) \hookrightarrow H^1(Q_p, \mathcal{V}_{f^g}), \]
which is injective because $H^0(Q_p, \mathcal{V}^+_{f^g}/\mathcal{V}^+_{f^g}) = 0$. Thus $H^1(Q_p, \mathcal{V}^+_{f^g})$ is identified with a subgroup of $H^1(Q_p, \mathcal{V}_{f^g})$.

Define $\xi_{f^g} := 1 - \alpha_{f^g}(g) \in \Lambda_2$. For the remainder of the article, we replace the ring $\Lambda_2$ and all modules over it by their localization at the multiplicative set generated by the powers of $\xi_{f^g}$. Since this element vanishes at no classical point of weight $\ell \geq 1$, this modification does not affect any of the subsequent arguments concerning the specialization at these points.

**Lemma 3.7.** The local class $\kappa_p(f, g)$ belongs to $H^1(Q_p, \mathcal{V}^+_{f^g})$.

Proof. It follows from known properties of the étale regulator map, as explained in [Ne98] and [Fl, §2], that for all points $y$ of weight $\nu_{2,\epsilon}$, the specializations $\kappa_p(f, g_y)$ belong to the image of $H^1(Q_p, \mathcal{V}^+_{f^g})$ in $H^1(Q_p, \mathcal{V}_{f^g})$. Since these points form a dense subset of $\Omega_{f^g}$ for the rigid analytic topology, we conclude that $\kappa_p(f, g)$ belongs to the kernel of the map
\[ H^1(Q_p, \mathcal{V}_{f^g}) \longrightarrow H^1(I_p, \mathcal{V}_{f^g}/\mathcal{V}^+_{f^g}). \]

In order to prove the lemma, it thus suffices to show that the kernel of the restriction map
\[ H^1(Q_p, \mathcal{V}_{f^g}/\mathcal{V}^+_{f^g}) \longrightarrow H^1(I_p, \mathcal{V}_{f^g}/\mathcal{V}^+_{f^g}) \]
is trivial. Note that such kernel is a quotient of $H^1(Q_p^m/Q_p, (\mathcal{V}_{f^g}/\mathcal{V}^+_{f^g})^I_p)$, which by Lemma 3.6 (i) is isomorphic to $H^1(Q_p^m/Q_p, \Lambda_2(\psi_f \psi_g))$. This module is $\xi_{gh}$-torsion. In view of the redefinition of $\Lambda_2$, this implies the sought-after claim. □

Set $\Psi := \psi_f \psi_g^{-1} \chi$. The two previous lemmas allow us to define
\[ \kappa_p^f(f, g) \in H^1(Q_p, \Lambda_2(\Psi \cdot \epsilon_{\text{cyc}})) \]
as the projection of $\kappa_p(f, g)$ to the first factor $\mathcal{V}^f_{f^g}$.
Let $y \in \Omega_2$ be a classical point of weight $\nu_{\ell,e}$ with $\ell \geq 2$ and $\epsilon \neq \omega^{\ell-1}$, and let $K_y/\mathbb{Q}_p$ denote the residue field of $y$. Write $\Psi_y : G_{\mathbb{Q}_p} \rightarrow K_y^{\times}$ for the specialisation at $y$ of the unramified $\Lambda$-adic character $\Psi$. The classical eigenform $g_y \in S_\ell(Np^s, \chi \omega^{1-\ell})$ is new at $p^s$, and the specialization of $\kappa^\ell_p(f, g)$ at $y$ yields a local class

$$\kappa^\ell_p(f, g_y) \in H^1(\mathbb{Q}_p, K_y(\Psi_y \cdot \epsilon \cdot \epsilon^{\ell-1} \cdot \omega^{1-\ell})).$$

Lemma 3.8. For all arithmetic points $y$ of weight $\nu_{\ell,e}$ with $\ell \geq 2$, the local cohomology group $H^1(\mathbb{Q}_p, K_y(\Psi_y \cdot \epsilon \cdot \epsilon^{\ell-1} \cdot \omega^{1-\ell}))$ is one-dimensional over $K_y$ and equal to $H^1_{\text{exp}}(\mathbb{Q}_p, K_y(\Psi_y \cdot \epsilon \cdot \epsilon^{\ell-1} \cdot \omega^{1-\ell}))$.

Proof. Write simply $V = V_{f, g_y}$ for the specialisation at $y$ of $V_{f, g_y}$ and $D = D_{\text{dr}}(V_{f, g_y})$ for its de Rham Dieudonné module. By [BK, Def. 3.10], [Bel, Ex. 2.19], the Bloch-Kato logarithm map is an isomorphism

$$\log_p : H^1_{\text{exp}}(\mathbb{Q}_p, V) \rightarrow D/(\text{Fil}^0D + D^{\phi=1}).$$

By [Bel, §2.1.3], $\dim_{K_y} H^1(\mathbb{Q}_p, V) = e_0 + e_1 + 1$ where $e_0 = \dim H^0(\mathbb{Q}_p, V)$ and $e_1$ is the multiplicity of $K_y(1)$ as a quotient of $V$. Since $\ell \geq 2$, $e_0 = 0$. Since $\epsilon \neq \omega^{\ell-1}$, $e_1 = 0$ and thus $H^1(\mathbb{Q}_p, V)$ is one-dimensional over $K_y$. Moreover, since the Hodge-Tate weight of $V$ is $1 - \ell \leq -1$, the subspace of $D$ on which Frobenius acts as the identity is $D^{\phi=1} = 0$, and $\text{Fil}^{1-\ell}(D) \supseteq \text{Fil}^{2-\ell}(D)$. Since (3.8) is an isomorphism and $\dim H^1_{\text{exp}}(\mathbb{Q}_p, V) \leq \dim H^1(\mathbb{Q}_p, V) = 1$, it follows that in fact $\text{Fil}^{1-\ell}(D) = D \supseteq \text{Fil}^0(D) = 0$, and thus (3.8) gives rise to an isomorphism

$$\log_p : H^1_{\text{exp}}(\mathbb{Q}_p, V) = H^1(\mathbb{Q}_p, V) \rightarrow D$$

of one-dimensional $K_y$-vector spaces. The result follows. 

For each arithmetic point $y \in \Omega_2$ of weight $\nu_{\ell,e}$ with $\ell \geq 2$, the choice of a period $\mathcal{O}_y \in D_{\text{dr}}(V_{f, g_y}^{\text{any}})$ attached to the dual $V_{f, g_y}^{\text{any}}$ of $V_{f, g_y}$ determines an isomorphism

$$\log_{\mathcal{O}_y} : H^1(\mathbb{Q}_p, K_y(\Psi_y \cdot \epsilon \cdot \epsilon^{\ell-1} \cdot \omega^{1-\ell})) \rightarrow \mathcal{O}_y, \quad \log_{\mathcal{O}_y}(\kappa) := \log_p(\kappa)(\mathcal{O}_y).$$

Fix a choice of $\mathcal{O}_y$ by defining, for all arithmetic points $y$ of weight $\nu_{\ell,e}$ with $\epsilon \neq 1$ of conductor $p^s$,

$$\mathcal{O}_y := \alpha^s \cdot \mathcal{G}(\chi \omega^{-1})^{-1} \cdot (\eta_f \otimes \omega_{g_y}).$$

Ohta’s $\Lambda$-adic Eichler-Shimura isomorphism can be used to show that these periods agree with those arising in Perrin-Riou’s generalised Coleman maps attached to families of cyclotomic twists of a fixed Galois representations and their generalisations to twists by unramified $\Lambda$-adic characters, as described in [LZ, Thm. 4.15]. (Cf. the discussions in [DR2, §5.2, §5.3], or in [KLZ,
§7.4, §7.6, Thm. 7.6.1.) It follows that, for all \( \kappa \in H^1(\mathbb{Q}, V_{f,g}) \), the function \( y \mapsto \log_{\Omega_y}(\kappa) \) interpolates to an element of the fraction field of \( \Lambda_2 \), denoted \( \mathcal{L}(\kappa) \).

Recall the point \( y \in \Omega_g \) of weight-character \( \nu_{1,1} \) satisfying \( g_y = g_\alpha \), and let \( \Lambda_2' \) denote the localisation of \( \Lambda_2 \) at the prime ideal \( \ker(y) \).

**Theorem 3.9.** For the map \( \mathcal{L} \) as above,

\[
\mathcal{L}(\kappa^f_p(f,g)) = L_p(f,g) \pmod{(\Lambda_2')^\times}.
\]

**Proof.** Theorem 3.5 implies that the elements

\[
\mathcal{L}(\kappa^f_p(f,g)), \quad (1 - \chi(p)^{-1}\alpha_f^{-1}\alpha_g^{-1})^{-1}L_p(f,g)
\]

agree on all points \( y \in \Omega_g \) of weight \( \nu_{2,\epsilon} \), as \( \epsilon \) ranges over all the characters of \( p \)-power conductor. Since both elements belong to the fraction field of \( \Lambda_2 \), it follows that they must be equal. Theorem 3.9 follows after noting that the factor \( (1 - \chi(p)^{-1}\alpha_f^{-1}\alpha_g^{-1}) \) does not lie in the kernel of the weight one point \( y \) attached to \( g_\alpha \) (for reasons of weight, since \( \alpha_f \) has complex absolute \( \sqrt{p} \) while \( \alpha_g \) is a root of unity), and hence belongs to \( (\Lambda_2')^\times \). \( \square \)

At the weight one point \( y \) attached to \( g = g_\alpha \), we have \( H^1_{\mathrm{fin}}(\mathbb{Q}_p, V_{f,g}) = 0 \), and hence the class \( \kappa^f_p(f,g) \) can only be crystalline if it is 0. Let

\[
\exp^*: H^1(\mathbb{Q}_p, V_{f,g}) \longrightarrow \text{Fil}^0 D_{\text{dR}}(V_{f,g})
\]

denote the dual exponential map of Bloch and Kato.

**Theorem 3.10.** Let \( y = \in \Omega_g \) be the point over \( \nu_{1,1} \), corresponding to the classical \( p \)-stabilised weight one form \( g_y = g_\alpha \in S_1(Np, \chi) \). Then

\[
\exp^*(\kappa^f_p(f,g_\alpha)) \neq 0 \iff L(f \otimes g, 1) \neq 0.
\]

**Proof.** Perrin-Riou’s reciprocity law and its extension in [LZ, Thm. 4.15] asserts that, at the weight one point \( y \) attached to \( g_\alpha \),

\[
\mathcal{L}(\kappa_p(f,g))(y_0) \sim \exp^* \kappa^f_p(f,g_\alpha),
\]

where the symbol \( \sim \) denotes agreement up to an (explicit) non-zero element of \( \text{Fil}^0 D_{\text{dR}}(V_{f,g_\alpha}) \). (The Euler factor that arises in the \( p \)-adic interpolation process is non-vanishing at \( y \), since the associated Galois representation \( V_{f,g} \) is pure of weight \( -1 \).) The result now follows by combining Theorems 3.9 and 2.12. \( \square \)

Theorem 3.10 implies that the class \( \kappa(f,g_y) = \kappa(f,g_\alpha) \) of equation (3.7) satisfies the two properties listed in Theorem 1.2. This concludes the proof of Theorems A and B of the Introduction.
References


