Persistence and Stochastic Periodicity in the Intensity Dynamics of a Fiber Laser During the Transition to Optical Turbulence

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Many natural systems display transitions among different dynamical regimes, which are difficult to identify when the data is noisy and high dimensional. A technologically relevant example is a fiber laser, which can display complex dynamical behaviors that involve nonlinear interactions of millions of cavity modes. Here we study the laminar-turbulence transition that occurs when the laser pump power is increased. By applying various data analysis tools to empirical intensity time series we characterize their persistence and demonstrate that at the transition temporal correlations can be precisely represented by a surprisingly simple model.

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INTRODUCTION

Fibre lasers are technologically relevant laser systems that can display complex spatio-temporal dynamics which involve nonlinear interactions of a huge number of cavity modes \cite{1,2,3}. The transition to “optical turbulence” as the laser pump power is increased has received attention and experimental observations of spatio-temporal dynamical regimes have yield new light into the rich underlying nonlinear physics \cite{4,5,7}.

In \cite{4,5} the laser output intensity was investigated by exploiting the analogy with an spatially extended system: the evolution of the intensity during one cavity round-trip time occurs in a “space-like dimension”, while the evolution during many round-trips, occurs in a “temporal dimension”. This two-dimensional representation of the output of dynamical systems that have a well-defined characteristic time-scale (such as the cavity round-trip time) has proven to be very useful to uncover hidden space-like features, such as defects and dislocations \cite{8,9}.

In the fiber laser, long-range correlations from one round-trip to the next (i.e., in the temporal dimension) have been identified \cite{4,5,10,11}, and during the laminar-turbulence transition, shorter correlations (in the space-like dimension) have also been detected \cite{7}. They were identified with two time series analysis methods: the horizontal visibility graph (HVG) \cite{10,11} and ordinal analysis \cite{12,13,14} (described in Appendix A).

The HVG method maps a time series into a graph that keeps information about the temporal ordering of the data points in the time series. In \cite{7} the graph was characterized by Shannon entropy, \( S[P] \), computed from the degree distribution, \( P(k) \), that gives the probability that a node (i.e., a data point \( I_t \)) has \( k \) links. \( S[P] \) (referred to as HVG entropy) was computed for various laser pump powers. When the intensity time series were pre-processed such that only the intensity peaks higher than a threshold were analyzed, a sharp decrease of the HVG entropy was detected at the transition. In contrast, the HVG entropy decreased smoothly when all the data points were analyzed. Ordinal analysis transforms a time series into a sequence of symbols (ordinal patterns, OPs), also keeping the information about the temporal ordering of the data points in the time series. The OPs sequence is then characterized by the ordinal probabilities and the associated entropy (permutation entropy, PE). In \cite{7} it was shown that the PE displays, as the pump power increases, the same variation as the HVG entropy. In addition, by using a lag-time to define the OPs, short scale correlations (in the space-like dimension) were detected.

Temporally correlated signals have been observed in many fields (examples include EEG signals, stock market prices, water flows through rivers, earthquake inter-event intervals, rainfall and climatic time series, among many others). A lot of work has been devoted to characterize them (typically the persistence, measured by the Hurst exponent, \( \mathcal{H} \)) and to understand the physical mechanisms underlying such correlations.

Here we re-analyze the fiber laser empirical data studied in \cite{7} to address both issues. In the first part, by using the HVG method, we identify synthetic data with similar short correlations as the laser data. Specifically, we consider a stochastic processes with known \( \mathcal{H} \): \textit{fractional Brownian motion} (fBm) \cite{10,11,16,22}. When \( \mathcal{H} > 0.5 \) consecutive increments tend to have the same sign and the fBm process is persistent; in contrast, when \( \mathcal{H} < 0.5 \) consecutive increments tend to have opposite signs, and the process is anti-persistent. In the second part of our work we demonstrate, by using ordinal analysis, that a very simple model accounts for the correlations among lagged intensity values that were identified in \cite{7}.
EXPERIMENTAL SETUP AND DATASETS

The experimental setup and datasets are described in [7]: the laser is a Raman fiber (normal dispersion) of 1 km placed between two fiber Bragg gratings acting as cavity mirrors; the pump power is varied from below to above the transition (which occurs for 0.9 W), and for each pump power a time series with $5 \times 10^7$ data points was recorded, with resolution $dt = 0.0125$ ns.

Examples of intensity time traces and the corresponding Fourier spectra are shown in Fig. [1]. At the transition noisy oscillations are seen with a periodicity of about 2.5 ns, while in the broadband power spectrum there is a narrow peak at 0.4 GHz. As it will be shown latter, the appearance of this noisy periodicity can be understood in terms of a surprisingly simple model.

RESULTS

Horizontal Visibility Graph analysis

First, we compare the empirical intensity time series with synthetic series generated by fBm, which is a family of processes, $B_H(t)$, that is Gaussian, self-similar and endowed with stationary increments, $W_H(t) = B_H(t + 1) - B_H(t)$ (which are known as fractional Gaussian noise, fGn). fBm has tunable memory and $B_H = 0.5$ corresponds to ordinary, memory-less Brownian motion, for which successive increments, $W_H = 0.5$, represent Gaussian white noise.

Using the HVG method (described in Appendix A) we transform the empirical and synthetic time series into graphs, and then compare the graphs by comparing their degree distributions, $P(k)$. The simplest way to do this is to fit $P(k)$ to an exponential, $P(k) \propto \exp(-\lambda k)$, and compare the values of $\lambda$. A second way is based in comparing information measures computed from $P(k)$: the HVG entropy and the Fisher information [23].

Figure 2(a) displays the degree distribution, $P(k)$, obtained from the raw intensity time series below, at, and above the transition to turbulence, and also from Gaussian white noise. These distributions can be fitted to $\exp(-\lambda k)$ with $\lambda = 0.59$, 0.69 and 0.75 for pump power 0.8 W (before), 0.9 W (at) and 1.5 W (after the transition). By comparing these $\lambda$ values with those obtained from fBm generated with different $H$ values [23], and considering the same scaling region ($3 \leq k \leq 20$), we infer the Hurst exponent to be $H = 0.3$, 0.5 and 0.6 – 0.7 for 0.8 W, 0.9 W and 1.5 W, respectively. Therefore, the dynamics changes from anti-persistent (before the transition) to persistent (after the transition). At the transition, $\lambda$ is comparable to the one found for fBm with $H = 0.5$ that corresponds to pure Brownian motion.

In contrast, if we first threshold the raw data and keep only the intensity peaks that are higher than a certain threshold (as in [7], we use a threshold equal to the intensity mean value plus two standard deviations) we find that the thresholded data has very different properties: $\lambda$ values are consistent with those found for fGn [24] with $\lambda = 0.403$ ($H = 0.5$) before the transition, $\lambda = 0.43$ ($H = 0.7$) at the transition and $\lambda = 0.395$ ($H \sim 0.4-0.5$) after the transition. $\lambda = 0.43$ at the transition suggests a persistent fGn, while all other $\lambda$ values are close to the $\lambda$ expected for Gaussian white noise ($\lambda = 0.405$).
This effect of the threshold resembles the threshold sensitivity found in certain chaotic systems, where varying the threshold can lead from clustering to repelling of extreme events, or vice versa [21].

The fit of $P(k)$ to an exponential has the drawback that detailed information about the shape of $P(k)$ is lost, and also, that $\lambda$ depends on the scaling region. Therefore, to compare the distributions, $P(k)$, derived from empirical and from synthetic data we use information-theory measures: the HVG entropy, $S[P]$, and the Fisher information, $F[P]$ [25]. In this approach, each time series is represented as a dot in the $S \times F$ plane and “trajectories” are obtained when the experimental parameter (laser pump power) and the synthetic parameter (Hurst exponent $H$) are varied.

In Fig. 3 the results of the analysis of the raw and the thresholded intensity data are compared to synthetic data (fBm and fGn). In the raw data there is a good agreement to what was inferred with the $\lambda$ fit: the raw data is well modeled by the fBm process; before and at the transition is close to fBm with $H \sim 0.5$ (ordinary Brownian motion), while after the transition is close to fBm with $H > 0.5$, which indicates a persistent process. The thresholded intensity peaks are well represented by the fGn processes, also in good agreement with the $\lambda$ fit.

In Fig. 3 we note that the raw intensity datasets are located close to the fBm “trajectory”, but there is a more or less constant distance to this line, which indicates that fBm does not fully represent the empirical data. This is consistent with the analysis of [27], where the probabilities of ordinal patterns of length three were computed analytically for a fBm process, and it was shown that they have a particular symmetry [$P(012) = P(210) = p$, while the other four probabilities are equal to ($1/2 - p$)/2], which does not hold for the probabilities computed from the empirical data [as it can be seen in [7], Fig. 2(a)].

**Ordinal analysis**

Next, we show that the correlations uncovered in [7] by applying lagged ordinal analysis to the raw intensity data can be understood in terms of a surprisingly simple model. The ordinal method (described in Appendix A) is used to uncover order relations among three intensity values ($I_i, I_{i+\tau}, I_{i+2\tau}$), where $\tau$ is an integer that gives an effective sampling time of $\tau dt$. In this way, each symbol encodes information about the intensity evolution during an interval of $3\tau dt$.

The analysis of the ordinal probabilities vs. $\tau$, shown in Fig. 4 reveals that below and above the transition there are no long-range correlations, as in panels [4, a] and [4, d], for $\tau$ large enough, the six patterns are equally probable. In contrast, at the transition [Fig. 4(b)] the pattern probabilities oscillate regularly with periodicity of about 2.5 ns. For a pump power slightly above the transition, Fig. 4(c), there are also regular oscillations of the probabilities with $\tau$, but the oscillations are of smaller amplitude.

Let us next show that these oscillations are captured by a remarkably simple model: a phase equation describing an stochastic limit cycle.

Considering that the polar coordinates of a particle
moving in a limit cycle trajectory are \( a(t)e^{i\omega_0 t} \) and neglecting the amplitude variations, the dynamics is described by a single rate equation for the phase:

\[
d\phi/dt = \omega_0 + f(\phi, t) + \zeta(t),
\]

where \( \omega_0 \) is the angular rotation frequency, \( f(\phi, t) \) is a \( 2\pi \) periodic function \( [f(\phi, t) = f(\phi + 2\pi, t)] \) that represents the variability of the instantaneous frequency, and \( \zeta \) represents stochastic fluctuations.

By stroboscopic sampling every time interval \( \Delta T \), the limit cycle evolution is described by a circle map [25]:

\[
\phi(t + \Delta T) = \phi(t) + \omega_0 \Delta T + F(\phi, t) + \xi(t),
\]

where \( F(\phi, t) \) is a \( 2\pi \) periodic function that represents the phase accumulated over the time interval \( \Delta T \), due to the variability of the instantaneous frequency, and \( \xi \) represents the influence of the stochastic term. Assuming \( F(\phi, t) = K \sin(\phi) \) gives

\[
\phi_{i+1} = \phi_i + \epsilon \rho + (K/2\pi) \sin(2\pi \phi_i) + D \xi_i.
\]

where \( \phi_i = \phi(t) \), \( \phi_{i+1} = \phi(t + \Delta T) \), \( \rho = \nu_0 \Delta T \) with \( \nu_0 = \omega_0/2\pi \). We also include a parameter \( \epsilon = \pm 1 \) that determines the direction of the rotation (anticlockwise or clockwise). In the following we assume that \( \xi_i \) a Gaussian white noise and \( D \) is the noise strength.

Next, we apply ordinal analysis to phase increments, \( \Delta \phi_i = \phi_i - \phi_{i-1} \), generated from iterations of this map. We keep constant the strength of the nonlinearity, \( K \), and the strength of the noise, \( D \), and vary \( \rho \) as control parameter. We chose \( \rho \) because it is proportional to the stroboscopic sampling time, \( \Delta T \), which is analogous, in the experimental situation, to the effective sampling time of the laser intensity, \( \tau \) \( dt \), used to define ordinal patterns from lagged intensity values.

For appropriated values of \( K \) and \( D \) we find that the circle map gives a set of ordinal probabilities that are in remarkable agreement with those computed from the laser data at the transition.

Figure 5 allows a precise comparison: panels (a) and (b) display in detail the oscillatory behavior of the probabilities with the effective sampling time of the intensity time series, while panels (c) and (d) display the probabilities computed from iterations of the circle map. We observe an excellent agreement as the same hierarchically structure (more/less probable patterns) and clustered structure (pairs of patterns with the same probability) are seen when comparing the empirical and the synthetic data. We note that the ordinal probabilities at the transition, Fig. 5(a), are reproduced by the iterations of the circle map with \( \epsilon = 1 \), Fig. 5(c), while slightly above the transition, Fig. 5(b), with \( \epsilon = -1 \), Fig. 5(d). This suggests that immediately after the transition there is a change of rotation.

Contrasting similar situations in Figs. 5(a) and 5(c), we observe that \( \rho = 1 \) in the circle map data corresponds to \( \tau = 2.5 \) ns in the laser data (as indicated with arrows). Because \( \rho = \nu_0 \tau \), using \( \rho = 1 \) and \( \tau = 2.5 \) ns we can estimate the frequency of the rotation in the limit cycle as \( \nu_0 = 1/\tau = 0.4 \) GHz, in agreement with the narrow peak seen in the spectrum in Fig. 1. Also comparing similar situations immediately after the transition, in Figs. 5(b) and 5(d), we observe that \( \rho = 2 \) in the circle map data corresponds to \( \tau = 4.3 \) ns in the laser data (as indicated with arrows). The same argument gives \( \nu_0 = 2/\tau = 0.46 \) GHz, and the spectrum in Fig. 1 we see the peak at about 0.93 GHz, which is consistent with \( 2\nu_0 \).

The agreement found is unexpected because, as shown in Fig. 1 the spectrum is extremely broad and thus, it is surprising to find that, in the symbolic representation, the intensity temporal dynamics is described by an stochastic limit cycle with rotation frequency \( \nu_0 \). It is worth noticing that statistics of the intensity values are not described by the statistics of the phase increments, \( \Delta \phi_i = \phi_i - \phi_{i-1} \), which are positive or negative.

**CONCLUSIONS**

To summarize, we have applied two data analysis tools to characterize persistence and temporal correlations in the intensity dynamics of a fiber laser. To characterize the persistence, intensity time series (raw and thresholded data) were transformed to graphs through the horizontal visibility algorithm and then compared with well-known stochastic processes, fractional Brownian motions (fBm) and fractional Gaussian noises (fGn).

Two different techniques that use the graph degree distribution (fitting the distribution to an exponential, and computing, from the degree distribution, the Shannon entropy and the Fisher information) gave consistent re-
results, with the raw intensity data being modeled by a fBm processes, and the thresholded data, by a fGn. The analysis of the raw data revealed that the dynamics is very close to the fBm process, but there is a distance that suggests a degree of determinism in the dynamics. With respect to the thresholded data, the HVG analysis reveals that the empirical data is fully consistent with an stochastic fGn process.

Using ordinal analysis we have also demonstrated that at the transition, correlations among three lagged intensity values can be precisely represented by a surprisingly simple model: a circle map. The physics underlying the emergence of stochastic periodicity at the transition could be mode locking; however, is computationally unfeasible to simulate the fiber laser model with a realistic number of modes (up to a million in the turbulent region). Therefore, in Appendix B, in order to check the generality of our results, we analyze two multi-mode laser models and find, for certain parameters, a variation of the ordinal probabilities with the lag, similar to that found in the empirical laser data at the transition.

Because of the unavailability of a model that reproduces the fiber laser dynamics in a time scale shorter than the cavity round trip time, our work is limited to the characterization of the empirical data, and leaves several relevant open questions: the physical origin of the temporal correlations, the relation between the characteristic time scale and the physical laser parameters, and the reason why the statistics of the raw and thresholded time series are different. We hope that our results will motivate further experimental and theoretical studies to address these issues.

Extracting information from observed data is a main challenge in diverse areas of engineering and science, and the analysis tools used here can be very useful for investigating the output signals of other systems that undergo similar transitions to turbulent regimes [29, 30]. They provide complementary insights into the correlations present in the data: the HVG method is parameter-free and captures correlations whose range is limited only by the actual values of the data points; on the other hand, the ordinal method (that only considers the temporal order of the data points but not their actual values) has two parameters (the length of the pattern and the lag between the data points) which allow tuning the scale of the analysis.

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APPENDIX A: METHODS OF TIME-SERIES ANALYSIS

In this Appendix we describe the two methods used in the main text to uncover temporal correlations in the intensity dynamics: the Horizontal Visibility Graph [10] and Ordinal Analysis [12].

Horizontal Visibility Graph

This method transforms a real time-series, \( x = \{x_1, \ldots, x_N\} \), into a graph, by connecting pairs of data points if there is “horizontal visibility” between them: \( x_i \) and \( x_j \) are connected if \( x_i > x_k \) and \( x_j > x_k \) for all \( k \) such that \( i < k < j \).

As an example, we consider the following time series:

\[
    x = \{0.71, 0.53, 0.56, 0.89, 0.50, 0.77, 0.21, 0.6, 0.72, 0.35\}. \tag{4}
\]

The number of links, \( k_i \), that each data point, \( x_i \), has is

\[
    k = \{3, 2, 3, 4, 2, 5, 2, 3, 3, 1\}. \tag{5}
\]
The degree distribution, \( P(k) \), is the probability that a data point has \( k \) links. As an example, Fig. 9 displays a graph that is obtained from the empirical laser data.

**ordinal probabilities**

from where we obtain the following set of probabilities

\[
D = 3 \text{ consecutive values is } \{012, 021, 102, 120, 201, 210\}, \quad (6)
\]

The symbolic sequence obtained by considering the ordering of \( D = 3 \) consecutive values is

\[
s = \{201, 012, 120, 201, 012, 120\}, \quad (7)
\]

from where we obtain the following set of probabilities (ordinal probabilities)

\[
P(012) = 2/8, \ P(021) = 0, \ P(102) = 0,
\]

\[
P(120) = 3/8, \ P(201) = 3/8, \ P(210) = 0.
\]

By considering a lag the ordinal method can be used to analyze order relations among non-consecutive data values. In this case the ordinal patterns are defined from the temporal ordering of \( D \) lagged data points, \( x_i, x_{i+\tau}, \ldots, x_{i+D\tau} \), where \( \tau \) is an integer that, if \( dt \) is the sampling time, gives an effective sampling time of \( \tau dt \).

In order to gain insight into the probabilities computed when data points are not consecutive but have a lag between them, we analyze the case of a noisy periodic signal: we consider a synthetic time-series generated from a sinusoidal of period \( T = 1 \),

\[
x(t) = \sin(2\pi t) + \eta \xi, \quad (8)
\]

where \( \xi \) is a Gaussian white noise with zero-mean and unit variance, and \( \eta \) is the noise strength.

We first generate a time-series of \( N = 4000 \) data points with \( \eta = 0.01 \) (small noise in comparison with the oscillation amplitude, which is equal to 1). The data points are sampled with \( \Delta t = 1/20 \) (i.e., the time-series covers 200 periods). Figure 8 displays a short section of the time-series (five periods) and the probabilities of the six \( D = 3 \) ordinal patterns vs. the lag (the integer numbers between 1 and 6 label the six patterns shown in Fig. 7). We note that the probabilities are divided in two groups: in one group are the “V” patterns (102, 201) and the “lambda” patterns (120, 021), while the other group includes the two “trend” patterns (012 and 210). The groups are either more or less probable, depending on the lag. An inspection of the time-series allows to gain insight into this dependency: if three data points are lagged by \( \tau = T/2 \) they form either a “V” or a “lambda” pattern, while if the three data points are lagged by \( \tau = T \), they are equal if \( \eta = 0 \) (as we have a pure sinusoidal of period \( T \)), but when \( \eta \neq 0 \) the differences among the three data points are fully random and thus, their temporal order is random. Therefore, when \( \tau = T \) all patterns are equally probable.

Next, we consider stronger noise. As seen in Fig. 9 which is done in the same way but with \( \eta = 1 \), the variation of the ordinal probabilities with the lag is more smooth, and resembles that found in the empirical laser data in the main text.

**APPENDIX B: MULTI-MODE LASER MODELS**

In this Appendix we present simulations of different laser models, and compare the results with those obtained from the empirical laser data (presented in the main text).

We performed extensive simulations of the laser model used in [3], however, with a limited number of modes this model only gives a good description of the temporal
Three data points (filled circles) lagged by τ = T/2 give either a “V” pattern (102, 201) or a “lambda” pattern (120, 021). The filled squares indicate three data points lagged by τ = T, which give, due to the small added noise, any pattern with equal probability. Bottom: ordinal probabilities computed from the synthetic time series, vs. the lag. We note that for τ = T/2 and τ = 3T/2 the “V” and “lambda” patterns have high probability, while for τ = T all patterns are equally probable.

We first use a multi-longitudinal-mode model where modal competition leads to an oscillatory behavior of the total intensity \[ T \], and then we use a two-mode model, which is well-known to describe the dynamics of two orthogonal linear polarization modes of vertical cavity surface emitting lasers (VCSELs) \[ 5,2 \]. In both cases we show that, for appropriated parameters, the models predict an output intensity that has a similar variation of the ordinal probabilities with the lag, as that described for the fiber laser in the main text. Understanding the underlying mechanisms will require detailed investigations, as they will likely be different in the different models. However, our goal here is only to demonstrate that the correlations detected with ordinal analysis in the empirical data (fiber laser), can also be found in the output of two multimode laser models. A detailed analysis of the models’ predictions is in progress and will be reported elsewhere.

**Multi-longitudinal-mode model**

In Ref. \[ 31 \] mode switching in a multimode semiconductor laser was studied experimentally and it was observed that the intensity of each mode displays large amplitude oscillations, which obey a highly organized antiphase dynamics leading to an almost constant total intensity output. A multimode model was proposed that identified four-wave mixing as the dominant mechanism at the origin of the observed dynamics. Here we simulate the model rate equations that couple N modal complex fields, \[ \textbf{A}_m(t) \], with N spatial harmonics of the carrier density, \[ \textbf{F}_m(t) \sim \int N(z,t)\phi_m^2(z)dz \], that represent the grating in the carrier density created by the standing waves of the field. The model equations are

\[
\frac{dA_m}{dt} = (G_m - 1)A_m - i\sigma \sum_{k,p} A_k A_p A_{k+p-m}, \\
\frac{dF_m}{dt} = J - F_m \left( 1 + \sum_n \beta_{mn} |A_n|^2 \right).
\]

Here \[ G_m = F_m(1 - \epsilon \sum_n \tilde{\beta}_{mn} |A_n|^2) \], \[ \tilde{\beta}_{mn} \] and \[ \beta_{mn} \] are self- and cross-coupling coefficients, \[ \tilde{\beta}_{mn} = \beta_{mn} = 1, \tilde{\beta}_{mn} = (4/3)\beta_{mn} \]. \( \sigma \) is the strength of the four-wave-mixing term; the restriction in the double sum is that \( 1 \leq k + p - m \leq N \). Time is in units of the cavity round trip time, \( \tau_p \), and \( \eta \) is the ratio between \( \tau_p \) and the carrier life time, \( \tau_x \). \( J \) is the pump normalized to the threshold value. The model in Ref. \[ 31 \] also includes the well known \( \alpha \) factor that accounts for symmetry breaking leading to a modal switching sequence from the blue to the red side of the optical spectrum. Here for simplicity we take \( \alpha = 0 \).

The model equations were simulated with the same parameters as in \[ 31 \]: \( J = 1.5, \eta = 1000, \tau_p = 0.01 \text{ ns}, \).
\( \epsilon = 0.05 \), \( \beta_{mn} = 0.975 \) \((\forall \ m, n, m \neq n)\), and \( \sigma = 0.35 \). Results for 4 modes are presented in Fig. 1. The variation of the ordinal probabilities with the lag is qualitatively very similar to that shown in the main text. Similar results are found with a higher number of modes, as shown in Fig. 11. Moreover, the plot of the entropy (\( H = -\sum p_i \log p_i \)) computed from the ordinal probabilities (known as permutation entropy \[^{14}\]) as a function of the lag and of the number of modes, shown in Fig. 12, suggests that the length of the correlations is maximum for a particular number of modes (for the parameters in Fig. 12 with 8 modes the entropy displays the longest oscillatory behavior with the lag). A detailed characterization of this effect is out of the scope of the present work and is left for future work.

**VCSEL model**

Here we simulate a well-known two-mode VCSEL model, which represents the competition of two linearly polarized modes. The model equations are \[^{32}\]:

\[
\begin{align*}
\frac{dE_+}{dt} &= k(1 + i\alpha)(N \pm n - 1)E_\pm - (\gamma_a + i\gamma_p)E_\mp, \\
\frac{dN}{dt} &= \gamma_N \left( J - N - (N + n)\lvert E_+ \rvert^2 - (N - n)\lvert E_- \rvert^2 \right), \\
\frac{dn}{dt} &= -\gamma_sn - \gamma_N \left((N + n)\lvert E_+ \rvert^2 + (N - n)\lvert E_- \rvert^2 \right).
\end{align*}
\]

Here \( E_\pm \) are the slowly varying amplitudes of the left and right circularly polarized components of the optical field \((E_+ = E_x \pm iE_y \) with \( E_x \) and \( E_y \) being the orthogonal linearly polarized components), \( N \) is the total population difference between conduction and valence bands, and \( n \) is the population difference between the carrier densities with positive and negative spin values. \( k \) is the field decay rate, \( \gamma_N \) is the decay rate of the total carrier population, and \( \gamma_s \) is the decay rate which accounts for the mixing of the populations with different spins. \( \gamma_a \) and \( \gamma_p \) represent gain anisotropies and birefringence, respectively. \( \alpha \) is the linewidth enhancement factor and \( J \) is the normalized injection current.

We simulated the model equations with typical parameters, \( k = 300 \ \text{ns}^{-1}, \ \gamma_N = 1 \ \text{ns}^{-1}, \ \gamma_s = 50 \ \text{ns}^{-1}, \ \gamma_a = -0.1 \ \text{ns}^{-1} \) and \( \alpha = 3 \), choosing \( J \) and \( \gamma_s \) such that both linear polarizations are unstable. Results are presented in Fig. 13 where we note that mode competition leads to an oscillatory dynamics in the total intensity, \( \lvert E_+ \rvert^2 + \lvert E_- \rvert^2 \), which, when analyzed with ordinal patterns, has a similar variation of the ordinal probabilities with the lag, as that found in the main text. We present deterministic simulations but we have verified that the inclusion of spontaneous emission noise did not have any noticeable effect in the variation of the probabilities with the lag.
FIG. 10. Numerical results obtained from simulation of the multi-mode model with $N = 4$ modes. The modal intensities (top) and the total intensity (center) are plotted vs. time. The bottom panel displays the ordinal probabilities (computed from the total intensity time-series, with ordinal patterns determined by the ordering of 3 intensity data points separated a lag $\tau$) vs. the lag $\tau$.

FIG. 11. Influence of the number of modes in the total intensity dynamics of the multimode model. Top: 8 modes, bottom: 16 modes.

FIG. 12. Entropy computed from the ordinal probabilities in color code (normalized to the maximum entropy value, which corresponds to equally probable ordinal patterns) vs. the lag and the number of longitudinal modes.

FIG. 13. Results of simulations of the VCSEL model with parameters $J = 3$, $\gamma_\alpha = -0.1$ ns$^{-1}$ and $\gamma_\rho = 11.75$ rad ns$^{-1}$ (top row), $\gamma_\rho = 12$ rad ns$^{-1}$ (lower row). The modal intensities (left panels) are shifted vertically for clarity.