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Spatial behavior in high order partial differential equations

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In this paper we study the spatial behavior of solutions to the equations obtained by taking formal Taylor approximations to the heat conduction dual-phase-lag and three-phase-lag theories, reflecting Saint-Venant's principle. In a recent paper, two families of cases for high order partial differential equations were studied. Here we investigate a third family of cases which corresponds to the fact that a certain condition on the time derivative must be satisfied. We also study the spatial behavior of a thermoelastic problem. We obtain a Phragmén-Lindelöf alternative for the solutions in both cases. The main tool to handle these problems is the use of an exponentially weighted Poincaré inequality. Copyright © 2009 John Wiley & Sons, Ltd.

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1. Introduction

It is well known that the juxtaposition of the Fourier constitutive equation of heat flux vector with the classical energy equation

$$-\operatorname{div} q(x,t) = c\theta_t(x,t), \qquad c>0, \tag{1}$$

brings about paradoxical behavior on the solutions. In particular, it has been shown that the perturbations at one point of a solid can be observed in any other point of it instantly, however distant. This is a drawback of the model and to overcome it and, at the same times, to satisfy the principle of causality, several modifications of the model have been suggested recently (see, for example, the reviews [1, 2, 3]). These modifications gave rise to new thermoelastic theories which are still being the aim of study of mathematicians, physicists and engineers. The applicability of these alternative thermoelastic theories to the real world situations has been analised in many works, including the books of Ignaczak et al. [4], Straughan [5] and Wang et al. [6].

One of these modifications was suggested in 1995 by Tzou [7]. He proposed a theory where the thermal flux and the gradient of temperature have a delay. The constitutive equation is:

$$q(x, t+\tau_q) = -k\nabla\theta(x, t+\tau_\theta), \qquad k>0. \tag{2}$$

Here q is the heat flux vector, θ is the temperature and τ_q, τ_θ are the delay parameters which are assumed to be positive. This equation says that the temperature gradient established across a material volume at the position x at time $t+\tau_\theta$ results in a heat flux to flow at a different instant of time $t+\tau_q$. The delays are understood in terms of the microstructure of the material.

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Choudhuri [8] proposed an extension of Tzou's equation. His constitutive equation for the heat flux vector is

$$q(x, t + \tau_q) = -\left(k\nabla\theta(x, t + \tau_\theta) + k^*\nabla\nu(x, t + \tau_\nu)\right). \tag{3}$$

The new variable ν is the thermal displacement, and it satisfies $\nu_t = \theta$, k^* is the rate of thermal conductivity, a new parameter which is typical in the type II and III thermoelastic theories proposed by Green and Naghdi [9, 10], and τ_ν is another delay parameter which is also assumed to be positive.

These two theories are strongly based on an intuitive point of view, but there are no *a priori* thermomechanical foundations for them. Furthermore, it can be proved that, when the proposed constitutive equations are combined with the classical energy equation (1), a sequence of solutions of the form

$$\theta_n(x, t) = \exp(\omega_n t)\Phi_n(x)$$

can be found with the real part of ω_n tending to infinity [11]. Therefore continuous dependence on initial data cannot be obtained, and the associated mathematical problem is *ill posed* in the sense of Hadamard. This kind of result is not expected *a priori*. For this reason a big interest has been developed to understand different formal Taylor approximations to these equations [12, 13, 14, 15, 16, 17, 18, 19]. In fact, the literature concerning these topics is increasing quickly because many researchers are interested in these proposals. These new theories allow to obtain stability of solutions and the well-posedness of the problems, provided that certain conditions on the parameters hold.

In a couple of recent papers Quintanilla [20, 21] proposed a modification of the theories of Tzou and Choudhuri by means of the energy equation usual for the two temperature theories (see [22, 23, 24, 25]):

$$-\operatorname{div} q(x, t) = c\Theta_t(x, t), \qquad c > 0, \;\; \Theta = \theta - a\Delta\theta, \;\; a > 0. \tag{4}$$

In this case, the associated mathematical problem is well posed. It is also interesting to consider different formal Taylor approximations to the equations proposed at [26, 27]; in this way, we obtain models which are currently under study (see for instance [28, 29, 30, 31]).

We consider general Taylor approximations to the dual-phase-lag or three-phase-lag theories. Plugging these into the energy equation (1), we obtain the heat equation[†]:

$$a_0\theta + a_1\theta^{(1)} + a_2\theta^{(2)} + \cdots + a_n\theta^{(n)} = b_0\Delta\theta + b_1\Delta\theta^{(1)} + \cdots + b_m\Delta\theta^{(m)}. \tag{5}$$

where $a_0, \ldots, a_n, b_0, \ldots, b_m$ are constants. It is worth noting that the case $n = m$ arises naturally for the Taylor approximations to the theories proposed by Quintanilla [20, 21].

The study of the *spatial* behavior for partial differential equations is related to Saint-Venant's principle. This topic has been investigated from the mathematical and thermomechanical viewpoints. Spatial decay estimates have been obtained for elliptic [32], parabolic [33, 34], hyperbolic [35] equations and/or combinations of them [36] in the last years. The authors try to describe how the perturbations on a part of the boundary are damped far away from the place where they were applied. From a mathematical viewpoint, it is usual to consider a semi-infinite cylinder whose finite end is perturbed and to study how the solutions decay when the large spatial variable tends to infinity. From a mathematical perspective, the spatial behavior of the solutions is an important issue to be studied [37, 38, 39, 40, 41, 42]. It is worth noting that the uniqueness of solutions cannot be expected in this context. In fact, the problem is ill posed in the sense of Hadamard and only a Phragmén-Lindelöf alternative for the solutions can be obtained.

Although the spatial behavior for dual-phase-lag or three-phase-lag models have been studied in several cases [43, 44, 26, 27], only equations up to fourth order with respect to the time variable were analysed. Arguments for a general type of higher-order equations have been restricted to the case $n - m = 1, 2$ [45]. In this paper, we will obtain spatial estimates for solutions of higher order equations when $m \geq n$. In the second part of the paper we consider the particular case $n = m + 1$ in the thermoelastic

[†]Here and from now on, $g^{(k)}$ denotes the k-th derivative of the function g with respect to time.

context and we also obtain spatial decay estimates. It is worth recalling that the existence of solutions for the thermoelastic problem we consider here has been obtained by means of the semigroup approach [46].

The plain of the paper is as follows. In Section 2 we propose the basic problems we are going to work with and we recall the exponentially weighted Poincaré inequality because it will be a fundamental tool in our approach. We obtain a Phragmén-Lindelöf alternative in Section 3 for a certain heat equation. Later, in Section 4 we describe how to obtain an upper bound for the amplitude term when the solution decays in the spatial variable. In Section 5 we consider a thermoelastic problem and we obtain again a Phragmén-Lindelöf alternative for the solutions. Section 6 is devoted to compute the corresponding upper bound for the amplitude term in terms of the boundary data when the solution decays. Last section exposes the conclusions of the work.

2. Preliminaries

In this section we define the two problems that we will study, and we recall a fundamental tool that we use in our approach: the *exponentially weighted Poincaré inequality*. The spatial domain is the semi-infinite cylinder $R = [0, \infty) \times D$, where D is a bounded domain in the two-dimensional Euclidean space, being smooth enough to guarantee the use of the divergence theorem.

The first problem is defined by the equation

$$a_0\theta + a_1\theta^{(1)} + a_2\theta^{(2)} + \cdots + a_m\theta^{(m)} = b_0\Delta\theta + b_1\Delta\theta^{(1)} + \cdots + b_m\Delta\theta^{(m)}, \tag{6}$$

where m is an arbitrary positive natural number, b_m is strictly positive and $a_m \geq 0$.

We do not consider the existence of solutions question, but we will assume the existence of solutions as well as the necessary regularity required to carry out our calculations.

In addition to the differential equation (6), we impose the initial conditions

$$\theta(x, 0) = \theta^{(1)}(x, 0) = \cdots = \theta^{(m-1)}(x, 0) = 0, \quad x \in R, \tag{7}$$

and the boundary conditions

$$\theta(x_1, x_2, x_3, t) = 0, \quad (x_2, x_3) \in \partial D, \quad t \geq 0, \tag{8}$$

$$\theta(0, x_2, x_3, t) = f(x_2, x_3, t), \quad (x_2, x_3) \in D, \quad t \geq 0. \tag{9}$$

To assure the compatibility, we naturally assume

$$f(x_2, x_3, t) = 0, \quad (x_2, x_3) \in \partial D, \quad t \geq 0.$$

It is worth covering the case where the parameters can be negative. In this sense we allow the parameters a_i $(i = 0, \dots, m-1)$ and b_j $(j = 0, \dots, m-1)$ to be positive, zero or negative. But, we want to point out that our results are new even if all the coefficients are positive.

The second problem we study in this paper comes from a thermoelastic situation. The system of equations that we obtain, following the arguments used by Chandrasekharaiah [1], p.723, are

$$\rho\ddot{u}_i = (C_{ijkl}u_{k,l} - \beta_{ij}\theta)_{,j}, \tag{10}$$

$$c\frac{d}{dt}(a_0\theta + \cdots + a_m\theta^{(m)}) + \beta_{ij}(a_0 v_{i,j} + \cdots + a_m v_{i,j}^{(m)}) = (b_{ij}^0\theta_{,i} + b_{ij}^1\theta_{,i}^{(1)} + \cdots + b_{ij}^m\theta_{,i}^{(m)})_{,j}. \tag{11}$$

Here C_{ijkl} is the elasticity tensor satisfying the symmetry $C_{ijkl} = C_{klij}$, ρ is the mass density, β_{ij} is the coupling tensor, the tensors b_{ij}^l are symmetric for $l = 0, \dots, m$, the constant c is the thermal capacity, u_i is the displacement and $v_i = \dot{u}_i$.

To set the problem we also need to impose initial and boundary conditions. Apart from conditions (7), (8) and (9), we will impose

$$u_i(x,0)=\dot{u}_i(x,0)=\theta^{(m)}(x,0)=0, \quad x\in R, \tag{12}$$

and the boundary conditions

$$u_i(x_1,x_2,x_3,t)=0, \quad (x_2,x_3)\in\partial D, \quad t\geq 0, \tag{13}$$

$$u_i(0,x_2,x_3,t)=g_i(x_2,x_3,t), \quad (x_2,x_3)\in D, \quad t\geq 0. \tag{14}$$

To assure the compatibility, we also assume

$$g_i(x_2,x_3,t)=0, \quad (x_2,x_3)\in\partial D, \quad t\geq 0.$$

In all this paper Greek sub-indices are restricted to the values 2 and 3.

As we said before, a relevant tool in our analysis is the following result (see the appendix of [47] for a proof), the *exponentially weighted Poincaré inequality*:

Assume that $f:[0,t]\to\mathbb{R}$ is differentiable such that $f(0)=0$. Then the following inequality

$$\int_0^t \exp(-2\omega s)f^2(s)\,ds \leq \frac{4t^2}{\pi^2+4t^2\omega^2}\int_0^t \exp(-2\omega s)\left(f^{(1)}(s)\right)^2 ds, \tag{15}$$

holds, for every $\omega>0$. We note that $\varphi(t)=\dfrac{4t^2}{\pi^2+4t^2\omega^2}$ is a growing function, hence

$$\int_0^t \exp(-2\omega s)f^2(s)\,ds \leq \omega^{-2}\int_0^t \exp(-2\omega s)\left(f^{(1)}(s)\right)^2 ds. \tag{16}$$

As a consequence, we obtain for $n>k+1$ and for f satisfying $f^{(k)}(0)=\cdots=f^{(n-1)}(0)=0$, that the estimate

$$\int_0^t \exp(-2\omega s)|f^{(k)}(s)|^2\,ds \leq \omega^{-2(n-k-1)}\int_0^t \exp(-2\omega s)|f^{(n-1)}(s)|^2 ds, \tag{17}$$

holds. These inequalities will allow us to deal with lower-order time derivatives in a comparison with higher-order terms.

3. Spatial estimates

In this section we obtain an alternative of Phragmén-Lindelöf type for the solutions of the problem determined by equation (6), initial conditions (7) and boundary conditions (8), (9).

The function, the properties of which will describe the spatial behavior, is given, for $z\geq 0$, $t\geq 0$, by

$$G_\omega(z,t)=-\int_0^t\int_{D(z)} \exp(-2\omega s)\left(b_0\theta_{,1}+b_1\theta^{(1)}_{,1}+\cdots+b_m\theta^{(m)}_{,1}\right)\theta^{(m)}\,da\,ds, \tag{18}$$

where ω is a positive constant to be chosen later.

Using the divergence theorem we see that

$$\begin{aligned} &G_\omega(z+h,t)-G_\omega(z,t)\\ &=-\int_0^t\int_z^{z+h}\int_D \exp(-2\omega s)\left[Q+a_m|\theta^{(m)}|^2+b_m|\nabla\theta^{(m)}|^2)\right]dv\,ds, \end{aligned} \tag{19}$$

where

$$Q:=(b_0\nabla\theta+b_1\nabla\theta^{(1)}+\cdots+b_{m-1}\nabla\theta^{(m-1)})\nabla\theta^{(m)}+(a_0\theta+a_1\theta^{(1)}+\cdots+a_{m-1}\theta^{(m-1)})\theta^{(m)}. \tag{20}$$

Thus, we get

$$\frac{\partial G_\omega}{\partial z}(z,t) = -\int_0^t\int_D \exp(-2\omega s)\left[Q + a_m|\theta^{(m)}|^2 + b_m|\nabla\theta^{(m)}|^2)\right] da\,ds. \tag{21}$$

To control the function Q we can use a similar argument to the one proposed in [45]. If we consider the exponentially weighted Poincaré inequality, we have that

$$\begin{aligned}&\left|\int_0^t\int_D \exp(-2\omega s)b_k\nabla\theta^{(k)}\nabla\theta^{(m)}da\,ds\right|\\&\leq |b_k|\left(\int_0^t\int_D \exp(-2\omega s)|\nabla\theta^{(k)}|^2 da\,ds\right)^{1/2}\left(\int_0^t\int_D \exp(-2\omega s)|\nabla\theta^{(m)}|^2 da\,ds\right)^{1/2}\\&\leq |b_k|b_m^{-1}\omega^{k-m}\int_0^t\int_D \exp(-2\omega s)b_m|\nabla\theta^{(m)}|^2 da\,ds,\end{aligned} \tag{22}$$

for $k = 0\dots m-1$. In a similar way we see that

$$\left|\int_0^t\int_D \exp(-2\omega s)a_k\theta^{(k)}\theta^{(m)}da\,ds\right| \leq \lambda_1^{-1}|a_k|\omega^{k-m}\int_0^t\int_D \exp(-2\omega s)|\nabla\theta^{(m)}|^2 da\,ds, \tag{23}$$

for $k = 0\dots m-1$. Here, λ_1 is the first eigenvalue of the negative Laplace operator $-\Delta$ with Dirichlet boundary conditions (clamped membrane) in the domain D. It arises in estimating the integral of the square of $\theta^{(m)}$ by means of the integral of the square of $\nabla\theta^{(m)}$.

Therefore, we can obtain the existence of two polynomials Q_1 and Q_2 satisfying $Q_i(0) = 0$, $i = 1, 2$ such that

$$\frac{\partial G_\omega}{\partial z}(z,t) \leq -\int_0^t\int_D \exp(-2\omega s)\left[(b_m - \lambda_1^{-1}Q_1(\omega^{-1}) - Q_2(\omega^{-1}))|\nabla\theta^{(m)}|^2)\right] da\,ds. \tag{24}$$

We can see that, for ω large enough, the estimate

$$\frac{\partial G_\omega}{\partial z} \leq -\int_0^t\int_D \exp(-2\omega s)(b_m - \epsilon)|\nabla\theta^{(m)}|^2 da\,ds \tag{25}$$

holds. In particular, when $\epsilon = b_m/2$, we obtain

$$\frac{\partial G_\omega}{\partial z} \leq -\frac{1}{2}\int_0^t\int_D \exp(-2\omega s)b_m|\nabla\theta^{(m)}|^2 da\,ds \leq 0. \tag{26}$$

We want to evaluate now the absolute value of G_ω in terms of its spatial derivative. With a sufficiently small ϵ, obtained for ω large enough, we have that

$$\begin{aligned}\left|\int_0^t\int_D \exp(-2\omega s)b_k\theta^{(k)}_{,1}\theta^{(m)}da\,ds\right| \leq &|b_k|\left(\int_0^t\int_D \exp(-2\omega s)|\theta^{(k)}_{,1}|^2 da\,ds\right)^{1/2}\left(\int_0^t\int_D \exp(-2\omega s)|\theta^{(m)}|^2 da\,ds\right)^{1/2}\\ \leq &|b_k|(b_m-\epsilon)^{-1}\lambda_1^{-1/2}\omega^{k-m}\left(\int_0^t\int_D \exp(-2\omega s)(b_m-\epsilon)\theta^{(m)}_{,1}\theta^{(m)}_{,1}da\,ds\right)^{1/2}\\ &\times\left(\int_0^t\int_D \exp(-2\omega s)(b_m-\epsilon)\theta^{(m)}_{,\alpha}\theta^{(m)}_{,\alpha}da\,ds\right)^{1/2}\\ \leq &|b_k|(2(b_m-\epsilon))^{-1}\lambda_1^{-1/2}\omega^{k-m}\left(\int_0^t\int_D \exp(-2\omega s)(b_m-\epsilon)\theta^{(m)}_{,i}\theta^{(m)}_{,i}da\,ds\right).\end{aligned} \tag{27}$$

Therefore, we can obtain positive constants

$$D_k := |b_k|(4(b_m-\epsilon)^2\lambda_1)^{-1/2}, \qquad \Omega_\omega := \sum_{k=0}^m D_k\omega^{(k-m)}, \tag{28}$$

such that

$$|G_\omega| \leq -\Omega_\omega \frac{\partial G_\omega}{\partial z}. \tag{29}$$

This inequality is well known in the study of spatial estimates. It implies that

$$G_\omega \leq -\Omega_\omega \frac{\partial G_\omega}{\partial z} \tag{30}$$

and

$$-G_\omega \leq -\Omega_\omega \frac{\partial G_\omega}{\partial z}. \tag{31}$$

For fixed t, we distinguish two cases:

(I) If there exists $z_0 \geq 0$ such that $G_\omega(z_0, t) < 0$, it follows that $G_\omega(z, t) < 0$ for every $z \geq z_0$. From (31) we conclude that

$$-G_\omega(z, t) \geq -G_\omega(z_0, t) \exp\left(\frac{z - z_0}{\Omega_\omega}\right), \quad z \geq z_0. \tag{32}$$

(II) Otherwise, we see that $G_\omega(z, t) \geq 0$ for every $z \geq 0$. From (30), it follows the spatial decay estimate

$$G_\omega(z, t) \leq G_\omega(0, t) \exp\left(-\frac{z}{\Omega_\omega}\right), \quad z \geq 0. \tag{33}$$

We can summarize this result in the following way:

Theorem *Let θ be a solution of the initial-boundary-value problem (6)–(9). Then, for ω large enough, either the function $-G_\omega(z, t)$ satisfies the asymptotic condition (32), or the function*

$$0 \leq G_\omega(z, t) = \int_0^t \int_{R(z)} \exp(-2\omega s) \left[Q + a_m|\theta^{(m)}|^2 + b_m|\nabla\theta^{(m)}|^2)\right] dv\, ds \tag{34}$$

satisfies the decay estimate (33).

Remark *It is worth noting that when ω increases, the parameter $(\Omega_\omega)^{-1}$ tends to $2(\lambda_1)^{1/2}$. If a_m is strictly positive we can improve the value of the constants D_k and therefore the value of Ω_ω. At the same time we note that the proposed analysis can be adapted for a_m negative whenever $b_m + a_m\lambda_1^{-1}$ is positive.*

4. The amplitude term

In this section we will describe how to obtain an upper bound for the amplitude term obtained in the previous section, $G_\omega(0, t)$, in terms of the boundary data. To make the calculations easier we assume in this section that a_m is also strictly positive, but we want to point out that this is not a relevant assumption, because the analysis can be carried out, *mutatis mutandis*, for $a_m = 0$.

From now on, we restrict our attention to solutions satisfying the decay estimate (33) where ω is large enough to guarantee that

$$G_\omega(z, t) \geq \frac{1}{2}\int_0^t \int_{R(z)} \exp(-2\omega s) \left[a_m|\theta^{(m)}|^2 + b_m|\nabla\theta^{(m)}|^2)\right] dv\, ds. \tag{35}$$

We denote by $\xi = \xi(x, t)$ a function which tends uniformly to zero, rapidly, as $x_1 \to \infty$, and satisfies the same boundary conditions as θ. Typically, ξ is choosen as

$$\xi(x, t) := \exp(-dx_1)f(x_2, x_3, t),$$

where d is a positive constant.

Then we have

$$G_\omega(0, t) = -\int_0^t \int_{D(0)} \exp(-2\omega s)(b_0\theta_{,1} + \cdots + b_m\theta_{,1}^{(m)})\xi^{(m)}\, da ds. \tag{36}$$

After the use of the boundary, asymptotic and the initial conditions, and the divergence theorem, we see that

$$G_\omega(0,t) = \int_0^t \int_R \exp(-2\omega s)\left[b_0\nabla\theta + \cdots + b_m\nabla\theta^{(m)}\right]\nabla\xi^{(m)} dvds + \int_0^t \int_R \exp(-2\omega s)\left[a_0\theta + \cdots + a_m\theta^{(m)}\right]\xi^{(m)} dvds. \quad (37)$$

We obtain

$$G_\omega(0,t) = I_1 + I_2, \quad (38)$$

where

$$I_1 := \int_0^t \int_R \exp(-2\omega s)\left[b_0\nabla\theta + \cdots + b_m\nabla\theta^{(m)}\right]\nabla\xi^{(m)} dvds, \quad (39)$$

$$I_2 := \int_0^t \int_R \exp(-2\omega s)\left[a_0\theta + \cdots + a_m\theta^{(m)}\right]\xi^{(m)} dv\, ds. \quad (40)$$

By choosing ω large enough, we can take $\epsilon_i,\ i=1,2$ as small as we want and such that

$$I_1 \le \epsilon_1 \int_0^t \int_R \exp(-2\omega s) b_m |\nabla\theta^{(m)}|^2 dvds + C_1^* \int_0^t \int_R \exp(-2\omega s)|\nabla\xi^{(m)}|^2 dvds, \quad (41)$$

$$I_2 \le \epsilon_2 \int_0^t \int_R \exp(-2\omega s) a_m |\theta^{(m)}|^2 dvds + C_2^* \int_0^t \int_R \exp(-2\omega s)|\xi^{(m)}|^2 dvds. \quad (42)$$

Here C_i^*, $i = 1, 2$, are two constants which can be computed in terms of the data of the problem, ω and ϵ_i. Taking into account (35), it then follows that

$$G_\omega(0,t) \le 2(\epsilon_1 + \epsilon_2) G_\omega(0,t) + C_1^* J_1 + C_2^* J_2, \quad (43)$$

where

$$J_1 := \int_0^t \int_R \exp(-2\omega s)|\nabla\xi^{(m)}|^2 dvds, \quad (44)$$

$$J_2 := \int_0^t \int_R \exp(-2\omega s)|\xi^{(m)}|^2 dvds. \quad (45)$$

If we select ϵ_i such that $\epsilon_1 + \epsilon_2 < 1/4$, we obtain that

$$G_\omega(0,t) \le 2(C_1^* J_1 + C_2^* J_2). \quad (46)$$

To obtain a precise upper bound for the J_i, we recall the choice of the function ξ, given previously, and we note that

$$\xi^{(m)} = \exp(-dx_1) f^{(m)}(x_2, x_3, t), \quad (47)$$

and

$$\nabla\xi^{(m)} = \exp(-dx_1)\left(-df^{(m)}(x_2,x_3,t), f_{,2}^{(m)}(x_2,x_3,t), f_{,3}^{(m)}(x_2,x_3,t)\right). \quad (48)$$

We conclude that

$$J_1 \le \int_0^t \int_{D(0)} \left(\frac{d}{2}|f^{(m)}|^2 + \frac{1}{2d}(|f_{,2}^{(m)}|^2 + |f_{,3}^{(m)}|^2)\right) dads \quad (49)$$

and

$$J_2 \le \frac{1}{2d}\int_0^t \int_{D(0)} |f^{(m)}|^2 dads. \quad (50)$$

From the previous inequalities, we finally obtain

$$G_\omega(0,t) \le \left(dC_1^* + \frac{C_2^*}{d}\right)\int_0^t \int_{D(0)} \left|f^{(m)}\right|^2 dads + \frac{C_1^*}{d}\int_0^t \int_{D(0)} f_{,\alpha}^{(m)} f_{,\alpha}^{(m)} dads. \quad (51)$$

We remark that one could optimize the right-hand side by taking a suitable value of the parameter d, but it does not seem to be an easy task.

5. A thermoelastic system

In this section we investigate the spatial behavior of the solutions of the problem determined by the system (10)-(11) with the initial conditions (7), (12) and the boundary conditions (8), (9), (13) and (14).

In this section we assume that ρ is a positive constant and that the elasticity tensor is positive definite, that is, there exists a positive constant M_1 such that

$$C_{ijkl}\xi_{ij}\xi_{kl} \geq M_1\xi_{ij}\xi_{ij}, \text{ for every tensor } (\xi_{ij}). \tag{52}$$

We also assume that a_m is a positive constant and that the tensor $b_{ij}^{(m)}$ is positive definite, that is, there exists a positive constant b such that

$$b_{ij}^{m}\xi_i\xi_j \geq b\xi_i\xi_i, \text{ for every tensor } (\xi_i). \tag{53}$$

We note that the existence of solutions to the problem proposed here could be obtained by following the arguments proposed in [46].

If we introduce the notation

$$\tilde{g} = a_0 g + \cdots + a_m g^{(m)}, \tag{54}$$

therefore, equation (10) implies

$$\rho\ddot{\tilde{u}}_i = (C_{ijkl}\tilde{u}_{k,l} - \beta_{ij}\tilde{\theta})_{,j}. \tag{55}$$

We now define the function which will describe the spatial behavior. It is

$$G_\omega(z,t) = -\int_0^t \int_{D(z)} \exp(-2\omega s)(L_1 + L_2)\,da\,ds, \tag{56}$$

where

$$L_1 = (C_{i1kl}\tilde{u}_{k,l} - \beta_{i1}\tilde{\theta})\dot{\tilde{u}}_i, \tag{57}$$

and

$$L_2 = (b_{i1}^0\theta_{,i} + b_{i1}^1\theta_{,i}^{(1)} + \cdots + b_{i1}^m\theta_{,i}^{(m)})\tilde{\theta}, \tag{58}$$

again ω is a positive constant to be chosen later. After the use of the divergence theorem we see that

$$\begin{aligned} G_\omega(z+h,t) - G_\omega(z,t) = &- \frac{\exp(-2\omega t)}{2}\int_z^{z+h}\int_D \left(c(\tilde{\theta})^2 + \rho\dot{\tilde{u}}_i\dot{\tilde{u}}_i + C_{ijkl}\tilde{u}_{i,j}\tilde{u}_{k,l}\right)dv \\ &- \omega\int_0^t\int_z^{z+h}\int_D \exp(-2\omega s)\left(c(\tilde{\theta})^2 + \rho\dot{\tilde{u}}_i\dot{\tilde{u}}_i + C_{ijkl}\tilde{u}_{i,j}\tilde{u}_{k,l}\right)dv\,ds \\ &- \int_0^t\int_z^{z+h}\int_D \exp(-2\omega s)\left(Q^* + b_{ij}^m a_m\theta_{,i}^{(m)}\theta_{,j}^{(m)}\right)dv\,ds, \end{aligned} \tag{59}$$

where

$$\begin{aligned} Q^* = &\, a_0 b_{ij}^0\theta_{,i}\theta_{,j} + a_1 b_{ij}^0\theta_{,i}\theta_{,j}^{(1)} + \cdots + a_m b_{ij}^0\theta_{,i}\theta_{,j}^{(m)} \\ &+ a_0 b_{ij}^1\theta_{,i}^{(1)}\theta_{,j} + a_1 b_{ij}^1\theta_{,i}^{(1)}\theta_{,j}^{(1)} + \cdots + a_m b_{ij}^1\theta_{,i}^{(1)}\theta_{,j}^{(m)} \\ &\ldots \\ &+ a_0 b_{ij}^m\theta_{,i}^{(m)}\theta_{,j} + a_1 b_{ij}^m\theta_{,i}^{(m)}\theta_{,j}^{(1)} + \cdots + a_{m-1} b_{ij}^m\theta_{,i}^{(m)}\theta_{,j}^{(m-1)}. \end{aligned} \tag{60}$$

Thus, we get

$$\begin{aligned} \frac{\partial G_\omega}{\partial z}(z,t) = &- \frac{\exp(-2\omega t)}{2}\int_D \left(c(\tilde{\theta})^2 + \rho\dot{\tilde{u}}_i\dot{\tilde{u}}_i + C_{ijkl}\tilde{u}_{i,j}\tilde{u}_{k,l}\right)da \\ &- \omega\int_0^t\int_D \exp(-2\omega s)\left(c(\tilde{\theta})^2 + \rho\dot{\tilde{u}}_i\dot{\tilde{u}}_i + C_{ijkl}\tilde{u}_{i,j}\tilde{u}_{k,l}\right)da\,ds - \int_0^t\int_D \exp(-2\omega s)\left(Q^* + b_{ij}^m a_m\theta_{,i}^{(m)}\theta_{,j}^{(m)}\right)da\,ds. \end{aligned} \tag{61}$$

A similar argument to the one proposed in Section 3 shows that

$$\left|\int_0^t\int_D \exp(-2\omega s)a_l b_{ij}^k\theta_{,i}^{(k)}\theta_{,j}^{(l)}\,da\,ds\right| \leq N_{kl}\omega^{k+l-2m}\int_0^t\int_D \exp(-2\omega s)b_{ij}^m a_m\theta_{,i}^{(m)}\theta_{,j}^{(m)}\,da\,ds, \tag{62}$$

where N_{kl} are calculable constants depending on the constitutive parameters and tensors, but independent of the time and the parameter ω and whenever $k+l<2m$. We obtain the existence of a polynomial Q_1^*, vanishing at zero, such that

$$\begin{aligned}\frac{\partial G_\omega}{\partial z}(z,t) \leq &-\frac{\exp(-2\omega t)}{2}\int_D \left(c(\tilde\theta)^2+\rho\dot{\tilde u}_i\dot{\tilde u}_i + C_{ijkl}\tilde u_{i,j}\tilde u_{k,l}\right)da\\ &-\omega\int_0^t\int_D \exp(-2\omega s)\left(c(\tilde\theta)^2+\rho\dot{\tilde u}_i\dot{\tilde u}_i + C_{ijkl}\tilde u_{i,j}\tilde u_{k,l}\right)da\,ds\\ &-\int_0^t\int_D \exp(-2\omega s)\left(b_{ij}^m a_m(1-Q_1^*(\omega^{-1}))\theta_{,i}^{(m)}\theta_{,j}^{(m)}\right)dv\,ds.\end{aligned} \tag{63}$$

Again, we want to evaluate the absolute value of G_ω by means of its spatial derivative. We note that

$$\left|\int_0^t\int_{D(z)} \exp(-2\omega s)L_1(a,s)\,da\,ds\right| \leq K_1\int_0^t\int_{D(z)} \exp(-2\omega s)\left(\rho\dot{\tilde u}_i\dot{\tilde u}_i + C_{ijkl}\tilde u_{i,j}\tilde u_{k,l}+c(\tilde\theta)^2\right)da\,ds, \tag{64}$$

and

$$\left|\int_0^t\int_{D(z)} \exp(-2\omega s)L_2(a,s)\,da\,ds\right| \leq K_2\int_0^t\int_{D(z)} \exp(-2\omega s)\left(a_m b_{ij}^m(1-\epsilon)\theta_{,i}^{(m)}\theta_{,j}^{(m)}+c(\tilde\theta)^2\right)da\,ds, \tag{65}$$

with ϵ sufficiently small, obtained for sufficiently large ω. Here, K_1 and K_2 can be determined in terms of the constitutive tensors and ϵ, but independent of the time. By taking ω large enough we can determine a positive constant K^* (which depends on ω) such that

$$|G_\omega| \leq -K^*\frac{\partial G_\omega}{\partial z}. \tag{66}$$

It implies that

$$G_\omega \leq -K^*\frac{\partial G_\omega}{\partial z}, \tag{67}$$

and

$$-G_\omega \leq -K^*\frac{\partial G_\omega}{\partial z}. \tag{68}$$

For fixed t, we distinguish two cases:
(I) If there exists $z_0\geq 0$ such that $G_\omega(z_0,t)<0$, it follows that $G_\omega(z,t)<0$ for every $z\geq z_0$. We conclude that

$$-G_\omega(z,t)\geq -G_\omega(z_0,t)\exp\left(\frac{z-z_0}{K^*}\right),\quad z\geq z_0. \tag{69}$$

(II) Otherwise, we see that $G_\omega(z,t)\geq 0$ for every $z\geq 0$. It then follows the spatial decay estimate

$$G_\omega(z,t)\leq G_\omega(0,t)\exp\left(-\frac{z}{K^*}\right),\quad z\geq 0. \tag{70}$$

If the estimate (70) holds, the function $G_\omega(z,t)$ can be written as

$$\begin{aligned}G_\omega(z,t) = &\frac{\exp(-2\omega t)}{2}\int_{R(z)}\left(c(\tilde\theta)^2+\rho\dot{\tilde u}_i\dot{\tilde u}_i + C_{ijkl}\tilde u_{i,j}\tilde u_{k,l}\right)dv\\ &+\omega\int_0^t\int_{R(z)} \exp(-2\omega s)\left(c(\tilde\theta)^2+\rho\dot{\tilde u}_i\dot{\tilde u}_i + C_{ijkl}\tilde u_{i,j}\tilde u_{k,l}\right)dv\,ds\\ &+\int_0^t\int_{R(z)} \exp(-2\omega s)\left(Q^*+b_{ij}^m a_m\theta_{,i}^{(m)}\theta_{,j}^{(m)}\right)dv\,ds.\end{aligned} \tag{71}$$

In particular, we obtain that for ω large enough

$$\int_0^t \int_{R(z)} \exp(-2\omega s)\left(b_{ij}^m a_m(1-\epsilon)\theta_{,i}^{(m)}\theta_{,j}^{(m)}\right) dv ds \leq G_\omega(0,t)\exp\left(-\frac{z}{K^*}\right), \tag{72}$$

which gives a decay estimate for the gradient of the temperature θ and its time derivatives.

We also need to obtain an estimate for the displacement. We point out that

$$\left|\int_0^t \int_{R(z)} \exp(-2\omega s)\rho a_k a_l u_i^{(k)} u_i^{(l)} dv ds\right| \leq P_{kl}\omega^{k+l-2m} \int_0^t \int_{R(z)} \exp(-2\omega s)\rho a_m^2 u_i^{(m+1)} u_i^{(m+1)} dv ds, \tag{73}$$

for $0 \leq k+l < 2m$, where $P_{kl} = |a_k a_l| a_m^{-2}$. In a similar way we have that

$$\left|\int_0^t \int_{R(z)} \exp(-2\omega s) a_k a_l C_{ijrn} u_{i,j}^{(k)} u_{r,n}^{(l)} dv ds\right| \leq Q_{kl}\omega^{k+l-2m} \int_0^t \int_{R(z)} \exp(-2\omega s) a_m^2 C_{ijrn} u_{i,j}^{(m)} u_{r,n}^{(m)} dv ds, \tag{74}$$

where Q_{kl} depends on the constitutive constants and tensors, but does not depend on ω neither on the time. By taking ω large enough we see that

$$\int_0^t \int_{R(z)} \exp(-2\omega s)\left(\rho\dot{\tilde{u}}_i\dot{\tilde{u}}_i + C_{ijkl}\tilde{u}_{i,j}\tilde{u}_{k,l}\right) dv ds \geq a_m^2(1-\epsilon)\int_0^t \int_{R(z)} \exp(-2\omega s)\left(\rho u_i^{(m+1)} u_i^{(m+1)} + C_{ijkl} u_{i,j}^{(m)} u_{k,l}^{(m)}\right) dv ds. \tag{75}$$

We then obtain that

$$\int_0^t \int_{R(z)} \exp(-2\omega s)\left(\rho\dot{u}_i^{(m)}\dot{u}_i^{(m)} + C_{ijkl} u_{i,j}^{(m)} u_{k,l}^{(m)}\right) dv ds \leq (a_m^2(1-\epsilon))^{-1} G_\omega(0,t)\exp\left(-\frac{z}{K^*}\right), \tag{76}$$

which gives a description of the spatial behavior of the mechanical part. Thus, we have obtained the following theorem.

Theorem *Let us consider (u_i, θ) be a solution of the problem determined by the system (10)-(11) with the initial conditions (7), (12) and the boundary conditions (8), (9), (13) and (14). Then, for ω large enough, either the solution satisfies the condition (69) or it satisfies the decay estimates (72) and (76).*

6. The amplitude term for the thermoelastic problem

In this section we obtain an upper bound for the amplitude term $G_\omega(0,t)$ in terms of the boundary conditions when the solution satisfies the estimate (70). We assume that ω is large enough to guarantee that the polynomial considered in the previous section, $Q_1^*(\omega^{-1})$, is less than ϵ, where ϵ is a positive real number much smaller than one.

We first note that

$$G_\omega(0,t) = -\int_0^t \int_{D(0)} \exp(-2\omega s)(R_1 + R_2) da ds, \tag{77}$$

where

$$R_1 = (C_{i1kl}\tilde{u}_{k,l} - \beta_{i1}\tilde{\theta})\dot{\tilde{\zeta}}_i, \tag{78}$$

and

$$R_2 = (b_{i1}^0\theta_{,i} + b_{i1}^1\theta_{,i}^{(1)} + \cdots + b_{i1}^m\theta_{,i}^{(m)})\tilde{\xi}. \tag{79}$$

Here, $\zeta_i(x,t)$ and $\xi(x,t)$ are functions which tend uniformly to zero, rapidly, as x_1 tends to ∞ and satisfying the boundary conditions for u_i and θ respectively.

After the use of the boundary, asymptotic and the initial conditions, we find that

$$G_\omega(0,t) = I_1 + I_2 + I_3 + I_4, \tag{80}$$

where

$$I_1 = \int_0^t \int_R \exp(-2\omega s) \left(C_{ijkl}\tilde{u}_{k,l} - \beta_{ij}\tilde{\theta} \right) \dot{\tilde{\zeta}}_{i,j} dvds, \qquad I_2 = \int_0^t \int_R \exp(-2\omega s)\rho\ddot{\tilde{u}}_i\dot{\tilde{\zeta}}_i dvds, \tag{81}$$

$$I_3 = \int_0^t \int_R \exp(-2\omega s) \left(b_{ij}^0\theta_{,i} + \cdots + b_{ij}^m\theta_{,i}^{(m)} \right) \tilde{\xi}_{,j} dvds, \qquad I_4 = \int_0^t \int_R \exp(-2\omega s) \left(c\dot{\tilde{\theta}} + \beta_{ij}\tilde{v}_{i,j} \right) \tilde{\xi} dvds. \tag{82}$$

It is worth noting that $I_2 = I_{21} + I_{22} + I_{23}$ and $I_4 = I_{41} + I_{42} + I_{43}$, where

$$I_{21} = \exp(-2\omega t) \int_R \rho\dot{\tilde{u}}_i\dot{\tilde{\zeta}}_i dv, \quad I_{22} = 2\omega \int_0^t \int_R \exp(-2\omega s)\rho\dot{\tilde{u}}_i\dot{\tilde{\zeta}}_i dv, \quad I_{23} = -\int_0^t \int_R \exp(-2\omega s)\rho\dot{\tilde{u}}_i\ddot{\tilde{\zeta}}_i dv, \tag{83}$$

$$I_{41} = \exp(-2\omega t) \int_R \left(c\tilde{\theta} + \beta_{ij}\tilde{u}_{i,j} \right) \tilde{\xi} dv, \quad I_{42} = 2\omega \int_0^t \int_R \exp(-2\omega s) \left(c\tilde{\theta} + \beta_{ij}\tilde{u}_{i,j} \right) \tilde{\xi} dvds, \tag{84}$$

and

$$I_{43} = -\int_0^t \int_R \exp(-2\omega s) \left(c\tilde{\theta} + \beta_{ij}\tilde{u}_{i,j} \right) \dot{\tilde{\xi}} dvds. \tag{85}$$

We see that

$$|I_1| \leq \epsilon_1\omega^{-1}G_\omega(0,t) + C_1^* \int_0^t \int_R \exp(-2\omega s)\dot{\tilde{\zeta}}_{i,j}\dot{\tilde{\zeta}}_{i,j} dvds, \tag{86}$$

$$|I_{21}| \leq \epsilon_2 G_\omega(0,t) + C_{21}^* \exp(-2\omega t) \int_R \rho\dot{\tilde{\zeta}}_i\dot{\tilde{\zeta}}_i dv, \tag{87}$$

$$|I_{22}| \leq \epsilon_3 G_\omega(0,t) + C_{22}^*\omega \int_0^t \int_R \exp(-2\omega s)\rho\dot{\tilde{\zeta}}_i\dot{\tilde{\zeta}}_i dvds, \tag{88}$$

$$|I_{23}| \leq \epsilon_4\omega^{-1}G_\omega(0,t) + C_{23}^* \int_0^t \int_R \exp(-2\omega s)\rho\ddot{\tilde{\zeta}}_i\ddot{\tilde{\zeta}}_i dvds, \tag{89}$$

$$|I_3| \leq \epsilon_5 G_\omega(0,t) + C_3^*(\omega^{-1}) \int_0^t \int_R \exp(-2\omega s)\tilde{\xi}_i\tilde{\xi}_i dvds, \tag{90}$$

$$|I_{41}| \leq \epsilon_6 G_\omega(0,t) + C_{41}^* \exp(-2\omega t) \int_R |\tilde{\xi}|^2 dv, \tag{91}$$

$$|I_{42}| \leq \epsilon_7 G_\omega(0,t) + C_{42}^*\omega \exp(-2\omega t) \int_0^t \int_R \exp(-2\omega s)|\tilde{\xi}|^2 dvds, \tag{92}$$

$$|I_{43}| \leq \epsilon_8\omega^{-1}G_\omega(0,t) + C_{43}^*\omega \exp(-2\omega t) \int_0^t \int_R \exp(-2\omega s)|\dot{\tilde{\xi}}|^2 dvds. \tag{93}$$

Here, the parameters C_i^* and C_{ij}^* are calculable positive constants or functions depending on ω^{-1}. We then obtain that

$$\begin{aligned} G_\omega(0,t) \leq\ & (\epsilon_1\omega^{-1} + \epsilon_2 + \epsilon_3 + \epsilon_4\omega^{-1} + \epsilon_5 + \epsilon_6 + \epsilon_7 + \epsilon_8\omega^{-1})G_\omega(0,t) \\ & + C_1^* \int_0^t \int_R \exp(-2\omega s)\dot{\tilde{\zeta}}_{i,j}\dot{\tilde{\zeta}}_{i,j} dvds + C_{21}^* \exp(-2\omega t) \int_R \rho\dot{\tilde{\zeta}}_i\dot{\tilde{\zeta}}_i dv \\ & + C_{22}^*\omega \int_0^t \int_R \exp(-2\omega s)\rho\dot{\tilde{\zeta}}_i\dot{\tilde{\zeta}}_i dvds + C_{23}^* \int_0^t \int_R \exp(-2\omega s)\rho\ddot{\tilde{\zeta}}_i\ddot{\tilde{\zeta}}_i dvds \\ & + C_3^*(\omega^{-1}) \int_0^t \int_R \exp(-2\omega s)\tilde{\xi}_{,i}\tilde{\xi}_{,i} dvds + C_{41}^* \exp(-2\omega t) \int_R |\tilde{\xi}|^2 dv \\ & + C_{42}^*\omega \exp(-2\omega t) \int_0^t \int_R \exp(-2\omega s)|\tilde{\xi}|^2 dvds \\ & + C_{43}^*\omega \exp(-2\omega t) \int_0^t \int_R \exp(-2\omega s)|\dot{\tilde{\xi}}|^2 dvds. \end{aligned} \tag{94}$$

We can select ϵ_i and ω such that $\epsilon_1\omega^{-1} + \epsilon_2 + \epsilon_3 + \epsilon_4\omega^{-1} + \epsilon_5 + \epsilon_6 + \epsilon_7 + \epsilon_8\omega^{-1} = 1/2$. It follows that

$$
\begin{aligned}
G_\omega(0,t) \leq\ & 2C_1^* \int_0^t \int_R \exp(-2\omega s)\dot{\zeta}_{i,j}\dot{\zeta}_{i,j}\,dv\,ds + 2C_{21}^* \exp(-2\omega t)\int_R \rho\dot{\zeta}_i\dot{\zeta}_i\,dv \\
& + 2C_{22}^*\omega \int_0^t \int_R \exp(-2\omega s)\rho\dot{\zeta}_i\dot{\zeta}_i\,dv\,ds + 2C_{23}^* \int_0^t \int_R \exp(-2\omega s)\rho\ddot{\zeta}_i\ddot{\zeta}_i\,dv\,ds \\
& + 2C_3^*\omega^{-1} \int_0^t \int_R \exp(-2\omega s)\xi_{,i}\xi_{,i}\,dv\,ds + 2C_{41}^* \exp(-2\omega t)\int_R |\xi|^2\,dv \\
& + 2C_{42}^*\omega \int_0^t \int_R \exp(-2\omega s)|\xi|^2\,dv\,ds + 2C_{43}^*\omega \int_0^t \int_R \exp(-2\omega s)|\dot{\xi}|^2\,dv\,ds.
\end{aligned}
\tag{95}
$$

Now, we select

$$
\zeta_i(x,t) = g_i(x_2,x_3,t)\exp(-\sigma x_1) \quad \text{and} \quad \xi(x,t) = f(x_2,x_3,t)\exp(-\sigma x_1), \tag{96}
$$

where σ is a positive parameter. We see that

$$
\int_0^t \int_R \exp(-2\omega s)\dot{\zeta}_{i,j}\dot{\zeta}_{i,j}\,dv\,ds = \int_0^t \int_{D(0)} \exp(-2\omega s)\left(\frac{\dot{g}_{i,\alpha}\dot{g}_{i,\alpha}}{2\sigma} + \frac{\sigma\dot{g}_i\dot{g}_i}{2}\right)da\,ds, \tag{97}
$$

$$
\int_R \rho\dot{\zeta}_i\dot{\zeta}_i\,dv = \rho\int_{D(0)} \frac{\dot{g}_i\dot{g}_i}{2\sigma}\,da, \tag{98}
$$

$$
\int_0^t \int_R \exp(-2\omega s)\rho\dot{\zeta}_i\dot{\zeta}_i\,dv\,ds = \rho\int_0^t \int_{D(0)} \exp(-2\omega s)\frac{\dot{g}_i\dot{g}_i}{2\sigma}\,da\,ds, \tag{99}
$$

$$
\int_0^t \int_R \exp(-2\omega s)\rho\ddot{\zeta}_i\ddot{\zeta}_i\,dv\,ds = \rho\int_0^t \int_{D(0)} \exp(-2\omega s)\frac{\ddot{g}_i\ddot{g}_i}{2\sigma}\,da\,ds, \tag{100}
$$

$$
\int_0^t \int_R \exp(-2\omega s)\xi_{,i}\xi_{,i}\,dv\,ds = \int_0^t \int_{D(0)} \exp(-2\omega s)\left(\frac{f_{,\alpha}f_{,\alpha}}{2\sigma} + \frac{\sigma(|f|^2)}{2}\right)da\,ds, \tag{101}
$$

$$
\int_R |\xi|^2\,dv = \int_{D(0)} \frac{|f|^2}{2\sigma}\,da, \tag{102}
$$

$$
\int_0^t \int_R \exp(-2\omega s)|\xi|^2\,dv\,ds = \int_0^t \int_{D(0)} \exp(-2\omega s)\frac{|f|^2}{2\sigma}\,da\,ds, \tag{103}
$$

$$
\int_0^t \int_R \exp(-2\omega s)|\dot{\xi}|^2\,dv\,ds = \int_0^t \int_{D(0)} \exp(-2\omega s)\frac{|\dot{f}|^2}{2\sigma}\,da\,ds. \tag{104}
$$

We then obtain that

$$
\begin{aligned}
G_\omega(0,t) \leq\ & \left(C_1^*\sigma + \frac{\rho C_{22}^*\omega}{\sigma}\right)\int_0^t \int_{D(0)} \exp(-2\omega s)\dot{g}_i\dot{g}_i\,da\,ds \\
& + \frac{C_1^*}{\sigma}\int_0^t \int_{D(0)} \exp(-2\omega s)\dot{g}_{i,\alpha}\dot{g}_{i,\alpha}\,da\,ds + \frac{\rho C_{12}^*}{\sigma}\exp(-2\omega t)\int_{D(0)} \dot{g}_i\dot{g}_i\,da \\
& + \frac{\rho C_{23}^*}{\sigma}\int_0^t \int_{D(0)} \exp(-2\omega s)\ddot{g}_i\ddot{g}_i\,da\,ds \\
& + \left(C_3^*\omega^{-1}\sigma + \frac{C_{42}^*\omega}{\sigma}\right)\int_0^t \int_{D(0)} \exp(-2\omega s)|f|^2\,da\,ds \\
& + \frac{C_3^*\omega^{-1}}{\sigma}\int_0^t \int_{D(0)} \exp(-2\omega s)f_{,\alpha}f_{,\alpha}\,da\,ds \\
& + \frac{C_{41}^*}{\sigma}\exp(-2\omega t)\int_{D(0)} |f|^2\,da + \frac{C_{43}^*\omega}{\sigma}\int_0^t \int_{D(0)} \exp(-2\omega s)|\dot{f}|^2\,da\,ds.
\end{aligned}
\tag{105}
$$

We could obtain an estimate independent of σ after an optimization. However it seems a very cumbersome task.

7. Conclusions

In this paper we have analysed two thermomechanical situations. One of them comes from considering the heat conduction theory proposed by the equation (6) and the other one from the thermoelastic system proposed by the equations (10)-(11). For both cases, we have obtained a Phragmén-Lindelöf alternative for the solutions, which is the mathematical counterpart of the well known Saint-Venant's principle in thermomechanics. That is, we have seen that the thermomechanical perturbations considered in a part of the boundary are strongly damped for the points which are far away from the place where the perturbations occur. The decay estimate is given (in both cases) by means of a negative exponential. Moreover, to have a measure of the damping, we also give an upper bound for the amplitude term by means of the disturbances.

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