On the evaluation of the Tutte polynomial at the points $(1, -1)$ and $(2, -1)$

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Abstract. C. Merino proved recently the following identity between evaluations of the Tutte polynomial of complete graphs: $t(K_{n+2}; 1, -1) = t(K_n; 2, -1)$. In this work we extend this result by giving a large class of graphs with this property, that is, graphs G such that there exist two vertices u and v with $t(G; 1, -1) = t(G - \{u, v\}; 2, -1)$. The class is described in terms of forbidden induced subgraphs and it contains in particular threshold graphs.

Key words: Tutte polynomial, generating function, threshold graph

1 Introduction

The Tutte polynomial is one of the most studied polynomial graph invariants. For a graph $G = (V, A)$, it is given by

$$t(G; x, y) = \sum_{A \subseteq E} (x - 1)^{r(G)-r(A)} (y - 1)^{|A|-r(A)},$$

where $r(A)$ is the *rank* of A, defined as $|V| - c(G|_A)$, where $c(G|_A)$ is the number of connected components of the spanning subgraph $G|_A = (V, A)$ induced by A.

We refer to [5] for details about the many combinatorial interpretations of the evaluations of the Tutte polynomial of a graph in different points of the plane and also along several algebraic curves. For example, $t(G; 1, 1)$ is the number of spanning trees of G when G is connected and $t(G; 2, 1)$ is the number of spanning forests of G. The hyperbolae $H_q = \{(x, y) : (x-1)(y-1) = q\}$ play a significant role in the theory of the Tutte polynomial. In particular, for $q \in \mathbb{N}$ the Tutte polynomial specializes on H_q to the partition function

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of the q-state Potts model. A pair of interpretations especially related to our work are that $t(G;2,0)$ is the number of acyclic orientations of G and that $t(G;1,0)$ is the number of acyclic orientations of G with a unique fixed source. With this in mind, it follows that $t(K_{n+1};1,0) = t(K_n;2,0)$ (in fact, the same is true of any graph G with a universal vertex).

In this paper we shall be concerned with evaluations of the Tutte polynomial at the points $(1,-1)$ and $(2,-1)$. A combinatorial interpretation for these evaluations is given in [1]. Merino [4] proved the following identity, which is the starting point of our work:

$$t(K_{n+2};1,-1) = t(K_n;2,-1).$$

Non-trivial relationships between evaluations of the Tutte polynomial at points on different hyperbolae are uncommon. Here, the point $(2,-1)$ lies on the hyperbola H_{-2} and $(1,-1)$ on the hyperbola H_0. We wonder whether there are other graphs with this property, that is, we search graphs G with a pair of vertices u,v such that $t(G;1,-1) = t(G-\{u,v\};2,-1)$. Merino's proof used generating functions. It is not very difficult to adapt his proof to show the property for complete bipartite graphs and for graphs that are the sum of a clique and a coclique. (By a clique we mean a complete graph, and by a coclique a graph with no edges; the sum operation adds all edges between the two summands.) Our main result (Theorem 1 below) generalizes these examples by giving sufficient conditions for a graph to have this property; moreover, it describes graphs for which the property holds when each vertex is replaced by a clique or a coclique of arbitrary order.

The rest of the paper is organized as follows. In the next section we give the notation needed to state the main theorem, we state it, and we discuss its consequences. Section 3 is devoted to the proof, including some intermediate results. We end with some remarks and open questions.

2 Main results

All graphs in this paper are simple. The graph with n vertices and no edges is denoted by $\overline{K}_n$, and usually referred to as a coclique. Let $\mathbb{N}$ denote the set of non-negative integers. Given a connected graph $G = (V,E)$, $\mathbf{n} \in \mathbb{N}^V$ and $\mathbf{c} \in \{0,1\}^V$, define $G(\mathbf{c};\mathbf{n})$ to be the graph obtained from G by replacing each vertex $k \in V$ by a clique on n_k vertices if $c_k = 1$ or by a coclique of n_k vertices if $c_k = 0$, and for each edge $ij \in E$ join the (co)clique on n_i vertices to the (co)clique on n_j vertices, joining each of the $n_i n_j$ pairs of vertices by an edge in $G(\mathbf{c};\mathbf{n})$. For example, $K_1(1;n) = K_n$, $K_1(0;n) = \overline{K}_n$ and $K_2((0,0);(m,n)) = K_{m,n}$. Note that $K_r((1,1,\ldots,1);(n_1,\ldots,n_r)) = K_1(1;n_1+\cdots+n_r) = K_{n_1+n_2+\cdots+n_r}$ since the join of two cliques is a clique.

We are looking for parameters $G,\mathbf{c}$ with the property that for all $\mathbf{n} \in \mathbb{N}^V$ there exist vertices u,v of $G(\mathbf{c};\mathbf{n})$ such that $t(G(\mathbf{c};\mathbf{n});1,-1) = t(G(\mathbf{c},\mathbf{n}) -$

$\{u, v\}; 2, -1)$, where $n_i, n_j \geq 1$ if u, v belong to the (co)cliques at vertices i, j of G.

In fact, we will find $i, j \in V$ such that for all $\mathbf{n} \in \mathbb{N}^V$ with $n_i, n_j \geq 1$ we have

$$t(G(\mathbf{c}; \mathbf{n}); 1, -1) = t(G(\mathbf{c}; \mathbf{n}'); 2, -1) \tag{1}$$

where $\mathbf{n}'$ is obtained from $\mathbf{n}$ by subtracting 1 from the ith and jth components. In other words, the vertices u, v of $G(\mathbf{c}; \mathbf{n})$ are taken from the fixed (co)cliques that replace the vertices i and j of G in making the graph $G(\mathbf{c}; \mathbf{n})$.

Our first result (Theorem 2 in Section 3) characterizes pairs $(G, \mathbf{c})$ for which this holds. This can in turn be rewritten in terms of induced subgraphs. See Figure 1 for an illustration of the statement of the following theorem.

Theorem 1. *Let $G = (V, E)$ be a simple graph and i and j distinct vertices of G such that $\{i, j\}$ is a vertex cover of G. Let $A = \{v \in V \setminus \{i, j\} : vi \in E, vj \notin E\}, B = \{v \in V \setminus \{i, j\} : vi \notin E, vj \in E\}$ and $C = \{v \in V \setminus \{i, j\} : vi \in E, vj \in E\}$.*

Then $t(G; 1, -1) = t(G - i - j; 2, -1)$ if the following conditions hold:

(i) $G[A]$ and $G[B]$ are cocliques, and $G[C \cup \{i, j\}]$ is a clique (in particular, $ij \in E$);
(ii) there is no induced pair of disjoint edges $2P_2$ with endpoints in $A \cup B$ or induced path of length four P_4 with both endpoints in A or both endpoints in B;
(iii) there is no induced path of length three P_3 with one endpoint in A and the other in B, nor the complement of such a path.

Furthermore, if G satisfies these conditions then so does any graph obtained from G by replacing a vertex of $A \cup B \cup \{i, j\}$ by a coclique of twin vertices, or a vertex of $C \cup \{i, j\}$ by a clique of twin vertices.

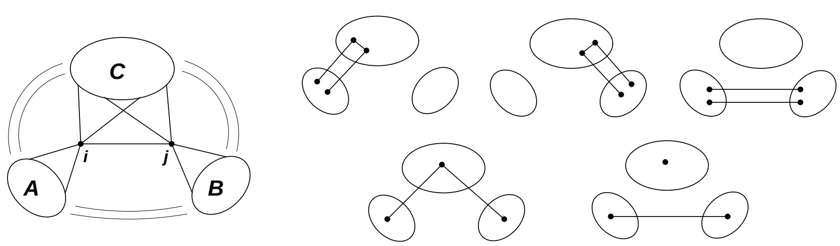

Fig. 1. On the left, structure of the graph described in Theorem 1; A and B induce cocliques, and $C \cup \{i, j\}$ induces a clique. On the right, the five forbidden induced subgraphs.

Since K_2 satisfies the conditions of the theorem (it is the simplest case $A = B = C = \emptyset$), we recover complete graphs, complete bipartite graphs and the sum of a clique and a coclique. If we take $G = K_3$, we have $A = B = \emptyset$ and $|C| = 1$. This means that we cannot replace the three vertices of a K_3 by cocliques, but all other possibilities are fine.

The case $B = \emptyset$ gives a much richer class of graphs, threshold graphs [3]. They are those graphs with no induced P_4, C_4 or $2P_2$. They are also the graphs that can be constructed by starting from K_1 by repeteadly adding new vertices that are either isolated or universal.

The next section is devoted to the proof of Theorem 1, which follows from Theorem 2. The proof follows a generating function approach. The key point is that it is possible to find the generating function for the Tutte polynomials of the family $G(\mathbf{c}, \mathbf{n})$. Then, the relationship between evaluations at $(1, -1)$ and $(2, -1)$ is expressed as a differential equation, from whose solutions the statement of Theorem 2 is read. Then Theorem 1 is deduced from it.

3 Proof

We begin by recalling some well-known properties of the Tutte polynomial: $t(\overline{K}_n; x, y) = 1$ and if G has blocks $G_1, \ldots, G_k$, then $t(G; x, y) = t(G_1; x, y) \cdots t(G_k; x, y)$.

Let us fix a connected graph G with two distinguished vertices i, j and a $\{0,1\}$-labelling of the vertices, that is, $\mathbf{c} \in \{0,1\}^V$. We look for conditions so that $G(\mathbf{c}; \mathbf{n})$ satisfies (1) for all $\mathbf{n} \in \mathbb{N}^V$ with $n_i, n_j \geq 1$.

We start by observing that every vertex $k \in V \backslash \{i, j\}$ is adjacent to either i or j. Indeed, suppose k is not adjacent to either i or j, and choose a neighbour l of k. Then it is easy to check that equation (1) does not hold if we take $\mathbf{n}$ to be zero everywhere except $n_i = n_j = n_k = n_l = 1$. So from now on we assume that i and j together cover V.

The main tool in the proof are generating functions. More concreteley, let $\mathbf{u} = (u_k : k \in V)$ and define

$$T(x, y; \mathbf{u}) = \sum_{\mathbf{n} \in \mathbb{N}^V} t(G(\mathbf{c}; \mathbf{n}); x, y) \frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!}, \qquad \mathbf{u}^{\mathbf{n}} = \prod_k u_k^{n_k}, \quad \mathbf{n}! = \prod_k n_k!,$$

taking $t(G(\mathbf{c}, \mathbf{0}); x, y) = t(\emptyset; x, y) = 1$. Equation (1) holds if and only if

$$\frac{\partial^2 T(1, -1; \mathbf{u})}{\partial u_i \partial u_j} = T(2, -1; \mathbf{u}). \tag{2}$$

Lemma 1. *Let $G = (V, E)$ be a connected graph containing vertices i and j such that $ki \in E$ or $kj \in E$ for every $k \in V \setminus \{i, j\}$. Define*

$$S(z, w; \mathbf{u}) = \sum_{\mathbf{n} \in \mathbb{N}^V} \frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!} \sum_{A \subseteq E(G(\mathbf{c}; \mathbf{n}))} z^{|A|} w^{c(A)}.$$

Then

$$\frac{\partial^2 T(x,y;\mathbf{u})}{\partial u_i \partial u_j} = \frac{1}{x-1}\frac{\partial^2 S(y-1,(x-1)(y-1);\frac{\mathbf{u}}{y-1})}{\partial u_i \partial u_j}, \tag{3}$$

and

$$T(2,y;\mathbf{u}) = S(y-1,y-1;\frac{\mathbf{u}}{y-1}). \tag{4}$$

Proof. Letting $|\mathbf{n}| = \sum_k n_k$ denote the number of vertices of $G(\mathbf{c};\mathbf{n})$,

$$\begin{aligned} t(G(\mathbf{c};\mathbf{n});x,y) &= \sum_{A\subseteq E(G(\mathbf{c};\mathbf{n}))} (x-1)^{c(A)-c(G(\mathbf{c};\mathbf{n}))}(y-1)^{|A|-|\mathbf{n}|+c(A)} \\ &= (x-1)^{-c(G(\mathbf{c};\mathbf{n}))} \sum_{A\subseteq E(G(\mathbf{c};\mathbf{n}))} [(x-1)(y-1)]^{c(A)}(y-1)^{|A|-|\mathbf{n}|}. \end{aligned}$$

Hence

$$\begin{aligned} T(x,y;\mathbf{u}) &= \sum_{\mathbf{n}\in\mathbb{N}^V} t(G(\mathbf{c};\mathbf{n});x,y)\frac{\mathbf{u}^\mathbf{n}}{\mathbf{n}!} \\ &= \frac{1}{x-1}S(y-1,(x-1)(y-1);\frac{\mathbf{u}}{y-1}) \\ &+ \sum_{\substack{\mathbf{n}\in\mathbb{N}^V \\ c(G(\mathbf{c};\mathbf{n}))\neq 1}} \left(\frac{1}{(x-1)^{c(G(\mathbf{c};\mathbf{n}))}} - \frac{1}{x-1}\right)\frac{\mathbf{u}^\mathbf{n}}{\mathbf{n}!} \times \\ &\qquad \sum_{A\subseteq E(G(\mathbf{c};\mathbf{n}))} [(x-1)(y-1)]^{c(A)}(y-1)^{|A|-|\mathbf{n}|}. \end{aligned}$$

Recall that the graph $G(\mathbf{c};\mathbf{n})$ is connected if $n_i \geq 1$ and $n_j \geq 1$. It follows that the second summand on the right-hand side of the above equation for $T(x,y;\mathbf{u})$ is non-zero only if $n_i = 0$ or $n_j = 0$ and hence vanishes upon differentiating with respect to u_i and u_j. This second term also vanishes when $x = 2$ because in this case $(x-1)^{-c} = 1 = (x-1)^{-1}$ for any c. □

In order to express $S(z,w;\mathbf{u})$, we introduce

$$q(\mathbf{n}) := \sum_{kl\in E} n_k n_l + \sum_{k\in V} c_k \binom{n_k}{2},$$

that is, the number of edges of $G(\mathbf{c};\mathbf{n})$.

Lemma 2.

$$S(z,w;\mathbf{u}) = F(z;\mathbf{u})^w \ \textit{where}\ F(z;\mathbf{u}) = \sum_{\mathbf{n}\in\mathbb{N}^V} (1+z)^{q(\mathbf{n})}\frac{\mathbf{u}^\mathbf{n}}{\mathbf{n}!}.$$

Proof. The key observation is that the connected components of a spanning subgraph of $G(\mathbf{c};\mathbf{n})$ are connected spanning subgraphs of graphs in the family $\{G(\mathbf{c};\mathbf{n}) : \mathbf{n} \in \mathbb{N}^V\}$. (For instance, a spanning subgraph of a complete bipartite graph is the union of connected spanning subgraphs of complete bipartite graphs.) From this and general properties of generating functions it follows that $S(z,w;\mathbf{u}) = e^{C(z;\mathbf{u})w}$, where $C(z;\mathbf{u})$ is the exponential generating function for connected spanning subgraphs of $\{G(\mathbf{c};\mathbf{n}) : \mathbf{n} \in \mathbb{N}^V\}$ (by number of edges). Now $F(z;\mathbf{u}) = e^{C(z;\mathbf{u})}$ is the exponential generating function of spanning subgraphs of $\{G(\mathbf{c};\mathbf{n}) : \mathbf{n} \in \mathbb{N}^V\}$, which is given by the expression in the statement of the theorem. □

Let $f(\mathbf{u}) = F(-2;\mathbf{u})$. By combining Lemmas 1 and 2, equation (2) becomes

$$\frac{\partial f(\mathbf{u})}{\partial u_i}\frac{\partial f(\mathbf{u})}{\partial u_j} - f(\mathbf{u})\frac{\partial^2 f(\mathbf{u})}{\partial u_i \partial u_j} = 2. \tag{5}$$

Solving the differential equation (5) will put conditions on the quadratic form $q(\mathbf{n})$ that translate to structural conditions on the graph G and the clique/coclique parameter $\mathbf{c}$ that specify the graph $G(\mathbf{c};\mathbf{n})$. This is Theorem 2 below.

We use $\mathbb{I}(P)$ to denote the indicator function, equal to 1 when the statement P is true and 0 otherwise. For a subset of vertices $U \subseteq V$, $G[U]$ denotes the subgraph of G induced by the vertices U.

Theorem 2. *A pair G and $\mathbf{c}$ satisfies equation (1) if and only if the following conditions hold:*

(i) $ij \in E$;
(ii) *for each* $k \in V \setminus \{i,j\}$, $\mathbb{I}(ki \in E) + \mathbb{I}(kj \in E) = c_k + 1$;
(iii) *for all* $U \subseteq V \setminus \{j\}$, *either* j *has odd degree in* $G[U \cup \{j\}]$ *or there is a vertex* $k \in U$ *whose degree in the induced subgraph* $G[U]$ *has the same parity as* c_k.

Proof. Note that have already observed that each $k \in V \setminus \{i,j\}$ is adjacent to at least one of i and j. We now wish to find f that solve equation (5).

Define the relation $k \sim l$ to hold if and only if (i) $kl \in E$, or (ii) $k = l$ and $c_k = 1$. The graph $\widetilde{G}$ with edges kl when $k \sim l$ is equal to the graph $G = (V,E)$ with loops added to each vertex k such that $c_k = 1$. We have

$$2q(\mathbf{n}) = \sum_{\substack{(k,l)\in V\times V\\ k\sim l}} n_k n_l - \sum_{\substack{k\in V\\ k\sim k}} n_k. \tag{6}$$

We have also

$$f(\mathbf{u}) = \sum_{\mathbf{n}\in\mathbb{N}^V} (-1)^{q(\mathbf{n})}\frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!}$$

from which we calculate

$$\frac{\partial f(\mathbf{u})}{\partial u_i} = \sum_{\mathbf{n}\in\mathbb{N}^V} (-1)^{q(\mathbf{n})+\Delta_i q(\mathbf{n})} \frac{\mathbf{u}^\mathbf{n}}{\mathbf{n}!},$$

where $\Delta_i q(\mathbf{n}) = q(\ldots, n_i+1, \ldots) - q(\ldots, n_i, \ldots)$ is the forward difference of $q(\mathbf{n})$ in the ith component, and

$$\frac{\partial^2 f(\mathbf{u})}{\partial u_i \partial u_j} = \sum_{\mathbf{n}\in\mathbb{N}^V} (-1)^{q(\mathbf{n})+\Delta_{i,j} q(\mathbf{n})} \frac{\mathbf{u}^\mathbf{n}}{\mathbf{n}!},$$

where $\Delta_{i,j} q(\mathbf{n}) = q(\ldots, n_i+1, \ldots, n_j+1, \ldots) - q(\ldots, n_i, \ldots, n_j, \ldots)$.

Multiplying power series we find that

$$\frac{\partial f(\mathbf{u})}{\partial u_i}\frac{\partial f(\mathbf{u})}{\partial u_j} - f(\mathbf{u})\frac{\partial^2 f(\mathbf{u})}{\partial u_i \partial u_j} =$$

$$\sum_{\mathbf{n}\in\mathbb{N}^V} \sum_{\mathbf{m}\leq\mathbf{n}} (-1)^{q(\mathbf{m})+q(\mathbf{n}-\mathbf{m})} \left((-1)^{\Delta_i q(\mathbf{m})+\Delta_j q(\mathbf{n}-\mathbf{m})} - (-1)^{\Delta_{i,j} q(\mathbf{m})} \right) \prod_k \binom{n_k}{m_k} \frac{\mathbf{u}^\mathbf{n}}{\mathbf{n}!}. \tag{7}$$

Here we write $\mathbf{m} \leq \mathbf{n}$ to mean $m_k \leq n_k$ for each $k \in V$.

After some manipulation, we get that the relative parity of $\Delta_i q(\mathbf{m}) + \Delta_j q(\mathbf{n}-\mathbf{m})$ and $\Delta_{i,j} q(\mathbf{m})$ is given by

$$\Delta_i q(\mathbf{m}) + \Delta_j q(\mathbf{n}-\mathbf{m}) + \Delta_{i,j} q(\mathbf{m}) \equiv \sum_{k\sim j} n_k + \mathbb{I}(i \sim j) \pmod 2, \tag{8}$$

which is independent of $\mathbf{m}$. If the right-hand side of equation (8) is zero then the coefficient of $\mathbf{u}^\mathbf{n}$ in equation (7) is equal to zero. Since the constant term ($\mathbf{n} = \mathbf{0}$) should be equal to 2 it is necessary that $i \sim j$. Given $i \sim j$, for any $\mathbf{n}$, if

$$\sum_{k\sim j} n_k \equiv 1 \pmod 2$$

then the coefficient of $\mathbf{u}^\mathbf{n}$ in equation (7) is zero.

Therefore, we need only bother about the coefficients of $\mathbf{u}^\mathbf{n}$ where $\sum_{k\sim j} n_k \equiv 0 \pmod 2$. The coefficient

$$\frac{1}{\mathbf{n}!}[\mathbf{u}^\mathbf{n}] \left(\frac{\partial f(\mathbf{u})}{\partial u_i}\frac{\partial f(\mathbf{u})}{\partial u_j} - f(\mathbf{u})\frac{\partial^2 f(\mathbf{u})}{\partial u_i \partial u_j} \right)$$

is given by

$$2 \sum_{\mathbf{m}\leq\mathbf{n}} (-1)^{q(\mathbf{m})+q(\mathbf{n}-\mathbf{m})+\Delta_{i,j} q(\mathbf{m})} \prod_k \binom{n_k}{m_k}.$$

So we wish to find necessary and sufficient conditions for this coefficient of $\frac{1}{\mathbf{n}!}\mathbf{u}^\mathbf{n}$, to equal zero for all $\mathbf{n} \neq \mathbf{0}$ subject to $\sum_{k\sim j} n_j \equiv 0 \pmod 2$ and $i \sim j$.

So what are the conditions? Again after some easy manipulation, we find that the coefficient we are interested in vanishes if and only if:

$$
\begin{aligned}
0 &= \sum_{\mathbf{m}\leq\mathbf{n}} (-1)^{\sum_k m_k \sum_{l\sim k}[n_l+\mathbb{I}(l=i)+\mathbb{I}(l=k)+\mathbb{I}(l=k)]} \prod_k \binom{n_k}{m_k} \\
&= \prod_k \sum_{m_k\leq n_k} (-1)^{\left[\sum_{l\sim k} n_l+\mathbb{I}(i\sim k)+\mathbb{I}(j\sim k)+\mathbb{I}(k\sim k)\right]m_k} \binom{n_k}{m_k} \\
&= \prod_k \left[1+(-1)^{\sum_{l\sim k} n_l+\mathbb{I}(i\sim k)+\mathbb{I}(j\sim k)+\mathbb{I}(k\sim k)}\right]^{n_k}. \qquad (9)
\end{aligned}
$$

By taking each n_k to be even, for the expression (9) to be zero it is necessary that, for each $k \in V$,

$$\mathbb{I}(i \sim k) + \mathbb{I}(j \sim k) + \mathbb{I}(k \sim k) \equiv 1 \pmod 2. \qquad (10)$$

Thus if $c_k = 1$ in $G(\mathbf{c};\mathbf{n})$ (a clique) the vertex k must be adjacent to both i and j, whereas if $c_k = 0$ (a coclique) then the vertex k must be adjacent to exactly one of i, j. Since by assumption $i \sim j$ and $\sum_{l\sim j} n_l \equiv 0 \pmod 2$ we can assume $n_j = 0$, otherwise we have a zero factor and we are done.

Since the expression (9) depends only on the parity of each n_k, if the coefficients subject to $\sum_{k\sim j} n_k \equiv 0 \pmod 2$ and $n_k \in \{0,1\}$ are all zero apart from the constant term then the coefficients of $\mathbf{u}^{\mathbf{n}}$ are zero for all $\mathbf{n} \neq \mathbf{0}$. In terms of the graph G, this is to say we may assume each vertex k is either deleted ($n_k = 0$) or is present as a single vertex ($n_k = 1$); if this graph satisfies the required conditions then so does $G(\mathbf{c};\mathbf{n})$ for all $\mathbf{n} \in \mathbb{N}^V$.

Define $U \subseteq V\backslash\{j\}$ by $U = \{k \in V : n_k \neq 0\}$. Since we assume $\sum_{k\sim j} n_k \equiv 0 \pmod 2$ we restrict attention to U such that the induced subgraph $G[U]$ of G has the property that the number of vertices $k \in U$ such that $kj \in E$ is even. A necessary and sufficient condition that the expression (9) is zero (under the assumption that $i \sim j$, $n_j = 0$ and $\sum_{k\sim j} n_k \equiv 0 \pmod 2$) is that for any such choice of U there is a vertex k of $G[U]$ of odd degree if k has a loop or even degree if k does not have a loop (i.e., a vertex k of degree of the same parity as c_k in the induced subgraph on U). □

From this theorem we wish now to deduce the induced-subgraph characterization of Theorem 1. First we need to give some properties of the pairs $(G, \mathbf{c})$ that satisfy the conditions in Theorem 2. For the rest of this section, we will use the following notation:

$$
\begin{aligned}
A &= \{k \in V \setminus \{i,j\} : ki \in E, c_k = 0\}, \\
B &= \{k \in V \setminus \{i,j\} : kj \in E, c_k = 0\}, \\
C &= \{k \in V \setminus \{i,j\} : ki, kj \in E, c_k = 1\}.
\end{aligned}
$$

Clearly by condition (ii) these sets partition $V \setminus \{i,j\}$ (recall Figure 1). The next lemma says that the values of c_i and c_j can be chosen freely (that is, whether they are replaced by cliques or cocliques does not affect the validity of equation (1)).

Lemma 3. *If $G=(V,E)$, $i,j \in V$ and $\mathbf{c} \in \{0,1\}^V$ satisfy the conditions of Theorem 2, then so does G and $\mathbf{c}'$ where $\mathbf{c}'$ is $\mathbf{c}$ with c_i replaced by $1-c_i$ or with c_j replaced by $1-c_j$ (or both).*

Proof. This is easy to see for j, since the conditions of Theorem 2 are independent of the value of c_j.

To see that the value of c_i does not matter, suppose that on the contrary that there is an induced subgraph $G[U]$ with $i \in U \subseteq V \setminus \{j\}$ which satisfies the conditions of the theorem but that i is the only vertex of degree congruent to $c_i \pmod 2$ in $G[U]$ as required by condition (iii).

Suppose first that $c_i = 0$, so that i has even degree in $G[U]$. Set $A' = A \cap U$, $B' = B \cap U$ and $C' = C \cap U$. By assumption all vertices $k \in U \setminus \{i\}$ have degree congruent to $c_k + 1 \pmod 2$. Since any graph has an even number of vertices of odd degree we must have $|A'| + |B'|$ even. Also, the vertex j is adjacent to an even number of vertices in U, including i, so that $1+|B'|+|C'|$ is even. This makes $|A'|+|C'|$ odd, but this is the degree of i in $G[U]$, which by assumption is even. Hence, when $c_i = 0$ the vertex i cannot be the only vertex of degree congruent to $c_i \pmod 2$ in $G[U]$.

The case $c_i = 1$ is treated by a similar parity argument. □

Note that if G and $\mathbf{c}$ satisfy the conditions of Theorem 2 then so for any $U \supseteq \{i,j\}$ must the induced subgraph $G[U]$ with $\mathbf{c}$ restricted to U.

Corollary 1. *The induced subgraphs $G[A]$ and $G[B]$ are cocliques and the induced subgraph $G[C \cup \{i,j\}]$ is a clique.*

Proof. By Lemma 3, we may assume $c_i = c_j = 0$. Deleting all vertices in C, leaves a graph that must satisfy the conditions of Theorem 2. Let k, k' be two vertices in A. By taking $U = \{k, k'\}$, condition (iii) implies that k and k' must have even degree in the subgraph they induce, so they cannot be adjacent. An analogous argument shows that B is also a coclique.

To show that $C \cup \{i,j\}$ induce a clique, argue similarly assuming that $c_i = c_j = 1$. □

Lemma 3 and Corollary 1 imply that condition (iii) of Theorem 2 is satisfied if and only if:

($\star$) for all $U \subseteq V \setminus \{i,j\}$ such that $U \cap (B \cup C)$ is even, the induced subgraph $G[U]$ contains either a vertex in $A \cup B$ of odd degree or a vertex in C of even degree.

Proof of Theorem 1. Finally we show how to deduce the characterization of Theorem 1 from this condition ($\star$). Clearly if G contains as induced subgraph any of the subgraphs described in conditions (ii) and (iii) in Theorem 1, then this subgraph contradicts condition ($\star$). Next we show the converse. (Recall that the forbidden induced subgraphs are pictured in Figure 1.)

Suppose there is $U \subseteq V \setminus \{i,j\}$ such that condition ($\star$) fails. Let us focus first on the case $U \subseteq A \cup B$ and let us call $A' = A \cap U$ and $B' = B \cap U$. For ($\star$) to fail, all vertices must have odd degree. We need to show that $G[U]$ contains a $2P_2$ (two disjoint edges) as an induced subgraph (condition (ii) in Theorem 1). Since $|B'|$ is even, the total number of edges in the bipartite graph $G[A' \cup B']$ is even, so $|A'|$ must also be even. Now let $x, y \in A'$ and let $N(x)$ and $N(y)$ be their respective neighbourhoods in B'. If there exist $z \in N(x) \setminus N(y)$ and $w \in N(y) \setminus N(x)$, then the edges xz and yw are a copy of an induced $2P_2$, and we are done. Therefore, we can restrict to the case where either $N(x) \subseteq N(y)$ or $N(y) \subseteq N(x)$ for every pair of vertices x and y of A'. This implies that the neighbourhoods of the vertices of A' are nested. Since every vertex in B' has at least one neighbour, we deduce that there is some $v \in A'$ such that $B' = N(v)$. But this is a contradiction, since $|N(v)|$ is odd and $|B'|$ is even.

The cases where $U \subseteq A \cup C$ and $U \subseteq B \cup C$ are dealt with in a completely analogous manner. So let us assume now that U contains vertices from all of A, B, and C; let $A' = A \cap U$, $B' = B \cap U$ and $C' = C \cap U$, all of them non-empty. We need to show that if condition ($\star$) fails then U contains as induced subgraph one of the five graphs described in Theorem 1.

We start by analysing the neighbourhoods of vertices of C' in A' and B'. We will assume initially that all vertices of C' have neighbours in both A' and B'. For $x \in C'$, let A_x and B_x be its neighbourhoods in A' and B'. If $A_x \cup B_x$ does not induce a bipartite graph, then $G[U]$ contains an induced P_3 with endpoints in A and B, one of the forbidden induced subgraphs; so we assume that $A_x \cup B_x$ induces a bipartite graph, otherwise we are done. Now, for another $y \in C'$, if the neighbourhoods A_x and A_y are not comparable, we can find a P_4 with both endpoints in A, which is also one of the forbidden subgraphs. The same holds for the neighbourhoods with respect to B'. From all this we can conclude that there is an ordering $x_1, \ldots, x_r$ of the vertices in C' such that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_r$, where $A_i = A_{x_i}$. We can assume that the ordering of the neighbourhoods in B' is the reverse one, that is, $B_r \subseteq B_{r-1} \subseteq \cdots \subseteq B_1$. Indeed, if $A_i \subset A_j$ and $B_i \subset B_j$ then we could find as induced subgraph the last configuration shown in Figure 1, and we would be done.

Now let us consider a vertex $z \in A_1$. By construction and the observations above, z is adjacent to $C' \cup B_1$. If $B' = B_1$, the degree of z would be even (since $|C' \cup B'|$ is even), so U would not satisfy condition ($\star$). Hence, $B'' = B' \setminus B_1$ is non-empty and of odd cardinality. Moreover, we can assume all neighbours of vertices in B'' are in A_1, because otherwise x_1 and one edge between $A' \setminus A_1$ and B'' would give a forbidden induced subgraph. Now we restrict our attention to the subgraph induced by A_1 and B''. Observe that, since the degree of the vertices of A_1 in U is odd, the degree in the induced subgraph $G[A_1 \cup B'']$ is of opposite parity as $|C' \cup B_1|$, which is equivalent to being of opposite parity as $|B''|$ (since $|B' \cup C''|$ is even). Also, the degrees of the vertices in B'' are odd in $G[U]$, and since all of its neighbours are in A_1, their degrees are also odd in the subgraph induced by A_1 and B''. Now, if $|B''|$ is even, this subgraph satisfies condition ($\star$), and we have already proved that in this case we can find an induced copy of $2P_2$. The case $|B''|$ odd is impossible, because it gives a graph with an odd number of vertices of odd degree.

To conclude the proof, we should analyse now the case that not all vertices in C' have neighbours in both A' and B'. This adds some more case-analysis in the discussion of the previous two paragraphs, but it follows essentially along the same lines. We omit it for the sake of brevity.

□

4 Concluding remarks

Theorem 2 (or its equivalent induced subgraph version) characterizes those graphs for which we can replace every vertex by either a clique or a coclique of arbitrary size and obtain a graph G' satisfying $t(G'; 1, -1) = t(G' - \{i, j\}; 2, -1)$. But this does not imply that all graphs G for which there exist two vertices $\{u, v\}$ such that $t(G; 1, -1) = t(G - \{u, \}; 2, -1)$ arise in this way. For instance, taking G to be a cycle of length 6 and u, v two vertices at distance two in the cycle yields such a graph. Moreover, if the vertices are, cyclically, u, x_1, v, x_2, w, x_3, one can prove that replacing x_1 by a clique and x_2, x_3 by a coclique, the result satisfies the equation, yet it is not of the form described in Theorem 2 (in particular, i and j are not adjacent, and they do not cover all vertices). Characterizing *all* graphs for which there are two vertices for which $t(G; 1, -1) = t(G - \{u, v\}; 2, -1)$ is probably a too ambitious problem.

Finally, let us say a few words about the combinatorial meaning of $t(G; 1, -1)$ and $t(G; 2, -1)$. For an arbitrary graph G, Gessel and Sagan [1] give intepretations in terms of spanning trees and forests. The case $G = K_n$ is more interesting. The evaluation $t(K_n; 1, y)$ is the inversion polynomial of trees, that is the generating function of rooted trees with n vertices counted by number of inversions (see for instance [2] and the references therein). It is well-known that the inversion polynomial at $y = -1$ is the number of alternating permutations, which is also the number of increasing trees (that is, trees without inversions). It follows from [1] that the evaluation $t(K_n; 2, -1)$ is the number of increasing forests on n vertices. These interpretations are the basis of a combinatorial proof of $t(K_{n+2}; 1, -1) = t(K_n; 2, -1)$ obtained by the authors of this paper. Although similar interpretations can be found for, say, a bipartite graph, we know of no combinatorial proof for that case.

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