FORBIDDEN SUBGRAPHS IN THE NORM GRAPH

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Abstract. We show that the norm graph with $n$ vertices about $\frac{1}{2} n^{2-1/t}$ edges, which contains no copy of the complete bipartite graph $K_{t,(t-1)!+1}$, does not contain a copy of $K_{t+1,(t-1)!-1}$.

1. Introduction

Let $H$ be a fixed graph. The Turán number of $H$, denoted $ex(n,H)$, is the maximum number of edges a graph with $n$ vertices can have, which contains no copy of $H$. The Erdős-Stone theorem from [7] gives an asymptotic formula for the Turán number of any non-bipartite graph, and this formula depends on the chromatic number of the graph $H$.

When $H$ is a complete bipartite graph, determining the Turán number is related to the “Zarankiewicz problem” (see [3], Chap. VI, Sect.2, and [9] for more details and references). In many cases even the question of determining the right order of magnitude for $ex(n,H)$ is not known.

Let $K_{t,s}$ denote the complete bipartite graph with $t$ vertices in one class and $s$ vertices in the other. Kővari, Sós and Turán [14] proved that for $s \geq t$

\[(1.1) \quad ex(n, K_{t,s}) \leq \frac{1}{2} (s-1)^{1/t} n^{2-1/t} + \frac{1}{2} (t-1) n.\]

The norm graph $\Gamma(t)$, which we will define the next section, has $n$ vertices and about $\frac{1}{2} n^{2-1/t}$ edges. In [1] (based on results from [13]) it was proven that the graph $\Gamma(t)$ contains no copy of $K_{t,(t-1)!+1}$, thus proving that for $s \geq (t-1)!+1$,

\[ex(n, K_{t,s}) > cn^{2-1/t}\]

for some constant $c$.

In [2], it was shown that $\Gamma(4)$ contains no copy of $K_{5,5}$, which improves on the probabilistic lower bound of Erdős and Spencer [6] for $ex(n, K_{5,5})$. In this article, we will generalise

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this result and prove that $\Gamma(t)$ contains no copy of $K_{t+1,(t-1)!-1}$. For $t \geq 5$, this does not improve the probabilistic lower bound of Erdős and Spencer,

$$ex(n, K_{t,s}) \geq cn^{2-(s+t-2)/(st-1)}.$$  

As far as we are aware, it is however the deterministic construction of a graph with $n$ vertices containing no $K_{t+1,(t-1)!-1}$ with the most edges.

2. The norm graph

Suppose that $q = p^h$, where $p$ is a prime, and denote by $\mathbb{F}_q$ the finite field with $q$ elements. We will use the following properties of finite fields. For any $a, b \in \mathbb{F}_q$, $(a+b)^p = a^p + b^p$, for any $i \in \mathbb{N}$. For all $a \in \mathbb{F}_q$, $a^q = a$ if and only if $a \in \mathbb{F}_q$. Finally $N(a) = a^{1+q+\cdots+q^{k-1}} \in \mathbb{F}_q$, for all $a \in \mathbb{F}_q^*$, since $N(a)^q = N(a)$.

Let $\mathbb{F}$ denote an arbitrary field. We denote by $\mathbb{P}_n(\mathbb{F})$ the projective space arising from the $(n+1)$-dimensional vector space over $\mathbb{F}$. Throughout dim will refer to projective dimension. A point of $\mathbb{P}_n(\mathbb{F})$ (which is a one-dimensional subspace of the vector space) will often be written as $\langle u \rangle$, where $u$ is a vector in the $(n+1)$-dimensional vector space over $\mathbb{F}$.

Let $\Gamma(t)$ be the graph with vertices $(a, \alpha) \in \mathbb{F}_{q^t-1} \times \mathbb{F}_q$, $\alpha \neq 0$, where $(a, \alpha)$ is joined to $(a', \alpha')$ if and only if $N(a + a') = \alpha \alpha'$. The graph $\Gamma(t)$ was constructed in [13], where it was shown to contain no copy of $K_{t,t!+1}$. In [1] Alon, Rónyai and Szabó proved that $\Gamma(t)$ contains no copy of $K_{t,(t-1)!}$. Our aim here is to show that it also contains no $K_{t+1,(t-1)!-1}$, generalizing the same result for $t = 5$ presented in [2].

Let

$$V = \{(1,a) \otimes (1,a^q) \otimes \cdots \otimes (1,a^{q^{t-2}}) \mid a \in \mathbb{F}_{q^t-1}\} \subset \mathbb{P}_{2^{t-1}}(\mathbb{F}_{q^t-1}).$$

The set $V$ is the affine part of an algebraic variety that is in turn a subvariety of the Segre variety

$$\Sigma = \mathbb{P}_1 \times \mathbb{P}_1 \times \cdots \times \mathbb{P}_1,$$

where $\mathbb{P}_1 = \mathbb{P}_1(\mathbb{F}_q)$.

The affine point $(1,a) \otimes (1,a^q) \otimes \cdots \otimes (1,a^{q^{t-2}})$ has coordinates indexed by the subsets of $T := \{0, 1, \ldots, t-1\}$, where the $S$-coordinate is

$$\left(\prod_{i \in S} a^i\right),$$

for any non-empty subset $S$ of $T$ and

$$\prod_{i \in S} a^i = 1$$

when $S = \emptyset$ (see [15]).
Let \( n = 2^t - 1 \).

We order the coordinates of \( \mathbb{P}_n(\mathbb{F}_{q^t-1}) \) so that if the \( i \)-th coordinate corresponds to the subset \( S \), then the \((n - i)\)-th coordinate corresponds to the subset \( T \setminus S \).

Embed the \( \mathbb{P}_n(\mathbb{F}_{q^t-1}) \) containing \( V \) as a hyperplane section of \( \mathbb{P}_{n+1}(\mathbb{F}_{q^t-1}) \) defined by the equation \( x_{n+1} = 0 \).

Let \( \beta \) be the symmetric bilinear form on the \((n + 2)\)-dimensional vector space over \( \mathbb{F}_{q^t-1} \) defined by

\[
\beta(u, v) = \sum_{i=0}^{n} u_i v_{n+1} - u_{n+1} v_{n+1}.
\]

Let \( \perp \) be defined in the usual way, so that given a subspace \( \Pi \) of \( \mathbb{P}_{n+1}(\mathbb{F}_{q^t-1}) \), \( \Pi \perp \) is the subspace of \( \mathbb{P}_{n+1}(\mathbb{F}_{q^t-1}) \) defined by \( \Pi \perp = \{ v \mid \beta(u, v) = 0, \text{ for all } u \in \Pi \} \).

We wish to define the same graph \( \Gamma(t) \), so that adjacency is given by the bilinear form. Let \( P_\infty = (0, 0, 0, \ldots, 1) \). Let \( \Gamma' \) be a graph with vertex set the set of points on the lines joining the points of \( V \) to \( P_\infty \) obtained using only scalars in \( \mathbb{F}_q \), distinct from \( P_\infty \) and not contained in the hyperplane \( x_{n+1} = 0 \). Join two vertices \( \langle u \rangle \) and \( \langle u' \rangle \) in \( \Gamma' \) if and only if \( \beta(u, u') = 0 \). It is a simple matter to verify that the graph \( \Gamma' \) is isomorphic to the graph \( \Gamma(t) \) since

\[
N(a + b) = \sum_{S \subseteq T} \prod_{i \in S, j \in T \setminus S} a^i b^j = \beta(u, v) + u_{n+1} v_{n+1},
\]

where

\[
u = (1, a) \otimes (1, a^q) \otimes \cdots \otimes (1, a^{q-2}),
\]

and

\[
v = (1, b) \otimes (1, b^q) \otimes \cdots \otimes (1, b^{q-2}).
\]

We shall refer to \( \Gamma' \) as \( \Gamma(t) \) from now on.

We recall some known properties of \( \Sigma \) and its subvariety

\[
\mathcal{V} = \{ (a, b) \otimes (a^q, b^q) \otimes \cdots (a^{q-2}, b^{q-2}) \mid (a, b) \in \mathbb{P}_1(\mathbb{F}_{q^t-1}) \}
\]

and prove a new one in Theorem 2.5.

Let \( \overline{\mathbb{F}}_q \) denote the algebraic closure of \( \mathbb{F}_q \) and consider \( \Sigma \) as the Segre variety over \( \overline{\mathbb{F}}_q \).

**Theorem 2.1.** \( \Sigma \) is a smooth irreducible variety.

**Theorem 2.2.** The dimension of \( \Sigma \) (as algebraic variety) is \( t - 1 \) and its degree is \( (t - 1)! \).

**Proof.** The (Segre) product \( X \times Y \) of two varieties \( X \) and \( Y \) of dimension \( d \) and \( e \) has dimension \( d + e \), see, for example [12], page 138. The Hilbert polynomial of \( X \times Y \) is the product of the Hilbert polynomials of \( X \) and \( Y \) (see [12, Chapter 18]).
polynomial \( h(m) \) of \( \mathbb{P}_1 \) is \( m + 1 \), hence the Hilbert polynomial of \( \Sigma = \mathbb{P}_1 \times \mathbb{P}_1 \times \cdots \times \mathbb{P}_1 \) is \( h_\Sigma(m) = (m + 1)^{t-1} \). Since the leading term of \( h_\Sigma \) is 1 and the dimension of \( \Sigma \) is \( t - 1 \), we have that the degree of \( \Sigma \) is \( (t - 1)! \).

**Theorem 2.3.** [15] Any \( t \) points of \( \mathcal{V} \) are in general position.

**Theorem 2.4.** [11] If \( t + 1 \) points span a \((t - 1)\)-dimensional projective space, then that space contains \( q + 1 \) points of \( \mathcal{V} \).

**Theorem 2.5.** If a subspace of codimension \( t \) contains a finite number of points of \( \Sigma \) then it contains at most \((t - 1)! - 2\) points of \( \Sigma \).

**Proof.** By Theorem 2.1, \( \Sigma \) is smooth, so it is regular at each of its points, i.e., if \( T_P\Sigma \) is the tangent space of \( \Sigma \) at a point \( P \in \Sigma \), then \( \dim T_P\Sigma = t - 1 \).

Let \( \Pi \) be a subspace of codimension \( t \) containing a finite number of points of \( \Sigma \). Let \( P \in \Pi \cap \Sigma \). Then \( \dim \langle T_P\Sigma, \Pi \rangle \leq n - 1 \). Therefore, there is a hyperplane \( H \) containing \( \langle T_P\Sigma, \Pi \rangle \).

Suppose that \( H \) contains another tangent space \( T_R\Sigma \), with \( R \in \Pi \cap \Sigma \). The algebraic variety \( H \cap \Sigma \) has dimension \( t - 2 \) (since \( \Sigma \) is irreducible) and it has two singular points, \( P \) and \( R \). Since \( \dim H \cap \Sigma = t - 2 \) as an algebraic variety, there must be a linear subspace \( \Pi_1 \) of codimension \( t - 2 \) in \( H \) containing \( \Pi \) and such that \( \Pi_1 \cap H \cap \Sigma \) consists of \( \deg(H \cap \Sigma) \leq (t - 1)! \) points of \( \Sigma \) counted with their multiplicity. Since \( \Pi_1 \) contains \( P \) and \( R \), which are singular points and so with multiplicity at least 2, we have that

\[
|\Pi \cap \Sigma| \leq |\Pi_1 \cap \Sigma| \leq (t - 1)! - 2.
\]

Suppose now that \( H \) does not contain any other tangent space \( T_R\Sigma \), with \( R \in \Pi \cap \Sigma \), \( R \neq P \). Then take \( R \in \Pi \cap \Sigma \) and consider a hyperplane \( H' \neq H \) containing \( \langle T_R\Sigma, \Pi \rangle \).

Then the tangent spaces of \( P \) and \( R \) with respect to \( H \cap H' \cap \Sigma \) are \( T_P\Sigma \cap H' \) and \( T_R\Sigma \cap H \), and they both have dimension \( t - 2 \) (as linear spaces).

If \( \dim H \cap H' \cap \Sigma = t - 3 \) as an algebraic variety, then \( P \) and \( R \) are two singular points of \( H \cap H' \cap \Sigma \) and we can find, as before, a linear subspace \( \Pi_1 \) of codimension \( t - 3 \) in \( H \cap H' \) such that it contains \( \Pi \) and intersects \( H \cap H' \cap \Sigma \) in \( \deg(H \cap H' \cap \Sigma) \leq (t - 1)! \) points, counted with their multiplicity. Since \( P \) and \( R \) have multiplicity at least 2, we have

\[
|\Pi \cap \Sigma| \leq |\Pi_1 \cap \Sigma| \leq (t - 1)! - 2.
\]

If \( \dim H \cap H' \cap \Sigma = t - 2 \) as an algebraic variety, then \( H \cap \Sigma \) is reducible. Hence, we have

\[
H \cap \Sigma = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_r,
\]

where \( \mathcal{V}_i \) is an irreducible variety of dimension \( t - 2 \), for all \( i = 1, \ldots, r \). So we have

\[
H \cap H' \cap \Sigma = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_s \cup \mathcal{W}_{s+1} \cup \mathcal{W}_{s+2} \cup \cdots \cup \mathcal{W}_r,
\]
where $\mathcal{W}_i$ is a hyperplane section of $\mathcal{V}_i$, for all $i = s + 1, \ldots, r$. We observe that also $H' \cap \Sigma$ has to be reducible and, since the decomposition in irreducible components is unique, we have
\[
H' \cap \Sigma = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_s \cup \mathcal{V}_{s+1}' \cup \mathcal{V}_{s+2}' \cup \cdots \cup \mathcal{V}_r',
\]
where $\mathcal{V}_i$ and $\mathcal{V}_j'$ are irreducible varieties of dimension $t - 2$.

We have, by hypothesis, that $T_P \Sigma \subset H$ and $P \in \Pi$. So either $P \in \mathcal{V}_i$ and it is singular for $\mathcal{V}_i$, for some $i \in \{1, 2, \ldots, r\}$, or it is not singular for $\mathcal{V}_i$, for any $\ell \in \{1, 2, \ldots, r\}$.

Suppose we are in the first case. We know that $\mathcal{V}_i \subset H'$, then $P$ is singular for an irreducible component of $H' \cap \Sigma$ and so $T_P \Sigma \subset H'$, contradicting our hypothesis, so $\mathcal{V}_i$ is not contained in $H'$ and $H' \cap \mathcal{V}_i = \mathcal{W}_i$. We have that $\dim T_P \Sigma \cap H' = t - 2$ (as linear subspace) and $\dim \mathcal{W}_i = t - 3$ (as algebraic variety), so $P$ is singular for $\mathcal{W}_i$.

Suppose now that $P$ is not singular for any $\mathcal{V}_i$, so the dimension of $T_P \mathcal{V}_i$, as a subspace, is $t - 2$. If $P \notin \mathcal{V}_j$, for any $i \neq j$, then
\[
T_P (H \cap \Sigma) = T_P (\mathcal{V}_i) = T_P (\Sigma),
\]
a contradiction since the dimension of $T_P (\Sigma)$ is $t - 1$. Hence $P \in \mathcal{V}_i \cap \mathcal{V}_j$, and so $P$ is contained in the intersection of two components of $H' \cap \Sigma$, so it is again a singular (or multiple) point. The same is true for the point $R$ such that $T_R \Sigma \subset H'$, so in
\[
\mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_s \cup \mathcal{W}_{s+1} \cup \mathcal{W}_{s+2} \cup \cdots \cup \mathcal{W}_r
\]
there are at least two multiple points and when we sum up all the degrees, we count at least two points twice, hence, by
\[
\sum_{i=1}^{s} \deg \mathcal{V}_i + \sum_{j=s+1}^{r} \deg \mathcal{W}_j \leq (t - 1)!,
\]
we get that the number of points in
\[
\Pi \cap (\mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_s \cup \mathcal{W}_{s+1} \cup \mathcal{W}_{s+2} \cup \cdots \cup \mathcal{W}_r),
\]
is at most $(t - 1)! - 2$. \qed

Remark One could wonder whether one could try with one more hyperplane $H''$ such that $T_Q \Sigma \subset H''$, $T_Q \Sigma \nsubseteq H$, $T_Q \Sigma \nsubseteq H'$ and $Q \in \Pi$. However, it can happen that $H \cap H' \cap H'' = H \cap H'$, so $\dim T_Q \Sigma \subset H \cap H' \cap H'' = t - 2$ (as a linear space) and $\dim H \cap H' \cap H'' \cap \Sigma = t - 2$, so $Q$ would not be a singular point of
\[
H \cap H' \cap H'' \cap \Sigma = H \cap H' \cap \Sigma.
\]
The locus of hyperplanes containing a tangent space to a variety $X$ of $\mathbb{P}^n$ is a variety $X^*$ of the dual space $(\mathbb{P}_n)^*$ (see, e.g., [12, Chapter 15]). Let $\Sigma^*$ be the dual variety of $\Sigma$. From [16], we know that $\Sigma^*$ is a hypersurface, hence, if $d$ is the degree of $\Sigma^*$, then the number of points of $\Sigma^*$ on a general line of $(\mathbb{P}_n)^*$ is $d$. Suppose that the line of $(\mathbb{P}_n)^*$ defined by $H \cap H'$ is general, hence if $|\Pi \cap \Sigma| > d$, then we could find a point $Q \in \Pi \cap \Sigma$ such that $T_Q \Sigma \subset H''$ and $H''$ is a hyperplane not containing $H \cap H'$. If $d > (t - 1)! - 2$ then we
would not be able to get a better bound than the bound in Theorem 2.5. The degree of $\Sigma^*$ is found in [10], where it is given by $N_{t-1}$, where $N_r$ is defined by the generating function

$$\sum_{r \geq 0} N_r \frac{z^r}{r!} = \frac{e^{-2z}}{(1-z)^2}.$$

Hence $d = \deg \Sigma^*$, is the evaluation of

$$\left( \frac{e^{-2z}}{(1-z)^2} \right)^{(t-1)}$$

at $z = 0$, where we denote by $f^{(n)}$ the $n$–th derivative of the function $f$.

Let $F = fg$, where $f$ and $g$ are two functions, then

$$F^{(n)} = \sum_{i=0}^{n} \binom{n}{i} f^{(i)} g^{(n-i)}.$$

Let

$$f = e^{-2z} \text{ and } g = (1-z)^{-2}.$$ 

It is easy to see that

$$f^{(i)} = (-2)^i f \text{ and } g^{(i)} = (i+1)!(1-z)^{-(i+2)}.$$ 

Since $f(0) = 1$, we have that $F^{(n)}$, evaluated at $z = 0$, is

$$\sum_{i=0}^{n} \binom{n}{i} (-2)^i (n+1-i)!.$$ 

When $n = t - 1$ and we have

$$d = N_{t-1} = \sum_{i=0}^{t-1} \binom{t-1}{i} (-2)^i (t-i)!.$$ 

Now

$$\sum_{i=0}^{t-1} \binom{t-1}{i} (-2)^i (t-i)! = (t-1)! \sum_{i=0}^{t-1} \frac{(-2)^i}{i!} (t-i).$$

Note that

$$\sum_{i=0}^{t-1} \frac{(-2)^i}{i!} (t-i) = 1$$

for $t = 5$ and

$$\sum_{i=0}^{t} \frac{(-2)^i}{i!} (t+1-i) - \sum_{i=0}^{t-1} \frac{(-2)^i}{i!} (t-i) = \sum_{i=0}^{t} \frac{(-2)^i}{i!}.$$
Since $\sum_{i=0}^{5} \frac{(-2)^i}{i!} = \frac{1}{15}$ and

$$\frac{(-2)^{n-1}}{(n-1)!} - \frac{(-2)^n}{n!} = \frac{2^{n-1}(n-2)}{n!} > 0$$

when $n \geq 3$ is odd,

$$\sum_{i=0}^{t} \frac{(-2)^i}{i!} > 0$$

for all $t \geq 4$ and so

$$\sum_{i=0}^{t-1} \frac{(-2)^i}{i!} (t - i)$$

is an increasing function. Thus, for $t \geq 5$,

$$\sum_{i=0}^{t-1} \frac{(-2)^i}{i!} (t - i) \geq 1,$$

and so

$$(t - 1)! \sum_{i=0}^{t-1} \frac{(-2)^i}{i!} (t - i) \geq (t - 1)!$$

and hence $d = N_{t-1} > (t - 1)! - 2$.

**Theorem 2.6.** For $q \geq (t - 1)! + 1$ the graph $\Gamma(t)$ contains no $K_{t+1,(t-1)!-1}$.

**Proof.** Let $X = \{x_1, x_2, \ldots, x_{t+1}\}$ be $t + 1$ distinct vertices of $\Gamma(t)$. The set of common neighbours of the elements of $X$ is $\Pi^\perp \cap \Gamma(t)$, where $\Pi$ is the subspace spanned by $X$. If any two elements of $X$ project from $P_\infty$ onto the same point of $V$, then $P_\infty \in \Pi$ and hence $\Pi^\perp \subset P_\infty^\perp$. Since $P_\infty^\perp$ is the hyperplane $x_{n+1} = 0$, $\Pi^\perp \cap \Gamma(t) = \emptyset$, and the elements of $X$ have no common neighbour.

Therefore, we assume now that all the points in $X$ project from $P_\infty$ onto distinct points of $V$. Then, by Theorem 2.3, $\dim \Pi \geq t - 1$.

If $\dim \Pi = t - 1$, then by Theorem 2.3, the projection of $\Pi$ onto $V$ contains at least $q$ points of $V$. Therefore, there are at least $q$ points $Y$ of $\Pi$ on the lines joining $P_\infty$ to the points of $V$. We wish to prove that the points of $Y$ are vertices of the graph $\Gamma(t)$. To do this, we have to show that the points of $Y$, which are of the form $\langle (v, \lambda) \rangle$, where $v \in V$ and $\lambda \in F_q$, are of the form $\langle (v, \lambda) \rangle$, where $v \in V$ and $\lambda \in F_q$. Assuming that the vertices in $X$ have at least two common neighbours, we can suppose that there is a common neighbour of the elements of $X$ of the form $\langle (u, \mu) \rangle$, where $u \in V$, $u \neq -v$ and $\mu \in F_q$, is a common neighbour of the elements of $X$. Then $\langle (u, \mu) \rangle$ is in $\Pi^\perp$ and since $Y \subset \Pi$,

$$N(u + v) = \lambda \mu.$$
Since \( N(u + v) \in \mathbb{F}_q \) and \( \mu \in \mathbb{F}_q \), we have that \( \lambda \in \mathbb{F}_q \) and so the points of \( Y \) are vertices of the graph \( \Gamma(t) \). Therefore, the vertices of \( X \) have at least \( q \) common neighbours. Since \( \Gamma \) contains no \( K_{t,(t-1)!+1} \), if \( q \geq (t - 1)! + 1 \), then this case cannot occur.

If \( \dim \Pi = t \) then \( \dim \Pi^\perp = n - t \). Let \( Y \) be the points of \( \Pi^\perp \) which project from \( P_\infty \) onto \( V \). Arguing as in the previous paragraph, the points \( Y \) are vertices of the graph \( \Gamma(t) \). Since the vertices of \( X \) have at most \( (t - 1)! + 1 \) common neighbours, there are a finite number of points in \( Y \) and so a finite number of points in the projection of \( \Pi^\perp \) onto \( V \). By Theorem 2.5, this projection contains at most \( (t - 1)! - 2 \) points of \( V \), so there are at most \( (t - 1)! - 2 \) points in \( Y \). Therefore, the vertices in \( X \) have at most \( (t - 1)! - 2 \) common neighbours. \( \square \)

References


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